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Research article

Convergence analysis of general parallel S-iteration process for system of mixed generalized Cayley variational inclusions

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Abstract: This work is concentrated on the study of a system of mixed generalized Cayley variational inclusions. Parallel Mann iteration process is defined in order to achieve the solution. We define an altering point problem which is equivalent to our system and then we construct general parallel S-iteration process. Finally, we discuss convergence criteria and provide an example.

Keywords: convergence; Lipschitz; process; Cayley; solution
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1. Introduction

Stampacchia [1] initiated the study of variational inequalities in 1964, which evolved in many applications related to nonlinear analysis, economics, physics, operations research, optimization, image recovery, signal processing, control theory, game theory, transportation theory, etc., see for example [2–7]. Hassouni and Moudafi [8] originated variational inclusions and proposed an scheme to solve them. System of variational inclusions are the generalized forms of variational inclusions, see [9–12]. In particular, Pang [13] showed that traffic equilibria, spatial equilibria, Nash equilibria and general equilibria can be transformed into a system of variational inequalities. Cayley operator is defined in terms of a resolvent operator and has many applications in Quaternion homography, Real homography, Complex homography, etc., see for example [14, 15]. The S-iteration process was introduced by Agarwal, O'Regan and Sahu [16] gives faster rate of convergence than Mann iteration process [17] as well as Ishikawa iteration process [18]. Sahu [19] introduced and studied the parallel S-iteration process and applied it to solve a system of operator equations in Banach space. For more details of the subject and related topics, see [20–33].

Since variational inclusions, system of variational inclusions, Cayley operator and Yosida approximation operator all have useful applications in applicable sciences, by combining all these concepts, we consider a system of mixed genearlized Cayley variational inclusions. We define parallel Mann iteration process with an equivalent altering point problem and general parallel S-iteration process to obtain the solution. Convergence criteria is also discussed with a numerical example.

2. Prelimilaries

We denote a real Banach space by $\widehat{E}$ and its dual by $\widehat{E}^*$. We denote norm by $\|\cdot\|$ and duality pairing by $\langle\cdot,\cdot\rangle$. The normalized duality mapping $\widehat{\mathcal{J}} : \widehat{E} \to 2^{\widehat{E}^*}$ is defined by

$$\widehat{\mathcal{J}}(x) = \left\{f \in \widehat{E}^* : \langle x, f\rangle = \|x\|\|f\|,\ \|f\| = \|x\|\right\}, x \in \widehat{E}.$$

The space $\widehat{E}$ is called uniformly smooth if

$$\lim_{t\to 0} \frac{\tau_{\widehat{E}}(t)}{t} = 0,$$

where $\tau_{\widehat{E}}(t)$ is the modulus of smoothness.

Definition 2.1. *For any $x, y \in \widehat{E}$ and $\hat{j}(x-y) \in \widehat{\mathcal{J}}(x-y)$, a single-valued mapping $A : \widehat{E} \to \widehat{E}$ is called*

(i) *accretive, if*

$$\langle A(x) - A(y), \hat{j}(x-y)\rangle \geq 0,$$

(ii) *strictly accretive, if*

$$\langle A(x) - A(y), \hat{j}(x-y)\rangle > 0,$$

and the equality holds if and only if $x = y$,

(iii) *strongly accretive, if there exist a constant $r > 0$ such that*

$$\langle A(x) - A(y), \hat{j}(x-y)\rangle \geq r\|x-y\|^2,$$

(iv) *Lipschitz continuous, if there exist a constant $\delta_A > 0$ such that*

$$\|A(x) - A(y)\| \leq \delta_A\|x-y\|,$$

Definition 2.2. *[34] Let $D_1(\neq \phi) \subset \widehat{E}$ and $D_2(\neq \phi) \subset \widehat{E}$. Then $x^* \in D_1$ and $y^* \in D_2$ are altering points of mappings $S_1 : D_1 \to D_2$ and $S_2 : D_2 \to D_1$, if $S_1(x^*) = y^*$ and $S_2(y^*) = x^*$.*

$Alt(S_1, S_2)$ means the set of altering points of mappings S_1 and S_2 such that

$$Alt(S_1, S_2) = \{(x^*, y^*) \in D_1 \times D_2 : S_1(x^*) = y^*, S_2(y^*) = x^*\}.$$

Definition 2.3. *[35] Let $A : \widehat{E} \to \widehat{E}$ be single-valued mapping. A multi-valued mapping $M : \widehat{E} \to 2^{\widehat{E}}$ is said to be A-accretive if M is accretive and $[A + \lambda M](\widehat{E}) = \widehat{E}$, for all $\lambda > 0$.*

Definition 2.4. *[35] Let $A : \widehat{E} \to \widehat{E}$ be single-valued mapping and $M : \widehat{E} \to 2^{\widehat{E}}$ be A-accretive multi-valued mapping. The generalized resolvent operator $\mathcal{R}^M_{A,\lambda} : \widehat{E} \to \widehat{E}$ is defined by*

$$\mathcal{R}^M_{A,\lambda}(x) = [A + \lambda M]^{-1}(x), \text{ for all } x \in \widehat{E}. \tag{2.1}$$

Definition 2.5. *The generalized Yosida approximation operator $\widehat{\mathcal{J}}^M_{A,\lambda} : \widehat{E} \to \widehat{E}$ is defined by*

$$\widehat{\mathcal{J}}^M_{A,\lambda}(x) = \frac{1}{\lambda}\left[A - \mathcal{R}^M_{A,\lambda}\right](x), \text{ for all } x \in \widehat{E} \text{ and } \lambda > 0. \tag{2.2}$$

Definition 2.6. *The generalized Cayley operator $C^M_{A,\lambda} : \widehat{E} \to \widehat{E}$ is defined by*

$$C^M_{A,\lambda}(x) = \left[2\mathcal{R}^M_{A,\lambda} - A\right](x), \text{ for all } x \in \widehat{E} \text{ and } \lambda > 0. \tag{2.3}$$

Lemma 2.1. *[36] Consider $\widehat{E}$ to be uniformly smooth Banach space and $\widehat{\mathcal{J}} : \widehat{E} \to 2^{\widehat{E}^*}$ to be normalized duality mapping. Then, following (i) and (ii) hold:*

(i) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, \hat{j}(x + y)\rangle$, *for all* $\hat{j}(x + y) \in \widehat{\mathcal{J}}(x + y)$,

(ii) $\langle x - y, \hat{j}(x) - \hat{j}(y)\rangle \le 2d^2\rho_{\widehat{E}}\left(\frac{4\|x-y\|}{d}\right)$, *where* $d = \sqrt{\frac{(\|x\|^2+\|y\|^2)}{2}}$.

Proposition 2.1. *[35] Let $A : \widehat{E} \to \widehat{E}$ be strongly accretive mapping with constant r and $M : \widehat{E} \to 2^{\widehat{E}}$ be A-accretive multi-valued mapping. Then generalized resolvent operator $\mathcal{R}^M_{A,\lambda} : \widehat{E} \to \widehat{E}$ satisfy,*

$$\left\|\mathcal{R}^M_{A,\lambda}(x) - \mathcal{R}^M_{A,\lambda}(y)\right\| \le \frac{1}{r}\|x - y\|, \text{ for all } x, y \in \widehat{E}.$$

Proposition 2.2. *[35] The generalized Yosida approximation operator is*

(i) θ_1*-Lipschitz continuous, where* $\theta_1 = \dfrac{\delta_A r + 1}{\lambda r}$; δ_A, r, $\lambda > 0$, *if* $A : \widehat{E} \to \widehat{E}$ *is Lipschitz continuous with constant* δ_A,

(ii) θ_2*-strongly accretive, where* $\theta_2 = \dfrac{r^2 - 1}{\lambda r}$; $r > 1$, $\lambda > 0$, *if* $A : \widehat{E} \to \widehat{E}$ *is strongly accretive with constant* r.

Proposition 2.3. *[35] The generalized Cayley operator is θ_3-Lipschitz continuous, where $\theta_3 = \dfrac{2 + \delta_A r}{r}$; $\delta_A, r > 0$, if $A : \widehat{E} \to \widehat{E}$ is Lipschitz continuous with constant δ_A.*

3. Problem and its solution

Let $A, B : \widehat{E} \to \widehat{E}$ be single-valued mappings and $M, N : \widehat{E} \to 2^{\widehat{E}}$ be multi-valued mappings. Let $\mathcal{R}^M_{A,\lambda} : \widehat{E} \to \widehat{E}$ and $C^M_{A,\lambda} : \widehat{E} \to \widehat{E}$ be generalized resolvent operator and generalized Cayley operator, respectively. We study the following problem:

For each $\lambda, \rho > 0$, find $(x, y) \in \widehat{E} \times \widehat{E}$ satisfying

$$\begin{cases} 0 \in A(x) - B(y) + \lambda(C^M_{A,\lambda}(R^M_{A,\lambda}(y)) + M(x)) \\ 0 \in B(y) - A(x) + \rho(C^N_{B,\rho}(\mathcal{R}^N_{B,\rho}(x)) + N(y)). \end{cases} \tag{3.1}$$

The following lemma is fixed point formulation for system (3.1).

Lemma 3.1. *Let $A, B : \widehat{E} \to \widehat{E}$ be single-valued mappings and $M, N : \widehat{E} \to 2^{\widehat{E}}$ be multi-valued mappings such that A is δ_A-Lipschitz continuous and r_1-strongly accretive maaping, B is δ_B-Lipschitz continuous and r_2-strongly accretive maaping, M and N are A-accretive and B-accretive multi-valued mappings, respectively. Let $\mathcal{R}^M_{A,\lambda}, \widehat{\mathcal{J}}^M_{A,\lambda}, C^M_{A,\lambda} : \widehat{E} \to \widehat{E}$ be generalized resolvent operator, generalized Yosida approximation operator and Cayley operator, respectively. Then, system of mixed generalized Cayley variational inclusions* (3.1) *admits a solution $(x, y) \in \widehat{E} \times \widehat{E}$, if and only if it satisfies the following equations:*

$$
\begin{aligned}
x &= \mathcal{R}^M_{A,\lambda}\left[\mathcal{R}^N_{B,\rho}(y) + \left(\rho\widehat{\mathcal{J}}^N_{B,\rho}(y) - \lambda C^M_{A,\lambda}\left(\mathcal{R}^M_{A,\lambda}(y)\right)\right)\right] \\
y &= \mathcal{R}^N_{B,\rho}\left[\mathcal{R}^M_{A,\lambda}(x) + \left(\lambda\widehat{\mathcal{J}}^M_{A,\lambda}(x) - \rho C^N_{B,\rho}\left(\mathcal{R}^N_{B,\rho}(x)\right)\right)\right],
\end{aligned}
\tag{3.2}
$$

where $\lambda, \rho > 0$ are constants.

Proof. Proof is easy and hence omitted. □

Using Lemma 3.1, we suggest following parallel Mann iteration process to solve system of mixed generalized Cayley variational inclusions (3.1).

Parallel Mann iteration process 3.1. Let $x_1, y_1 \in \widehat{E}$, then compute $\{x_n\}$ and $\{y_n\}$ by the iterative process:

$$
\begin{cases}
x_{n+1} = (1-\alpha_n)x_n + \alpha_n\mathcal{R}^M_{A,\lambda}\left[\mathcal{R}^N_{B,\rho}(y_n) + \left(\rho\widehat{\mathcal{J}}^N_{B,\rho}(y_n) - \lambda C^M_{A,\lambda}\left(\mathcal{R}^M_{A,\lambda}(y_n)\right)\right)\right] \\
y_{n+1} = (1-\beta_n)y_n + \beta_n\mathcal{R}^N_{B,\rho}\left[\mathcal{R}^M_{A,\lambda}(x_n) + \left(\lambda\widehat{\mathcal{J}}^M_{A,\lambda}(x_n) - \rho C^N_{B,\rho}\left(\mathcal{R}^N_{B,\rho}(x_n)\right)\right)\right],
\end{cases}
\tag{3.3}
$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $n = 1, 2, \cdots$.

Theorem 3.1. *Let $\widehat{E}$ be real uniformly smooth Banach space with modulus of smoothness $\tau_{\widehat{E}}(t) \leq Ct^2$ for some $C > 0$. Let $A, B : \widehat{E} \to \widehat{E}$ be single-valued mappings and $M, N : \widehat{E} \to 2^{\widehat{E}}$ be multi-valued mappings such that A is δ_A-Lipschitz continuous and r_1-strongly accretive maaping, B is δ_B-Lipschitz continuous and r_2-strongly accretive maaping, M and N are A-accretive and B-accretive multi-valued mappings, respectively. Suppose that generalized resolvent operators $\mathcal{R}^M_{A,\lambda}$ and $\mathcal{R}^N_{B,\rho}$ are Lipschitz continuous with constants $\frac{1}{r_1}$ and $\frac{1}{r_2}$; generalized Yosida approximation operators $\widehat{\mathcal{J}}^M_{A,\lambda}$ and $\widehat{\mathcal{J}}^N_{B,\rho}$ are Lipschitz continuous with constant θ_1 and θ_1', and strongly accretive with constants θ_2 and θ_2', and generalized Cayley operators $C^M_{A,\lambda}$ and $C^N_{B,\rho}$ are Lipschitz continuous with constant θ_3 and θ_3', respectively. Let for some $\lambda, \rho > 0$, the following conditions are satisfied:*

$$
\begin{cases}
\left|\lambda - \frac{r_1 r_2(r_1-\rho)-r_1}{\theta_3 r_2}\right| < \sqrt{\frac{\rho^2 r_1^2(1-2\theta_2'+64C{\theta_1'}^2)}{\theta_3^2}}, \\
\left|\rho - \frac{r_1 r_2(r_2-\lambda)-r_2}{\theta_3' r_1}\right| < \sqrt{\frac{\lambda^2 r_2^2(1-2\theta_2+64C{\theta_1}^2)}{{\theta_3'}^2}}, \\
2\theta_2' < 1 + 64C{\theta_1'}^2 \text{ and } 2\theta_2 < 1 + 64C\theta_1^2,
\end{cases}
\tag{3.4}
$$

where $\theta_1 = \frac{\delta_A r_1+1}{\lambda r_1}$, $\theta_1' = \frac{\delta_B r_2+1}{\rho r_2}$, $\theta_2 = \frac{r_1^2-1}{\lambda r_1}$, $\theta_2' = \frac{r_2^2-1}{\rho r_2}$, $\theta_3 = \frac{2+\delta_A r_1}{r_1}$ and $\theta_3' = \frac{2+\delta_B r_2}{r_2}$.

Then, the iterative sequences $\{x_n\}$ *and* $\{y_n\}$ *generated by process 3.1 strongly converge to the solution* $(x, y) \in \widehat{E} \times \widehat{E}$ *of our system* (3.1).

Proof. Applying parallel Mann iteration process 3.1, Propositions 2.1–2.3, we have

$$
\begin{aligned}
\|x_{n+1} - x^*\| = & \Big\|\Big((1-\alpha_n)x_n + \alpha_n \mathcal{R}^M_{A,\lambda}[\mathcal{R}^N_{B,\rho}(y_n) + (\rho\widehat{\mathcal{J}}^N_{B,\rho}(y_n) - \lambda C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}(y_n)))]\Big) \\
& -\Big((1-\alpha_n)x^* + \alpha_n \mathcal{R}^M_{A,\lambda}[\mathcal{R}^N_{B,\rho}(y^*) + (\rho\widehat{\mathcal{J}}^N_{B,\rho}(y^*) - \lambda C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}(y^*)))]\Big)\Big\| \\
\leq \;& (1-\alpha_n)\|x_n - x^*\| + \alpha_n \Big\|\mathcal{R}^M_{A,\lambda}[\mathcal{R}^N_{B,\rho}(y_n) + (\rho\widehat{\mathcal{J}}^N_{B,\rho}(y_n) - \lambda C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}(y_n)))] \\
& -\mathcal{R}^M_{A,\lambda}[\mathcal{R}^N_{B,\rho}(y^*) + (\rho\widehat{\mathcal{J}}^N_{B,\rho}(y^*) - \lambda C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}(y^*)))]\Big\| \\
\leq \;& (1-\alpha_n)\|x_n - x^*\| + \alpha_n \frac{1}{r_1}\Big\|[\mathcal{R}^N_{B,\rho}(y_n) + (\rho\widehat{\mathcal{J}}^N_{B,\rho}(y_n) - \lambda C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}(y_n)))] \\
& -[\mathcal{R}^N_{B,\rho}(y^*) + (\rho\widehat{\mathcal{J}}^N_{B,\rho}(y^*) - \lambda C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}(y^*)))]\Big\| \\
\leq \;& (1-\alpha_n)\|x_n - x^*\| + \alpha_n \frac{1}{r_1}\Big(\Big\|\mathcal{R}^N_{B,\rho}(y_n) - \mathcal{R}^N_{B,\rho}(y^*)\Big\| + \rho\Big\|\widehat{\mathcal{J}}^N_{B,\rho}(y_n) - \widehat{\mathcal{J}}^N_{B,\rho}(y^*)\Big\| \\
& +\lambda\Big\|C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}(y_n)) - C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}(y^*))\Big\|\Big) \\
\leq \;& (1-\alpha_n)\|x_n - x^*\| + \alpha_n\Big(\frac{1}{r_1 r_2} + \frac{\rho}{r_1} + \frac{\lambda\theta_3}{r_1^2}\Big)\|y_n - y^*\| \\
& +\alpha_n \frac{\rho}{r_1}\Big\|(y_n - y^*) - (\widehat{\mathcal{J}}^N_{B,\rho}(y_n) - \widehat{\mathcal{J}}^N_{B,\rho}(y^*))\Big\|.
\end{aligned} \tag{3.5}
$$

Using Lemma 2.1, we evaluate

$$
\begin{aligned}
&\Big\|(y_n - y^*) - \Big(\widehat{\mathcal{J}}^N_{B,\rho}(y_n) - \widehat{\mathcal{J}}^N_{B,\rho}(y^*)\Big)\Big\|^2 \\
&\quad \leq \|y_n - y^*\|^2 + 2\Big\langle -\Big(\widehat{\mathcal{J}}^N_{B,\rho}(y_n) - \widehat{\mathcal{J}}^M_{A,\lambda}(y^*)\Big), \\
&\qquad \widehat{j}\Big((y_n - y^*) - \Big(\widehat{\mathcal{J}}^N_{B,\rho}(y_n) - \widehat{\mathcal{J}}^N_{B,\rho}(y^*)\Big)\Big)\Big\rangle \\
&\quad = \|y_n - y^*\|^2 + 2\Big\langle -\Big(\widehat{\mathcal{J}}^N_{B,\rho}(y_n) - \widehat{\mathcal{J}}^N_{B,\rho}(y^*)\Big), \widehat{j}(y_n - y^*)\Big\rangle \\
&\qquad +2\Big\langle -\Big(\widehat{\mathcal{J}}^N_{B,\rho}(y_n) - \widehat{\mathcal{J}}^N_{B,\rho}(y^*)\Big), \\
&\qquad \widehat{j}\Big((y_n - y^*) - \Big(\widehat{\mathcal{J}}^N_{B,\rho}(y_n) - \widehat{\mathcal{J}}^N_{B,\rho}(y^*)\Big)\Big) - \widehat{j}(y_n - y^*)\Big\rangle \\
&\quad \leq \|y_n - y^*\|^2 - 2\theta_2'\|y_n - y^*\|^2 \\
&\qquad +4d^2\tau_{\widehat{E}}\left(\frac{4\Big\|\widehat{\mathcal{J}}^N_{B,\rho}(y_n) - \widehat{\mathcal{J}}^N_{B,\rho}(y^*)\Big\|}{d}\right) \\
&\quad \leq \|y_n - y^*\|^2 - 2\theta_2'\|y_n - y^*\|^2 + 64C\Big\|\widehat{\mathcal{J}}^N_{B,\rho}(y_n) - \widehat{\mathcal{J}}^N_{B,\rho}(y^*)\Big\|^2 \\
&\quad \leq (1 - 2\theta_2' + 64C{\theta_1'}^2)\|y_n - y^*\|^2.
\end{aligned} \tag{3.6}
$$

Using (3.6), (3.5) becomes

$$
\|x_{n+1} - x^*\| \;\leq\; (1-\alpha_n)\|x_n - x^*\| + \alpha_n\Big(\frac{1}{r_1 r_2} + \frac{\rho}{r_1} + \frac{\lambda\theta_3}{r_1^2}\Big)\|y_n - y^*\|
$$

$$
\begin{aligned}
& +\alpha_n \frac{\rho\sqrt{(1-2\theta'_2+64C\theta'^2_1)}}{r_1}\|y_n-y^*\| \\
= \;& (1-\alpha_n)\|x_n-x^*\|+\alpha_n\Omega'\|y_n-y^*\|, \qquad (3.7)
\end{aligned}
$$

where

$$\Omega' = \Big(\frac{1}{r_1r_2}+\frac{\rho}{r_1}+\frac{\lambda\theta_3}{r_1^2}\Big)+\frac{\rho\sqrt{(1-2\theta'_2+64C\theta'^2_1)}}{r_1}.$$

Using the similar arguments as for (3.5)–(3.7), we compute

$$
\begin{aligned}
\|y_{n+1}-y^*\| \;\le\;& (1-\alpha_n)\|y_n-y^*\|+\alpha_n\Big(\frac{1}{r_1r_2}+\frac{\lambda}{r_2}+\frac{\rho\theta'_3}{r_2^2}\Big)\|x_n-x^*\| \\
& +\alpha_n\frac{\lambda\sqrt{(1-2\theta_1+64C\theta_2^2)}}{r_2}\|x_n-x^*\| \\
=\;& (1-\alpha_n)\|y_n-y^*\|+\alpha_n\Omega''\|x_n-x^*\|, \qquad (3.8)
\end{aligned}
$$

where

$$\Omega'' = \Big(\frac{1}{r_1r_2}+\frac{\lambda}{r_2}+\frac{\rho\theta'_3}{r_2^2}\Big)+\frac{\lambda\sqrt{(1-2\theta_2+64C\theta_1^2)}}{r_2}.$$

Combining (3.7) and (3.8), we get

$$
\begin{aligned}
\|x_{n+1}-x^*\|+\|y_{n+1}-y^*\| \;\le\;& (1-\alpha_n)\|x_n-x^*\|+\alpha_n\Omega'\|y_n-y^*\| \\
& +(1-\alpha_n)\|y_n-y^*\|+\alpha_n\Omega''\|x_n-x^*\| \\
\le\;& (1-\alpha_n)(\|x_n-x^*\|+\|y_n-y^*\|) \\
& +\alpha_n\max\{\Omega',\Omega''\}(\|x_n-x^*\|+\|y_n-y^*\|),
\end{aligned}
$$

which implies that

$$\|x_{n+1}-x^*\|+\|y_{n+1}-y^*\| \;\le\; (1-\alpha_n(1-\Omega))(\|x_n-x^*\|+\|y_n-y^*\|), \qquad (3.9)$$

where $\Omega = \max\{\Omega',\Omega''\}$. Now, we define the norm $\|.\|_*$ on $\widehat{E}\times\widehat{E}$ by $\|(x,y)\|_* = \|x\|+\|y\|$, for all $(x,y)\in\widehat{E}\times\widehat{E}$. Using (3.9), we have

$$
\begin{aligned}
\|(x_{n+1},y_{n+1})-(x^*,y^*)\|_* \;=\;& \|(x_{n+1}-x^*,y_{n+1}-y^*)\|_* \\
=\;& \|x_{n+1}-x^*\|+\|y_{n+1}-y^*\|_* \\
\le\;& (1-\alpha_n(1-\Omega))(\|x_n-x^*\|+\|y_n-y^*\|) \\
=\;& (1-\alpha_n(1-\Omega))\|(x_n,y_n)-(x^*,y^*)\|_*.
\end{aligned}
$$

From condition (3.4), it is clear that $\Omega < 1$ and consequently $\{(x_n,y_n)\}$ is a Cauchy sequence which strongly converges to $(x,y)\in\widehat{E}\times\widehat{E}$. As $A,\ B,\ M,\ N,\ \mathcal{R}^M_{A,\lambda},\ \widehat{\mathcal{J}}^M_{A,\lambda},\ C^M_{A,\lambda}$ are all continuous, the conclusion follows from Lemma 3.1. Hence, $(x,y)\in\widehat{E}\times\widehat{E}$ is the solution of our system (3.1). □

4. Convergence analysis applying altering points problem

By using the altering points problem we obtain strong convergence of system of mixed generalized Cayley variational inclusions (3.1). We suggest general parallel S-iteration process for altering points problem associated with the system (3.1).

Let $D_1(\neq \phi) \subset \widehat{E}$ and $D_2(\neq \phi) \subset \widehat{E}$ such that D_1 and D_2 are closed and convex. Let $S_1 : D_1 \to D_2$, $S_2 : D_2 \to D_1$ be mappings such that

$$S_1 = \mathcal{R}^N_{B,\rho}[\mathcal{R}^M_{A,\lambda} + (\lambda\widehat{\mathcal{J}}^M_{A,\lambda} - \rho C^N_{B,\rho}(\mathcal{R}^N_{B,\rho}))] \tag{4.1}$$

$$S_2 = \mathcal{R}^M_{A,\lambda}[\mathcal{R}^N_{B,\rho} + (\rho\widehat{\mathcal{J}}^N_{B,\rho} - \lambda C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}))], \tag{4.2}$$

where $\lambda, \rho > 0$ are constants. Using Lemma 3.1, it is clear that the system (3.1) is equivalent to following altering point problem:

Find $(x, y) \in D_1 \times D_2$ satisfying

$$\begin{cases} x = S_2(y) = \mathcal{R}^N_{B,\rho}[\mathcal{R}^M_{A,\lambda} + (\lambda\widehat{\mathcal{J}}^M_{A,\lambda} - \rho C^N_{B,\rho}(\mathcal{R}^N_{B,\rho}))](y) \\ \\ y = S_1(x) = \mathcal{R}^M_{A,\lambda}[\mathcal{R}^N_{B,\rho} + (\rho\widehat{\mathcal{J}}^N_{B,\rho} - \lambda C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}))](x). \end{cases} \tag{4.3}$$

We construct the general parallel S-iteration process to solve system (3.1).

Parallel S-iteration process 4.1. For any given $(x_1, y_1) \in D_1 \times D_2$, let $\{(x_n, y_n)\}$ be an iterative sequence in $D_1 \times D_2$ defined by

$$\begin{aligned} x_{n+1} &= S_2[(1-\alpha_n)y_n + \alpha_n S_1(x_n)] \\ \\ y_{n+1} &= S_1[(1-\alpha_n)x_n + \alpha_n S_2(y_n)], \end{aligned} \tag{4.4}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. The following result is needed in continuation.

Theorem 4.1. *Let $\widehat{E}$ be real uniformly smooth Banach space with modulus of smoothness $\tau_{\widehat{E}}(t) \leq Ct^2$ for some $C > 0$. Let $A, B : \widehat{E} \to \widehat{E}$ be single-valued mappings and $M, N : \widehat{E} \to 2^{\widehat{E}}$ be multi-valued mappings such that A is δ_A-Lipschitz continuous and r_1-strongly accretive maaping, B is δ_B-Lipschitz continuous and r_2-strongly accretive maaping, M and N are A-accretive and B-accretive multi-valued mappings, respectively. Suppose that generalized resolvent operators $\mathcal{R}^M_{A,\lambda}$ and $\mathcal{R}^N_{B,\rho}$ are Lipschitz continuous with constants $\frac{1}{r_1}$ and $\frac{1}{r_2}$; generalized Yosida approximation operators $\widehat{\mathcal{J}}^M_{A,\lambda}$ and $\widehat{\mathcal{J}}^N_{B,\rho}$ are Lipschitz continuous with constant θ_1 and θ_1', and strongly accretive with constants θ_2 and θ_2', and generalized Cayley operators $C^M_{A,\lambda}$ and $C^N_{B,\rho}$ are Lipschitz continuous with constant θ_3 and θ_3' with $2\theta_2' < 1 + 64C{\theta_1'}^2$ and $2\theta_2 < 1 + 64C\theta_1^2$, where $\theta_1 = \frac{\delta_A r_1+1}{\lambda r_1}$, $\theta_1' = \frac{\delta_B r_2+1}{\rho r_2}$, $\theta_2 = \frac{r_1^2-1}{\lambda r_1}$, $\theta_2' = \frac{r_2^2-1}{\rho r_2}$, $\theta_3 = \frac{2+\delta_A r_1}{r_1}$ and $\theta_3' = \frac{2+\delta_B r_2}{r_2}$, respectively. Then mappings S_1 and S_2 are Lipschitz continuous with constants Ω' and Ω'', respectively, where $\Omega' = \left(\frac{1}{r_1 r_2} + \frac{\rho}{r_1} + \frac{\lambda\theta_3}{r_1^2}\right) + \frac{\rho\sqrt{(1-2\theta'_2+64C{\theta'}_1^2)}}{r_1}$ and $\Omega'' = \left(\frac{1}{r_1 r_2} + \frac{\lambda}{r_2} + \frac{\rho\theta_3'}{r_2^2}\right) + \frac{\lambda\sqrt{(1-2\theta_2+64C\theta_1^2)}}{r_2}$.*

Proof. Let $x, y \in \widehat{E}$. Then, we have

$$\|S_1(x) - S_1(y)\| = \Big\|\mathcal{R}^M_{A,\lambda}[\mathcal{R}^N_{B,\rho} + (\rho\widehat{\mathcal{J}}^N_{B,\rho} - \lambda C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}))](x)$$

$$
\begin{aligned}
& -\mathcal{R}^M_{A,\lambda}[\mathcal{R}^N_{B,\rho} + (\rho\widehat{\mathcal{J}}^N_{B,\rho} - \lambda C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}))](y)\Big\| \\
\leq\ & \frac{1}{r_1}\Big\|[\mathcal{R}^N_{B,\rho}(x) + (\rho\widehat{\mathcal{J}}^N_{B,\rho}(x) - \lambda C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}(x)))] \\
& -[\mathcal{R}^N_{B,\rho}(y) + (\rho\widehat{\mathcal{J}}^N_{B,\rho}(y) - \lambda C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}(y)))]\Big\| \\
\leq\ & \frac{1}{r_1}\Big(\Big\|\mathcal{R}^N_{B,\rho}(x) - \mathcal{R}^N_{B,\rho}(y)\Big\| + \rho\Big\|\widehat{\mathcal{J}}^N_{B,\rho}(x) - \widehat{\mathcal{J}}^N_{B,\rho}(y)\Big\| \\
& +\lambda\Big\|C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}(x)) - C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}(y))\Big\|\Big)
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{1}{r_1}\Big(\frac{1}{r_2}\|x-y\| + \rho\Big\|(x-y) - (\widehat{\mathcal{J}}^N_{B,\rho}(x) - \widehat{\mathcal{J}}^N_{B,\rho}(y))\Big\| \\
& +\rho\|x-y\| + \frac{\lambda\theta_3}{r_1}\|x-y\|\Big) \\
\leq\ & \Big(\frac{1}{r_1 r_2} + \frac{\rho}{r_1} + \frac{\lambda\theta_3}{r_1^2}\Big)\|x-y\| \\
& +\frac{\rho}{r_1}\Big\|(x-y) - (\widehat{\mathcal{J}}^N_{B,\rho}(x) - \widehat{\mathcal{J}}^N_{B,\rho}(y))\Big\|. \qquad (4.5)
\end{aligned}
$$

Using same arguments as used for (3.6), we obtain

$$
\Big\|(x-y) - \Big(\widehat{\mathcal{J}}^N_{B,\rho}(x) - \widehat{\mathcal{J}}^N_{B,\rho}(y)\Big)\Big\|^2 \leq (1 - 2\theta_2' + 64C{\theta_1'}^2)\|x-y\|^2.
$$

Combining (4.5) and (4.6), we obtain

$$
\|S_1(x) - S_1(y)\| \leq \left[\Big(\frac{1}{r_1 r_2} + \frac{\rho}{r_1} + \frac{\lambda\theta_3}{r_1^2}\Big) + \frac{\rho\sqrt{(1 - 2\theta_2' + 64C{\theta_1'}^2)}}{r_1}\right]\|x-y\|.
$$

Hence S_1 is Ω'-Lipschitz continuous, where $\Omega' = \Big(\frac{1}{r_1 r_2} + \frac{\rho}{r_1} + \frac{\lambda\theta_3}{r_1^2}\Big) + \frac{\rho\sqrt{(1-2\theta_2'+64C{\theta_1'}^2)}}{r_1}$. In the same manner, it follows that S_2 is Ω''-Lipschitz continuous, where $\Omega'' = \Big(\frac{1}{r_1 r_2} + \frac{\lambda}{r_2} + \frac{\rho\theta_3'}{r_2^2}\Big) + \frac{\lambda\sqrt{(1-2\theta_2+64C\theta_1^2)}}{r_2}$. □

The convergence criteria is established for system (3.1) by applying general parallel S-iteration process.

Theorem 4.2. *Let D_1 and D_2 be same as in parallel S-iteration process* (4.1)*. Let $A, B, M, N, \mathcal{R}^M_{A,\lambda}, \widehat{\mathcal{J}}^M_{A,\lambda}$ and $C^M_{A,\lambda}$ be same as in Theorem 3.1 such that all the conditions of Theorem* 3.1 *are satisfied. Let S_1 and S_2 be same as in Theorem 4.1. Then (I) and (II) hold.*

(I) *There exists a point $(x^*, y^*) \in D_1 \times D_2$, which solves altering point problem (4.3) associated with the system* (3.1).

(II) *The sequence $\{(x_n, y_n)\}$ generated by general parallel S-iteration process (4.4) converges strongly to point $(x^*, y^*) \in D_1 \times D_2$.*

Proof. (**I**). Applying Lemma 3.1 and (4.3), (I) follows.
(**II**). Using parallel S-iteration process 4.1 and Theorem 4.1, we have

$$\begin{aligned} \|y_{n+1} - y^*\| &= \left\|S_1[(1-\alpha_n)x_n + \alpha_n S_2(y_n)] - S_1(x^*)\right\| \\ &\leq \Omega^{'}\|[(1-\alpha_n)x_n + \alpha_n S_2(y_n)] - x^*\| \\ &\leq \Omega^{'}((1-\alpha_n)\|x_n - x^*\| + \alpha_n\|S_2(y_n) - x^*\|) \\ &\leq \Omega^{'}((1-\alpha_n)\|x_n - x^*\| + \alpha_n\|S_2(y_n) - S_2(y^*)\|) \\ &\leq \Omega^{'}((1-\alpha_n)\|x_n - x^*\| + \alpha_n\Omega^{''}\|y_n - y^*\|) \\ &\leq (1-\alpha_n)\Omega^{'}\|x_n - x^*\| + \alpha_n\Omega^{'}\Omega^{''}\|y_n - y^*\|. \end{aligned} \tag{4.6}$$

Similarly,

$$\begin{aligned} \|x_{n+1} - x^*\| &= \left\|S_2[(1-\alpha_n)y_n + \alpha_n S_1(x_n)] - S_2(y^*)\right\| \\ &\leq \Omega^{''}\|[(1-\alpha_n)y_n + \alpha_n S_1(x_n)] - y^*\| \\ &\leq \Omega^{''}((1-\alpha_n)\|y_n - y^*\| + \alpha_n\|S_1(x_n) - y^*\|) \\ &\leq \Omega^{''}((1-\alpha_n)\|y_n - y^*\| + \alpha_n\|S_1(x_n) - S_1(x^*)\|) \\ &\leq \Omega^{''}((1-\alpha_n)\|y_n - y^*\| + \alpha_n\Omega^{'}\|x_n - x^*\|) \\ &\leq (1-\alpha_n)\Omega^{''}\|y_n - y^*\| + \alpha_n\Omega^{'}\Omega^{''}\|x_n - x^*\|. \end{aligned} \tag{4.7}$$

Combining (4.6) with (4.7), we have

$$\begin{aligned} \|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| &\leq (1-\alpha_n)\Omega^{'}\|x_n - x^*\| + \alpha_n\Omega^{'}\Omega^{''}\|y_n - y^*\| \\ &\quad +(1-\alpha_n)\Omega^{''}\|y_n - y^*\| + \alpha_n\Omega^{'}\Omega^{''}\|x_n - x^*\| \\ &\leq \Omega^{'}\{(1-\alpha_n)\|x_n - x^*\|\alpha_n\Omega^{''}\|x_n - x^*\|\} \\ &\quad +\Omega^{''}\{(1-\alpha_n)\|y_n - y^*\|\alpha_n\Omega^{'}\|y_n - y^*\|\} \\ &\leq \max\{\Omega^{'},\Omega^{''}\}\{(1-\alpha_n)(\|x_n - x^*\| + \|y_n - y^*\|) \\ &\quad +\alpha_n\max\{\Omega^{'},\Omega^{''}\}(\|x_n - x^*\| + \|y_n - y^*\|)\} \\ &\leq \Omega(1-\alpha_n(1-\Omega))(\|x_n - x^*\| + \|y_n - y^*\|), \end{aligned} \tag{4.8}$$

where $\Omega = \max\{\Omega^{'},\Omega^{''}\}$. The norm $\|.\|_*$ on $E \times E$ is defined by $\|(x,y)\|_* = \|x\| + \|y\|$. Using (4.9), we have

$$\begin{aligned} \|(x_{n+1},y_{n+1}) - (x^*,y^*)\|_* &= \|(x_{n+1} - x^*, y_{n+1} - y^*)\|_* \\ &= \|(x_{n+1} - x^*\| + \|y_{n+1} - y^*)\| \\ &\leq \Omega(1-\alpha_n(1-\Omega))\|(x_n,y_n) - (x^*,y^*)\|_*. \end{aligned} \tag{4.9}$$

Since $\Omega(1-\alpha_n(1-\Omega)) \leq \Omega < 1$, we obtain that $\lim_{n\to\infty}\|(x_n,y_n)-(x^*,y^*)\|_* = 0$. Thus, we get $\lim_{n\to\infty}\|x_n-x^*\| = \lim_{n\to\infty}\|y_n - y^*\| = 0$ and hence the sequence $\{(x_n,y_n)\}$ converges strongly to point (x^*,y^*). □

5. Numerical example

For illustration, we provide the following example. All codes are written in MATLAB R2019a.

Example 5.1. *Let $\widehat{E} = \mathbb{R}, D_1 = [0,20]$, and $D_2 = [0,23]$. Let $A, B : \widehat{E} \to \widehat{E}$ and $M, N : \widehat{E} \to 2^{\widehat{E}}$ be such that*

$$A(x) = \frac{6}{5}x,$$
$$B(x) = \frac{4}{3}x,$$
$$M(x) = \left\{\frac{1}{10}x\right\},$$
$$N(x) = \left\{\frac{1}{5}x\right\}.$$

Clearly, A is $\frac{11}{10}$-strongly accretive and $\frac{13}{10}$-Lipschtiz continuous mapping and B is $\frac{7}{6}$-strongly accretive and $\frac{3}{2}$-Lipschtiz continuous mapping and M and N are accretive mappings. For $\lambda = 1, [A + \lambda M](\widehat{E}) = \widehat{E}$ and for $\rho = 1, [B + \rho N](\widehat{E}) = \widehat{E}$, the multi-valued mappings M and N are A-accretive and B-accretive mappings.

The generalized resolvent operators, generalized Yosida approximation operators and generalized Cayley operators are defined below:

$$\mathcal{R}^M_{A,\lambda}(x) = [A + \lambda M]^{-1}(x) = \frac{10}{13}x,\ \mathcal{R}^N_{B,\rho}(x) = [B + \rho N]^{-1}(x) = \frac{15}{23}x,$$
$$\mathcal{J}^M_{A,\lambda}(x) = \frac{1}{\lambda}[A - \mathcal{R}^M_{A,\lambda}](x) = \frac{28}{65}x,\ \ \mathcal{J}^N_{B,\rho}(x) = \frac{1}{\rho}[B - \mathcal{R}^N_{B,\rho}](x) = \frac{47}{69}x,$$
$$C^M_{A,\lambda}(x) = 2\mathcal{R}^M_{A,\lambda}(x) - A(x) = \frac{22}{65}x,\ C^N_{B,\rho}(x) = 2\mathcal{R}^N_{B,\rho}(x) - B(x) = -\frac{2}{69}x, \text{ for all } x \in \widehat{E}.$$

The above defined generalized resolvent operators $\mathcal{R}^M_{A,\lambda}$ and $\mathcal{R}^N_{B,\rho}$ are Lipschitz continuous with constants $\frac{1}{r_1} = \frac{10}{11}$ and $\frac{1}{r_2} = \frac{6}{7}$, respectively, and generalized Yosida approximation operators $\mathcal{J}^M_{A,\lambda}$ and $\mathcal{J}^N_{B,\rho}$ are Lipschitz continuous with constants $\theta_1 = \frac{\delta_A r_1 + 1}{\lambda r_1} = \frac{22}{10}$ and $\theta'_1 = \frac{\delta_B r_2 + 1}{\rho r_2} = \frac{33}{14}$, respectively and strongly accretive with constants $\theta_2 = \frac{r_1^2 - 1}{\lambda r_1} = \frac{19}{100}$ and $\theta'_2 = \frac{r_2^2 - 1}{\rho r_2} = \frac{13}{42}$, respectively and generalized Cayley operators $C^M_{A,\lambda}$ and $C^N_{B,\rho}$ are Lipschitz continuous with constants $\theta_3 = \frac{2 + \delta_A r_1}{r_1} = \frac{31}{10}$ and $\theta'_3 = \frac{2 + \delta_B r_2}{r_2} = \frac{45}{14}$, respectively.

For $\lambda = 1$ and $\rho = 1$, we evaluate the mappings S_1 and S_2 such that

$$\begin{aligned} S_1(x) &= \mathcal{R}^M_{A,\lambda}[\mathcal{R}^N_{B,\rho} + (\rho\mathcal{J}^N_{B,\rho} - \lambda\, C^M_{A,\lambda} \circ \mathcal{R}^M_{A,\lambda})](x) = 0.8254x, \\ S_2(x) &= \mathcal{R}^N_{B,\rho}[\mathcal{R}^M_{A,\lambda} + (\lambda\mathcal{J}^M_{A,\lambda} - \rho\, C^N_{B,\rho} \circ \mathcal{R}^N_{B,\rho})](x) = 0.8308x, \end{aligned}$$

which are Lipschitz continuous with constants 15 *and* 19, *respectively.*

Also, the conditions considered in Theorem 3.1,
$\left|\lambda - \frac{r_1 r_2(r_1 - \rho) - r_1}{\theta_3 r_2}\right| < \sqrt{\frac{\rho^2 r_1^2(1 - 2\theta'_2 + 64C{\theta'_1}^2)}{\theta_3^2}}$ *and* $\left|\rho - \frac{r_1 r_2(r_2 - \lambda) - r_2}{\theta'_3 r_1}\right| < \sqrt{\frac{\lambda^2 r_2^2(1 - 2\theta_2 + 64C\theta_1^2)}{{\theta'}_3^2}}$,
$r_1, r_2 > 1,\ \ 2\theta'_2 < 1 + 64C{\theta'_1}^2$ *and* $2\theta_2 < 1 + 64C\theta_1^2$ *are satisfied for all the values considered above. Thus, Theorem 3.1 is satisfied.*

For arbitrary $x_1 \in D_1$ and $y_1 \in D_2$, the common terms of $\{x_n\}$ and $\{y_n\}$ produced by process 3.1 are given by

$$\begin{aligned}
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n \mathcal{R}^M_{A,\lambda}[\mathcal{R}^N_{B,\rho}(y_n) + (\rho \mathcal{J}^N_{B,\rho}(y_n) - \lambda C^M_{A,\lambda}(\mathcal{R}^M_{A,\lambda}(y_n)))] \\
&= \frac{x_n}{n+1} + \frac{n(0.8254)y_n}{n+1}, \\
y_{n+1} &= (1-\alpha_n)y_n + \alpha_n \mathcal{R}^N_{B,\rho}[\mathcal{R}^M_{A,\lambda}(x_n) + (\lambda \mathcal{J}^M_{A,\lambda}(x_n) - \rho C^N_{B,\rho}(\mathcal{R}^N_{B,\rho}(x_n)))] \\
&= \frac{y_n}{n+1} + \frac{n(0.8308)x_n}{n+1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
x_{n+1} &= \frac{x_n}{n+1} + \frac{n(0.8254)y_n}{n+1}, \\
y_{n+1} &= \frac{y_n}{n+1} + \frac{n(0.8308)x_n}{n+1}.
\end{aligned}$$

For arbitrary $x_1 \in D_1$ and $y_1 \in D_2$, the common terms of $\{x_n\}$ and $\{y_n\}$ produced by process 4.1 are given by

$$\begin{aligned}
x_{n+1} &= S_2[(1-\alpha_n)y_n + \alpha_n S_1(x_n)] \\
&= \frac{(0.8308)y_n}{n+1} + \frac{n(0.6857)x_n}{n+1}, \\
y_{n+1} &= S_1[(1-\alpha_n)x_n + \alpha_n S_2(y_n)] \\
&= \frac{(0.8254)x_n}{n+1} + \frac{n(0.6857)y_n}{n+1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
x_{n+1} &= \frac{(0.8308)y_n}{n+1} + \frac{n(0.6857)x_n}{n+1}, \\
y_{n+1} &= \frac{(0.8254)x_n}{n+1} + \frac{n(0.6857)y_n}{n+1}.
\end{aligned}$$

Taking different initial values $x_1 = 3, y_1 = -3, x_1 = 5$ and $y_1 = -7$, the sequences $\{x_n\}$ and $\{y_n\}$ converge to the unique solution $(x^*, y^*) = (0, 0)$ of system (3.1).

In Table 1, we compare both the processes under consideration for $x = 3, x = -3$. Convergence of parallel Mann iteration process for different initial values of x is shown in Figure 1. In Table 2, comparison of parallel Mann iteration process and parallel S-iteration process for $x = 5$ and $x = -7$ is shown and in Figure 2, convergence of parallel S-iteration process is shown for different initial values of x. In all cases, the iteration process will terminate for $\|x_{n+1} - x_n\| \leq 10^{-5}$ and $\|y_{n+1} - y_n\| \leq 10^{-5}$.

Table 1. Comparison table: For initial values $x_1 = 3$ and $y_1 = -3$.

No. of iterations (n)	Parallel Mann iteration process x_n	Parallel Mann iteration process y_n	Parallel S-iteration process x_n	Parallel S-iteration process y_n
1	3	-3	3	-3
2	2.05722	-2.05722	0.90219	-0.90219
3	1.41072	-1.41073	0.27131	-0.27131
4	0.96739	-0.96739	0.08159	-0.08159
5	0.66338	-0.66338	0.02453	-0.02453
6	0.45491	-0.45491	0.00737	-0.00737
7	0.31195	-0.31195	0.00221	-0.00221
8	0.21391	-0.21391	0.00066	-0.00066
9	0.14669	-0.14669	0.00020	-0.00020
10	0.10059	-0.10059	0.00006	-0.00006
13	0.03244	-0.03243	0	0
16	0.01046	-0.01046	0	0
20	0.00231	-0.00231	0	0
23	0.00074	-0.00074	0	0
26	0	0	0	0
27	0	0	0	0

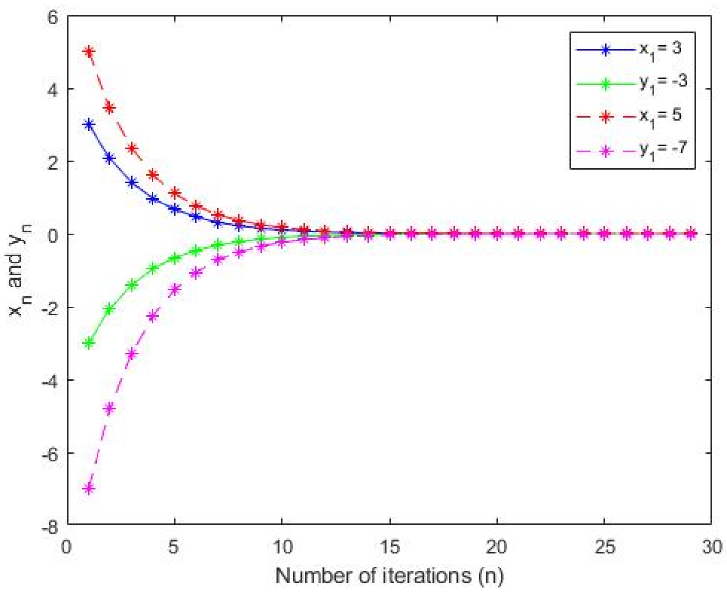

Figure 1. Convergence graph of $\{x_n\}$ and $\{y_n\}$ produced by parallel Mann iteration process (3.3) taking different initial values.

Table 2. Comparison table: For initial values $x_1 = 5$ and $y_1 = -7$.

No. of iterations (n)	Parallel Mann iteration process x_n	Parallel Mann iteration process y_n	Parallel S-iteration process x_n	Parallel S-iteration process y_n
1	5	-7	5	-7
2	3.42871	-4.80019	1.50366	-2.10512
3	2.35121	-3.29169	0.45219	-0.63307
4	1.61232	-2.25725	0.13599	-0.19038
5	1.10564	-1.54789	0.04089	-0.05725
6	0.75818	-1.06145	0.01229	-0.01721
7	0.51991	-0.72788	0.00369	-0.00517
8	0.35653	-0.49914	0.00111	-0.00155
9	0.24448	-0.34228	0.00033	-0.00046
10	0.16765	-0.23471	0.00010	-0.00014
13	0.05406	-0.07568	0	0
16	0.01743	-0.02441	0	0
20	0.00385	-0.00539	0	0
23	0.00124	-0.00174	0	0
26	0.00040	-0.00056	0	0
30	0	0	0	0

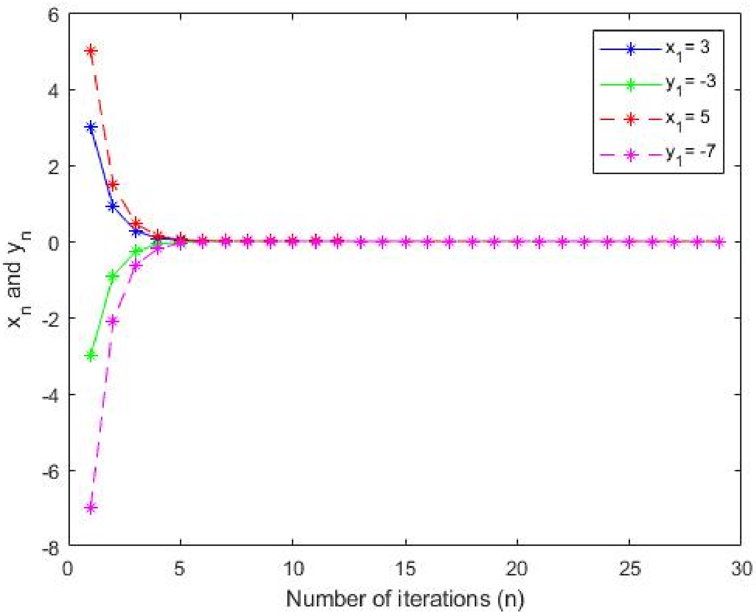

Figure 2. Convergence graph of $\{x_n\}$ and $\{y_n\}$ produced by parallel S-iteration process (4.4) taking different initial values.

6. Conclusions

The general parallel S-iteration process is established to discuss the convergence criteria for the problem (3.1). We apply parallel Mann iteration process to obtain the solution of our system. We provide a numerical example applying Matlab program.

Further, we remark that our result can be extended in other higher dimensional spaces.

Conflict of interest

The authors declare that they have no conflict of interest.

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