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Research article

On the convergence of an iterative process for enriched Suzuki nonexpansive mappings in Banach spaces

Thabet Abdeljawad[1,2,*], Kifayat Ullah[3], Junaid Ahmad[4,*], Muhammad Arshad[4] and Zhenhua Ma[5,*]

[1] Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia

[2] Department of Medical Research, China Medical University, Taichung 40402, Taiwan

[3] Department of Mathematical Sciences, University of Lakki Marwat, Lakki Marwat 28420, Khyber Pakhtunkhwa, Pakistan

[4] Department of Mathematics and Statistic, International Islamic University, Islamabad 44000, Pakistan

[5] Department of Mathematics and Physics, Hebei University of Architecture, Zhangjiakou 075024, China

* **Correspondence:** Email: tabdeljawad@psu.edu.sa, ahmadjunaid436@gmail.com, mazhenghua_1981@163.com.

Abstract: We study the existence and approximation of fixed points for the recently introduced class of mappings called enriched Suzuki nonexpansive mappings in the setting of Banach spaces. We use the modified K-iteration process to establish the main results of the paper. The class of enriched Suzuki nonexpansive operators is an important class of nonlinear operators that includes properly the class of Suzuki nonexpansive operators as well as enriched nonexpansive operators. Various assumptions are imposed on the domain or on the operator to establish the main convergence theorems. Eventually, a numerical example of enriched Suzuki nonexpansive operators is used to show the effectiveness of the studied iteration scheme. The main outcome of the paper is new and essentially suggests a new direction for researchers who are working on fixed point problems in a Banach space setting. Our results improve and extend some main results due to Hussain et al. (*J. Nonlinear Convex Anal.* 2018, *19*, 1383–1393.), Ullah et al. (*Axioms* 2022, 1.) and others.

Keywords: Banach space; enriched mapping; K-iteration; condition (I); rate of convergence
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1. Introduction

Real world problems are normally connected with the concept of fixed points. The main reason for this is that many problems that naturally arise in the applied sciences are very difficult to solve analytically or even impossible to solve by analytical approaches. Fixed point theory suggests in such cases alternative techniques. It is well known that if a problem has solution, then the sought solution can be set in the form of a fixed point of a certain mapping; and in many cases, the mapping is nonexpansive on a certain subset of a Banach space. Thus, we note that the study of fixed points of nonexpansive mappings on Banach spaces is an important area of reasearch on its own. Notice that a mapping G on a subset $\mathcal{P}$ of a Banach space is called nonexpansive if

$$\|Gp - Gp'\| \leq \|p - p'\|,$$

for every $p, p' \in \mathcal{P}$.

Since the class of nonexpansive mappings is important, it is natural to investigate some extensions of these mappings. Among other things, in 2008, Suzuki [1] imposed a condition on mappings and the condition is named as "condition (C)". Notice that a mapping G on a subset $\mathcal{P}$ of a Banach space is said to satisfy the condition (C) (or said to be Suzuki nonexpansive) if

$$\frac{1}{2}\|p - Gp\| \leq \|p - p'\| \Rightarrow \|Gp - Gp'\| \leq \|p - p'\|,$$

for every $p, p' \in \mathcal{P}$.

From the definition of Suzuki nonexpansive mappings, one concludes obviously that all nonexpansive mappings are in fact Suzuki nonexpansive. In [1], Suzuki proved that there are some mapping that are Suzuki nonexpansive but not nonexpansive. It follows that the class of Suzuki nonexpansive mappings includes properly the class of all nonexpansive mappings.

The following facts are due to Suzuki [1].

Proposition 1.1. *Suppose we have a subset $\mathcal{P}$ of a Banach space $\mathcal{W}$ and $G : \mathcal{P} \to \mathcal{P}$, a selfmap.*

(i) If the selfmap G is Suzuki nonexpansive, and p, p' are any elements of $\mathcal{P}$, then

$$\|p - Gp'\| \leq 3\|p - Gp\| + \|p - p'\|.$$

(ii) Suppose $\mathcal{W}$ has the well-known Opial's property (see definition in the next section), and the mapping G is Suzuki nonexpansive. If $\{p_i\}$ is weakly convergent to y^*, then $Gy^* = y^*$ provided that $\lim_{i\to\infty} \|p_i - Gp_i\| = 0$.

As many know, a point $y^* \in \mathcal{P}$ is called a fixed point of $G : \mathcal{P} \to \mathcal{P}$ if and only if $Gy^* = y^*$, and the set of fixed points of G is denoted precisely by $F_G := \{y^* \in \mathcal{P} : Gy^* = y^*\}$. The study of nonexpansive mappings becomes an important class of mappings when Browder [2] and Gohde [3] independently obtained a fixed point result for these mappings on the setting of a uniformly convex Banach space [4] (UCBS, for short). The Browder–Gohde result motivated many authors to study the extensions of nonexpansive mappings (see e.g., [5–15] and others). Among the other things, in 2019, Berinde [16] suggested a new and different extension of nonexpansive operators. Any mappings in this new class of mappings he called enriched nonexpansive mapping. Berinde [16] proved that the class of nonexpansive mappings is properly contained in the class of all enriched nonexpansive mappings. Inspired by Berinde [16] work, Ullah et al. [17] introduced the concept of enriched Suzuki nonexpansive mappings as follows.

Definition 1.2. *[17] Suppose $\mathcal{P}$ is a subset of a Banach space and $G : \mathcal{P} \to \mathcal{P}$. The mapping G is said to be enriched Suzuki nonexpansive if one has a constant $b \in [0, \infty)$ such that*

$$\frac{1}{2}\|p - Gp\| \leq (b + 1)\|p - p'\| \Rightarrow \|b(p - p') + Gp - Gp'\| \leq (b + 1)\|p - p'\|,$$

for every $p, p' \in \mathcal{P}$.

Remark 1.3. *If G is enriched nonexpansive, then it immediately follows from the above definition that G is enriched Suzuki nonexpansive. Interestingly, if G is a Suzuki nonexpansive mapping, then G is also an enriched Suzuki nonexpansive mapping, but the converse may not valid in general as shown by a numerical example in this paper. Accordingly, the class of enriched Suzuki nonexpansive mappings properly includes the class of enriched nonexpansive mappings.*

Example 1.4. *[18] Suppose $\mathcal{P} = \{p \in l^\infty : \|p\|_\infty \leq 1\}$ and $G : \mathcal{P} \to \mathcal{P}$ be defined by $Gp = (0, p_1^2, p_2^2, ...)$ for all $p = (p_1^2, p_2^2, p_3^2...)$. Then for $p = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, ...)$ and $p' = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, ...)$, we get $\|b(p-p')+Gp-Gp'\|_\infty > (b + 1)\|p - p'\|_\infty$ for all $b \in [0, \infty)$. On the other hand, G is enriched Suzuki nonexpansive.*

In [17], Ullah et al. suggested some existence and approximation theorems for Suzuki nonexpansive operators on the Hilbert space setting. The existence and approximation of a fixed point once established on a Hilbert space setting then the extension of such findings on a Banach space setting is always desirable. The reason behind this fact is that the scope of many problems naturally falls within the Banach space setting. As always, the Banach Fixed Point Theorem [19] (BFPT, for short) uses the iterative scheme due to Picard [20] for a certain class of nonlinear mappings. However, the Picard iterative scheme sometimes may not converge to a fixed point in the class of nonexpansive operators. Alternatively, there are several iterative schemes in the literature having various steps, for example: Mann iteration [21] (one-step iteration), Ishikawa [22] (two-step iteration), Noor [23] (three-step iteration), Agarwal [24] (two-step iteration that is a slight modification of the Ishikawa iteration), Abbas iteration [25] (three-step iteration) for approximating fixed points for nonexpansive operators Hilbert and Banch spaces.

In 2018, Hussain et al. [26] suggested K-iteration for Suzuki nonexpansive mappings. This iteration was numerically compared with some previously published iteration processes and they observed that the high accuracy of this scheme is superfluous among many other iterative schemes. We study the K-iteration in the new setting of mappings as: Let $\mathcal{P}$ be a closed convex subset of a Banach space such that $G : \mathcal{P} \to \mathcal{P}$ be enriched Suzuki nonexpansive with $b \in [0, \infty)$. Set $G_\lambda : \mathcal{P} \to \mathcal{P}$ as $G_\lambda p = (1-\lambda)p+\lambda Gp$ where $\lambda = \frac{1}{b+1}$. Obviously, the fixed point set of G_λ is same as that of G. In this case, the Agarwal [24], Thakur [27] and K-iteration of Hussain et al. [26] respectively read as follows.

$$\begin{cases} p_1 \in \mathcal{P}, \\ q_i = (1 - \beta_i)p_i + \beta_i G_\lambda p_i, \\ p_{i+1} = (1 - \alpha_i)G_\lambda p_i + \alpha_i G_\lambda q_i, i \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\alpha_i, \beta_i \in (0, 1)$ and $\lambda = \frac{1}{b+1}$.

$$\begin{cases} p_1 \in \mathcal{P}, \\ u_i = (1-\beta_i)p_i + \beta_i G_\lambda p_i, \\ q_i = G_\lambda((1-\alpha_i)p_i + \alpha_i u_i), \\ p_{i+1} = G_\lambda q_i, i \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\alpha_i, \beta_i \in (0,1)$ and $\lambda = \frac{1}{b+1}$.

$$\begin{cases} p_1 \in \mathcal{P}, \\ u_i = (1-\beta_i)p_i + \beta_i G_\lambda p_i, \\ q_i = G_\lambda((1-\alpha_i)G_\lambda p_i + \alpha_i G_\lambda u_i), \\ p_{i+1} = G_\lambda q_i, i \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\alpha_i, \beta_i \in (0,1)$ and $\lambda = \frac{1}{b+1}$.

Hussain et al. [26] proved several weak and strong convergence theorems of the K-iteration for Suzuki nonexpansive mappings and proved that this iteration gives better approximation results, comparatively other iterations like Thakur [27] and Agarwal [24] iterations. However, a natural question is that: Can we improve and extend their main results to the new setting of enriched Suzuki nonexpansive mappings? In this research we suggest an affirmative answer to this question. In fact we prove some weak and strong convergence results using iteration (1.3) for enriched Suzuki nonexpansive mappings. Using an example that is enriched Suzuki nonexpansive and exceeds the class of Suzuki nonexpansive mappings and show that the high accuracy of the K iteration is still very effective.

2. Preliminaries

This section precisely contains some well-known facts and results that are needed in the main outcome.

Suppose we are given a closed convex subset $\mathcal{P}$, a Banach space $\mathcal{W}$ and $p \in \mathcal{W}$ is any fixed element of $\mathcal{W}$. Then we can set $r(p, \{u_i\})$ as $r(p, \{p_i\}) := \limsup_{i\to\infty} \|p - p_i\|$.

We shall denote in this paper the asymptotic radius of $\{p_i\}$ corresponding to $\mathcal{P}$ by $A(\mathcal{P}, \{p_i\})$ and define by the formula: $r(\mathcal{P}, \{p_i\}) := \inf\{r(p, \{p_i\}) : p \in \mathcal{P}\}$. We shall denote in this paper the asymptotic center of $\{p_i\}$ corresponding to $\mathcal{P}$ by $A(\mathcal{P}, \{p_i\})$ and define by the formula: $A(\mathcal{P}, \{p_i\}) := \{p \in \mathcal{P} : r(p, \{p_i\}) = r(\mathcal{P}, \{p_i\})\}$. As many knows that the set $A(\mathcal{P}, \{p_i\})$ contain a single point in the UCBS setting (see, e.g., [28, 29] and others).

Definition 2.1. *[30] Suppose $\mathcal{W}$ have the property that every weakly convergent $\{p_i\}$ in $\mathcal{W}$ with limit $p \in \mathcal{W}$, then one has*

$$\limsup_{i\to\infty} \|p_i - p\| < \limsup_{i\to\infty} \|p_i - p'\| \ \ \forall p' \in \mathcal{Z} - \{p\}.$$

In this case, we call the space $\mathcal{W}$ as a Banach space endowed with Opial's condition.

The following facts can be found in [17]. For the sake of completeness, we include the details.

Lemma 2.2. *Suppose $\mathcal{P}$ is closed and convex in $\mathcal{W}$, and $G : \mathcal{P} \to \mathcal{P}$. If G is enriched Suzuki nonexpansive with a constant $b \in [0, \infty)$. Then G_λ is Suzuki nonexpansive, where $\lambda = \frac{1}{b+1}$.*

Proof. Put $b = \frac{1-\lambda}{\lambda}$. Then we see that $\lambda = \frac{1}{b+1}$. Now, from the definition of enriched Suzuki nonexpansive mappings, we have

$$\frac{1}{2}\|p - GP\| \le \frac{1}{\lambda}\|p - p'\| \Rightarrow \|(\frac{1-\lambda}{\lambda})(p - p') + Gp - Gp'\| \le \frac{1}{\lambda}\|p - p'\|.$$

It follows that:

$$\frac{1}{2}\|\lambda p - \lambda Gp\| \le \|p - p'\| \Rightarrow \|(1-\lambda)(p - p') + \lambda Gp - \lambda Gp'\| \le \|p - p'\|.$$

Using the definition of G_λ, we have

$$\frac{1}{2}\|p - G_\lambda p\| \le \|p - p'\| \Rightarrow \|G_\lambda p - G_\lambda p'\| \le \|p - p'\|, \text{for all } p, p' \in \mathcal{P}.$$

Thus, G_λ forms a Suzuki nonexpansive mapping, where $\lambda = \frac{1}{b+1}$. □

Among the other things, Schu [31] established the following lemma.

Lemma 2.3. *Suppose $\mathcal{W}$ is a UCBS, $i \in \mathbb{N}$ and $0 < a \le \alpha_i \le b < 1$.Suppose $z \ge 0$ is given, $\{p_i\}$ and $\{q_i\}$ in $\mathcal{W}$ are two sequences with $\limsup_{i\to\infty} \|p_i\| \le z$, $\limsup_{i\to\infty} \|q_i\| \le z$ and $\lim_{i\to\infty} \|\alpha_i p_i + (1-\alpha_i)q_i\| = z$, then $\lim_{i\to\infty} \|p_i - q_i\| = 0$.*

3. Main outcome

This section suggests some new results on the K-iteration scheme (1.3) for the class of enriched nonexpansive mappings. We start the section with a very basic lemma.

Lemma 3.1. *Let $\mathcal{P}$ be a closed convex subset of a UCBS $\mathcal{W}$ and $G : \mathcal{P} \to \mathcal{P}$. If G is enriched Suzuki nonexpansive $F_G \neq \emptyset$, and $\{p_i\}$ is obtained from (1.3), then $\lim_{i\to\infty} \|p_i - y^*\|$ exists for all $y^* \in F_G$.*

Proof. We may fix any $y^* \in F_G$. Accordingly, we have $y^* \in F_{G_\lambda}$. By Lemma 2.2, G_λ is enriched Suzuki nonexpansive, in particular, $\frac{1}{2}\|y^* - G_\lambda y^*\| \le \|y^* - p\|$ implies $\|G_\lambda y^* - G_\lambda p\| \le \|y^* - p\|$ for all $p \in \mathcal{P}$. Using this, we get

$$\begin{aligned}
\|u_i - y^*\| &= \|(1-\beta_i)p_i + \beta_i G_\lambda p_i - y^*\| \\
&\le (1-\beta_i)\|p_i - y^*\| + \beta_i\|G_\lambda p_i - y^*\| \\
&\le (1-\beta_i)\|p_i - y^*\| + \beta_i\|p_i - y^*\| \\
&= \|p_i - y^*\|,
\end{aligned}$$

and

$$\begin{aligned}
\|q_i - y^*\| &= \|G_\lambda((1-\alpha_i)G_\lambda p_i + \alpha_i G_\lambda u_i) - y^*\| \\
&\le \|(1-\alpha_i)G_\lambda p_i + \alpha_i G_\lambda u_i - y^*\| \\
&\le (1-\alpha_i)\|G_\lambda p_i - y^*\| + \alpha_i\|G_\lambda u_i - y^*\| \\
&\le (1-\alpha_i)\|p_i - y^*\| + \alpha_i\|u_i - y^*\|.
\end{aligned}$$

While using the above inequalities, we have

$$\begin{aligned} \|p_{i+1} - a^*\| &= \|G_\lambda q_i - y^*\| \\ &\leq \|q_i - y^*\| \\ &\leq (1-\alpha_i)\|p_i - y^*\| + \alpha_i\|u_i - y^*\| \\ &\leq (1-\alpha_i)\|p_i - y^*\| + \alpha_i\|p_i - y^*\| \\ &= \|p_i - y^*\|. \end{aligned}$$

Now we can see that $\|p_{i+1} - y^*\| \leq \|p_i - y^*\|$. So the sequence $\{\|p_i - y^*\|\}$ is nonincreasing and also bounded. It follows $\lim_{i\to\infty}\|p_i - y^*\|$ exists, where $y^* \in F_{G_\lambda} = F_G$ is any point. □

After this, we establish another key lemma.

Lemma 3.2. *Let $\mathcal{P}$ be a closed convex subset of a UCBS $\mathcal{W}$, and $G : \mathcal{P} \to \mathcal{P}$. If G is enriched Suzuki nonexpansive, and $\{p_i\}$ is obtained from (1.3).Then $F_G \neq \emptyset$ if and only if the sequence $\{p_i\}$ is bounded in $\mathcal{P}$ and* $\lim_{i\to\infty}\|G_\lambda p_i - p_i\| = 0$ *where* $\lambda = \frac{1}{b+1}$.

Proof. According to Lemma 3.1 if $y^* \in F_G$ is any point, then $\lim_{i\to\infty}\|p_i - y^*\|$ essentially exists, and the sequence $\{p_i\}$ must be bounded. Hence we set

$$\lim_{i\to\infty}\|p_i - y^*\| = z. \tag{3.1}$$

But we have proved in Lemma 3.1 that

$$\|u_i - y^*\| \leq \|p_i - y^*\|.$$

Accordingly, we get

$$\Rightarrow \limsup_{i\to\infty}\|u_i - y^*\| \leq \limsup_{i\to\infty}\|p_i - y^*\| = z. \tag{3.2}$$

Now by Lemma 2.2, G_λ is enrichd Suzuki nonexpansive and $\frac{1}{2}\|y^* - G_\lambda y^*\| \leq \|p_i - y^*\|$, therefore, $\|G_\lambda p_i - y^*\| \leq \|p_i - y^*\|$. Hence

$$\limsup_{i\to\infty}\|G_\lambda p_i - y^*\| \leq \limsup_{i\to\infty}\|p_i - y^*\| = z. \tag{3.3}$$

Again, from the proof of Lemma 3.1,

$$\|p_{i+1} - y^*\| \leq (1-\alpha_i)\|p_i - y^*\| + \alpha_i\|u_i - y^*\|$$

It follows that $\|p_{i+1} - y^*\| - \|p_i - y^*\| \leq \frac{\|p_{i+1} - y^*\| - \|p_i - y^*\|}{\alpha_i} \leq \|u_i - y^*\| - \|p_i - y^*\|$. Hence

$$z \leq \liminf_{i\to\infty}\|u_i - y^*\|. \tag{3.4}$$

From (3.2) and (3.4), we get

$$z = \lim_{i\to\infty}\|u_i - y^*\|. \tag{3.5}$$

From (3.5), we have

$$z = \lim_{i\to\infty}\|u_i - y^*\| = \lim_{i\to\infty}\|(1-\beta_i)(p_i - y^*) + \beta_i(G_\lambda p_i - y^*)\|.$$

Applying Lemma 2.3, we obtain

$$\lim_{i\to\infty} \|G_\lambda p_i - p_i\| = 0.$$

Conversely, we may assume that $\{p_i\} \subseteq \mathcal{P}$ is bounded such that $\lim_{i\to\infty} \|p_i - G_\lambda p_i\| = 0$. If we suppose $y^* \in A(\mathcal{P}, \{p_i\})$ any point. Need is to prove that $G_\lambda y^* \in A(\mathcal{P}, \{p_i\})$. For this, by Lemma 2.2, G_λ is Suzuki nonexpansive. Hence using Proposition 1.1(i), we have

$$\begin{aligned} r(G_\lambda y^*, \{p_i\}) &= \limsup_{i\to\infty} \|p_i - G_\lambda y^*\| \\ &= \limsup_{i\to\infty}(3\|p_i - G_\lambda p_i\| + \|p_i - y^*\|) \\ &= \limsup_{i\to\infty} \|p_i - y^*\| = r(y^*, \{p_i\}). \end{aligned}$$

We have seen that $G_\lambda y^* \in A(\mathcal{P}, \{p_i\})$. From the property that $A(\mathcal{P}, \{p_i\})$ contains only element, we must have $y^* = G_\lambda y^*$. But $F_{G_\lambda} = F_G$, it follows that F_G is a nonempty set. □

Now we are able to establish a strong convergence theorem as follows.

Theorem 3.3. *Let $\mathcal{P}$ be a compact convex subset of a UCBS $\mathcal{W}$ and $G : \mathcal{P} \to \mathcal{P}$. If G is enriched Suzuki nonexpansive with $F_G \neq \emptyset$, and $\{p_i\}$ is obtained from (1.3). Then, $\{p_i\}$ converges strongly to a fixed point of G.*

Proof. The subset $\mathcal{P}$ of $\mathcal{W}$ is given as compact. Thus, there exists a subsequence $\{p_{i_j}\}$ that satisfies $\lim_{j\to\infty} \|p_{i_j} - p_0\| = 0$, for some $p_0 \in \mathcal{P}$. Also, in the view of Theorem 3.2, $\lim_{i\to\infty} \|p_{i_j} - G_\lambda p_{i_j}\| = 0$. According to Lemma 2.2, G_λ is Suzuki nonexpansive. Subsequently, we can use Proposition 1.1(ii), to get

$$\|p_{i_j} - G_\lambda p_0\| \leq 3\|p_{i_j} - G_\lambda p_{i_j}\| + \|p_{i_j} - p_0\|.$$

Applying strong limit, we get $G_\lambda p_0 = p_0$, and this means that p_0 is a fixed point for G_λ. It follows that p_0 is also a fixed point of G. In the view of Lemma 3.1, $\lim_{i\to\infty} \|p_i - p_0\|$ must exist. Eventually, p_0 is a strong limit for $\{p_i\}$. □

Theorem 3.4. *Let $\mathcal{P}$ be a closed convex subset of a UCBS $\mathcal{W}$, and $G : \mathcal{P} \to \mathcal{P}$. If G is enriched Suzuki nonexpansive with $F_G \neq \emptyset$, and $\{p_i\}$ is obtained from (1.3). Then, $\{p_i\}$ converges strongly to a fixed point of G if* $\liminf_{i\to\infty} dist(p_i, F_{G_\lambda}) = 0$.

Proof. The proof is elementary and, hence, omitted. □

We now write a definition of condition (I) that was introduced by Senter and Dotson [32].

Definition 3.5. *[32] A selfmap G on a subset $\mathcal{P}$ of a UCBS $\mathcal{W}$ is called a mapping with condition (I) when there is a function f that has properties $f(a) = 0$ for $a = 0$, $f(a) > 0$ for $a > 0$ and $\|p - Gp\| \geq f(dist(p, F_G))$ for any $p \in \mathcal{P}$.*

Theorem 3.6. *Let $\mathcal{P}$ be a closed convex subset of a UCBS $\mathcal{W}$ and $G : \mathcal{P} \to \mathcal{P}$. If G is enriched Suzuki nonexpansive with $F_G \neq \emptyset$, and $\{p_i\}$ is obtained from (1.3), then $\{p_i\}$ converges strongly to a fixed point of G if G_λ satisfies condition (I).*

Proof. To prove this result, we may write from Theorem 3.2 that $\liminf_{i\to\infty}\|p_i - G_\lambda p_i\| = 0$. It now follows from condition (I) of G_λ and the last equation that $\liminf_{i\to\infty}\operatorname{dist}(p_i, F_{G_\lambda}) = 0$. Applying Theorem 3.4, we get $\{p_i\}$ converges to a fixed point of G. □

The final result of the paper is the following weak convergence theorem.

Theorem 3.7. *Let $\mathcal{P}$ be a closed convex subset of a UCBS $\mathcal{W}$ and $G : \mathcal{P} \to \mathcal{P}$. If G is enriched Suzuki nonexpansive with $F_G \neq \emptyset$, and $\{p_i\}$ is obtained from (1.3), then $\{p_i\}$ converges strongly to a fixed point of G if $\mathcal{W}$ satisfies the Opial's condition.*

Proof. We complete the proof of this result as follows. The given space $\mathcal{W}$ is reflexive due to its uniform convexity. Also, Theorem 3.2 provides that $\{p_i\}$ is a bounded sequence in $\mathcal{W}$. Bearing reflexiveness of $\mathcal{W}$ in mind, we can select a subsequence $\{p_{i_j}\}$ of the given sequence $\{p_i\}$ that converges to some $u_1 \in \mathcal{P}$. By Theorem 3.2, $\lim_{i\to\infty}\|G_\lambda p_{i_j} - p_{i_j}\| = 0$. According to Lemma 2.2, G_λ is Suzuki nonexpansive. Subsequently, we can use Proposition 1.1(ii), and hence $p_1 \in F_{G_\lambda} = F_G$.

Claim. We claim that p_1 itself is the weak limit of the given sequence $\{p_i\}$.

If one assumes that $p_2 \in \mathcal{P}$ a weak limit of $\{p_i\}$ then it means we can select a weakly convergent subsequence, namely, $\{p_{i_t}\}$ with limit $p_1 \in \mathcal{P}$. But using the same calculations as previous, we have $p_2 \in F_{G_\lambda} = F_G$. By Lemma 3.1 and Opial's condition of the space, we get

$$\begin{aligned}\lim_{i\to\infty}\|p_i - p_1\| &= \lim_{j\to\infty}\|p_{i_j} - p_1\| < \lim_{j\to\infty}\|p_{i_j} - p_2\| \\ &= \lim_{i\to\infty}\|p_i - p_2\| = \lim_{t\to\infty}\|p_{i_t} - p_2\| \\ &< \lim_{t\to\infty}\|p_{i_t} - p_1\| = \lim_{i\to\infty}\|p_i - p_1\|.\end{aligned}$$

Consequently, we proved $\lim_{i\to\infty}\|p_i - p_1\| < \lim_{i\to\infty}\|p_i - p_1\|$. But $p_1 \neq p_2$, so this is a contradiction, and hence our claim is true. This finishes the required proof. □

4. Numerical example and numerical computations

Once, the theoretical outcome is established, then one needs to verify the claims and findings by using numerical examples and experiments. Thus, here we wish to perform a comparative numerical experiment as follows.

Example 4.1. *[17] Set a selfmap $G : [0.5, 2] \to [0.5, 2]$ by the rule $Gp = p^{-1}$. Then the following hold:*

(i) $F_G \neq \emptyset$.
(ii) G is not Suzuki nonexpansive.
(iii) G is enriched Suzuki nonexpansive.

Proof. Since $G1 = 1$, so we proved (i). For (ii), since $\frac{1}{2}|1-G1| \leq |1-0.5|$ implies that $|G1-G0.5| > |1-0.5|$, so (ii) holds. Next, we select $b = 1.5$, in this case, $\frac{1}{2}|p-Gp| \leq (b+1)|p-p'| \Rightarrow |Gp-Gp'| \leq |p-p'|$, that is, (iii) also holds. □

We now consider the operator G that is defined in Example 4.1, and assume that $\lambda = \frac{1}{1+1.5} = 0.4$. We obtained Table 1 and Figure 1. Clearly, our novel defined scheme of G suggests better approximation results.

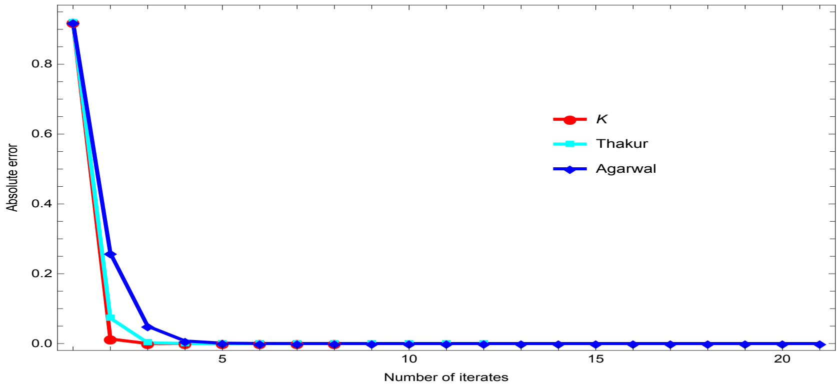

Figure 1. Convergence behaviors of K (1.3), Thakur (1.2) and Agarwal (1.1) iterations towards the fixed point 1 of the mapping G.

Table 1. Comparison of K and other iterations.

i	**K** (1.3)	**Thakur** (1.2)	**Agarwal** (1.1)
1	1.92	1.92	1.92
2	1.01302070007745	1.07277650160287	1.25862510144471
3	1.00005956858732	1.00230054769915	1.0495816518778
3	1.00000026594432	1.00006414040039	1.00743470290856
5	1.00000000118717	1.00000178075291	1.00104460080111
6	1.0000000000050	1.00000004943386	1.00014524130111
7	1.00000000000002	1.00000000137228	1.00002016434127
8	**1**	1.00000000003809	1.00000279890403
9	1	1.00000000000105	1.0000003884896
10	1	1.00000000000002	1.00000005392240
11	1	**1**	1.0000000074844
12	1	1	1.00000000103883
13	1	1	1.00000000014419
14	1	1	1.00000000002001
15	1	1	1.00000000000277
16	1	1	1.00000000000038
17	1	1	1.00000000000005
18	1	1	**1**

5. Conclusions

In this research, we studied the existence and approximation of fixed points for a recently introduced novel class of mapping called the class of enriched Suzuki nonexpansive mappings under a highly accurate approximation scheme. The strong and weak convergence theorems are obtained under various assumptions. The main outcome is well supported by numerical computations. Since the class of enriched Suzuki nonexpansive operators is more general than the class of Suzuki nonexpansive operators as shown in this paper, it follows that the main outcome of this paper properly extends the main outcome of Hussain et al. [17] from Suzuki nonexpansive mappings to the context of more general class of mappings called enriched Suzuki nonexpansive mappings.

Conflict of interest

The authors declare that they have no conflicts of interest.

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