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Research article

Qualitative study of linear and nonlinear relaxation equations with ψ-Riemann-Liouville fractional derivatives

Muath Awadalla[1,*], **Mohammed S. Abdo**[2], **Hanan A. Wahash**[3] **and Kinda Abuasbeh**[1]

[1] Department of Mathematics and Statistics, College of Science, King Faisal University, Hafuf, Al Ahsa 31982, Saudi Arabia

[2] Department of Mathematics, Hodeidah University, Al-Hudaydah, Yemen

[3] Department of Mathematics, Albaydha University, Albaydha, Yemen

* **Correspondence:** Email: mawadalla@kfu.edu.sa.

Abstract: In the present paper, we consider the linear and nonlinear relaxation equation involving ψ-Riemann-Liouville fractional derivatives. By the generalized Laplace transform approach, the guarantee of the existence of solutions for the linear version is shown by Ulam-Hyer's stability. Then by establishing the method of lower and upper solutions along with Banach contraction mapping, we investigate the existence and uniqueness of iterative solutions for the nonlinear version with the non-monotone term. A new condition on the nonlinear term is formulated to ensure the equivalence between the solution of the nonlinear problem and the corresponding fixed point. Moreover, we discuss the maximal and minimal solutions to the nonlinear problem at hand. Finally, we provide two examples to illustrate the obtained results.

Keywords: relaxation equations; fractional derivative; lower and upper solution method; fixed point theorem
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1. Introduction

As an extension of the current development and generalizations in the field of fractional calculus [1–3], the investigation of solution behaviors and qualitative properties of the solution in classical or fractional differential equations has become a matter of intense interest for researchers. This reflects the extent of its uses in several applied and engineering aspects. It draws amazing applications in nonlinear oscillations of seismic tremors, the detection of energy transport rate, and energy generation rate. The importance of fractional equations has been recognized in many physical phenomena, in addition to its importance in the mathematical modeling of diseases and viruses to limit and reduce their spread.

The relaxation differential equation gives as $u'(\varkappa) + u(\varkappa) = f(t)$; $u(0^+) = u_0$, whose solution is

$$u(\varkappa) = u_0 \exp(-\varkappa) + \int_0^{\varkappa} f(t-\tau)\exp(-\tau)d\tau.$$

Some recent contributions to the theory of FDEs can be seen in [4]. In [5], the authors studied the following problem

$$\begin{aligned} D_{0+}^{\kappa}u(\varkappa) &= f(\varkappa, u(\varkappa)), \quad \varkappa \in (0,1), \\ u(0) &= 0, \end{aligned}$$

where $0 < \kappa < 1$, $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $f(\varkappa, \cdot)$ is non-decreasing for $\varkappa \in [0,1]$, by lower and upper (LU) solution method. The existence and uniqueness of solutions of the FDE

$$D_{0+}^{\kappa}u(\varkappa) = f(\varkappa, u(\varkappa)), \ (0 < \kappa < 1;\ \varkappa > 0), \tag{1.1}$$

$$D_{0+}^{\kappa-1}u(0+) = u_0, \tag{1.2}$$

were obtained in [1, 2, 6], by using the fixed point theorem (FPT) of Banach.

In [7], the authors discussed the existence and uniqueness of solutions of the following FDE

$$\begin{aligned} D_{0+}^{\kappa}u(\varkappa) &= f(\varkappa, u(\varkappa)), \quad \varkappa \in (0,\varkappa], \\ \varkappa^{1-\kappa}u(\varkappa)\big|_{\varkappa=0} &= u_0, \end{aligned} \tag{1.3}$$

by using the LU solution method and its associated monotone iterative (MI) method. The problem (1.3) with non-monotone term has been studied by Bai et al. [8].

In [9], a new appraoch of the maximum principle was presented by using the completely monotonicity of the Mittag-Leffler (ML) function.

On the other hand, there are several definitions and generalizations of the fractional operators (FOs) that contributed a lot to the development of this field. The generalization of RL's FOs based on a local kernel containing a differentiable function was first introduced by Osler [3]. Next, Kilbas et al. [1] dealt with some of the properties of this operator. Then, the interesting properties for this operator have been discussed by Agarwal [10]. Recently, Jarad and Abdeljawad [11] achieved some properties in accordance with the generalized Laplace transform with respect to another function.

In this regard, most of the results similar to our current work are covered under the generalized FOs of Caputo [12] and Hilfer [13], for instance, see [14–19], whereas, very few considered results related to the dependence on generalized RL's definition. The authors in [20, 21], investigated the existence and uniqueness of positive solutions of the fractional Cauchy problem in the frame of generalized RL and Caputo, respectively.

For this end, as an additional contribution and enrichment to this active field, we consider the following linear and nonlinear relaxation equations with non-monotone term under ψ-RL fractional derivatives (ψ-RLFD):

$$D_{0+}^{\kappa;\psi}u(\varkappa) + \lambda D_{0+}^{\delta;\psi}u(\varkappa) = f(\varkappa), \quad \varkappa \in (0,h], \tag{1.4}$$

$$(\psi(\varkappa) - \psi(0))^{1-\kappa}\, u(\varkappa)\big|_{\varkappa=0} = u_0 \neq 0, \tag{1.5}$$

where $0 < h < +\infty$, $0 < \kappa, \delta < 1$, $\lambda \geq 0$, $f \in C([0,h],\mathbb{R})$ and

$$D_{0+}^{\kappa;\psi} u(\varkappa) + \lambda u(\varkappa) = f(\varkappa, u(\varkappa)), \quad \varkappa \in (0,h], \tag{1.6}$$
$$(\psi(\varkappa) - \psi(0))^{1-\kappa} u(\varkappa)\Big|_{\varkappa=0} = u_0 \neq 0, \tag{1.7}$$

where $f \in C([0,h] \times \mathbb{R}, \mathbb{R})$, $D_{0+}^{\kappa;\psi}$ and $D_{0+}^{\delta;\psi}$ are RL fractional derivatives of order κ and δ, respectively, with respect to another function $\psi \in C^1([0,h],\mathbb{R})$, which is increasing, and $\psi'(\varkappa) \neq 0$ for all $[0,h]$. The main contributions of this work stand out as follows:

i) With a new version of Laplace transform, we obtain Hyers-Ulam (HU) and generalized Hyers-Ulam (GHU) stabilities on the finite time interval to check whether the approximate solution is near the exact solution for a ψ-RL linear FDEs (1.4) and (1.5).

ii) We establish a condition to derive the existence and uniqueness of solutions for ψ-RL nonlinear FDEs (1.6) and (1.7), by using LU solution method along with the Banach contraction map (this generalizes the results in [7]).

iii) We formulate a new condition on the nonlinear term to ensure the equivalence between the solution of the proposed problem and the corresponding fixed point. Then in light of that, we discuss the maximal and minimal solutions for (1.6) and (1.7).

Remark 1.1.

(1) Our results remain valid if $\lambda = 0$ on problems (1.4)–(1.7), which reduce to

$$D_{0+}^{\kappa;\psi} u(\varkappa) = f(\varkappa), \quad \varkappa \in (0,h], \tag{1.8}$$
$$(\psi(\varkappa) - \psi(0))^{1-\kappa} u(\varkappa)\Big|_{\varkappa=0} = u_0 \neq 0. \tag{1.9}$$

and

$$D_{0+}^{\kappa;\psi} u(\varkappa) = f(\varkappa, u(\varkappa)), \quad \varkappa \in (0,h], \tag{1.10}$$
$$(\psi(\varkappa) - \psi(0))^{1-\kappa} u(\varkappa)\Big|_{\varkappa=0} = u_0 \neq 0, \tag{1.11}$$

(2) If $\psi(\varkappa) = \varkappa$, the problems (1.10) and (1.11) reduces to problem (1.3) considered in [7].
(3) The linear versions (1.4) and (1.5) generalizes that given in Theorem 5.1 by Jarad et al. [11].

Observe that in the preceding works, the nonlinear term needs to fulfill the monotone or other control conditions. Indeed, the nonlinear FDE with a non-monotone term can respond better to generic regulation, so it is vital to debilitate the control states of the nonlinear term.

This work is coordinated as follows. Section 2 provides some consepts of ψ-fractional calculus. Section 3 studies the stability results for the ψ-RL linear FDEs (1.4) and (1.5). In Section 4, we investigate of the the existence and uniqueness results for the ψ-RL nonlinear FDEs (1.6) and (1.7). Moreover, the existence of maximal and minimal solutions is also obtaind. At the end, we provide some examples in the last section.

2. Preliminaries

Given $0 \le a < b < +\infty$ and $s > 0$, and let $\psi_s(\varkappa, a) := (\psi(\varkappa) - \psi(a))^s$. Define a set

$$C_{s;\psi}[a,b] = \{u : u \in C(a,b],\ \psi_s(\varkappa, a)u(\varkappa) \in C[a,b]\}.$$

Clearly, $C_{s;\psi}[a,b]$ is a Banach space with the norm

$$\|u\|_{C_{s;\psi}} = \|\ \psi_s(\varkappa, a)u(\varkappa)\|_C = \max_{\varkappa\in[a,b]} \psi_s(\varkappa, a)|u(\varkappa)|.$$

Definition 2.1. *[1]* Let $\theta > 0$, and $f : [a,b] \to \mathbb{R}$ be an integrable function. Then the generalized RL fractional integral and derivative with respect to ψ is given by

$$I^{\kappa;\psi}_{a+} f(\varkappa) = \frac{1}{\Gamma(\kappa)} \int_a^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa, s) f(\zeta) d\zeta,$$

and

$$D^{\kappa;\psi}_{a+} f(\varkappa) = \Big[\frac{1}{\psi'(\varkappa)}\frac{d}{d\varkappa}\Big]^n I^{n-\kappa;\psi}_{a+} f(\varkappa),$$

respectively, where $n = [\kappa] + 1$, and $\psi : [a,b] \to \mathbb{R}$ is an increasing with $\psi'(\varkappa) \neq 0$, for all $t \in [a,b]$.

Lemma 2.1. *([11],Theorem 5.1) Let* $0 < \kappa < 1$, $\lambda \in \mathbb{R}$ *is a constant, and* $\phi \in L(0,h)$. *Then the linear version*

$$\begin{cases} D^{\kappa;\psi}_{a+} u(\varkappa) - \lambda u(\varkappa) = \phi(\varkappa),\ \varkappa > a, \\ I^{1-\kappa;\psi}_{a+} u(\varkappa)\big|_{\varkappa=a} = c \in \mathbb{R}, \end{cases} \tag{2.1}$$

has the following solution

$$\begin{aligned} u(\varkappa) &= c\ \psi_{\kappa-1}(\varkappa, a) E_{\kappa,\kappa}(\lambda\psi_\kappa(\varkappa, a)) \\ &\quad + \int_0^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa,\zeta) E_{\kappa,\kappa}(\lambda\psi_\kappa(\varkappa,\zeta))\phi(\zeta) d\zeta. \end{aligned}$$

Lemma 2.2. *[1] For* $0 < \kappa \le 1$, *the ML function* $E_{\kappa,\kappa}(-\lambda\,(\psi(\varkappa) - \psi(0))^\kappa)$ *satisfies*

$$0 \le E_{\kappa,\kappa}(-\lambda\,(\psi(\varkappa) - \psi(0))^\kappa) \le \frac{1}{\Gamma(\kappa)}, \quad \varkappa \in [0,\infty),\ \lambda \ge 0.$$

Lemma 2.3. *(see [11], Lemma 4.2) For* $Re(s) > |\lambda|^{\frac{1}{\kappa-\delta}}$, *we have*

$$\int_0^\infty e^{-s\left[\psi(\varkappa)-\psi(0)\right]} \left[\psi(\varkappa) - \psi(0)\right]^{\beta-1} E_{\alpha,\beta}(-\lambda\left[\psi(\varkappa) - \psi(0)\right]^\alpha) d\varkappa = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}.$$

Lemma 2.4. *[22] Assume that* U *is an ordered Banach space,* $u_0, v_0 \in U$, $u_0 \le v_0$, $D = [u_0, v_0]$, $Q : D \to U$ *is an increasing completely continuous map and* $u_0 \le Qu_0$, $v_0 \ge Qv_0$. *Then,* Q *has* u^* *and* v^* *are minimal and maximal fixed point, respectively. If we set*

$$u_n = Qu_{n-1}, \quad v_n = Qv_{n-1}, \quad n = 1, 2, \ldots,$$

then

$$\begin{gathered} u_0 \le u_1 \le u_2 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_2 \le v_1 \le v_0, \\ u_n \to u^*, \quad v_n \to v^*. \end{gathered}$$

Definition 2.2. We say that $v(\varkappa) \in C_{1-\kappa;\psi}[0,h]$ is a lower solution of (1.6) and (1.7), if it satisfies

$$D_{0+}^{\kappa;\psi} v(\varkappa) + \lambda v(\varkappa) \leq f(\varkappa, v(\varkappa)), \quad \varkappa \in (0,h), \tag{2.2}$$

$$\psi_{1-\kappa}(\varkappa, 0)\, v(\varkappa)\Big|_{\varkappa=0} \leq u_0. \tag{2.3}$$

Definition 2.3. We say that $w(\varkappa) \in C_{1-\kappa;\psi}[0,h]$ is an upper solution of (1.6) and (1.7), if it satisfies

$$D_{0+}^{\kappa;\psi} w(\varkappa) + \lambda w(\varkappa) \geq f(\varkappa, w(\varkappa)), \quad \varkappa \in (0,h), \tag{2.4}$$

$$\psi_{1-\kappa}(\varkappa, 0)\, w(\varkappa)\Big|_{\varkappa=0} \geq u_0. \tag{2.5}$$

Theorem 2.4. *[11] Let $0 < \kappa < 1$. Then, the generalized Laplace transform of ψ-RL fractional derivative is given by*

$$\mathscr{L}_\psi \left[D_{0+}^{\kappa;\psi} u(\varkappa) \right] = s^\kappa \mathscr{L}_\psi \left[u(\varkappa) \right] - I_{0+}^{1-\kappa;\psi} u(\varkappa)\Big|_{\varkappa=0},$$

where

$$\mathscr{L}_\psi\{f(t)\} = \int_a^\infty e^{-s[\psi(t)-\psi(a)]} \psi'(t) f(t) dt.$$

3. Stability analysis for a linear problem

Here, we discuss the HU and GHU stability of ψ-RL linear problems (1.4) and (1.5), by using the ψ-Laplace transform. Before proceeding to prove the results, we will provide the following auxiliary lemmas:

Lemma 3.1. *Let $0 < \kappa < 1$, and $u \in C_{1-\kappa;\psi}[0,h]$. If*

$$\lim_{\varkappa \to 0^+} \left(\psi(\varkappa) - \psi(0)\right)^{1-\kappa} u(\varkappa) = u_0, \; u_0 \in \mathbb{R},$$

then

$$I_{0+}^{1-\kappa;\psi} u(0^+) := \lim_{\varkappa \to 0^+} I_{0+}^{1-\kappa;\psi} u(\varkappa) = u_0 \Gamma(\kappa).$$

Proof. The proof is obtained by the same technique presented in Lemma 3.2, see [1], taking into account the properties of the ψ function.

Lemma 3.2. *Let $0 < \kappa, \delta < 1$, $\lambda \geq 0$ is a constant, and $f : [0,h] \to \mathbb{R}$ is a continuous function. Then the linear problems (1.4) and (1.5) has the following solution*

$$\begin{aligned} u(\varkappa) &= \left[\Gamma(\kappa) + \lambda \Gamma(\delta)\right] u_0 \psi_{\kappa-1}(\varkappa, 0) E_{\kappa-\delta,\kappa}(-\lambda \psi_{\kappa-\delta}(\varkappa, 0)) \\ &\quad + \int_0^\varkappa \psi'(\zeta) \psi_{\kappa-1}(\varkappa, \zeta) E_{\kappa-\delta,\kappa}(-\lambda \psi_{\kappa-\delta}(\varkappa, \zeta)) f(\zeta) d\zeta. \end{aligned} \tag{3.1}$$

Proof. Taking the generalized Laplace transform of (1.4) as

$$\mathscr{L}_\psi \left[D_{0+}^{\kappa;\psi} u(\varkappa) \right] + \lambda \mathscr{L}_\psi \left[D_{0+}^{\delta;\psi} u(\varkappa) \right] = \mathscr{L}_\psi \left[f(\varkappa) \right].$$

Via Theorem 2.4, we have

$$s^\kappa \mathscr{L}_\psi \left[u(\varkappa) \right] - I_{0+}^{1-\kappa;\psi} u(\varkappa)\Big|_{\varkappa=0} + \lambda \left[s^\delta \mathscr{L}_\psi \left[u(\varkappa) \right] - I_{0+}^{1-\delta;\psi} u(\varkappa)\Big|_{\varkappa=0} \right]$$

$$= \mathscr{L}_\psi \left[f(\varkappa) \right].$$

From (1.5) and Lemma 3.1, we have $I_{0+}^{1-\kappa;\psi} u(\varkappa)\Big|_{\varkappa=0} = \Gamma(\kappa) u_0$. It follows that

$$s^\kappa \mathscr{L}_\psi \left[u(\varkappa) \right] - \Gamma(\kappa) u_0 + \lambda \left[s^\delta \mathscr{L}_\psi \left[u(\varkappa) \right] - \Gamma(\delta) u_0 \right] = \mathscr{L}_\psi \left[f(\varkappa) \right].$$

One has,

$$\mathscr{L}_\psi \left[u(\varkappa) \right] = \frac{s^{-\delta}}{s^{\kappa-\delta} + \lambda} \Big[\left[\Gamma(\kappa) + \lambda \Gamma(\delta) \right] u_0 + \mathscr{L}_\psi \left[f(\varkappa) \right] \Big]. \tag{3.2}$$

Taking $\mathscr{L}_\psi^{-1}$ to both sides of (3.2), it follow from Lemma 2.3 that

$$\begin{aligned} u(\varkappa) &= \left[\Gamma(\kappa) + \lambda \Gamma(\delta) \right] u_0 \psi_{\kappa-1}(\varkappa, 0) E_{\kappa-\delta,\kappa}(-\lambda \psi_{\kappa-\delta}(\varkappa, 0)) \\ &\quad + \int_0^\varkappa \psi'(\zeta) \psi_{\kappa-1}(\varkappa, \zeta) E_{\kappa-\delta,\kappa}(-\lambda \psi_{\kappa-\delta}(\varkappa, \zeta)) f(\zeta) d\zeta, \end{aligned}$$

which is (3.1).

Theorem 3.1. *Let* $0 < \kappa, \delta < 1$, $\lambda \geq 0$, *and* $f : [0, h] \to \mathbb{R}$ *is a continuous function. If* $u \in C_{1-\kappa;\psi}[0, h]$ *satisfies the inequality*

$$\left| D_{0+}^{\kappa;\psi} u(\varkappa) + \lambda D_{0+}^{\delta;\psi} u(\varkappa) - f(\varkappa) \right| \leq \epsilon, \tag{3.3}$$

for each $\varkappa \in (0, h]$ *and* $\epsilon > 0$, *then there exists a solution* $u_a \in C_{1-\kappa;\psi}[0, h]$ *of (1.4) such that*

$$|u(\varkappa) - u_a(\varkappa)| \leq \frac{\psi_\kappa(h, 0)}{\Gamma(\kappa + 1)} \epsilon.$$

Proof. Let

$$\Upsilon(\varkappa) := D_{0+}^{\kappa;\psi} u(\varkappa) + \lambda D_{0+}^{\delta;\psi} u(\varkappa) - f(\varkappa), \quad \varkappa \in (0, h]. \tag{3.4}$$

As per (3.3), $|\Upsilon(\varkappa)| \leq \epsilon$. Taking the ψ-Laplace transform of (3.4) via Theorem 2.4, we have

$$\begin{aligned} \mathscr{L}_\psi \left[\Upsilon(\varkappa) \right] &= \mathscr{L}_\psi \left[D_{0+}^{\kappa;\psi} u(\varkappa) \right] + \lambda \mathscr{L}_\psi \left[D_{0+}^{\delta;\psi} u(\varkappa) \right] - \mathscr{L}_\psi \left[f(\varkappa) \right] \\ &= s^\kappa \mathscr{L}_\psi \left[u(\varkappa) \right] - I_{0+}^{1-\kappa;\psi} u(\varkappa)\Big|_{\varkappa=0} \\ &\quad + \lambda \left[s^\delta \mathscr{L}_\psi \left[u(\varkappa) \right] - I_{0+}^{1-\delta;\psi} u(\varkappa)\Big|_{\varkappa=0} \right] - \mathscr{L}_\psi \left[f(\varkappa) \right]. \end{aligned}$$

From (1.5) and Lemma 3.1, $I_{0+}^{1-\kappa;\psi} u(\varkappa)\Big|_{\varkappa=0} = \Gamma(\kappa) u_0$. It follows that

$$\mathscr{L}_\psi \left[\Upsilon(\varkappa) \right] = s^\kappa \mathscr{L}_\psi \left[u(\varkappa) \right] - \Gamma(\kappa) u_0 + \lambda s^\delta \mathscr{L}_\psi \left[u(\varkappa) \right] - \lambda \Gamma(\delta) u_0 - \mathscr{L}_\psi \left[f(\varkappa) \right].$$

One has,

$$\mathscr{L}_\psi \left[u(\varkappa) \right] = \frac{s^{-\delta}}{s^{\kappa-\delta} + \lambda} \Big[\left[\Gamma(\kappa) + \lambda \Gamma(\delta) \right] u_0 + \mathscr{L}_\psi \left[\Upsilon(\varkappa) \right] + \mathscr{L}_\psi \left[f(\varkappa) \right] \Big]. \tag{3.5}$$

Set

$$\begin{aligned} u_a(\varkappa) &= \left[\Gamma(\kappa) + \lambda \Gamma(\delta) \right] u_0 \psi_{\kappa-1}(\varkappa, 0) E_{\kappa-\delta,\kappa}(-\lambda \psi_{\kappa-\delta}(\varkappa, 0)) \\ &\quad + \int_0^\varkappa \psi'(\zeta) \psi_{\kappa-1}(\varkappa, \zeta) E_{\kappa-\delta,\kappa}(-\lambda \psi_{\kappa-\delta}(\varkappa, \zeta)) f(\zeta) d\zeta. \end{aligned} \tag{3.6}$$

Taking the Laplace transform of (3.6), It follow from Lemma 2.3 that

$$\mathscr{L}_{\psi}\left[u_a(\varkappa)\right] = \frac{s^{-\delta}}{s^{\kappa-\delta}+\lambda}\left[\left[\Gamma(\kappa)+\lambda\Gamma(\delta)\right]u_0+\mathscr{L}_{\psi}\left[f(\varkappa)\right]\right]. \tag{3.7}$$

Note that

$$\begin{aligned}
&\mathscr{L}_{\psi}\left[D_{0+}^{\kappa;\psi}u_a(\varkappa)\right]+\lambda\mathscr{L}_{\psi}\left[D_{0+}^{\delta;\psi}u_a(\varkappa)\right]\\
&=\mathscr{L}_{\psi}\left[\Upsilon(\varkappa)\right]+\mathscr{L}_{\psi}\left[f(\varkappa)\right]\\
&=s^{\kappa}\mathscr{L}_{\psi}\left[u_a(\varkappa)\right]-\Gamma(\kappa)u_0+\lambda s^{\delta}\mathscr{L}_{\psi}\left[u_a(\varkappa)\right]-\lambda\Gamma(\delta)u_0\\
&=\left(s^{\kappa}+\lambda s^{\delta}\right)\mathscr{L}_{\psi}\left[u_a(\varkappa)\right]-\left(\Gamma(\kappa)+\lambda\Gamma(\delta)\right)u_0.
\end{aligned} \tag{3.8}$$

Substituting (3.7) into (3.8), we get

$$\mathscr{L}_{\psi}\left[D_{0+}^{\kappa;\psi}u_a(\varkappa)\right]+\lambda\mathscr{L}_{\psi}\left[D_{0+}^{\delta;\psi}u_a(\varkappa)\right]=\mathscr{L}_{\psi}\left[f(\varkappa)\right],$$

which implies that $u_a(\varkappa)$ is a solution of (1.4) and (1.5) due to $\mathscr{L}_{\psi}$ is one-to-one. It follow from (3.5) and (3.7) that

$$\mathscr{L}_{\psi}\left[u(\varkappa)-u_a(\varkappa)\right](s)=\frac{s^{-\delta}}{s^{\kappa-\delta}+\lambda}\mathscr{L}_{\psi}\left[\Upsilon(\varkappa)\right],$$

which implies

$$\begin{aligned}
u(\varkappa)-u_a(\varkappa) &= \mathscr{L}_{\psi}\left\{\psi_{\kappa-1}(\varkappa,0)E_{\kappa-\delta,\kappa}(-\lambda\psi_{\kappa-\delta}(\varkappa,0))\right\}\mathscr{L}_{\psi}\left[\Upsilon(\varkappa)\right]\\
&= \left\{\psi_{\kappa-1}(\varkappa,0)E_{\kappa-\delta,\kappa}(-\lambda\psi_{\kappa-\delta}(\varkappa,0))\right\}*_{\psi}\Upsilon(\varkappa).
\end{aligned}$$

Thus, from (Definition 2.6, [21]) and Lemma 2.2, we obtain

$$\begin{aligned}
|u(\varkappa)-u_a(\varkappa)| &= \left|\left\{\psi_{\kappa-1}(\varkappa,0)E_{\kappa-\delta,\kappa}(-\lambda\psi_{\kappa-\delta}(\varkappa,0))\right\}*_{\psi}\Upsilon(\varkappa)\right|\\
&\le \int_0^{\varkappa}\psi'(\zeta)\psi_{\kappa-1}(\varkappa,\zeta)\left|E_{\kappa-\delta,\kappa}(-\lambda\psi_{\kappa-\delta}(\varkappa,\zeta))\right||\Upsilon(\zeta)|\,d\zeta\\
&\le \epsilon\int_0^{\varkappa}\psi'(\zeta)\psi_{\kappa-1}(\varkappa,\zeta)\left|E_{\kappa-\delta,\kappa}(-\lambda\psi_{\kappa-\delta}(\varkappa,\zeta))\right|d\zeta\\
&\le \frac{\epsilon}{\Gamma(\kappa)}\int_0^{\varkappa}\psi'(\tau)\psi_{\kappa-1}(\varkappa,\zeta)d\zeta\\
&\le \frac{\psi_{\kappa}(h,0)}{\Gamma(\kappa+1)}\epsilon.
\end{aligned}$$

Remark 3.1. If $h<\infty$, then (1.4) is HU stable with the constant $K:=\frac{\psi_{\kappa}(h,0)}{\Gamma(\kappa+1)}$.

Corollary 3.1. *On Theorem 3.1, let $\varphi:[0,\infty)\to[0,\infty)$ is continuous function. If we set $\varphi(\epsilon)=\frac{\psi_{\kappa}(h,0)}{\Gamma(\kappa+1)}\epsilon$, which satisfies $\varphi(0)=0$, then (1.4) is GHU stable.*

4. Existence results for a nonlinear problem

In this section, we prove the existence and uniqueness results for ψ-RL nonlinear FDEs (1.6) and (1.7), by using the LU solution method and the Banach contraction mapping. Moreover, we discuss the maximal and minimal solutions for the problem at hand. The following hypotheses will be used in our forthcoming analysis:

(A1) There exist constants $A, B \geq 0$ and $0 < s_1 \leq 1 < s_2 < 1/(1-\kappa)$ such that for $\varkappa \in [0, h]$,

$$|f(\varkappa, u) - f(\varkappa, v)| \leq A|u - v|^{s_1} + B|u - v|^{s_2}, \quad u, v \in \mathbb{R}. \tag{4.1}$$

(A2) $f : [0, h] \times \mathbb{R} \to \mathbb{R}$ satisfies

$$f(\varkappa, u) - f(\varkappa, v) + \lambda(u - v) \geq 0, \quad for\ \hat{u} \leq v \leq u \leq \tilde{u},$$

where $\lambda \geq 0$ is a constant and $\hat{u}, \tilde{u}$ are lower and upper solutions of problems (1.6) and (1.7) respectively.

(A3) There exist constant $\aleph > 0$ such that

$$|f(\varkappa, u) - f(\varkappa, v)| \leq \aleph|u - v|, \quad \varkappa \in [0, h], u, v \in \mathbb{R}.$$

Remark 4.1. Suppose that $f(\varkappa, u) = a(\varkappa)g(u)$ with g is a Hölder continuous and $a(\varkappa)$ is bounded, then (4.1) holds.

Theorem 4.1. *Suppose (A1) holds. The function u solves problems* (1.6) *and* (1.7) *iff it is a fixed-point of the operator* $Q : C_{1-\kappa;\psi}[0, h] \to C_{1-\kappa;\psi}[0, h]$ *defined by*

$$\begin{aligned}(Qu)(\varkappa) = &\Gamma(\kappa)u_0\psi_{\kappa-1}(\varkappa, 0)E_{\kappa,\kappa}(-\lambda\psi_\kappa(\varkappa, 0)) \\ &+ \int_0^\varkappa \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)E_{\kappa,\kappa}(-\lambda(\psi_\kappa(\varkappa, \zeta))f(\zeta, u(\zeta))d\zeta.\end{aligned} \tag{4.2}$$

Proof. At first, we show that the operator Q is well defined. Indeed, for every $u \in C_{1-\kappa;\psi}[0, h]$ and $\varkappa > 0$, the integral

$$\int_0^\varkappa \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)E_{\kappa,\kappa}(-\lambda(\psi_\kappa(\varkappa, \zeta))f(\zeta, u(\zeta))d\zeta,$$

belongs to $C_{1-\kappa;\psi}[0, h]$, due to

$$\psi_{1-\kappa}(\varkappa, 0)f(\zeta, u(\zeta)) \in C[0, h], \text{ and } \psi_{1-\kappa}(\varkappa, 0)u(\zeta) \in C[0, h],$$

bearing in mind that

$$\Phi(\varkappa) := \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)E_{\kappa,\kappa}(-\lambda(\psi_\kappa(\varkappa, \zeta)) = \sum_{m=0}^{\infty} \psi'(\zeta)\frac{(-\lambda)^m \ \psi_{(2\kappa-1)(m+1)}(\varkappa, \zeta)}{\Gamma(\kappa(m+1))}$$

is continuous on $[0, h]$.

By the condition (4.1), we have

$$|f(\varkappa, u)| \leq A|u|^{s_1} + B|u|^{s_2} + C, \tag{4.3}$$

where $C = \max_{\varkappa\in[0,h]} f(\varkappa, 0)$.

By Lemma 2.2, for $u(\varkappa) \in C_{1-\kappa;\psi}[0, h]$, we have

$$\begin{aligned}
&\left|\psi_{1-\kappa}(\varkappa, 0)\int_0^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)E_{\kappa,\kappa}(-\lambda(\psi_\kappa(\varkappa, \zeta))f(\zeta, u(\zeta))d\zeta\right| \\
&\leq \psi_{1-\kappa}(\varkappa, 0)\int_0^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)E_{\kappa,\kappa}(-\lambda(\psi_\kappa(\varkappa, \zeta))\,|f(\zeta, u(\zeta))|\,d\zeta \\
&\leq \psi_{1-\kappa}(\varkappa, 0)\int_0^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)E_{\kappa,\kappa}(-\lambda(\psi_\kappa(\varkappa, \zeta))(A|u|^{s_1} + B|u|^{s_2} + C)d\zeta \\
&\leq \psi_{1-\kappa}(\varkappa, 0)\int_0^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)E_{\kappa,\kappa}(-\lambda(\psi_\kappa(\varkappa, \zeta))\Big\{A\psi_{(\kappa-1)s_1}(\zeta, 0)[\psi_{1-\kappa}(\zeta, 0)|u(\zeta)|]^{s_1} \\
&\quad + B\psi_{(\kappa-1)s_2}(\zeta, 0)[\psi_{1-\kappa}(\zeta, 0)|u(\zeta)|]^{s_2} + C\Big\}d\zeta \\
&\leq \frac{A\left(\|u\|_{C_{1-\kappa;\psi}}\right)^{s_1}\ \psi_{1-\kappa}(\varkappa, 0)}{\Gamma(\kappa)}\int_0^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)\psi_{(\kappa-1)s_1}(\zeta, 0)d\zeta \\
&\quad + \frac{B\left(\|u\|_{C_{1-\kappa;\psi}}\right)^{s_2}\ \psi_{1-\kappa}(\varkappa, 0)}{\Gamma(\kappa)}\int_0^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)\psi_{(\kappa-1)s_2}(\zeta, 0)d\zeta + \frac{C}{\Gamma(\kappa+1)}\psi_1(\varkappa, 0) \\
&\leq A\left(\|u\|_{C_{1-\kappa;\psi}}\right)^{s_1}\ \frac{\Gamma((\kappa-1)s_1+1)}{\Gamma((\kappa-1)s_1+\kappa+1)}\psi_{(\kappa-1)s_1+\kappa+1-\kappa}(\varkappa, 0) \\
&\quad + B\left(\|u\|_{C_{1-\kappa;\psi}}\right)^{s_2}\ \frac{\Gamma((\kappa-1)s_2+1)}{\Gamma((\kappa-1)s_2+\kappa+1)}\psi_{(\kappa-1)s_2+\kappa+1-\kappa}(\varkappa, 0) + \frac{C}{\Gamma(\kappa+1)}\psi_1(\varkappa, 0) \\
&\leq \frac{\Gamma[(\kappa-1)s_1+1]A\psi_{(\kappa-1)s_1+1}(h, 0)}{\Gamma[(\kappa-1)s_1+\kappa+1]}\left(\|u\|_{C_{1-\kappa;\psi}}\right)^{s_1} \\
&\quad + \frac{\Gamma[(\kappa-1)s_2+1]B\psi_{(\kappa-1)s_2+1}(h, 0)}{\Gamma[(\kappa-1)s_2+\kappa+1]}\left(\|u\|_{C_{1-\kappa;\psi}}\right)^{s_2} + \frac{C}{\Gamma(\kappa+1)}\psi_1(h, 0).
\end{aligned}$$

Thus, the integral exists and belongs to $C_{1-\kappa;\psi}[0, h]$.

The previous inequality and the hypothesis $0 < s_1 \leq 1 < s_2 < 1/(1-\kappa)$ imply that

$$\lim_{\varkappa\to 0+}\psi_{1-\kappa}(\varkappa, 0)\int_0^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)E_{\kappa,\kappa}(-\lambda(\psi_\kappa(\varkappa, \zeta))f(\zeta, u(\zeta))d\zeta = 0.$$

Since $\lim_{\varkappa\to 0+} E_{\kappa,\kappa}(-\lambda\psi_\kappa(\varkappa, 0)) = E_{\kappa,\kappa}(0) = 1/\Gamma(\kappa)$ it follows that

$$\lim_{\varkappa\to 0+}\psi_{1-\kappa}(\varkappa, 0)(Qu)(\varkappa) = u_0.$$

.

The above arguments concerted along with Lemma 2.1 yields that the fixed-point of Q solves (1.6) and (1.7). And the vice versa. The proof is complete.

Next, we consider the compactness of $C_{s;\psi}[0, h]$. Let $F \subset C_{s;\psi}[0, h]$ and $X = \{g(\varkappa) = \psi_s(\varkappa, 0)h(\varkappa) \mid h(\varkappa) \in F\}$, then $X \subset C[0, h]$. It is obvious that F is a bounded set of $C_{s;\psi}[0, h]$ iff X is a bounded set of $C[0, h]$.

Thus, to prove that $F \subset C_{s;\psi}[0, h]$ is a compact set, it is sufficient to show that $X \subset C[0, h]$ is a bounded and equicontinuous set.

Theorem 4.2. *Let $f : [0, h] \times \mathbb{R} \to \mathbb{R}$ is a continuous and (A1) holds. Then Q is a completely continuous.*

Proof. Given $u_n \to u \in C_{1-\kappa;\psi}[0, h]$, with the definition of Q and condition (A1), we get

$$\begin{aligned}
&\|Qu_n - Qu\|_{C_{1-\kappa;\psi}} \\
&= \|\psi_{1-\kappa}(\varkappa, 0)(Qu_n - Qu)\|_\infty \\
&= \max_{0\le\varkappa\le h} \left|\psi_{1-\kappa}(\varkappa, 0) \int_0^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)E_{\kappa,\kappa}(-\lambda(\psi_\kappa(\varkappa, \zeta))[f(\zeta, u_n) - f(\zeta, u)]d\zeta\right| \\
&\le \frac{1}{\Gamma(\kappa)} \max_{0\le\varkappa\le h} \psi_{1-\kappa}(\varkappa, 0) \int_0^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)[A|u_n - u|^{s_1} + B|u_n - u|^{s_2}]d\zeta \\
&\le \frac{1}{\Gamma(\kappa)}\Big[A \max_{0\le\varkappa\le h} \psi_{1-\kappa}(\varkappa, 0) \int_0^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)\psi_{-s_1(1-\kappa)}(\zeta, 0)\, \psi_{s_1(1-\kappa)}(\zeta, 0)|u_n - u|^{s_1}d\zeta \\
&\quad + B \max_{0\le\varkappa\le h} \psi_{1-\kappa}(\varkappa, 0) \int_0^{\varkappa} \int_0^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)\psi_{-s_2(1-\kappa)}(\zeta, 0)\, \psi_{s_2(1-\kappa)}(\zeta, 0)|u_n - u|^{s_2}d\zeta\Big] \\
&\le \frac{1}{\Gamma(\kappa)}\Big[A\left(\|u_n - u\|_{C_{1-\kappa;\psi}}\right)^{s_1} \max_{0\le\varkappa\le h} \psi_{1-\kappa}(\varkappa, 0) \int_0^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)\psi_{-s_1(1-\kappa)}(\zeta, 0)d\zeta \\
&\quad + B\left(\|u_n - u\|_{C_{1-\kappa;\psi}}\right)^{s_2} \max_{0\le\varkappa\le h} \psi_{1-\kappa}(\varkappa, 0) \int_0^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta)\psi_{s_2(1-\kappa)}(\zeta, 0)d\zeta\Big] \\
&\le \frac{A\left(\|u_n - u\|_{C_{1-\kappa;\psi}}\right)^{s_1} \Gamma[1 - s_1(1-\kappa)]}{\Gamma[1 - s_1(1-\kappa) + \kappa]}\psi_{1-s_1(1-\kappa)}(h, 0) \\
&\quad + \frac{B\left(\|u_n - u\|_{C_{1-\kappa;\psi}}\right)^{s_1} \Gamma[1 - s_2(1-\kappa)]}{\Gamma[1 - s_2(1-\kappa) + \kappa]}\, \psi_{1-s_2(1-\kappa)}(h, 0) \\
&\to 0, \quad (n \to \infty).
\end{aligned}$$

Thus, Q is continuous.

Assume that $F \subset C_{1-\kappa;\psi}[0, h]$ is a bounded set. Theorem 4.1 shows that $Q(F) \subset C_{1-\kappa;\psi}[0, h]$ is bounded.

Finally, we show the equicontinuity of $Q(F)$. Given $\epsilon > 0$, for every $u \in F$ and $\varkappa_1, \varkappa_2 \in [0, h], \varkappa_1 \le \varkappa_2$,

$$\begin{aligned}
&\left|[\psi_{1-\kappa}(\varkappa, 0)(Qu)(\varkappa)]_{\varkappa=\varkappa_2} - [\psi_{1-\kappa}(\varkappa, 0)(Qu)(\varkappa)]_{\varkappa=\varkappa_1}\right| \\
&\le [\Gamma(\kappa)u_0E_{\kappa,\kappa}(-\lambda\psi_\kappa(\varkappa, 0))]_{\varkappa_1}^{\varkappa_2} + \Big[\psi_{1-\kappa}(\varkappa, 0) \int_0^{\varkappa} \psi'(\zeta)\psi_{\kappa-1}(\varkappa, \zeta) \\
&\times E_{\kappa,\kappa}(-\lambda\psi_\kappa(\varkappa, \zeta))f(\zeta, u(\zeta))d\zeta\Big]_{\varkappa_1}^{\varkappa_2} \\
&\le [\Gamma(\kappa)u_0E_{\kappa,\kappa}(-\lambda\psi_\kappa(\varkappa, 0))]_{\varkappa_1}^{\varkappa_2} \\
&+ \frac{\psi_{1-\kappa}(\varkappa_2, 0)}{\Gamma(\kappa)} \int_{\varkappa_1}^{\varkappa_2} \psi'(\zeta)\psi_{\kappa-1}(\varkappa_2, \zeta)|f(\zeta, u(\zeta))|d\zeta \\
&+ \frac{\psi_{1-\kappa}(\varkappa_2, 0) - \psi_{1-\kappa}(\varkappa_1, 0)}{\Gamma(\kappa)} \int_0^{\varkappa_1} \psi'(\zeta)\left[\psi_{\kappa-1}(\varkappa_2, \zeta) - \psi_{\kappa-1}(\varkappa_1, \zeta)\right]|f(\zeta, u(\zeta))|\, d\zeta \\
&:= [\Gamma(\kappa)u_0E_{\kappa,\kappa}(-\lambda\psi_\kappa(\varkappa, 0))]_{\varkappa_1}^{\varkappa_2} + I_1 + I_2.
\end{aligned}$$

As $E_{\kappa,\kappa}(-\lambda\psi_\kappa(\varkappa,0))$ is uniformly continuous on $[0,h]$. Thus

$$[\Gamma(\kappa)u_0E_{\kappa,\kappa}(-\lambda\psi_\kappa(\varkappa,0))]_{\varkappa_1}^{\varkappa_2} \to 0; \text{ as } \varkappa_2 \to \varkappa_1,$$

$$\begin{aligned}
I_1 \quad : \quad &= \frac{\psi_{1-\kappa}(\varkappa_2,0)}{\Gamma(\kappa)}\int_{\varkappa_1}^{\varkappa_2}\psi'(\zeta)\psi_{\kappa-1}(\varkappa_2,\zeta)|f(\zeta,u(\zeta))|d\zeta \\
&\leq \frac{\psi_{1-\kappa}(\varkappa_2,0)}{\Gamma(\kappa)}\int_{\varkappa_1}^{\varkappa_2}\psi'(\zeta)\psi_{\kappa-1}(\varkappa_2,\zeta)(A|u|^{s_1}+B|u|^{s_2}+C)d\zeta \\
&\leq \frac{p_1A\left(\|u\|_{C_{1-\kappa;\psi}}\right)^{s_1}}{\Gamma(p_1+\kappa)}\psi_{1-\kappa}(\varkappa_2,0)\psi_{p_1}(\varkappa_2,\varkappa_1) \\
&+ \frac{p_2B\left(\|u\|_{C_{1-\kappa;\psi}}\right)^{s_2}}{\Gamma(p_2+\kappa)}\psi_{1-\kappa}(\varkappa_2,0)\psi_{p_2}(\varkappa_2,\varkappa_1) \\
&+ \frac{C}{\Gamma(\kappa+1)}\psi_1(\varkappa_2,\varkappa_1)\psi_{1-\kappa}(\varkappa_2,0) \\
\to \quad & 0, \text{ as } \varkappa_2 \to \varkappa_1,
\end{aligned}$$

where $p_1 = (\kappa-1)s_1+1$ and $p_2 = (\kappa-1)s_2+1$,

$$\begin{aligned}
I_2 \quad : \quad &= \frac{\psi_{1-\kappa}(\varkappa_2,0)-\psi_{1-\kappa}(\varkappa_1,0)}{\Gamma(\kappa)}\int_0^{\varkappa_1}\psi'(\zeta)\left[\psi_{\kappa-1}(\varkappa_2,\zeta)-\psi_{\kappa-1}(\varkappa_1,\zeta)\right]|f(\zeta,u(\zeta))|\,d\zeta \\
&\leq \frac{\psi_{1-\kappa}(\varkappa_2,0)-\psi_{1-\kappa}(\varkappa_1,0)}{\Gamma(\kappa)}\int_0^{\varkappa_1}\psi'(\zeta)\left[\psi_{\kappa-1}(\varkappa_2,\zeta)-\psi_{\kappa-1}(\varkappa_1,\zeta)\right] \\
&\times(A|u|^{s_1}+B|u|^{s_2}+C)d\zeta \\
&:= J_1+J_2+J_3-J_4-J_5-J_6 \\
&\to 0,
\end{aligned}$$

where

$$\begin{aligned}
J_1 &\leq \frac{\psi_{1-\kappa}(\varkappa_2,0)-\psi_{1-\kappa}(\varkappa_1,0)}{\Gamma(\kappa)}A\left(\|u\|_{C_{1-\kappa;\psi}}\right)^{s_1}\int_0^{\varkappa_1}\psi'(\zeta)\psi_{\kappa-1}(\varkappa_2,\zeta)\psi_{p_1}(\zeta,0)d\zeta \to 0 \\
J_2 &\leq \frac{\psi_{1-\kappa}(\varkappa_2,0)-\psi_{1-\kappa}(\varkappa_1,0)}{\Gamma(\kappa)}B\left(\|u\|_{C_{1-\kappa;\psi}}\right)^{s_2}\int_0^{\varkappa_1}\psi'(\zeta)\psi_{\kappa-1}(\varkappa_2,\zeta)\psi_{p_2}(\zeta,0)d\zeta \to 0 \\
J_3 &\leq \frac{\psi_{1-\kappa}(\varkappa_2,0)-\psi_{1-\kappa}(\varkappa_1,0)}{\Gamma(\kappa)}C\int_0^{\varkappa_1}\psi'(\zeta)\psi_{\kappa-1}(\varkappa_2,\zeta)d\zeta \to 0 \\
J_4 &\leq \frac{\psi_{1-\kappa}(\varkappa_2,0)-\psi_{1-\kappa}(\varkappa_1,0)}{\Gamma(\kappa)}A\left(\|u\|_{C_{1-\kappa;\psi}}\right)^{s_1}\int_0^{\varkappa_1}\psi'(\zeta)\psi_{\kappa-1}(\varkappa_1,\zeta)\psi_{p_1}(\zeta,0)d\zeta \to 0 \\
J_5 &\leq \frac{\psi_{1-\kappa}(\varkappa_2,0)-\psi_{1-\kappa}(\varkappa_1,0)}{\Gamma(\kappa)}B\left(\|u\|_{C_{1-\kappa;\psi}}\right)^{s_2}\int_0^{\varkappa_1}\psi'(\zeta)\psi_{\kappa-1}(\varkappa_1,\zeta)\psi_{p_2}(\zeta,0)d\zeta \to 0 \\
J_6 &\leq \frac{\psi_{1-\kappa}(\varkappa_2,0)-\psi_{1-\kappa}(\varkappa_1,0)}{\Gamma(\kappa)}C\int_0^{\varkappa_1}\psi'(\zeta)\psi_{\kappa-1}(\varkappa_1,\zeta)d\zeta \to 0
\end{aligned}$$

as $\varkappa_2 \to \varkappa_1$ along with the continuity of ψ. To summarise,

$$\left|[\psi_{1-\kappa}(\varkappa,0)(Qu)(\varkappa)]_{\varkappa=\varkappa_2} - [\psi_{1-\kappa}(\varkappa,0)(Qu)(\varkappa)]_{\varkappa=\varkappa_1}\right| \to 0, \text{ as } \varkappa_2 \to \varkappa_1.$$

Thus, $Q(F)$ is equicontinuous. The proof is complete.

Theorem 4.3. *Let $f:[0,h]\times\mathbb{R}\to\mathbb{R}$ is a continuous, (A1) and (A2) hold, and $v,w\in C_{1-\kappa;\psi}[0,h]$ are lower and upper solutions of* (1.6) *and* (1.7)*, respectively, such that*

$$v(\varkappa)\le w(\varkappa),\quad 0\le\varkappa\le h. \tag{4.4}$$

Then, the problems (1.6) *and* (1.7) *has x^* and y^* as minimal and maximal solution, respectively, such that*

$$x^*=\lim_{n\to\infty}Q^n v,\quad y^*=\lim_{n\to\infty}Q^n w.$$

Proof. Obviously, if functions v,w are lower and upper solutions of problems (1.6) and (1.7), then there are $v\le Qv$, and $w\ge Qw$. Indeed, by the definition of the lower solution, there exist $\underline{q}(\varkappa)\ge 0$ and $\epsilon\ge 0$ such that

$$\begin{aligned}&D_{0+}^{\kappa;\psi}v(\varkappa)+\lambda v(\zeta)=f(\varkappa,v(\varkappa))-\underline{q}(\varkappa),\quad \varkappa\in(0,h),\\&\psi_{1-\kappa}(\varkappa,0)v(\varkappa)=u_0-\epsilon.\end{aligned}$$

Using Theorem 4.1 and Lemma 2.2, we obtain

$$\begin{aligned}v(\varkappa)&=\Gamma(\kappa)(u_0-\epsilon)\psi_{\kappa-1}(\varkappa,0)E_{\kappa,\kappa}(-\lambda\psi_\kappa(\varkappa,0))\\&\quad+\int_0^{\varkappa}\psi'(\zeta)\psi_{\kappa-1}(\varkappa,\zeta)E_{\kappa,\kappa}(-\lambda\psi_\kappa(\varkappa,\zeta))[f(\zeta,v(\zeta))-\underline{q}(\zeta)]d\zeta\\&\le(Qv)(\varkappa).\end{aligned}$$

By the definition of the upper solution, there exist $\overline{q}(\varkappa)\ge 0$ such that

$$\begin{aligned}&D_{0+}^{\kappa;\psi}w(\varkappa)+\lambda w(\zeta)=f(\varkappa,w(\varkappa))+\overline{q}(\varkappa),\quad \varkappa\in(0,h),\\&\psi_{1-\kappa}(\varkappa,0)w(\varkappa)=u_0+\epsilon.\end{aligned}$$

Similarly, there is $w\ge Qw$.

By Theorem 4.2, $Q:C_{1-\kappa;\psi}[0,h]\to C_{1-\kappa;\psi}[0,h]$ is increasing and completely continuous. Setting $D:=[v,w]$, by the use of Lemma 2.4, the existence of x^*,y^* is gotten. The proof is complete.

Theorem 4.4. *Let $f:[0,h]\times\mathbb{R}\to\mathbb{R}$ is a continuous and (A3) hold. Then problems* (1.6) *and* (1.7) *has a unique solution $\widetilde{u}$ in the sector $[v_0,w_0]$ on $[0,h]$, provided*

$$\frac{\aleph}{\Gamma(\kappa+1)}\psi_\kappa(h,0)<1, \tag{4.5}$$

where v_0,w_0 are lower and upper solutions, respectively, of (1.6) *and* (1.7)*, and $v_0(\varkappa)\le w_0(\varkappa)$.*

Proof. Let $\widetilde{u}$ is a solution of (1.6) and (1.7). Then $v_0\le\widetilde{u}\le w_0$. Consider the operator $Q:C_{1-\kappa;\psi}[0,h]\to C_{1-\kappa;\psi}[0,h]$ defined by (4.2). For any $u_1,u_2\in C_{1-\kappa;\psi}[0,h]$, we have

$$\begin{aligned}&\|Qu_1-Qu_2\|_{C_{1-\kappa;\psi}}\\&=\|\psi_{1-\kappa}(\varkappa,0)(Qu_1-Qu_2)\|_\infty\\&\times\max_{0\le\varkappa\le h}\left|\psi_{1-\kappa}(\varkappa,0)\int_0^{\varkappa}\psi'(\zeta)\psi_{\kappa-1}(\varkappa,\zeta)E_{\kappa,\kappa}(-\lambda(\psi_\kappa(\varkappa,\zeta))\right.\end{aligned}$$

$$\times [f(\zeta, u_1) - f(\zeta, u_2)]d\zeta\Big|$$
$$\leq \frac{1}{\Gamma(\kappa)} \max_{0\leq\varkappa\leq h} \psi_{1-\kappa}(\varkappa, 0) \int_0^\varkappa \psi'(\zeta)\psi_{\kappa-1}(\varkappa,\zeta)\aleph|u_1 - u_2|d\zeta$$
$$\leq \max_{0\leq\varkappa\leq h} \frac{\aleph}{\Gamma(\kappa)} \int_0^\varkappa \psi'(\zeta)\psi_{\kappa-1}(\varkappa,\zeta)\, \|u_1 - u_2\|_{C_{1-\kappa;\psi}} d\zeta$$
$$\leq \frac{\aleph}{\Gamma(\kappa+1)}\psi_\kappa(h,0)\|u_1 - u_2\|_{C_{1-\kappa;\psi}}.$$

From (4.5), we obtain
$$\|Qu_1 - Qu_2\|_{C_{1-\kappa;\psi}} < \|u_1 - u_2\|_{C_{1-\kappa;\psi}}.$$
According to Banach's contraction mapping [23], Q has a unique fixed point, which is unique solution.

5. Examples

In this part, we provide two examples to illustrate main results.

Example 5.1. *Consider the following problem*

$$D_{0+}^{\kappa;\psi} u(\varkappa) + \frac{1}{10}u(\varkappa) = \frac{1+\varkappa^{-\kappa}}{\Gamma(1-\kappa)} + \frac{\sin\pi\varkappa}{\Gamma(1-\kappa)\sqrt{\pi}}\left(|u(\varkappa)|^{0.5} + |u(\varkappa)|^{1.5}\right),$$
$$\psi_{1-\kappa}(\varkappa,0)u(\varkappa)\Big|_{\varkappa=0} = \frac{1}{2} \neq 0.$$

Here $\varkappa \in (0,1]$*,* $\lambda = \frac{1}{10}$*,* $D_{0+}^{\kappa;\psi}$ *is the* ψ*-RL fractional derivative of order* $0<\kappa<1$*. Obviously,*

$$f(\varkappa,u) = \frac{1+\varkappa^{-\kappa}}{\Gamma(1-\kappa)} + \frac{\sin\pi\varkappa}{\Gamma(1-\kappa)\sqrt{\pi}}\left(|u|^{0.5} + |u|^{1.5}\right),$$

and for $\varkappa \in [0,1]$*,* $u, \widetilde{u} \in [0,\infty)$*, we have*

$$\begin{aligned} |f(\varkappa,u) - f(\varkappa,\widetilde{u})| &\leq \frac{|\sin\pi\varkappa|}{\Gamma(1-\kappa)\sqrt{\pi}}\left[|u|^{0.5} + |u|^{1.5} - |\widetilde{u}|^{0.5} - |\widetilde{u}|^{1.5}\right] \\ &\leq \frac{1}{\Gamma(1-\kappa)\sqrt{\pi}}\left[|u|^{0.5} - |\widetilde{u}|^{0.5} + |u|^{1.5} - |\widetilde{u}|^{1.5}\right] \\ &\leq \frac{1}{\Gamma(1-\kappa)\sqrt{\pi}}\left[|u-\widetilde{u}|^{0.5} + |u-\widetilde{u}|^{1.5}\right] \\ &= A|u-\widetilde{u}|^{0.5} + B|u-\widetilde{u}|^{1.5}, \end{aligned}$$

for $0 < s_1 = 0.5 < s_2 = 1.5 < 1/(1-\kappa) = 2$*, for* $\kappa = \frac{1}{2}$*, here* $A = B = \frac{1}{\Gamma(1-\kappa)\sqrt{\pi}}$*. Moreover, we have*

$$\begin{aligned} |f(\varkappa,u)| &\leq \frac{|\sin\pi\varkappa|}{\Gamma(1-\kappa)\sqrt{\pi}}\left[|u|^{0.5} + |u|^{1.5}\right] \\ &\leq \frac{1}{\Gamma(1-\kappa)\sqrt{\pi}}\left[1 + |u|^{0.5} + |u|^{1.5}\right] \end{aligned}$$

$$= \quad A|u|^{0.5} + B|u|^{1.5} + C,$$

where $C = \max_{\varkappa\in[0,1]} f(\varkappa, 0) = \frac{1}{\Gamma(1-\kappa)\sqrt{\pi}}$. *Also, for* $\varkappa \in [0, 1]$, $u, \widetilde{u} \in [0, \infty]$, *we have*

$$\begin{aligned} f(\varkappa, u) - f(\varkappa, \widetilde{u}) &= \frac{\sin \pi\varkappa}{\Gamma(1-\kappa)\sqrt{\pi}} \left(|u|^{0.5} - |\widetilde{u}|^{0.5} + |u|^{1.5} - |\widetilde{u}|^{1.5}\right) \\ &\geq \frac{\sin \pi\varkappa}{\Gamma(1-\kappa)\sqrt{\pi}} \left(|u|^{1.5} - |\widetilde{u}|^{1.5}\right) \geq \frac{-1}{\Gamma(1-\kappa)\sqrt{\pi}} |u - \widetilde{u}|, \end{aligned}$$

where $\lambda = \frac{1}{\Gamma(1-\kappa)\sqrt{\pi}} > 0$. *From the foregoing, we conclude that (A1), (A2) and (4.3) are satisfied. Hence, problem (3.4) has a solution on* $[0, 1]$.

Example 5.2. *Consider the following problem*

$$\begin{cases} D_{0+}^{\kappa;\psi} u(\varkappa) + \frac{1}{10} u(\varkappa) = \frac{1}{2}\varkappa^3(\varkappa - u(\varkappa))^3 - \frac{1}{4}\varkappa^4, \\ \psi_{1-\kappa}(\varkappa, 0) u(\varkappa)\big|_{\varkappa=0} = 1 \neq 0. \end{cases} \tag{5.1}$$

Here $\varkappa \in (0, 1]$, $\lambda = \frac{1}{10}$, $D_{0+}^{\kappa;\psi}$ *is the* ψ*-RL fractional derivative of order* $0 < \kappa < 1$. *Obviously,*

$$f(\varkappa, u) = \frac{1}{2}\varkappa^3(\varkappa - u)^3 - \frac{1}{4}\varkappa^4,$$

and for $\varkappa \in [0, 1]$, $u, \widetilde{u} \in [0, \infty)$,*we have*

$$\begin{aligned} |f(\varkappa, u) - f(\varkappa, \widetilde{u})| &\leq \frac{1}{2}\varkappa^3 \left|(\varkappa - u)^3 - (\varkappa - \widetilde{u})^3\right| \\ &\leq \frac{1}{2}\varkappa^3 \left|-\left(u^3 - \widetilde{u}^3\right) - 3\varkappa\left(\widetilde{u}^2 - u^2\right) + 3\varkappa^2 (\widetilde{u} - u)\right| \\ &\leq \frac{3}{2}\varkappa^5 |u - \widetilde{u}| \leq \frac{3}{2} |u - \widetilde{u}| = B|u - \widetilde{u}|^{1.5}, \end{aligned}$$

for $s_2 = 1.5 < 1/(1-\kappa) = 2$, *for* $\kappa = \frac{1}{2}$, *here* $A = 0$, $B = \frac{3}{2}$. *Take* $v_0(\varkappa) = 0$, $w_0(\varkappa) = \psi_2(\varkappa, 0) = [\psi(\varkappa) - \psi(0)]^2$, *it is not difficult to verify that* $v_0(\varkappa)$, $w_0(\varkappa)$ *be lower and upper solutions, respectively, of (5.1), and* $v_0(\varkappa) \leq w_0(\varkappa)$. *Then for* $\varkappa \in [0, 1]$,

$$\begin{aligned} D_{0+}^{\kappa;\psi} v_0(\varkappa) + \frac{1}{10} v_0(\varkappa) &= 0 \leq \frac{1}{2}\varkappa^6 - \frac{1}{4}\varkappa^4 = f(\varkappa, v_0(\varkappa)) \\ D_{0+}^{\kappa;\psi} w_0(\varkappa) + \frac{1}{10} w_0(\varkappa) &= D_{0+}^{\kappa;\psi} [\psi(\varkappa) - \psi(0)]^2 + \frac{1}{10} [\psi(\varkappa) - \psi(0)]^2 \\ &= \frac{\Gamma(3)}{\Gamma(3-\kappa)} [\psi(\varkappa) - \psi(0)]^{2-\kappa} + \frac{1}{10} [\psi(\varkappa) - \psi(0)]^2 \\ &= \frac{8}{3\sqrt{\pi}}\varkappa^{\frac{3}{2}} + \frac{1}{10}\varkappa^2 \\ &\geq \frac{1}{2}\varkappa^6(1-\varkappa)^3 - \frac{1}{4}\varkappa^4 = f(\varkappa, w_0(\varkappa)), \end{aligned}$$

where we used $\psi(\varkappa) = \varkappa$. *In addition, let* $\epsilon > 0$, $\underline{q}(\varkappa) = \frac{\varkappa}{2}$, *and* $\overline{q}(\varkappa) = \varkappa^2$, *and consider*

$$\begin{cases} D_{0+}^{\kappa;\psi} v(\varkappa) + \frac{1}{10} v(\varkappa) = \frac{1}{2}\varkappa^6 - \frac{1}{4}\varkappa^4 - \underline{q}(\varkappa), \quad \varkappa \in (0, h), \\ \psi_{1-\kappa}(\varkappa, 0) v(\varkappa) = u_0 - \epsilon. \end{cases} \tag{5.2}$$

and

$$\begin{cases} D_{0+}^{\kappa;\psi} w(\varkappa) + \frac{1}{10} w(\varkappa) = \frac{1}{2}\varkappa^6(1-\varkappa)^3 - \frac{1}{4}\varkappa^4 + \overline{q}(\varkappa), \quad \varkappa \in (0,h), \\ \psi_{1-\kappa}(\varkappa, 0) w(\varkappa) = u_0 + \epsilon. \end{cases} \tag{5.3}$$

By Lemma 4.1, we have

$$\begin{aligned} v(\varkappa) = {} & \Gamma(\frac{1}{2})\,(u_0 - \epsilon)\,\frac{1}{\sqrt{\varkappa}} E_{\kappa,\kappa}\left(-\frac{\sqrt{\varkappa}}{10}\right) \\ & + \int_0^{\varkappa} \frac{1}{\sqrt{(\varkappa - \zeta)}} E_{\frac{1}{2},\frac{1}{2}}\left(-\frac{\sqrt{(\varkappa - \zeta}}{10}\right)\left(\frac{\zeta^6}{2} - \frac{\zeta^4}{4} - \frac{\zeta}{2}\right) d\zeta, \end{aligned}$$

and

$$\begin{aligned} w(\varkappa) = {} & \Gamma(\frac{1}{2})\,(u_0 + \epsilon)\,\frac{1}{\sqrt{\varkappa}} E_{\kappa,\kappa}\left(-\frac{\sqrt{\varkappa}}{10}\right) \\ & + \int_0^{\varkappa} \frac{1}{\sqrt{(\varkappa - \zeta)}} E_{\frac{1}{2},\frac{1}{2}}(-\frac{1}{10}\sqrt{(\varkappa - \zeta)}\left(\frac{\zeta^6(1-\zeta)^3}{2} - \frac{\zeta^4}{4} - \frac{\zeta^2}{10}\right) d\zeta. \end{aligned}$$

Thus, all assumptions of Theorem 4.2 are fulfilled. As per Theorem 4.3, problem (5.1) has minimal and maximal solutions $u^* \in [v_0, w_0]$, $\widetilde{u}^* \in [v_0, w_0]$ *, which can be obtained by*

$$u^* = \lim_{n\to\infty} v_n, \quad \widetilde{u}^* = \lim_{n\to\infty} w_n,$$

where

$$\begin{aligned} v_n(\varkappa) = {} & \Gamma(\frac{1}{2}) u_0 \frac{1}{\sqrt{\varkappa}} E_{\frac{1}{2},\frac{1}{2}}\left(-\frac{\sqrt{\varkappa}}{10}\right) \\ & + \int_0^{\varkappa} \frac{1}{\sqrt{(\varkappa - \zeta)}} E_{\frac{1}{2},\frac{1}{2}}\left(-\frac{\sqrt{(\varkappa - \zeta}}{10}\right)\left(\frac{1}{2}\zeta^3(\zeta - v_{n-1}(\zeta))^3 - \frac{1}{4}\zeta^4\right) d\zeta, \ n \geq 1 \end{aligned}$$

and

$$\begin{aligned} w_n(\varkappa) = {} & \Gamma(\frac{1}{2}) u_0 \frac{1}{\sqrt{\varkappa}} E_{\frac{1}{2},\frac{1}{2}}\left(-\frac{\sqrt{\varkappa}}{10}\right) \\ & + \int_0^{\varkappa} \frac{1}{\sqrt{(\varkappa - \zeta)}} E_{\frac{1}{2},\frac{1}{2}}\left(-\frac{\sqrt{(\varkappa - \zeta}}{10}\right)\left(\frac{1}{2}\zeta^3(\zeta - w_{n-1}(\zeta))^3 - \frac{1}{4}\zeta^4\right) d\zeta, \ n \geq 1. \end{aligned}$$

6. Conclusions

In the current work, we have investigated two classes of fractional relaxation equations. Our results were based on generalized Laplace transform, fixed point theorem due to lower and upper solutions method, and functional analysis approaches. The *psi*- RL fractional operator, which is connected with numerous well-known fractional operators, has been used in our study. Ulam-Hyer's stability of solutions for the linear version has been shown by the generalized Laplace transform approach. Then by establishing the method of lower and upper solutions along with Banach's fixed point technique, we have investigated the existence and uniqueness of iterative solutions for the nonlinear version with the non-monotone term $f(\varkappa, u(\varkappa))$, which permits the nonlinearity f to manage the condition (A1) to $|f(\varkappa, u)| \leq A|u|^{s_1} + B|u|^{s_2} + C$. Besides, we have also discussed the maximal and minimal solutions to the nonlinear version. Then, some known results in the literature have been extended. Finally, two examples to illustrate the obtained results have been provided.

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Conflict of interest

No conflicts of interest are related to this work.

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