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Research article

Noncyclic contractions and relatively nonexpansive mappings in strictly convex fuzzy metric spaces

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Abstract: A concept of fuzzy projection operator is introduced and use to investigate the nonemptiness of the fuzzy proximal pairs. We then consider the classes of noncyclic contractions and noncyclic relatively nonexpansive mappings and survey the existence of best proximity pairs for such mappings. In the case that the considered mapping is noncyclic relatively nonexpansive, we need a geometric notion of fuzzy proximal normal structure defined on a nonempty and convex pair in a convex fuzzy metric space. We also prove that every nonempty, compact and convex pair of subsets of a strictly convex fuzzy metric space has the fuzzy proximal normal structure.

Keywords: strictly convex fuzzy metric space; fuzzy proximal normal structure; best proximity pair
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1. Introduction

The notion of normal structure was introduced by Brodskil and Milman [1] in 1948 in order to study the existence of common fixed points of certain sets of isometries. Later, the notion of normal structure was generalized for the weak topology.

Definition 1.1. *A Banach space X is said to have normal structure (NS) (res., weak normal structure (WNS)) if for every bounded, closed (res., weakly compact) and convex subset K of X such that* $\operatorname{diam}(K) := \sup\{\|x - y\| : x, y \in K\} > 0$*, there is a point $p \in K$ which is not a diametral point, that is,* $\sup\{\|p - x\| : x \in K\} < \operatorname{diam}(K)$.

It is well-known that compact and convex subset of a Banach space X has normal structure. In 1965, Kirk proved the following important celebrated result.

Theorem 1.2. *(Kirk's fixed point theorem [2]) Let A be a nonempty, bounded, closed (res., weakly compact) and convex subset of a Banach space X and $T : A \to A$ be a nonexpansive self-mapping, that is,*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in A.$$

If X is a reflexive Banach space with NS (res., a Banach space with WNS), then T has a fixed point.

There are many interesting extensions of Kirk's fixed point theorem. One of them is due to Ješić, where he generalized Kirk's fixed point theorem to fuzzy metric spaces [3]. To present the main theorem of [3], we recall some related concepts as below.

In 1965, Zadeh [4] introduced the notion of fuzzy set. After this pioneering work, the concept of a fuzzy metric space which is closely related to the class of probabilistic metric spaces was introduced by Kramosil and Michalek in 1975 [5]. George and Veeramani [6] modified the notion of fuzzy metric space given in [5] with the help of continuous t-norm and described a Hausdorff topology on the modified fuzzy metric space (see also [7, 8] for more general discussions).

In [3], Ješić defined strict convexity and normal structure in fuzzy metric space and proved a fixed point theorem for a nonexpansive self-mapping on a strictly convex fuzzy metric space. In this work, we study the existence of best proximity pairs by considering noncyclic contractions defined on a union of two nonempty subset of a strictly convex fuzzy metric space. We also consider the noncyclic relatively nonexpansive mappings and obtain a best proximity pair theorem which is a real extension of the aforesaid theorem in [3]. In this way, we provide an important basis for the existence of the best proximity pairs in the setting of strictly convex fuzzy metric spaces. To this end, we need the following definitions and notions.

Definition 1.3. *(Schweizer and Sklar [9]) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm (t-norm) if $*$ satisfies the following conditions:*

(i) *$*$ is commutative and associative;*
(ii) *$*$ is continuous;*
(iii) *$a * 1 = a$ for all $a \in [0, 1]$;*
(iv) *$a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.*

Some of continuous t-norms are as below:

$$\begin{aligned} a * b &= \min\{a, b\}, \\ a * b &= ab, \\ a * b &= \begin{cases} \min\{a, b\}, & \text{if } \max\{a, b\} = 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Definition 1.4. *(George and Veeramani [6]) A 3-tuple $(X, M, *)$ is said to be an fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions: For all $x, y, z \in X$, $s, t > 0$,*

(a) *$M(x, y, t) > 0$;*

(b) $M(x, y, t) = 1$ *if and only if* $x = y$;
(c) $M(x, y, t) = M(y, x, t)$;
(d) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(e) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ *is continuous.*

It is worth noticing that if (X, d) is a metric space and we define $a * b = ab$ for any $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$, then $(X, M, *)$ is a fuzzy metric space (see [6] for more information).

Remark 1.5. (Mariusz [10]) *In a fuzzy metric space* $(X, M, *)$, *the function* $M(x, y, \cdot)$ *is non-decreasing for all* $x, y \in X$.

Remark 1.6. ([6]) *A sequence* $\{x_n\}$ *in a fuzzy metric space* $(X, M, *)$ *is said to be convergent to* x *in* X *if and only if* $\lim_{n\to\infty} M(x_n, x, t) = 1$ *for all* $t > 0$.

The next lemma will be used in our coming discussions.

Lemma 1.7. ([11]) If $(X, M, *)$ is a fuzzy metric space and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then

$$\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t).$$

Definition 1.8. ([6]) *Let* $(X, M, *)$ *be a fuzzy metric space and* $r \in (0, 1)$, $t > 0$ *and* $x \in X$. *The set*

$$\mathcal{B}(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\},$$

is called an open ball with center x *and radius* r *with respect to* t.

The closed ball with center x and radius r with respect to t is given by

$$\mathcal{B}[x, r, t] = \{y \in X : M(x, y, t) \geq 1 - r\}.$$

It was announced in [6] that every fuzzy metric space $(X, M, *)$ generates a Hausdorff first countable topology, where its basis is the family of $\{B(x, r, t) : x \in X, r \in (0, 1), t > 0\}$.

Definition 1.9. ([11]) *Let* $(X, M, *)$ *be a fuzzy metric space and* A *be a nonempty subset of* X. *The mappings* $\delta_A(t) : (0, \infty) \to [0, 1]$ *is defined as*

$$\delta_A(t) := \inf_{x,y\in A} \sup_{\varepsilon<t} M(x, y, \varepsilon).$$

The constant $\delta_A = \sup_{t>0} \delta_A(t)$ *is called fuzzy diameter of nearness of the set* A.

Lemma 1.10. ([12]) Let $(X, M, *)$ be a fuzzy metric space. A subset A of X is said to be fuzzy bounded (F-bounded) if there exist $t > 0$ and $r \in (0, 1)$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Proposition 1.11. ([6]) Every compact subset of a fuzzy metric space is F-bounded.

Definition 1.12. ([3]) *Let* A *be a nonempty subset of a fuzzy metric space* $(X, M, *)$. *A point* $p \in A$ *is called a diametral point if*

$$\inf_{y\in A} \sup_{\varepsilon<t} M(p, y, \varepsilon) = \delta_A(t), \quad \forall t > 0.$$

Therefore, a point $u \in A$ *is nondiametral whenever there exists* $t_0 > 0$ *such that*

$$\inf_{y\in A} \sup_{\varepsilon<t_0} M(u, y, \varepsilon) > \delta_A(t_0).$$

In 1970, Takahashi introduced a convex structure on metric spaces [13]. It was generalized by S.N. Ješić to fuzzy metric spaces as follows.

Definition 1.13. ([3]) *A fuzzy metric space* $(X, M, *)$ *possesses a convex structure if there exists a function* $\mathcal{W} : X \times X \times [0,1] \to X$*, satisfying* $\mathcal{W}(x, y, 0) = y$*,* $\mathcal{W}(x, y, 1) = x$ *and for all* $x, y, z \in X$*,* $\theta \in (0, 1)$ *and* $t > 0$

$$M(\mathcal{W}(x, y, \theta), z, 2t) \geq M\left(x, z, \frac{t}{\theta}\right) * M\left(y, z, \frac{t}{1-\theta}\right).$$

Throughout this article $(X, M, *; \mathcal{W})$ *stand to denote a fuzzy metric space equipped with a convex structure* $\mathcal{W} : X \times X \times [0,1] \to X$ *and we call it a* convex fuzzy metric space.

Definition 1.14. *A subset* K *of a convex fuzzy metric space* $(X, M, *; \mathcal{W})$ *is said to be a convex set if for every* $x, y \in K$ *and* $\theta \in [0, 1]$ *it follows that* $\mathcal{W}(x, y, \theta) \in K$*.*

Lemma 1.15. ([3]) Let $(X, M, *; \mathcal{W})$ be a convex fuzzy metric space and $\{K_\alpha\}_{\alpha\in\Lambda}$ be a family of convex subsets of X. Then the intersection $K = \bigcap_{\alpha\in\Lambda} K_\alpha$ is a convex set.

Definition 1.16. *A convex fuzzy metric space* $(X, M, *; \mathcal{W})$ *is said to have property* (C) *if every decreasing net consists of nonempty, F-bounded, closed and convex subsets of* X *has a nonempty intersection.*

For instance if a convex fuzzy metric space $(X, M, *; \mathcal{W})$ is compact, then it has the property (C). Furthermore, if X is a reflexive Banach space, $a*b = \min\{a, b\}$ and for any $x, y \in X$ and $t > 0, \theta \in (0, 1)$,

$$M(x, y, t) = \frac{t}{t + \|x - y\|}, \quad \mathcal{W}(x, y, \theta) = \theta x + (1-\theta)y,$$

then from the Eberlein-Šmulian's theorem $(X, M, *; \mathcal{W})$ is a convex fuzzy metric space having property (C).

Definition 1.17. ([3]) *A convex fuzzy metric space* $(X, M, *; \mathcal{W})$ *is said to be strictly convex provided that for every* $x, y \in X$*, and* $\theta \in (0, 1)$ *there exists a unique element* $z = \mathcal{W}(x, y, \theta) \in X$ *for which*

$$M(x, y, \frac{t}{\theta}) = M(z, y, t), \quad M(x, y, \frac{t}{1-\theta}) = M(x, z, t),$$

for all $t > 0$*.*

We will use the following useful lemmas in our coming discussions.

Lemma 1.18. Let $(X, M, *; \mathcal{W})$ be a convex fuzzy metric space. Suppose that for every $\theta \in (0, 1)$, $t > 0$ and $x, y, z \in X$ the following condition holds

$$M(\mathcal{W}(x, y, \theta), z, t) > \min\{M(z, x, t), M(z, y, t)\}, \qquad (\sharp)$$

If there exists $u \in X$ for which

$$M(\mathcal{W}(x, y, \theta), u, t) = \min\{M(u, x, t), M(u, y, t)\},$$

for all $t > 0$, then $\mathcal{W}(x, y, \theta) \in \{x, y\}$.

Proof. Since for any $t > 0$ we have

$$M(\mathcal{W}(x, y, \theta), u, t) = \min\{M(u, x, t), M(u, y, t)\}$$

for some $u \in X$, by using the condition ($\sharp$) we must have $\theta = 0$ or $\theta = 1$ which ensures that $\mathcal{W}(x, y, 0) = y$ or $\mathcal{W}(x, y, 1) = x$ and this completes the proof. □

Lemma 1.19. ([3]) Let $(X, M, *; \mathcal{W})$ be a strictly convex fuzzy metric space. Then for any $x, y \in X$ with $x \neq y$ there exists $\theta \in (0, 1)$ such that $\mathcal{W}(x, y, \theta) \notin \{x, y\}$.

Lemma 1.20. ([3]) Let $(X, M, *; \mathcal{W})$ be a fuzzy metric space which satisfies the condition ($\sharp$). Then the closed balls $\mathcal{B}[x, r, t]$ are convex sets.

The fuzzy version of the notion of normal structure was introduced in [3] as below.

Definition 1.21. *A convex fuzzy metric space $(X, M, *; \mathcal{W})$ is said to have fuzzy normal structure if for every closed, F-bounded and convex subset K of X which consists of at least two different points, there exists a point $p \in K$ which is a non-diametral point.*

Definition 1.22. *A self-mapping f defined on a fuzzy metric space $(X, M, *)$ is said to be nonexpansive provided that*

$$M(fx, fy, t) \geq M(x, y, t), \quad \forall x, y \in X, \ \forall t > 0.$$

Example 1.23. *For any $x, y \in \mathbb{N}$ and $t > 0$, let*

$$M(x, y, t) = \begin{cases} \frac{\min\{x,y\}}{\max\{x,y\}}, & \forall t > 0, \ x \neq y \\ 1, & \forall t > 0, \ x = y, \end{cases}$$

and define $a * b = ab$. Then $(\mathbb{N}, M, *)$ is a fuzzy metric space. Define a function $f : \mathbb{N} \to \mathbb{N}$ with $f(x) = x + 1$. Then f is nonexpansive:

If $x = y$, then $M(fx, fy, t) = 1 = M(x, y, t)$, for all $t > 0$.

If $x \neq y$ and $x < y$, then

$$M(fx, fy, t) = \frac{x + 1}{y + 1} > \frac{x}{y} = M(x, y, t).$$

Therefore, $M(fx, fy, t) \geq M(x, y, t)$ for all $x, y \in \mathbb{N}, t > 0$.

The next theorem is a main result of ([3]).

Theorem 1.24. ([3]) Let $(X, M, *; \mathcal{W})$ be a strictly convex fuzzy metric space which satisfies the condition ($\sharp$). Assume that K is a nonempty, compact and convex subset of X and $f : K \to K$ is a nonexpansive self-mapping. Then f has at least one fixed point in K.

It is remarkable to note that the proof of Theorem 1.24 is based on the fact that every nonempty, compact and convex subset of a strictly convex fuzzy metric space X satisfying the condition ($\sharp$) has the fuzzy normal structure.

The main purpose of this article is to extend Theorem 1.24 from nonexpansive self-mappings to noncyclic relatively nonexpansive mappings in order to study the existence of best proximity pairs.

This article is organized as follows: In Section 2, we define the fuzzy projection operators and survey the nonemptiness of fuzzy proximal pairs under some sufficient conditions. In Section 3, we consider the class of noncyclic contractions defined on a union of two nonempty subset of a fuzzy metric space and study the existence of best proximity pairs for such mappings. Finally, in Section 4, a concept of fuzzy proximal normal structure is introduced and used to investigate a best proximity pair theorem for noncyclic relatively nonexpansive mappings which is a real extension of Theorem 1.24. We also show that every nonempty, compact and convex pair of subsets of a strictly convex metric space which satisfies the condition (♯) has the fuzzy proximal normal structure.

2. Preliminaries

Let A and B be two nonempty subsets of a convex fuzzy metric space $(X, M, *; \mathcal{W})$. We shall say that a pair (A, B) in X satisfies a property if both A and B satisfy that property. For instance, (A, B) is convex if and only if both A and B are convex; $(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C,$ and $B \subseteq D$.

We shall also adopt the following notations:

$$\Delta_{(x,B)}(t) := \inf_{y\in B} \sup_{\varepsilon<t} M(x, y, \varepsilon), \quad \forall x \in X,\ \forall t > 0,$$
$$\Delta_{(A,B)}(t) := \inf_{(x,y)\in A\times B} \sup_{\varepsilon<t} M(x, y, \varepsilon), \quad \forall t > 0,$$
$$\Delta_{(A,B)} := \sup_{t>0} \Delta_{(A,B)}(t).$$

The *closed and convex hull* of a set A will be denoted by $\overline{\mathrm{con}}(A)$ and defined as below

$$\overline{\mathrm{con}}(A) := \bigcap \Big\{C : C \text{ is a closed and convex subset of } X \text{ such that } C \supseteq A\Big\}.$$

The *F-distance* between A and B is defined by

$$\varrho_{AB}(t) := \sup_{(x,y)\in A\times B} \sup_{\epsilon<t} M(x, y, \epsilon), \quad \forall t > 0.$$

Moreover, the F-distance between an element $x \in X$ and the set B will be denoted by $\varrho_{xB}(t)$ for all $t > 0$. A point $(x, y) \in A \times B$ is called *F-proximal* provided that

$$M(x, y, t) = \varrho_{AB}(t), \quad \forall t > 0.$$

The *F-proximal pair* of (A, B) is denoted by (A_0, B_0) which is defined as follows:

$$A_0 := \{x \in A : \ M(x, y, t) = \varrho_{AB}(t), \quad \text{for all } t > 0, \ \text{for some } y \in B\},$$
$$B_0 := \{y \in B : \ M(x, y, t) = \varrho_{AB}(t), \quad \text{for all } t > 0, \ \text{for some } x \in A\}.$$

It is remarkable to note that the F-proximal pairs may be empty. Next example illustrates this fact.

Example 2.1. *Consider the Banach space $X = \ell^p$, $(1 \le p < \infty)$ with the canonical basis $\{e_n\}$. Let $k \in (0, 1)$ be fixed and suppose $A = \Big\{((1 + k^{2n})e_{2n}) : n \in \mathbb{N}\Big\}$ and $B = \Big\{((1 + k^{2n+1})e_{2n+1}) : n \in \mathbb{N}\Big\}$. Assume that $M(x, y, t) = \frac{t}{t+\|x-y\|_p}$ for all $(x, y) \in A \times B$ and $t > 0$ and let $a * b = ab$ for any $a, b \in [0, 1]$. Then $(X, M, *)$ is a fuzzy metric space and the sets A and B are bounded, closed and we have*

$$\varrho_{AB}(t) = \sup_{(x,y)\in A\times B} \sup_{\varepsilon<t} \left(\frac{\varepsilon}{\varepsilon + \|x - y\|_p}\right) = \sup_{\varepsilon<t} \left(\frac{\varepsilon}{\varepsilon + (2)^{1/p}}\right).$$

In view of the fact that for any $(x, y) \in A \times B$ and $t > 0$ we have $M(x, y, t) < \varrho_{AB}(t)$, then $A_0 = B_0 = \emptyset$.

Definition 2.2. *Let $(X, M, *; \mathcal{W})$ be a convex fuzzy metric space and E be a nonempty subset of X. The* fuzzy projection operator (briefly F-projection operator) $\mathbb{P}_E : X \to 2^E$ *is defined as*

$$\mathbb{P}_E(x) := \Big\{y \in E : M(x, y, t) = \varrho_{x,E}(t),\ \forall t > 0\Big\},$$

where 2^E denotes the set of all subsets of E.

Proposition 2.3. Let E be a nonempty, F-bounded, closed and convex subset of a strictly convex fuzzy metric space $(X, M, *; \mathcal{W})$ which satisfies the condition ($\sharp$). If X has the property (C), then the F-projection operator $\mathbb{P}_E$ is single-valued.

Proof. Consider an arbitrary element $x \in X$ and let $r \in (0, 1)$ be fixed. Define

$$P_r = \Big\{y \in E : M(x, y, t) \geq \varrho_{x,E}(t) - (1 - r),\ \forall t > 0\Big\}.$$

If $y_1, y_2 \in P_r$ and $\theta \in (0, 1)$, then by the condition ($\sharp$) for all $t > 0$ we have

$$M(x, \mathcal{W}(y_1, y_2, \theta), t) > \min\{M(x, y_1, t), M(x, y_2, t)\} \geq \varrho_{x,E}(t) - (1 - r),$$

which deduces that $\mathcal{W}(y_1, y_2, \theta) \in P_r$, that is, P_r is convex. We also note that if $\{y_k\}_{k\geq 1}$ is a sequence in P_r which converges to an element $y \in X$, then from Lemma 1.7 we obtain

$$M(x, y, t) = \lim_{k\to\infty} M(x, y_k, t) \geq \varrho_{x,E}(t) - (1 - r), \quad \forall t > 0,$$

which ensures that $y \in P_r$, that is, P_r is closed. Thus $\{P_r\}_r$ is a descending chain of nonempty, F-bounded, closed and convex subsets of X. Since X has the property (C), $\bigcap_{r>0} P_r$ is nonempty. Let $p \in \bigcap_{r>0} P_r$. Then

$$M(x, p, t) \geq \varrho_{x,E}(t) - (1 - r), \quad \forall r \in (0, 1).$$

Now, if $r \to 1^-$, we obtain $M(x, p, t) = \varrho_{x,E}(t)$ and so $p \in \mathbb{P}_E(x)$. On the other hand, if $p' \in X$ is another member of $\mathbb{P}_E(x)$, then from the strict convexity of X there exists $\theta_0 \in (0, 1)$ such that $E \ni \mathcal{W}(p, p', \theta_0) \notin \{p, p'\}$. Using the condition ($\sharp$) for any $t > 0$ we obtain

$$M(x, \mathcal{W}(p, p', \theta_0), t) > \min\{M(x, p, t), M(x, p', t)\} = \varrho_{x,E}(t),$$

which is a contradiction. □

In what follows we present some sufficient conditions which guarantees the nonemptiness of F-proximal pairs.

Lemma 2.4. Let (A, B) be a nonempty, F-bounded, closed and convex pair in a strictly convex fuzzy metric space $(X, M, *; \mathcal{W})$ which satisfies the condition ($\sharp$). If X has the property (C), then (A_0, B_0) is also nonempty, F-bounded, closed and convex. Furthermore, $\varrho_{AB}(t) = \varrho_{A_0B_0}(t)$ for all $t > 0$.

Proof. Let $r \in (0, 1)$ be fixed and put

$$E_r = \Big\{x \in A : \varrho_{x,B}(t) \geq \varrho_{A,B}(t) - (1 - r),\ \forall t > 0\Big\}.$$

By the fact that $\varrho_{A,B}(t) = \sup_{u\in A} \varrho_{u,B}(t)$ for any $t > 0$, the set E_r is nonempty. Suppose $\{x_n\}$ is a sequence in E_r such that $x_n \longrightarrow x \in X$. Thus $\varrho_{x_n,B}(t) \geq \varrho_{A,B}(t) - (1-r)$ for all $n \in \mathbb{N}$. So there exists an element $y \in B$ such that $M(x_n, y, t) \geq \varrho_{A,B}(t) - (1-r)$ for all $n \in \mathbb{N}$. Using Lemma 1.7, we obtain

$$\varrho_{x,B}(t) \geq M(x, y, t) = \lim_{n\to\infty} M(x_n, y, t) \geq \varrho_{A,B}(t) - (1-r),$$

which concludes that $x \in E_r$, that is, E_r is closed. Moreover, if $u_1, u_2 \in E_r$ and $\theta \in (0, 1)$, then by the condition ($\sharp$) for any $y \in B$ we have

$$M(\mathcal{W}(u_1, u_2, \theta), y, t) > \min\{M(u_1, y, t), M(u_2, y, t)\}.$$

Taking supremum over all $y \in B$ in above relation, we deduce that

$$\varrho_{\mathcal{W}(u_1,u_2,\theta),B}(t) \geq \min\{\varrho_{u_1,B}(t), \varrho_{u_2,B}(t)\} \geq \varrho_{A,B}(t) - (1-r),$$

which implies that E_r is convex. Since A is F-bounded, E_r is F-bounded too. Thereby $\{E_r\}_{r>0}$ is a decreasing net consists of nonempty, F-bounded, closed and convex subsets of X. Since X has the property (C), $\bigcap_{r>0} E_r$ is nonempty. Using Proposition 2.3 we obtain $A_0 = \bigcap_{r>0} E_r$, which implies that A_0 is a nonempty, closed and convex subset of X. By a similar argument we can see that B_0 is a nonempty, closed and convex subsets of X. □

Definition 2.5. *A pair (A, B) in a convex fuzzy metric space $(X, M, *; \mathcal{W})$ is said to be* F-proximinal *if $A_0 = A$ and $B_0 = B$.*

Definition 2.6. *Let (A, B) be a nonempty pair in a convex fuzzy metric space $(X, M, *; \mathcal{W})$. A mapping $\mathcal{T} : A \cup B \longrightarrow A \cup B$ is said to be* noncyclic *if $\mathcal{T}(A) \subseteq A$ and $\mathcal{T}(B) \subseteq B$. Also, a point $(p, q) \in A \times B$ is said to be a* best proximity pair *for the noncyclic mapping $\mathcal{T}$ whenever*

$$\mathcal{T}p = p, \quad \mathcal{T}q = q, \quad M(p, q, t) = \varrho_{AB}(t), \ \forall t > 0.$$

The set of all best proximity pairs of the noncyclic mapping $\mathcal{T}$ is denoted by $\mathrm{Best}_{A\times B}(\mathcal{T})$.

3. Noncyclic contraction type mapping

We begin our main result of this section by introducing the following class of noncyclic mappings.

Definition 3.1. *Let (A, B) be a nonempty pair in a convex fuzzy metric space $(X, M, *; \mathcal{W})$. A mapping $\mathcal{T} : A \cup B \longrightarrow A \cup B$ is said to be a*

- noncyclic contraction mapping *if $\mathcal{T}$ is noncyclic and there exists $\lambda \in (0, 1)$ such that for all $(x, y) \in A \times B$ and $t > 0$*

$$M(\mathcal{T}x, \mathcal{T}y, t) \geq \lambda M(x, y, t) + (1-\lambda)\varrho_{AB}(t);$$

- noncyclic contraction type mapping *if $\mathcal{T}$ is noncyclic and there exists $\lambda \in (0, 1)$ such that for all $(x, y) \in A \times B$ and $t > 0$*

$$M(\mathcal{T}x, \mathcal{T}y, t) \geq \lambda \max\{M(x, y, t), M(x, \mathcal{T}y, t), M(\mathcal{T}x, y, t)\} + (1-\lambda)\varrho_{AB}(t);$$

- noncyclic relatively nonexpansive mapping, *if $\mathcal{T}$ is noncyclic and*

$$M(\mathcal{T}x, \mathcal{T}y, t) \geq M(x, y, t),$$

for all $(x, y) \in A \times B,\ t > 0$. In this case, if $A = B$, then $\mathcal{T}$ is called a nonexpansive self-mapping.

It is clear that every noncyclic contraction type mapping is a noncyclic contraction. Moreover, any noncyclic contraction type mapping is a relatively nonexpansive mapping, but the reverse is not true.

Example 3.2. *Let $X = \mathbb{R}$, $A = [0, 1]$ and $B = [3, 4]$. Suppose $M(x, y, t) = \frac{t}{t+|x-y|}$ and define $\mathcal{T} : A \cup B \to A \cup B$ by*

$$\mathcal{T}(x) = \begin{cases} x, & x \in A \\ 3, & x \in B. \end{cases}$$

Notice that $\mathcal{T}(A) \subseteq A$, $\mathcal{T}(B) \subseteq B$ and $\varrho_{AB}(t) = \frac{t}{t+2}$. Also, $\forall (x, y) \in A \times B$, $t > 0$,

$$M(\mathcal{T}x, \mathcal{T}y, t) = \frac{t}{t + |x - 3|} \geq \frac{t}{t + |x - y|} = M(x, y, t).$$

Now suppose $\mathcal{T}$ is a noncyclic contraction type mapping. Then for some $\lambda \in (0, 1)$ and for all $(x, y) \in A \times B$, we have

$$M(\mathcal{T}x, \mathcal{T}y, t) - \varrho_{AB}(t) \geq \lambda[M(x, y, t) - \varrho_{AB}(t)], \quad \forall t > 0.$$

Thus we have,

$$\frac{t}{t + |x - 3|} - \frac{t}{t + 2} \geq \lambda \left[\frac{t}{t + |x - y|} - \frac{t}{t + 2}\right], \quad \forall t > 0.$$

Besides for $x = 0$, $y = 3$ we have

$$\frac{t}{t + 3} - \frac{t}{t + 2} \geq \lambda \left[\frac{t}{t + 3} - \frac{t}{t + 2}\right], \quad \forall t > 0,$$

which implies that $\lambda \geq 1$ and this is a contradiction. Hence, $\mathcal{T}$ is not a noncyclic contraction type mapping.

It is worth noticing that the class of noncyclic relatively nonexpansive mappings may not be continuous in general.

Example 3.3. *Consider $X = \mathbb{R}$ and let $A = [-1, 0]$, $B = [0, 1]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$. Define $\mathcal{T} : A \cup B \to A \cup B$ by*

$$\mathcal{T}(x) = \begin{cases} -x - 1, & x \in A \cap [-1, -\frac{1}{2}] \\ \frac{x}{2}, & x \in A \cap (-\frac{1}{2}, 0] \\ x, & x \in B. \end{cases}$$

Clearly $\mathcal{T}$ is not continuous, $\mathcal{T}(A) \subseteq A$ and $\mathcal{T}(B) \subseteq B$. We claim that $\mathcal{T}$ is a noncyclic relatively nonexpansive mapping.

If $x \in A \cap [-1, -\frac{1}{2}]$, $y \in B$, $t > 0$, then

$$M(\mathcal{T}x, \mathcal{T}y, t) = \frac{t}{t + |\mathcal{T}x - \mathcal{T}y|} = \frac{t}{t + | - x - 1 - y|} \geq \frac{t}{t + |x - y|} = M(x, y, t).$$

If $x \in A \cap (-\frac{1}{2}, 0]$, $y \in B$, $t > 0$, then

$$M(\mathcal{T}x, \mathcal{T}y, t) = \frac{t}{t + |\mathcal{T}x - \mathcal{T}y|} = \frac{t}{t + |\frac{x}{2} - y|} \geq \frac{t}{t + |x - y|} = M(x, y, t).$$

Hence $\mathcal{T}$ is a noncyclic relatively nonexpansive mapping.

In 2005, Eldred et al. studied the existence of best proximity pairs for noncyclic relatively nonexpansive mappings defined on a union of two nonempty, weakly compact and convex subsets of a strictly convex Banach space X by using a geometric notion of proximal normal structure (see Theorem 2.2 of [14]). In the case that we restrict the considered mappings to noncyclic contractions, then the existence of best proximity pairs is guaranteed without the proximal normal structure [15, 16].

In what follows we present best proximity pair results in the framework of strictly convex fuzzy metric spaces. To this end we need the following useful lemmas.

Lemma 3.4. Let (A, B) be a nonempty, F-bounded, closed, convex pair in a strictly convex fuzzy metric space $(X, M, *; \mathcal{W})$ which satisfies the condition $(\sharp)$ and has the property (C). Let $\mathcal{T} : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping. Then there exists a pair $(G_1, G_2) \subseteq (A, B)$ which is minimal with respect to being nonempty, closed, convex and $\mathcal{T}$-invariant pair of subsets of (A, B) such that $\varrho_{G_1G_2}(t) = \varrho_{AB}(t)$, for all $t > 0$. Also (G_1, G_2) is F-proximinal.

Proof. It follows from Lemma 2.4 that (A_0, B_0) is a nonempty, closed and convex pair for which $\varrho_{AB}(t) = \varrho_{A_0B_0}(t)$ for all $t > 0$. Also, if $x \in A_0$, then there exists a point $y \in B_0$ such that $M(x, y, t) = \varrho_{AB}(t)$ for all $t > 0$. Since that $\mathcal{T}$ is relatively nonexpansive,

$$\varrho_{AB}(t) \geq M(\mathcal{T}x, \mathcal{T}y, t) \geq M(x, y, t) = \varrho_{AB}(t), \ \forall t > 0,$$

and so $\mathcal{T}x \in A_0$, that is, $\mathcal{T}(A_0) \subseteq A_0$. Equivalently, $\mathcal{T}(B_0) \subseteq B_0$ which concludes that (A_0, B_0) is $\mathcal{T}$-invariant. Assume that Ξ is a collection of all nonempty sets $G \subseteq A_0 \cup B_0$ such that $(G \cap A_0, G \cap B_0)$ is a nonempty, closed and convex pair which is F-proximinal, $\mathcal{T}$-invariant and

$$\varrho_{(G\cap A_0)(G\cap B_0)}(t) = \varrho_{A_0B_0}(t)(= \varrho_{AB}(t))$$

for all $t > 0$. Note that $A_0 \cup B_0 \in \Xi$ and so Ξ is nonempty. Suppose $\{U_j\}_{j\in J}$ is a descending chain in Ξ and set $U := \bigcap_{j\in J} U_j$. Since X has the property (C), we have

$$U \cap A_0 = (\bigcap_{j\in J} U_j) \cap A_0 = \bigcap_{j\in J}(U_j \cap A_0) \neq \emptyset.$$

Obviously, $U \cap A_0$ is closed and convex. Similarly, the set $U \cap B_0$ is also nonempty, closed, convex and it is easy to see that the pair $(U \cap A_0, U \cap B_0)$ is $\mathcal{T}$-invariant. We show that $\varrho_{(U\cap A_0)(U\cap B_0)}(t) = \varrho_{AB}(t)$ for all $t > 0$ and that $(U \cap A_0, U \cap B_0)$ is F-proximinal. Let $x \in U \cap A_0$. Then $x \in U_j \cap A_0$ for any $j \in J$. In view of the fact that the pair $(U_j \cap A_0, U_j \cap B_0)$ is F-proximinal, there exists $y \in U_j \cap B_0$ for which $M(x, y, t) = \varrho_{AB}(t)$ for all $t > 0$. Note that this element, y, is unique. Indeed, if there is another element $y' \in B_0$ such that $M(x, y', t) = \varrho_{AB}(t)$ for all $t > 0$ then from Lemma 1.19 there exists $\theta \in (0, 1)$ such that $\mathcal{W}(y, y', \theta) \notin \{y, y'\}$. It follows from the condition $(\sharp)$ that

$$M(\mathcal{W}(y, y', \theta), x, t) = \min\{M(y, x, \theta), x, t), M(y', x, \theta), x, t)\}$$

for all $t > 0$. Using Lemma 1.18 we obtain $\mathcal{W}(y, y', \theta) \in \{y, y'\}$ which is a contradiction. Hence $(U \cap A_0, U \cap B_0) \in \Xi$. Now using Zorn's lemma, Ξ has a minimal element, say G. If we set $G_1 = G \cap A_0$ and $G_2 = G \cap B_0$, then the result follows. It is worth noticing that since $G \in \Xi$ is minimal we must have (G_1, G_2) is F-proximinal. □

It is remarkable to note that if in Lemma 3.4 the pair (A, B) is compact, then the condition of property (C) of X can be dropped.

Notation: Under the hypothesis of Lemma 3.4, by $\mathcal{M}_{\mathcal{T}}(A, B)$ we denote the family of all nonempty, closed, convex, minimal and $\mathcal{T}$-invariant pair $(G_1, G_2) \subseteq (A, B)$ for which $\varrho_{G_1,G_2}(t) = \varrho_{AB}(t)$ for all $t > 0$.

Lemma 3.5. Let (A, B) be a nonempty pair in a convex fuzzy metric space $(X, M, *; \mathcal{W})$. Then

$$\Delta_{\left(\overline{\mathrm{con}}(A),\overline{\mathrm{con}}(B)\right)}(t) = \Delta_{(A,B)}(t), \quad \forall t > 0.$$

Proof. Since $(A, B) \subseteq (\overline{\mathrm{con}}(A), \overline{\mathrm{con}}(B))$, it is sufficient to verify that $\Delta_{(A,B)}(t) \leq \Delta_{(\overline{\mathrm{con}}(A),\overline{\mathrm{con}}(B))}(t)$ for all $t > 0$. Let $x \in A$ and $t > 0$ be arbitrary and fixed. Then for any $y \in B$ we have $M(x, y, t) \geq \Delta_{(x,B)}(t)$. Put $\Delta_{(x,B)}(t) := 1 - r_x$. Thus we have $B \subseteq \mathcal{B}[x, r_x, t]$ which implies that $\overline{\mathrm{con}}(B) \subseteq \mathcal{B}[x, r_x, t]$. Therefore, $\overline{\mathrm{con}}(B) \subseteq \bigcap_{x\in A} \mathcal{B}[x, r_x, t]$. Put $\Delta_{(A,B)}(t) := 1 - r$. Now if $v \in \overline{\mathrm{con}}(B)$, then $\overline{\mathrm{con}}(A) \subseteq \mathcal{B}[v, r, t]$. Indeed, for all $x \in A$ since $v \in \overline{\mathrm{con}}(B)$,

$$M(x, v, t) \geq 1 - r_x = \Delta_{(x,B)}(t) \geq \Delta_{(A,B)}(t) = 1 - r,$$

and so $x \in \mathcal{B}[v, r, t]$, that is, $A \subseteq \mathcal{B}[v, r, t]$ which concludes that $\overline{\mathrm{con}}(A) \subseteq \mathcal{B}[v, r, t]$. This implies that

$$\overline{\mathrm{con}}(A) \subseteq \bigcap_{v\in\overline{\mathrm{con}}(B)} \mathcal{B}[v, r, t],$$

which ensures that $\Delta_{\left(\overline{\mathrm{con}}(A),\overline{\mathrm{con}}(B)\right)}(t) \geq 1 - r = \Delta_{(A,B)}(t)$ and hence the lemma. □

Next theorem is the main result of this section.

Theorem 3.6. Let (A, B) be a nonempty, F-bounded, closed and convex pair in a strictly convex fuzzy metric space $(X, M, *; \mathcal{W})$ which satisfies the condition ($\sharp$) and has the property (C). Assume that $\mathcal{T}: A \cup B \to A \cup B$ is a noncyclic contraction type mapping in the sense of Definition 3.1. Then $\mathrm{Best}_{A\times B}(\mathcal{T}) \neq \emptyset$.

Proof. Lemma 3.4 guarantees that $\mathcal{M}_{\mathcal{T}}(A, B)$ is nonempty. Suppose $(G_1, G_2) \in \mathcal{M}_{\mathcal{T}}(A, B)$. In view of the fact that $\mathcal{T}$ is noncyclic, $(\overline{\mathrm{con}}(\mathcal{T}(G_1)), \overline{\mathrm{con}}(\mathcal{T}(G_2))) \subseteq (G_1, G_2)$ and so

$$\mathcal{T}(\overline{\mathrm{con}}(\mathcal{T}(G_1))) \subseteq \mathcal{T}(G_1) \subseteq \overline{\mathrm{con}}(\mathcal{T}(G_1)),$$

$$\mathcal{T}(\overline{\mathrm{con}}(\mathcal{T}(G_2))) \subseteq \mathcal{T}(G_2) \subseteq \overline{\mathrm{con}}(\mathcal{T}(G_2)),$$

which implies that the closed and convex pair $(\overline{\mathrm{con}}(\mathcal{T}(G_1)), \overline{\mathrm{con}}(\mathcal{T}(G_2)))$ is $\mathcal{T}$-invariant, that is,

$$(\overline{\mathrm{con}}(\mathcal{T}(G_1)), \overline{\mathrm{con}}(\mathcal{T}(G_2))) \in \mathcal{M}_{\mathcal{T}}(A, B).$$

It follows from the minimality of (G_1, G_2) that $\overline{\text{con}}(\mathcal{T}(G_1)) = G_1$ and $\overline{\text{con}}(\mathcal{T}(G_2)) = G_2$. Since (G_1, G_2) is F-proximinal, there exists an element $(x, y) \in G_1 \times G_2$ for which

$$M(x, y, t) = \varrho_{G_1G_2}(t)\Big(= \varrho_{AB}(t)\Big), \quad \forall t > 0.$$

Relatively nonexpansiveness of $\mathcal{T}$ deduces that

$$\varrho_{AB}(t) \geq \varrho_{\overline{\text{con}}(\mathcal{T}(G_1))\overline{\text{con}}(\mathcal{T}(G_2))}(t) \geq M(\mathcal{T}x, \mathcal{T}y, t) \geq M(x, y, t) = \varrho_{AB}(t),$$

for all $t > 0$ and so, $\varrho_{\overline{\text{con}}(\mathcal{T}(G_1))\overline{\text{con}}(\mathcal{T}(G_2))}(t) = M(\mathcal{T}x, \mathcal{T}y, t) = \varrho_{AB}(t)$. Let $u \in G_1$ and $t > 0$ be an arbitrary fixed number. If $v \in G_2$, because of the fact that $\mathcal{T}$ is a noncyclic contraction type mapping, we have

$$\begin{aligned} M(\mathcal{T}u, \mathcal{T}v, t) &\geq \lambda \max\{M(u, v, t), M(u, \mathcal{T}v, t), M(\mathcal{T}u, v, t)\} + (1 - \lambda)\varrho_{AB}(t) \\ &\geq \lambda \max\{M(u, v, t), M(u, \mathcal{T}v, t), M(\mathcal{T}u, v, t)\} + (1 - \lambda)M(u, v, t) \\ &\geq \lambda\Delta_{(G_1,G_2)}(t) + (1 - \lambda)\varrho_{AB}(t) \end{aligned}$$

where $\lambda \in (0, 1)$. Put

$$1 - r := \lambda\Delta_{(G_1,G_2)}(t) + (1 - \lambda)\varrho_{AB}(t).$$

Then $M(\mathcal{T}u, \mathcal{T}v, t) \geq 1 - r$, and hence $\mathcal{T}v \in \mathcal{B}[\mathcal{T}u, r, t]$ for all $v \in G_2$. Thus $\mathcal{T}(G_2) \subseteq \mathcal{B}[\mathcal{T}u, r, t]$ which concludes that

$$G_2 = \overline{\text{con}}(\mathcal{T}(G_2)) \subseteq \mathcal{B}[\mathcal{T}u, r, t].$$

Thus for any $w \in G_2$ we have $M(\mathcal{T}u, w, t) \geq 1 - r$ and so

$$\Delta_{(\mathcal{T}u,G_2)}(t) = \inf_{w\in G_2} M(\mathcal{T}u, w, t) \geq 1 - r, \quad \forall u \in G_1.$$

Therefore, $\Delta_{(\mathcal{T}(G_1),G_2)}(t) = \inf_{x\in G_1} \Delta_{(\mathcal{T}u,G_2)}(t) \geq 1 - r$. Equivalently, $\Delta_{(G_1,\mathcal{T}(G_2))}(t) \geq 1 - r$. Using Lemma 3.5 we obtain

$$\Delta_{(G_1,G_2)}(t) = \Delta_{\left(\overline{\text{con}}(\mathcal{T}(G_1)),G_2\right)}(t) = \Delta_{(\mathcal{T}(G_1),G_2)}(t) \geq 1 - r = \lambda\Delta_{(G_1,G_2)}(t) + (1 - \lambda)\varrho_{AB}(t).$$

Thereby $\Delta_{(G_1,G_2)}(t) = \varrho_{AB}(t)$, which leads us to

$$M(x, y, t) = \varrho_{AB}(t), \quad \forall (x, y) \in G_1 \times G_2.$$

We assert that both the sets G_1 and G_2 are singleton. Let x_1 and x_2 be two distinct elements of G_1. Since X is strictly convex, from Lemma 1.19, there exists $\theta \in (0, 1)$ for which $\mathcal{W}(x_1, x_2, \theta) \notin \{x_1, x_2\}$. According to the condition ($\sharp$), for any $y \in G_2$ we obtain

$$\varrho_{AB}(t) \geq M(\mathcal{W}(x_1, x_2, \theta), y, t) > \min\{M(x_1, y, t), M(x_2, y, t)\} \geq \Delta_{(G_1,G_2)}(t) = \varrho_{AB}(t), \ \forall t > 0,$$

which is a contradiction. So G_1 is singleton. Similarly, G_2 is singleton too. Let $G_1 = \{p\}$ and $G_2 = \{q\}$ for some $(p, q) \in G_1 \times G_2$. Then $(p, q) \in \text{Best}_{A\times B}(\mathcal{T})$ and the proof is completed. □

The following corollaries are straightforward consequences of Theorem 3.6.

Corollary 3.7. *Let (A, B) be a nonempty, F-bounded, closed and convex pair in a strictly convex fuzzy metric space $(X, M, *; \mathcal{W})$ which satisfies the condition $(\sharp)$ and has the property* (C). *If $\mathcal{T}: A \cup B \to A \cup B$ is a noncyclic contraction mapping, then* $\mathrm{Best}_{A\times B}(\mathcal{T}) \neq \emptyset$.

Corollary 3.8. *Let (A, B) be a nonempty, closed and convex pair in a strictly convex and compact fuzzy metric space $(X, M, *; \mathcal{W})$ which satisfies the condition $(\sharp)$. If $\mathcal{T}: A \cup B \to A \cup B$ is a noncyclic contraction type mapping, then* $\mathrm{Best}_{A\times B}(\mathcal{T}) \neq \emptyset$.

In the next section, we present an extension version of Corollary 3.8 for noncyclic relatively nonexpansive mappings. We do that by considering a geometric concept of *fuzzy proximal normal structure* which is defined on a nonempty and convex pair of subsets of a convex fuzzy metric space.

4. Noncyclic relatively nonexpansive mappings

The notion of proximal normal structure ($\mathcal{PNS}$ for brief) was first introduced in [14] in the setting of Banach spaces in order to study the existence of best proximity pairs for noncyclic relatively nonexpansive mappings. After that, in [17], a concept of *proximal quasi-normal structure* as a generalization of $\mathcal{PNS}$ was presented in the framework of convex metric spaces for the purpose of survey the existence of best proximity points for cyclic relatively nonexpansive mappings. We also mention that a characterization of $\mathcal{PNS}$ was given in [18] by using proximal diametral sequences. It was announced in [14] that every nonempty, compact and convex pair in a Banach space X has the $\mathcal{PNS}$ (see also Theorem 3.5 of [18] for a different approach to the same problem).

In what follows we present the concept of $\mathcal{PNS}$ in the setting of convex fuzzy metric spaces.

Definition 4.1. *A convex pair (A, B) in a convex fuzzy metric space $(X, M, *; \mathcal{W})$ is said to have Fuzzy proximal normal structure (F-$\mathcal{PNS}$ for brief) if for any F-bounded, closed, convex and proximinal pair $(G_1, G_2) \subseteq (A, B)$ such that $\varrho_{G_1G_2}(t) = \varrho_{AB}(t)$ and $\Delta_{(G_1,G_2)}(t) < \varrho_{AB}(t)$ for all $t > 0$, there exist a point $(u, v) \in G_1 \times G_2$ and $t_0 > 0$ such that*

$$\min\left\{\Delta_{(u,G_2)}(t_0), \Delta_{(G_1,v)}(t_0)\right\} > \Delta_{(G_1,G_2)}(t_0).$$

It is clear that under the assumptions of the above definition, the sets G_1 and G_2 are not singleton. Moreover, if we take $A = B$, then we get the notion of fuzzy normal structure which was introduced in [3].

Remark 4.2. It is remarkable to note that if X is a Banach space, $a * b = \min\{a, b\}$ and for any $x, y \in X$ and $t > 0, \theta \in (0, 1)$, we define

$$M(x, y, t) = \frac{t}{t + \|x - y\|}, \quad \mathcal{W}(x, y, \theta) = \theta x + (1 - \theta)y,$$

then the concepts of F-$\mathcal{PNS}$ and $\mathcal{PNS}$ coincide. In this way, every nonempty, bounded, closed and convex pair in a uniformly convex Banach space X has the F-$\mathcal{PNS}$ (see Proposition 2.1 of [14]). We refer to [18–20] for further information about the $\mathcal{PNS}$ in Banach spaces.

Definition 4.3. *We say that a convex fuzzy metric space $(X, M, *; \mathcal{W})$ is strongly convex provided that for any $x_1, x_2, y_1, y_2 \in X$ and $\theta \in (0, 1)$ we have*

$$M(\mathcal{W}(x_1, x_2, \theta), \mathcal{W}(y_1, y_2, \theta), t) \geq \theta M(x_1, y_1, t) + (1 - \theta)M(x_2, y_2, t), \quad \forall t > 0.$$

Example 4.4. *Suppose that $(X, \mathcal{W}, d)$ is a hyperbolic metric space in the sense of Kohlenbach ([21]). If we define $a * b = \min\{a, b\}$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$, then $(X, M, *; \mathcal{W})$ is a strongly convex fuzzy metric space.*

We are now ready to state the main conclusion of this section.

Theorem 4.5. Let (A, B) be a nonempty, F-bounded, closed and convex pair in a strongly convex fuzzy metric space $(X, M, *; \mathcal{W})$ which satisfies the condition $(\sharp)$ and has the property (C). Let $\mathcal{T} : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping. If moreover, X is strictly convex and (A, B) has F-$\mathcal{PNS}$, then $\mathrm{Best}_{A\times B}(\mathcal{T}) \neq \emptyset$.

Proof. It follows from Lemma 3.4 that $\mathcal{M}_{\mathcal{T}}(A, B)$ is nonempty. Assume that $(G_1, G_2) \in \mathcal{M}_{\mathcal{T}}(A, B)$. Equivalent reasoning of the proof of Theorem 3.6 concludes that $\overline{\mathrm{con}}(\mathcal{T}(G_1)) = G_1$ and $\overline{\mathrm{con}}(\mathcal{T}(G_2)) = G_2$. In the case that $\Delta_{(G_1,G_2)}(t) = \varrho_{AB}(t)$ for all $t > 0$, then by a similar argument of Theorem 3.6 we are finished. So assume that $\Delta_{(G_1,G_2)}(t) < \varrho_{AB}(t)$ for all $t > 0$. Since (A, B) has F-$\mathcal{PNS}$, there is a point $(u, v) \in G_1 \times G_2, t_0 > 0$ and $\nu \in (0, 1)$ such that

$$\nu \min\left\{\Delta_{(u,G_2)}(t_0), \Delta_{(G_1,v)}(t_0)\right\} \geq \Delta_{(G_1,G_2)}(t_0).$$

By the proximinality of the pair (G_1, G_2), there exists an element $(u', v') \in G_1\times G_2$ such that $M(u, v', t) = \sigma_{AB}(t) = M(u', v, t)$ for all $t > 0$. For $\theta \in (0, 1)$ put $u^* := \mathcal{W}(u, u', \theta) \in G_1$ and $v^* := \mathcal{W}(v', v, \theta) \in G_2$. Since X is strongly convex,

$$\begin{aligned} M(u^*, v^*, t) &= M(\mathcal{W}(u, u', \theta), \mathcal{W}(v', v, \theta), t) \\ &\geq \theta M(u, v', t) + (1-\theta)M(u', v, t) \\ &= \varrho_{AB}(t), \end{aligned}$$

which implies that $M(u^*, v^*, t) = \varrho_{AB}(t)$ for all $t > 0$. On the other hand for any $y \in G_2$ we have

$$\begin{aligned} M(u^*, y, t_0) &= M(\mathcal{W}(u, u', \theta), y, t_0) \\ &\geq \theta M(u, y, t_0) + (1-\theta)M(u', y, t_0). \end{aligned}$$

By taking infimum of two sides of the above inequality on $y \in G_2$, we conclude that

$$\begin{aligned} \Delta_{(u^*,G_2)}(t_0) &\geq \theta\Delta_{(u,G_2)}(t_0) + (1-\theta)M(u', G_2, t_0) \\ &\geq \frac{\theta}{\nu}\Delta_{(G_1,G_2)}(t_0) + (1-\theta)\Delta_{(G_1,G_2)}(t_0) \\ &> \Delta_{(G_1,G_2)}(t_0). \end{aligned}$$

Similarly, we can see that

$$\Delta_{(G_1,v^*)}(t_0) > \Delta_{(G_1,G_2)}(t_0).$$

Therefore, $\min\{\Delta_{(u^*,G_2)}(t_0), \Delta_{(G_1,v^*)}(t_0)\} > \Delta_{(G_1,G_2)}(t_0)$. Put $1 - r_1 := \Delta_{(u^*,G_2)}(t_0)$ and $1 - r_2 := \Delta_{(G_1,v^*)}(t_0)$. If we define

$$G_1^* = \left(\bigcap_{y\in G_2} \mathcal{B}[y, r_1, t_0]\right) \cap G_1 \quad \text{and} \quad G_2^* = \left(\bigcap_{x\in G_1} \mathcal{B}[x, r_2, t_0]\right) \cap G_2,$$

then (G_1^*, G_2^*) is a closed and convex subset of (G_1, G_2) and $(u^*, v^*) \in G_1^* \times G_2^*$ which implies that $\varrho_{G_1^*G_2^*}(t) = \varrho_{AB}(t)$ for any $t > 0$. We show that $\mathcal{T}$ is noncyclic on $G_1^* \cup G_2^*$. Suppose $x \in G_1^*$. Then $x \in G_1$ and for any $y \in G_2$ we have $M(x, y, t_0) \geq 1 - r_1$. Since $\mathcal{T}$ is relatively nonexpansive, $M(\mathcal{T}x, \mathcal{T}y, t_0) \geq M(x, y, t_0) \geq 1 - r_1$, that is, $\mathcal{T}y \in \mathcal{B}[\mathcal{T}x, r_1, t_0]$ for all $y \in G_2$. Thus $\mathcal{T}(G_2) \subseteq \mathcal{B}[\mathcal{T}x, r_1, t_0]$. Thereby, $G_2 = \overline{\text{con}}(\mathcal{T}(G_2)) \subseteq \mathcal{B}[\mathcal{T}x, r_1, t_0]$ and so $\mathcal{T}x \in G_1^*$ for all $x \in G_1^*$, which ensures that $\mathcal{T}(G_1^*) \subseteq G_1^*$. Equivalently, $\mathcal{T}(G_2^*) \subseteq G_2^*$, that is, $\mathcal{T}$ is noncyclic on $G_1^* \cup G_2^*$. Minimality of (G_1, G_2) deduces that $G_1 = G_1^*$ and $G_2 = G_2^*$. Hence

$$G_1 \subseteq \left(\bigcap_{y \in G_2} \mathcal{B}[y, r_1, t_0] \right) \quad \text{and} \quad G_2 \subseteq \left(\bigcap_{x \in G_1} \mathcal{B}[x, r_2, t_0] \right).$$

Thus, for any $(x, y) \in G_1 \times G_2$ we have $M(x, y, t_0) \geq \max\{1 - r_1, 1 - r_2\}$ and so

$$\min\{1 - r_1, 1 - r_2\} > \Delta_{(G_1,G_2)}(t_0) \geq \max\{1 - r_1, 1 - r_2\},$$

which is impossible. □

To obtain a real extension of Theorem 1.24, we need the following proposition.

Proposition 4.6. Every nonempty, compact and convex pair (A, B) in a strictly convex fuzzy metric space $(X, M, *; \mathcal{W})$ which satisfies the condition $(\sharp)$ has the F-$\mathcal{PNS}$.

Proof. Suppose the contrary, that is, there exists a closed, convex and proximinal pair $(G_1, G_2) \subseteq (A, B)$ such that

$$\begin{cases} \varrho_{G_1G_2}(t) = \varrho_{AB}(t) > \Delta_{(G_1,G_2)}(t), & \forall t > 0, \\ \Delta_{(u,G_2)}(t) = \Delta_{(G_1,G_2)}(t), & \forall u \in G_1, \forall t > 0. \end{cases}$$

Notice that if $G_2 = \{v\}$, for some $v \in X$, then from the proximinality of the pair (G_1, G_2), there exists an element $u' \in G_1$ such that $M(u', v, t) = \varrho_{AB}(t)$ for all $t > 0$. Then

$$\varrho_{AB}(t) = M(u', v, t) = \Delta_{(u',G_2)}(t) = \Delta_{(G_1,G_2)}(t),$$

which is impossible. So, assume that $v_1, v_2 \in G_2$. Strict convexity of X implies that there exists $\theta_0 \in (0, 1)$ for which $G_2 \ni \mathcal{W}(v_1, v_2, \theta_0) \notin \{v_1, v_2\}$. Proximinality of (G_1, G_2) deduces that there are $u_1, u_2 \in G_1$ for which $M(u_1, v_1, t) = \varrho_{AB}(t) = M(u_2, v_2, t)$ for any $t > 0$. If $u_1 = u_2$, then by the condition $(\sharp)$,

$$M(u_1, \mathcal{W}(v_1, v_2, \theta_0), t) > \min\{M(u_1, v_1, t), M(u_2, v_2, t)\} = \varrho_{AB}(t),$$

which is impossible. Thus $u_1 \neq u_2$. Again from the strict convexity of X there exists $\theta_1 \in (0, 1)$ such that $G_2 \ni \mathcal{W}(u_1, u_2, \theta_1) \notin \{u_1, u_2\}$. Since G_2 is compact and $M(\mathcal{W}(u_1, u_2, \theta_1), \,.\,, t)$ is continuous on G_2, there exists an element $v_3 \in G_2$ for which

$$M(\mathcal{W}(u_1, u_2, \theta_1), v_3, t) = \Delta_{\left(\mathcal{W}(u_1,u_2,\theta_1),G_2\right)}(t) = \Delta_{(G_1,G_2)}(t), \quad \forall t > 0.$$

Besides, from the condition $(\sharp)$ we have

$$\Delta_{(G_1,G_2)}(t) = M(\mathcal{W}(u_1, u_2, \theta_1), v_3, t) > \min\{M(u_1, v_3, t), M(u_2, v_3, t)\},$$

which is a contradiction.

By an equivalent manner if $\Delta_{(G_1,v)}(t) = \Delta_{(G_1,G_2)}(t)$ for all $v \in G_2$ and $t > 0$, then we get a contradiction and the result follows. □

The next result is a generalization of Theorem 1.24 and Corollary 3.8.

Corollary 4.7. Let (A, B) be a nonempty, compact, convex pair in a strongly convex fuzzy metric space $(X, M, *; \mathcal{W})$ which is strictly convex and satisfies the condition $(\sharp)$. Assume that $\mathcal{T}: A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive mapping. Then $\text{Best}_{A\times B}(\mathcal{T}) \neq \emptyset$.

As a consequence of Corollary 4.7 we obtain the following best proximity pair theorem which is a main result of [14].

Corollary 4.8. Let (A, B) be a nonempty, compact, convex pair in a strictly convex Banach space X. Assume that $\mathcal{T}: A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive mapping. Then $\text{Best}_{A\times B}(\mathcal{T}) \neq \emptyset$.

5. Conclusions

In this article, we have considered the concept of fuzzy projection operator and used to ensure the nonemptiness of proximal pairs of F-bounded, closed and convex pair of subsets of a strictly convex fuzzy metric space. Then we have established a best proximity pair theorem for noncyclic contractions. Finally by using a geometric property of fuzzy proximal normal structure, we have presented a new extension of Kirk's fixed point theorem ([2]) for noncyclic relatively nonexpansive mappings in the framework of strictly convex fuzzy metric spaces.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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