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Research article

Simultaneous characterizations of partner ruled surfaces using Flc frame

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Abstract: In this study, we introduce partner ruled surfaces according to the Flc frame that is defined on a polynomial curve. First, the conditions of each couple of two partner ruled surfaces to be simultaneously developable and minimal are investigated. Then, the asymptotic, geodesic and curvature lines of the parameter curves of the partner ruled surfaces are simultaneously characterized. Finally, the examples of the partner ruled surfaces are given, and their graphs are drawn.

Keywords: Flc frame; partner ruled surface; geodesic curve; asymptotic curve; polynomial curves
Mathematics Subject Classification: 53A04, 53A05

1. Introduction

Surface theory is an attractive research field, as it has applications in many disciplines. Among the surfaces, the ruled surfaces are the most interesting and were first introduced by G. Monge. The ruled surfaces are formed by moving a line along a curve, where this curve is called the base curve, and the straight line is called the generator line. Since it consists of an infinite number of straight lines, it is known as a surface of lines. That is, the ruled surfaces are surfaces created by families of straight lines. The importance of the theory of ruled surfaces in some engineering fields is evident. Especially, in kinematics and computer-aided design problems, these surfaces have widespread use. Cylinder and cone surfaces are the most well-known ruled surfaces. The ruled surfaces also arise in admirable architectural works. For example, the ruled surfaces are seen in many famous structures, such as Ciechanow Water Tower, Kobe Port Tower and Shuckhov Tower. In addition to the visibility of ruled surfaces in real-world applications, theoretical developments for these surfaces continue in depth. After its localization in the scientific literature, researchers started to question the characterizations of these surfaces as well as the isoparametric curves lying on them. For example, the relations between the cylindrical helix and Gaussian curvature and between the Bertrand curve and the mean curvature of a ruled surface were given in [1]. The invariants and kinematic/ geometric properties of non-developable ruled surfaces were examined by considering the structural functions of ruled surfaces in [2]. The ruled surfaces with directrix of a focal curve of a given curve were studied and characterized in [3]. Developable ruled surfaces on Bezier curves were constructed in [4]. Normal and binormal ruled surfaces based on W-direction curves were discussed in [5].

Motivated by these, in this study, we have introduced partner ruled surfaces based on a polynomial curve and ruled by the vector elements of a Frenet-like frame known as the Flc frame [6]. Then, we have simultaneously provided the conditions for each partner ruled surface to be developable or minimal by considering the main curvatures with the Flc frame invariants. Such conditions have also been linked to the characterizations of isoparametric curves such as asymptotic, geodesic or curvature lines. An example has been given at the end of the paper with the corresponding figures of the generated partner ruled surfaces.

2. Preliminaries

In this section, we present some basic concepts that will be used throughout the paper. Let $\alpha = \alpha(s)$ be a regular space curve satisfying non-degenerate condition $\alpha^{'}(s) \wedge \alpha^{''}(s) \neq 0$. Then, the orthonormal vector system called the Frenet frame is defined by

$$T(s) = \frac{\alpha^{'}(s)}{\|\alpha^{'}(s)\|}, \quad B(s) = \frac{\alpha^{'}(s) \wedge \alpha^{''}(s)}{\|\alpha^{'}(s) \wedge \alpha^{''}(s)\|}, \quad N(s) = B(s) \wedge T(s), \tag{2.1}$$

where T is tangent, N is principal normal and B is binormal vector field. The Frenet formulas are given by

$$T' = \kappa\eta N, N' = -\kappa\eta T + \tau\eta B, B' = -\tau\eta N, \|\alpha'\| = \eta, \tag{2.2}$$

where the curvature κ and torsion τ of the curve are [7],

$$\kappa = \frac{\left\|\alpha^{'}(s) \wedge \alpha^{''}(s)\right\|}{\|\alpha^{'}(s)\|^3}, \qquad \tau = \frac{\left\langle \alpha^{'}(s) \wedge \alpha^{''}(s), \alpha^{'''}(s)\right\rangle}{\|\alpha^{'}(s) \wedge \alpha^{''}(s)\|^2}. \tag{2.3}$$

The n^{th} degree polynomial with parameter s is defined as

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s^1 + a_0, \ a_n \neq 0,$$

where $n \in \mathbb{N}_0$, $a_i \in \mathbb{R}$, $(0 \leq i \leq n)$ [8].

Now, let us define a curve such that $\alpha : [a, b] \to E^n, \quad \alpha(s) = (\alpha_1(s), \alpha_2(s), ..., \alpha_n(s))$. If each α_is are polynomials for $1 \leq i \leq n$, then $a_s \in \mathbb{R}[s]$ is defined to be an $n-$dimensional polynomial curve [9]. The degree of such a polynomial curve as $\alpha(s)$ is given by

$$\deg \alpha(s) = max \{\deg(\alpha_1(s)), \deg(\alpha_2(s)), ..., \deg(\alpha_n(s))\} \ [8].$$

The definition of the Flc frame of a polynomial space curve $\alpha = \alpha(s)$ given by Dede in [6] is as follows:

$$T(s) = \frac{\alpha^{'}(s)}{\|\alpha^{'}(s)\|}, \quad D_1(s) = \frac{\alpha^{'}(s) \wedge \alpha^{(n)}(s)}{\left\|\alpha^{'}(s) \wedge \alpha^{(n)}(s)\right\|}, \quad D_2(s) = D_1(s) \wedge T(s), \tag{2.4}$$

where the prime $'$ indicates the differentiation with respect to s and $^{(n)}$ stands for the n^{th} derivative. The new vectors D_1 and D_2 are called the binormal-like vector and normal-like vector, respectively. The curvatures of the Flc-frame $d_1, d_2$and d_3 are given by

$$d_1 = \frac{\langle T', D_2\rangle}{\eta}, \;\; d_2 = \frac{\langle T', D_1\rangle}{\eta}, \;\; d_3 = \frac{\langle D_2', D_1\rangle}{\eta}, \tag{2.5}$$

where $\|\alpha'\| = \eta$. The local rate of change of the Flc-frame called as the Frenet-like formulas can be expressed in the following form:

$$\begin{bmatrix} T' \\ D_2' \\ D_1' \end{bmatrix} = \eta \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix}. \tag{2.6}$$

On the other hand, a ruled surface as a family of straight lines is defined as

$$\varphi(s,u) = \alpha(s) + ur(s), \tag{2.7}$$

where $\alpha(s)$ is the base curve, and the $r(s)$ is the generator. The Gaussian and mean curvatures of the ruled surface φ with the normal vector field N_φ are given as

$$K = \frac{eg - f^2}{EG - F^2}, \quad H = \frac{1}{2}\frac{Eg - 2Ef + Ge}{EG - F^2}. \tag{2.8}$$

Here, the coefficients of first and second fundamental forms are defined as

$$E = \langle \varphi_s, \varphi_s\rangle, \quad F = \langle \varphi_s, \varphi_u\rangle, \quad G = \langle \varphi_u, \varphi_u\rangle, \tag{2.9}$$

$$e = \left\langle \varphi_{ss}, N_\varphi\right\rangle, \quad f = \left\langle \varphi_{su}, N_\varphi\right\rangle, \quad \text{g} = \left\langle \varphi_{uu}, N_\varphi\right\rangle = 0, \tag{2.10}$$

respectively [7].

3. Simultaneous characterizations of partner ruled surfaces using Flc frame

In this section, we study simultaneously partner ruled surfaces constructed by the tangent, normal-like and binormal-like vectors of the Flc frame along a polynomial space curve.

3.1. TD_2-partner ruled surfaces

Definition 3.1. *Let α be a differentiable polynomial space curve and $\{T, D_2, D_1\}$ be the Flc frame of the polynomial space curve. The two ruled surfaces defined by*

$$\begin{cases} \varphi^{T}_{D_2}(s,u) = T(s) + uD_2(s), \\ \varphi^{D_2}_{T}(s,u) = D_2(s) + uT(s), \end{cases} \tag{3.1}$$

are called TD_2-partner ruled surfaces with the Flc frame of the polynomial curve.

Theorem 3.1. *Let* $\varphi^T_{D_2}$ *and* $\varphi^{D_2}_T$ *be* TD_2*-partner ruled surfaces, and then* TD_2*-partner ruled surfaces are simultaneously*

(i) developable surfaces if and only if $d_1 = 0,\ d_2 \neq 0$ *or* $d_3 \neq 0$*,*
(ii) minimal surfaces if and only if $d_2 = d_3 = 0$ *and* $d_1 \neq 0$.

Proof. By differentiating the first equation of (3.1) with respect to s and u, respectively and using Flc frame derivative formulas, one can obtain

$$\begin{aligned}(\varphi^T_{D_2})_s &= -\eta u d_1 T(s) + \eta d_1 D_2(s) + \eta (d_2 + u d_3) D_1(s),\\ (\varphi^T_{D_2})_u &= D_2(s).\end{aligned} \tag{3.2}$$

Then, by considering the partial derivatives of the surface $\varphi^T_{D_2}$ given by Eq (3.2) and the cross product of both vectors $(\varphi^T_{D_2})_s$ and $(\varphi^T_{D_2})_u$, the normal vector field of the surface $\varphi^T_{D_2}$ is found as

$$U^T_{D_2} = \frac{(\varphi^T_{D_2})_s \times (\varphi^T_{D_2})_u}{\left\|(\varphi^T_{D_2})_s \times (\varphi^T_{D_2})_u\right\|} = \frac{-(d_2 + u d_1) T(s) - u d_1 D_1(s)}{\sqrt{u^2 {d_1}^2 + (d_2 + u d_3)^2}}. \tag{3.3}$$

By applying the scalar product for both vectors in (3.2), we find the components of the first fundamental form of the ruled surface $\varphi^T_{D_2}$ as follows:

$$E_{TD_2} = \eta^2 \left(\left(1 + u^2\right) {d_1}^2 + (d_2 + u d_3)^2\right),\ F_{TD_2} = \eta d_1,\ G_{TD_2} = 1. \tag{3.4}$$

By differentiating Eq (3.2) with respect to s and u and making the scalar product with the normal vector field (3.3), we have the component of the second fundamental form of the ruled surface $\varphi^T_{D_2}$ as follows:

$$\begin{aligned} e_{TD_2} &= \frac{\eta\left(\left(1+u^2\right){d_1}^2 d_2 \eta + (d_2+u d_3)\left(d_2(d_2+u d_3)\eta + u {d_1}'\right) - u d_1\left({d_2}' + u {d_3}'\right)\right)}{\sqrt{u^2 {d_1}^2 + (d_2+u d_3)^2}},\\ f_{TD_2} &= \frac{\eta d_1 d_2}{\sqrt{u^2 {d_1}^2 + (d_2+u d_3)^2}},\ g_{TD_2} = 0. \end{aligned} \tag{3.5}$$

Thus, by substituting Eqs (3.4) and (3.5) into Eq (2.8), the Gaussian curvature and the mean curvature of the ruled surface $\varphi^T_{D_2}$ are calculated by

$$\begin{aligned} K_{TD_2} &= -\left(\frac{d_1 d_2}{u^2 {d_1}^2 + (d_2+u d_3)^2}\right)^2,\\ H_{TD_2} &= \frac{\left(\left(-1+u^2\right){d_1}^2 d_2 \eta + (d_2+u d_3)\left(d_2(d_2+u d_3)\eta + u {d_1}'\right) - u d_1\left({d_2}' + u {d_3}'\right)\right)}{2\eta\left(u^2 {d_1}^2 + (d_2+u d_3)\right)^{\frac{3}{2}}}. \end{aligned} \tag{3.6}$$

On the other hand, by differentiating the second equation of (3.1) with respect to s and u , respectively and using the Flc frame derivative formulae, one can obtain

$$\begin{aligned}(\varphi^{D_2}_T)_s &= -\eta d_1 T(s) + \eta u d_1 D_2(s) + \eta (d_3 + u d_2) D_1(s),\\ (\varphi^{D_2}_T)_u &= T(s).\end{aligned} \tag{3.7}$$

Then, by considering the partial derivatives of the surface $\varphi^{D_2}_T$ given by Eq (3.7) and the cross product of both vectors, the normal vector field of the surface $\varphi^{D_2}_T$ is found as

$$U^{D_2}_T = \frac{(\varphi^{D_2}_T)_s \times (\varphi^{D_2}_T)_u}{\left\|(\varphi^{D_2}_T)_s \times (\varphi^{D_2}_T)_u\right\|} = \frac{(d_3 + u d_2) D_2(s) - u d_1 D_1(s)}{\sqrt{u^2 {d_1}^2 + (d_3 + u d_2)^2}}. \tag{3.8}$$

By applying the scalar product for both vectors in (3.8), we have the components of the first fundamental form of the ruled surface $\varphi_T^{D_2}$ as follows:

$$E_{D_2T} = \eta^2\left(\left(1+u^2\right){d_1}^2 + (d_3+ud_2)^2\right),\ F_{D_2T} = -\eta d_1,\ G_{D_2T} = 1. \tag{3.9}$$

By differentiating Eq (3.7) with respect to s and u and making the scalar product with the normal vector field (3.8), we have the component of the second fundamental form of the ruled surface $\varphi_T^{D_2}$ as follows:

$$\begin{aligned} e_{D_2T} &= \frac{-\eta\left(\left(1+u^2\right){d_1}^2 d_3\eta+(d_3+ud_2)\left(d_3(d_3+ud_2)\eta-ud_1{}'\right)+ud_1\left(d_3{}'+ud_2{}'\right)\right)}{\sqrt{u^2{d_1}^2+(d_3+ud_2)^2}}, \\ f_{D_2T} &= \frac{\eta d_1 d_3}{\sqrt{u^2{d_1}^2+(d_3+ud_2)^2}},\ g_{D_2T} = 0. \end{aligned} \tag{3.10}$$

Thus, by substituting Eqs (3.9) and (3.10) into Eq (2.8), the Gaussian curvature K_{D_2T} and the mean curvature H_{D_2T} of the ruled surface $\varphi_T^{D_2}$ are calculated by

$$\begin{aligned} K_{D_2T} &= -\left(\frac{d_1d_3}{u^2{d_1}^2+(d_3+ud_2)^2}\right)^2, \\ H_{D_2T} &= \frac{\left(\left(1-u^2\right){d_1}^2 d_3\eta+(d_3+ud_2)\left(-d_3(d_3+ud_2)\eta+ud_1{}'\right)-ud_1\left(d_3{}'+ud_2{}'\right)\right)}{2\eta\left(u^2{d_1}^2+(d_3+ud_2)\right)^{\frac{3}{2}}}. \end{aligned} \tag{3.11}$$

Consequently, from Eqs (3.6) and (3.11), it can easily be said TD_2-partner ruled surfaces are simultaneously developable and minimal surfaces under the conditions stated in the hypothesis. □

Theorem 3.2. *Let $\varphi_{D_2}^T$ and $\varphi_T^{D_2}$ be TD_2-partner ruled surfaces, and then s-parameter curves of TD_2-partner ruled surfaces are simultaneously*

(i) not geodesic,
(ii) asymptotic if and only if $d_2 = d_3 = 0$ and $d_1 \neq 0$.

Proof. Let $\varphi_{D_2}^T$ and $\varphi_T^{D_2}$ be TD_2-partner ruled surfaces and the cross products of second partial derivates with the normal vector fields of the TD_2-partner ruled surfaces being found as

$$(\varphi_{D_2}^T)_{ss} \times U_{D_2}^T = \begin{pmatrix} \frac{ud_1\left(u{d_1}^2\eta^2+\eta\left(d_2d_3\eta+u{d_3}^2\eta-d'_1\right)-d_1\eta'\right)}{\sqrt{u^2{d_1}^2+(d_2+ud_3)^2}}, \\ \frac{-u{d_1}^3\eta^2-d_1\eta\left(d_2d_3+u\eta\left({d_3}^2\eta+ud'_1\right)\right)-u^2{d_1}^2\eta'-(d_2+ud_3)(\eta(d'_2+ud'_3)+(d_2+ud_3)\eta')}{\sqrt{u^2{d_1}^2+(d_2+ud_3)^2}}, \\ -\frac{(d_2+ud_3)\left(u{d_1}^2\eta^2+\eta\left(d_2d_3\eta+u{d_3}^2\eta-d'_1\right)-d_1\eta'\right)}{\sqrt{u^2{d_1}^2+(d_2+ud_3)^2}}, \end{pmatrix}$$

$$(\varphi_T^{D_2})_{ss} \times U_T^{D_2} = \begin{pmatrix} \frac{u{d_1}^3\eta^2+d_1\eta\left(u{d_2}^2\eta+d_2d_3\eta-u^2d'_1\right)-u^2{d_1}^2\eta'-(ud_2+d_3)(\eta(ud'_2+d'_3)+(ud_2+d_3)\eta')}{\sqrt{u^2{d_1}^2+(ud_2+d_3)^2}}, \\ -\frac{ud_1\left(u{d_1}^2\eta^2+\eta\left(u{d_2}^2\eta+d_2d_3\eta+d'_1\right)+d_1\eta'\right)}{\sqrt{u^2{d_1}^2+(ud_2+d_3)^2}}, \\ -\frac{(ud_2+d_3)\left(u{d_1}^2\eta^2+\eta\left(u{d_2}^2\eta+d_2d_3\eta+d'_1\right)+d_1d'_1\right)}{\sqrt{u^2{d_1}^2+(ud_2+d_3)^2}}. \end{pmatrix}$$

Since $(\varphi_{D_2}^{T})_{ss} \times U_{D_2}^{T} \neq 0$ and $(\varphi_{T}^{D_2})_{ss} \times U_{T}^{D_2} \neq 0$, s-parameter curves of the TD_2-partner ruled surfaces simultaneously are not geodesic. On the other hand, the scalar products of second partial derivates with the normal vector fields of the TD_2-partner ruled surfaces are calculated as

$$\left\langle (\varphi_{D_2}^{T})_{ss}, U_{D_2}^{T} \right\rangle = \frac{\eta\left(\left(1+u^2\right) {d_1}^2 d_2 \eta + (d_2 + ud_3)(d_2(d_2+ud_3)\eta + ud'_1) - ud_1(d'_2 + ud'_3)\right)}{\sqrt{u^2 {d_1}^2 + (d_2+ud_3)^2}},$$

$$\left\langle (\varphi_{T}^{D_2})_{ss}, U_{T}^{D_2} \right\rangle = \frac{-\eta\left(\left(1+u^2\right) {d_1}^2 d_3 \eta + (ud_2 + d_3)(d_3\eta(ud_2+d_3) - ud'_1) + ud_1(ud'_2 + d'_3)\right)}{\sqrt{u^2 {d_1}^2 + (ud_2+d_3)^2}}.$$

From here, if $d_2 = d_3 = 0$ and $d_1 \neq 0$, then $\left\langle (\varphi_{D_2}^{T})_{ss}, U_{D_2}^{T} \right\rangle = 0$ and $\left\langle (\varphi_{T}^{D_2})_{ss}, U_{T}^{D_2} \right\rangle = 0$. So, we can say that s-parameter curves of the TD_2-partner ruled surfaces simultaneously are asymptotic if and only if $d_2 = d_3 = 0$ and $d_1 \neq 0$. □

Theorem 3.3. *Let $\varphi_{D_2}^{T}$ and $\varphi_{T}^{D_2}$ be TD_2-partner ruled surfaces, and then u-parameter curves of TD_2-partner ruled surfaces are simultaneously*

(i) geodesic,
(ii) asymptotic.

Proof. Let $\varphi_{D_2}^{T}$ and $\varphi_{T}^{D_2}$ be TD_2- partner ruled surfaces. Since $(\varphi_{D_2}^{T})_{uu} \times U_{D_2}^{T} = 0$ and $(\varphi_{T}^{D_2})_{uu} \times U_{T}^{D_2} = 0$, u-parameter curves of the TD_2-partner ruled surfaces simultaneously are geodesic. On the other hand, since $\left\langle (\varphi_{D_2}^{T})_{uu}, U_{D_2}^{T} \right\rangle = 0$ and $\left\langle (\varphi_{T}^{D_2})_{uu}, U_{T}^{D_2} \right\rangle = 0$, u-parameter curves of the TD_2-partner ruled surfaces are simultaneously asymptotic. □

Theorem 3.4. *Let $\varphi_{D_2}^{T}$ and $\varphi_{T}^{D_2}$ be TD_2-partner ruled surfaces, and then s and u-parameter curves of TD_2-partner ruled surfaces are simultaneously lines of curvature if and only if $d_1 = 0$.*

Proof. Let $\varphi_{D_2}^{T}$ and $\varphi_{T}^{D_2}$ be TD_2-partner ruled surfaces. For $d_1 = 0$,

$$F_{TD_2} = \eta d_1 = 0, \quad f_{TD_2} = \frac{\eta d_1 d_2}{\sqrt{u^2 {d_1}^2 + (d_2+ud_3)^2}} = 0,$$

and

$$F_{D_2T} = -\eta d_1 = 0, \quad f_{D_2T} = \frac{\eta d_1 d_3}{\sqrt{u^2 {d_1}^2 + (d_3+ud_2)^2}} = 0.$$

Thus, we can easily say that s and u-parameter curves of TD_2-partner ruled surfaces are simultaneously lines of curvature if and only if $d_1 = 0$. □

3.2. *TD_1-partner ruled surfaces*

Definition 3.2. *Let α be a differentiable polynomial space curve and $\{T, D_2, D_1\}$ be the Flc frame of the polynomial space curve. The two ruled surfaces defined by*

$$\begin{cases} \varphi_{D_1}^{T}(s,u) = T(s) + uD_1(s), \\ \varphi_{T}^{D_1}(s,u) = D_1(s) + uT(s), \end{cases} \tag{3.12}$$

are called TD_1-partner ruled surfaces with the Flc frame of the polynomial curve.

Theorem 3.5. *Let $\varphi_{D_1}^{T}$ and $\varphi_{T}^{D_1}$ be TD_1-partner ruled surfaces, and then TD_1-partner ruled surfaces are simultaneously*

(i) developable surfaces if and only if $d_2 = 0$ and $d_1 \neq 0$ or $d_3 \neq 0$,
(ii) minimal surfaces if and only if $d_1 = d_3 = 0$ and $d_2 \neq 0$.

Proof. By differentiating the first equation of (3.12) with respect to s and u, respectively and using Flc frame derivative formulae, one can obtain

$$\begin{aligned}(\varphi_{D_1}^{T})_s &= -\eta u d_2 T(s) + \eta(d_1 - ud_3) D_2(s) + \eta d_2 D_1(s), \\ (\varphi_{D_1}^{T})_u &= D_1(s).\end{aligned} \tag{3.13}$$

Then, by considering the partial derivatives of the surface $\varphi_{D_1}^{T}$ given by Eq (3.13) and the cross product of both vectors, the normal vector field of the surface $\varphi_{D_1}^{T}$ is found as

$$U_{D_1}^{T} = \frac{(\varphi_{D_1}^{T})_s \times (\varphi_{D_1}^{T})_u}{\left\|(\varphi_{D_1}^{T})_s \times (\varphi_{D_1}^{T})_u\right\|} = \frac{(d_1 - ud_3) T(s) + ud_2 D_2(s)}{\sqrt{u^2 {d_2}^2 + (d_1 - ud_3)^2}}. \tag{3.14}$$

By applying the scalar product for both vectors in (3.13), we have the components of the first fundamental form of the ruled surface $\varphi_{D_1}^{T}$ as follows:

$$E_{TD_1} = \eta^2\left(\left(1 + u^2\right){d_2}^2 + (d_1 - ud_3)^2\right), \ F_{TD_1} = \eta d_2, \ G_{TD_1} = 1. \tag{3.15}$$

By differentiating Eq (3.13) with respect to s and u and making the scalar product with the normal vector field (3.14), we have the component of the second fundamental form of the ruled surface $\varphi_{D_1}^{T}$ as follows:

$$\begin{aligned} e_{TD_1} &= \frac{-\eta\left(\eta\left({d_1}^3 - 2u{d_1}^2 d_3\right) + d_1 {d_2}^2 \eta\left(\left(1+u^2\right) + u\left(u{d_3}^2\eta + d_2'\right)\right) - u\left(ud_3 d_2' + d_2\left(d_1' - ud_3'\right)\right)\right)}{\sqrt{u^2 {d_2}^2 + (d_1 - ud_3)^2}}, \\ f_{TD_1} &= \frac{-\eta d_1 d_2}{\sqrt{u^2 {d_2}^2 + (d_1 - ud_3)^2}}, \ g_{TD_1} = 0. \end{aligned} \tag{3.16}$$

Thus, by substituting Eqs (3.15) and (3.16) into Eq (2.8), the Gaussian curvature and the mean curvature of the ruled surface $\varphi_{D_1}^{T}$ are found by

$$\begin{aligned} K_{TD_1} &= -\left(\frac{d_1 d_2}{u^2 {d_2}^2 + (d_1 - ud_3)^2}\right)^2, \\ H_{TD_1} &= \frac{\left(\eta\left(-{d_1}^3 + 2u{d_1}^2 d_3\right) + d_1\left(\left(1-u^2\right){d_2}^3\eta - u\left(u{d_3}^2\eta + d_2'\right)\right) + u\left(ud_3 d_2' + d_2\left(d_1' - ud_3'\right)\right)\right)}{2\eta\left(u^2 {d_2}^2 + (d_1 - ud_3)\right)^{\frac{3}{2}}}. \end{aligned} \tag{3.17}$$

On the other hand, by differentiating the second equation of (3.12) with respect to s and u, respectively and using Flc frame derivative formulae, one can obtain

$$\begin{aligned}(\varphi_{T}^{D_1})_s &= -\eta d_2 T(s) + \eta(ud_1 - d_3) D_2(s) + u\eta d_2 D_1(s), \\ (\varphi_{T}^{D_1})_u &= T(s).\end{aligned} \tag{3.18}$$

Then, by considering the partial derivatives of the surface $\varphi_{T}^{D_1}$ given by Eq (3.18) and the cross product of both vectors, the normal vector field of the surface $\varphi_{T}^{D_1}$ is found as

$$U_{T}^{D_1} = \frac{(\varphi_{T}^{D_1})_s \times (\varphi_{T}^{D_1})_u}{\left\|(\varphi_{T}^{D_1})_s \times (\varphi_{T}^{D_1})_u\right\|} = \frac{ud_2 D_2(s) - (ud_1 - d_3) D_1(s)}{\sqrt{u^2 {d_2}^2 + (ud_1 - d_3)^2}}. \tag{3.19}$$

By applying the scalar product for both vectors in (3.19), we have the components of the first fundamental form of the ruled surface $\varphi_T^{D_1}$ as follows:

$$E_{D_1T} = \eta^2\left(\left(1+u^2\right){d_2}^2 + (d_3 - ud_1)^2\right),\ F_{D_1T}(s,u,t) = -\eta d_2,\ G_{D_1T} = 1. \tag{3.20}$$

The scalar products of differentiation of Eq (3.18) with respect to s and u with the normal vector field (3.19) gives us the component of the second fundamental form of the ruled surface $\varphi_T^{D_1}$ as follows:

$$\begin{aligned} e_{D_1T} &= \frac{-\eta\left(u^2{d_1}^2d_3\eta+\left(1+u^2\right){d_2}^2d_3\eta+{d_3}^3\eta-ud_3{d_2}'+ud_1\left(-2{d_3}^2\eta+ud_2'\right)+ud_2\left(-ud_1'+d_3'\right)\right)}{\sqrt{u^2{d_2}^2+(d_3-ud_1)^2}},\\ f_{D_1T} &= \frac{\eta d_2d_3}{\sqrt{u^2{d_2}^2+(d_3-ud_1)^2}},\ g_{D_1T} = 0. \end{aligned} \tag{3.21}$$

Thus, by substituting Eqs (3.20) and (3.21) into Eq (2.8), the Gaussian curvature K_{D_1T} and the mean curvature H_{D_1T} of the ruled surface $\varphi_T^{D_1}$ are calculated by

$$\begin{aligned} K_{D_1T} &= -\left(\frac{d_2d_3}{u^2{d_2}^2+(d_3-ud_1)^2}\right)^2,\\ H_{D_1T} &= \frac{\left(\left(1-u^2\right){d_2}^2d_3\eta-u^2{d_1}^2d_3\eta-{d_3}^2\eta+ud_3d_2'+ud_1\left(2\eta{d_3}^2-ud_2'\right)+ud_2\left(ud_1'-d_3'\right)\right)}{2\eta\left(u^2{d_2}^2+(d_3-ud_1)\right)^{\frac{3}{2}}}. \end{aligned} \tag{3.22}$$

Consequently, from Eqs (3.17) and (3.22), it can easily be said TD_1-partner ruled surfaces are simultaneously developable and minimal surfaces under the conditions stated in the hypothesis. □

Theorem 3.6. *Let $\varphi_{D_1}^T$ and $\varphi_T^{D_1}$ be TD_1-partner ruled surfaces, and then s-parameter curves of TD_1-partner ruled surfaces are simultaneously*

(i) not geodesic,
(ii) asymptotic if and only if $d_1 = d_3 = 0$ and $d_2 \neq 0$.

Proof. The proof is done in a similar way to the proof of the theorem given for TD_2-partner ruled surfaces. □

Theorem 3.7. *Let $\varphi_{D_1}^T$ and $\varphi_T^{D_1}$ be TD_1-partner ruled surfaces, and then u-parameter curves of TD_1-partner ruled surfaces are simultaneously*

(i) geodesic,
(ii) asymptotic.

Proof. The proof is done in a similar way to the proof of the theorem given for TD_2-partner ruled surfaces. □

Theorem 3.8. *Let $\varphi_{D_1}^T$ and $\varphi_T^{D_1}$ be TD_1-partner ruled surfaces, and then s and u-parameter curves of TD_1-partner ruled surfaces are simultaneously lines of curvature if and only if $d_2 = 0$.*

Proof. The proof is done in a similar way to the proof of the theorem given for TD_2-partner ruled surfaces. □

3.3. D_2D_1-partner ruled surfaces

Definition 3.3. *Let α be a differentiable polynomial space curve and $\{T, D_2, D_1\}$ be the Flc frame of the polynomial space curve. The two ruled surfaces defined by*

$$\begin{cases} \varphi_{D_1}^{D_2}(s,u) = D_2(s) + uD_1(s), \\ \\ \varphi_{D_2}^{D_1}(s,u) = D_1(s) + uD_2(s), \end{cases} \tag{3.23}$$

are called D_2D_1-partner ruled surfaces with the Flc frame of the polynomial curve.

Theorem 3.9. *Let $\varphi_{D_1}^{D_2}$ and $\varphi_{D_2}^{D_1}$ be a D_2D_1-partner ruled surfaces, and then D_2D_1-partner ruled surfaces are simultaneously*

(i) developable surfaces if and only if $d_3 = 0$ and $d_1 \neq 0$ or $d_2 \neq 0$,
(ii) minimal surfaces if and only if $d_1 = d_2 = 0$ and $d_3 \neq 0$.

Proof. By differentiating the first equation of (3.23) with respect to s and u, respectively and using Flc frame derivative formulae, one can obtain

$$\begin{aligned} (\varphi_{D_1}^{D_2})_s &= -\eta(d_1 + ud_2)T(s) - \eta d_3 D_2(s) + \eta d_3 D_1(s), \\ (\varphi_{D_1}^{D_2})_u &= D_1(s). \end{aligned} \tag{3.24}$$

Then, by considering the partial derivatives of the surface $\varphi_{D_1}^{D_2}$ given by Eq (3.24) and the cross product of both vectors, the normal vector field of the surface $\varphi_{D_1}^{D_2}$ is found as

$$U_{D_1}^{D_2} = \frac{(\varphi_{D_1}^{D_2})_s \times (\varphi_{D_1}^{D_2})_u}{\left\|(\varphi_{D_1}^{D_2})_s \times (\varphi_{D_1}^{D_2})_u\right\|} = \frac{-ud_3 D_2(s) + (d_1 + ud_2) D_1(s)}{\sqrt{u^2 {d_3}^2 + (d_1 + ud_2)^2}}. \tag{3.25}$$

By applying the scalar product for both vectors in (3.24), we have the components of the first fundamental form of the ruled surface $\varphi_{D_1}^{D_2}$ as follows:

$$E_{D_2D_1} = \eta^2\left(\left(1 + u^2\right){d_3}^2 + (d_1 + ud_2)^2\right), \; F_{D_2D_1} = \eta d_3, \; G_{D_2D_1} = 1. \tag{3.26}$$

By differentiating Eq (3.24) with respect to s and u and making the scalar product with the normal vector field (3.25), we have the component of the second fundamental form of the ruled surface $\varphi_{D_1}^{D_2}$ as follows:

$$\begin{aligned} e_{D_2D_1} &= \frac{-\eta\left({d_1}^3\eta + 2u{d_1}^2 d_2\eta + d_1\left({d_3}^2 + u^2\eta\left({d_2}^2 + {d_3}^2\right) + ud_3'\right) + u\left(-d_3\left(d_1' + ud_2'\right) + ud_2 d_3'\right)\right)}{\sqrt{u^2{d_3}^2 + (d_1 + ud_2)^2}}, \\ f_{D_2D_1} &= \frac{\eta d_1 d_3}{\sqrt{u^2{d_3}^2 + (d_1 + ud_2)^2}}, \; g_{D_2D_1} = 0. \end{aligned} \tag{3.27}$$

Thus, by substituting Eqs (3.26) and (3.27) into Eq (2.8), the Gaussian curvature $K_{D_2D_1}$ and the mean curvature $H_{D_2D_1}$ of the ruled surface $\varphi_{D_1}^{D_2}$ are calculated by

$$\begin{aligned} K_{D_2D_1} &= -\left(\frac{d_1 d_3}{u^2{d_3}^2 + (d_1 + ud_2)^2}\right)^2, \\ H_{D_2D_1} &= \frac{\left(-{d_1}^3\eta - 2u{d_1}^2 d_2\eta - d_1\left(u^2{d_2}^2\eta + \left(-1 + u^2\right){d_3}^2\eta + ud_3' + u\left(d_3\left(d_1' + ud_2'\right) - ud_2 d_3'\right)\right)\right)}{2\eta\left(u^2{d_3}^2 + (d_1 + ud_2)\right)^{\frac{3}{2}}}. \end{aligned} \tag{3.28}$$

On the other hand, by differentiating the second equation of (3.23) with respect to s and u, respectively and using Flc frame derivative formulae, one can obtain

$$\begin{aligned} (\varphi_{D_2}^{D_1})_s &= -\eta\left(d_2 + ud_1\right) T\left(s\right) - \eta d_3 D_2\left(s\right) + u\eta d_3 D_1\left(s\right), \\ (\varphi_{D_2}^{D_1})_u &= D_2\left(s\right). \end{aligned} \tag{3.29}$$

Then, by considering the partial derivatives of the surface $\varphi_{D_2}^{D_1}$ given by Eq (3.29) and the cross product of both vectors, the normal vector field of the surface $\varphi_{D_2}^{D_1}$ is found as

$$U_{D_2}^{D_1} = \frac{(\varphi_{D_2}^{D_1})_s \times (\varphi_{D_2}^{D_1})_u}{\left\|(\varphi_{D_2}^{D_1})_s \times (\varphi_{D_2}^{D_1})_u\right\|} = \frac{-ud_3 T\left(s\right) - \left(d_2 + ud_1\right) D_1\left(s\right)}{\sqrt{u^2 {d_3}^2 + \left(d_2 + ud_1\right)^2}}. \tag{3.30}$$

By applying the scalar product for both vectors in (3.30), we have the components of the first fundamental form of the ruled surface $\varphi_{D_2}^{D_1}$ as follows:

$$E_{D_1 D_2} = \eta^2\left(\left(1 + u^2\right){d_3}^2 + \left(d_2 + ud_1\right)^2\right),\ F_{D_1 D_2} = -\eta d_3,\ G_{D_1 D_2} = 1. \tag{3.31}$$

By differentiating Eq (3.29) with respect to s and u and making the scalar product with the normal vector field (3.30), we have the component of the second fundamental form of the ruled surface $\varphi_{D_2}^{D_1}$ as follows:

$$\begin{aligned} e_{D_1 D_2} &= \frac{\eta\left(u{d_1}^2 d_2\eta + {d_2}^3\eta + ud_3\left({d_2}' + u{d_1}'\right) + ud_1\left(2{d_2}^2\eta - u{d_3}'\right) + d_2\left(\left(1 + u^2\right){d_3}^2\eta - u{d_3}'\right)\right)}{\sqrt{u^2{d_3}^2 + \left(d_2 + ud_1\right)^2}}, \\ f_{D_1 D_2} &= -\frac{\eta d_2 d_3}{\sqrt{u^2{d_3}^2 + \left(d_2 + ud_1\right)^2}},\ g_{D_1 D_2} = 0. \end{aligned} \tag{3.32}$$

Thus, by substituting Eqs (3.31) and (3.32) into Eq (2.8), the Gaussian curvature $K_{D_1 D_2}$ and the mean curvature $H_{D_1 D_2}$ of the ruled surface $\varphi_{D_2}^{D_1}$ are calculated by

$$\begin{aligned} K_{D_1 D_2} &= -\left(\frac{d_2 d_3}{u^2{d_3}^2 + \left(d_2 + ud_1\right)^2}\right)^2, \\ H_{D_1 D_2} &= \frac{\left(u^2{d_1}^2 d_2\eta + {d_2}^3\eta + ud_3\left({d_2}' + u{d_1}'\right) + ud_1\left(2{d_2}^2\eta - u{d_3}'\right) + d_2\left(\left(-1 + u^2\right){d_3}^2\eta - u{d_3}'\right)\right)}{2\eta\left(u^2{d_3}^2 + \left(d_2 + ud_1\right)\right)^{\frac{3}{2}}}. \end{aligned} \tag{3.33}$$

Consequently, from Eqs (3.28) and (3.33), it can easily be said $D_2 D_1$-partner ruled surfaces are simultaneously developable and minimal surfaces under the conditions stated in the hypothesis. □

Theorem 3.10. *Let $\varphi_{D_1}^{D_2}$ and $\varphi_{D_2}^{D_1}$ be $D_2 D_1$-partner ruled surfaces, and then s-parameter curves of $D_2 D_1$-partner ruled surfaces are simultaneously*

(i) not geodesic,
(ii) asymptotic if and only if $d_1 = d_2 = 0$ and $d_3 \neq 0$.

Proof. The proof is done in a similar way to the proof of the theorem given for TD_2-partner ruled surfaces. □

Theorem 3.11. *Let $\varphi_{D_1}^{D_2}$ and $\varphi_{D_2}^{D_1}$ be D_2D_1-partner ruled surfaces, and then s and u-parameter curves of D_2D_1-partner ruled surfaces are simultaneously line of curvatures if and only if $d_3 = 0$.*

Proof. The proof is done in a similar way to the proof of the theorem given for TD_2-partner ruled surfaces. □

Example 3.1. *Let us consider a helical polynomial curve parameterized as $\alpha(s) = \left(6s, 3s^2, s^3\right)$. Then, the Flc frame elements of α are given by*

$$T(s) = \left(\frac{2}{2+s^2}, \frac{2s}{2+s^2}, \frac{s^2}{2+s^2}\right), D_1(s) = \left(\frac{s}{\sqrt{1+s^2}}, -\frac{1}{\sqrt{1+s^2}}, 0\right),$$

$$D_2(s) = \left(-\frac{s^2}{\sqrt{1+s^2}\left(2+s^2\right)}, -\frac{s^3}{\sqrt{1+s^2}\left(2+s^2\right)}, \frac{2\sqrt{1+s^2}}{2+s^2}\right)$$

and the corresponding curvatures according to the Flc frame are the following:

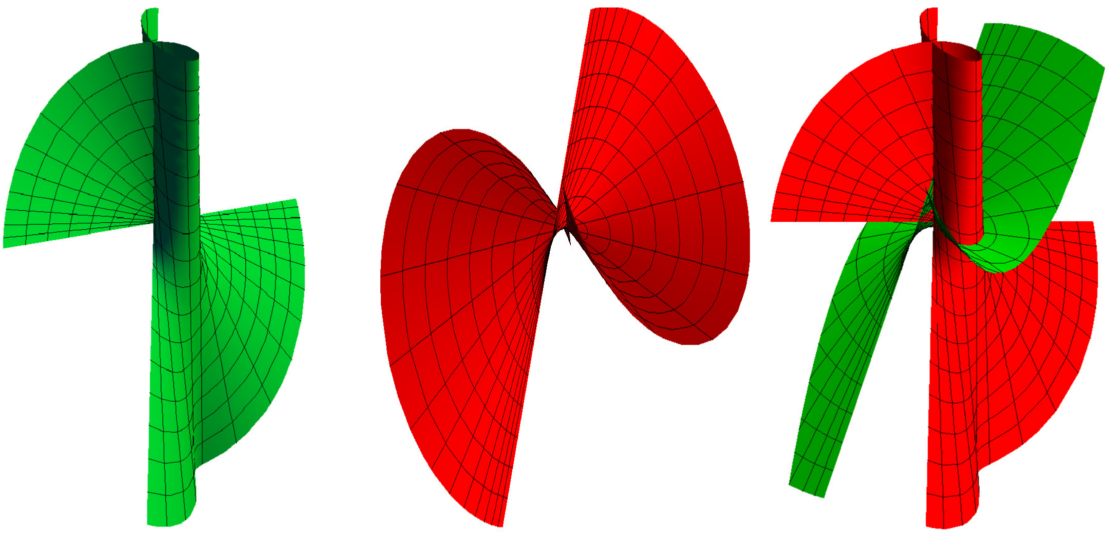
$$d_1(s) = \frac{s}{\sqrt{s^2+1}}, \; d_2(s) = \frac{-1}{\sqrt{s^2+1}}, \; d_3(s) = \frac{s^2}{2(s^2+1)}.$$

(1) Thus, we have the parametric forms for TD_2-partner ruled surfaces as follows:

$$\varphi_{D_2}^{T} = \left(\frac{2\sqrt{s^2+1}-s^2u}{\left(s^2+2\right)\sqrt{s^2+1}}, \frac{s\left(2\sqrt{s^2+1}-s^2u\right)}{\left(s^2+2\right)\sqrt{s^2+1}}, \frac{s^2+2\sqrt{1+s^2}u}{\left(s^2+2\right)}\right),$$

$$\varphi_{T}^{D_2} = \left(\frac{-s^2+2u\sqrt{s^2+1}}{\left(s^2+2\right)\sqrt{s^2+1}}, \frac{s\left(-s^2+2u\sqrt{s^2+1}\right)}{\left(s^2+2\right)\sqrt{s^2+1}}, \frac{2\sqrt{1+s^2}+s^2u}{\left(s^2+2\right)}\right).$$

See Figure 1.



(a) $\varphi_{D_2}^{T}$ surface **(b)** $\varphi_{T}^{D_2}$ surface **(c)** $\varphi_{D_2}^{T}$ and $\varphi_{T}^{D_2}$ surfaces

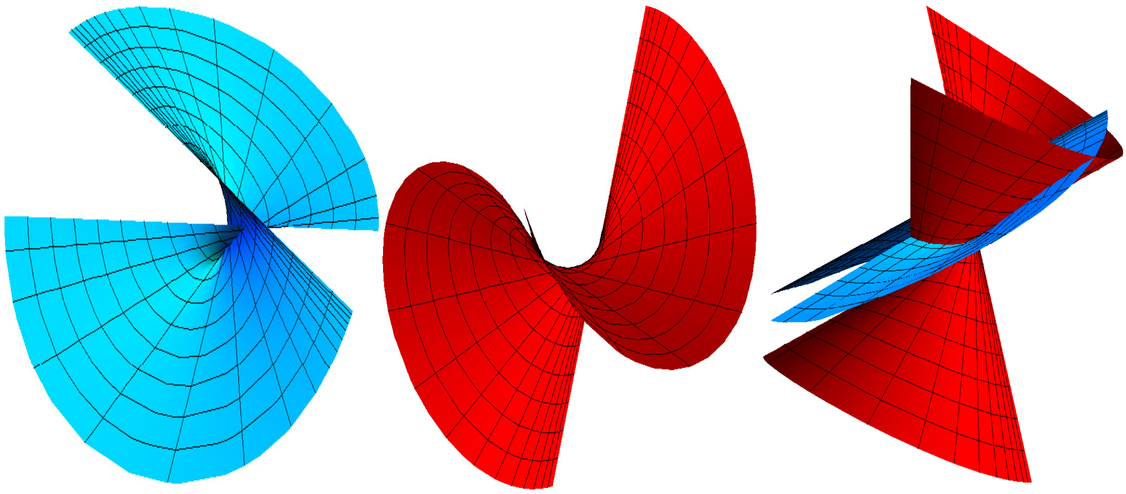
Figure 1. TD_2-partner ruled surfaces with $s \in [-2, 5]$ and $u \in [-5, 5]$.

(2) Thus, we have the parametric forms for TD_1-partner ruled surfaces as follows:

$$\varphi_{D_1}^{T} = \left(\frac{2}{2+s^2} + \frac{su}{\sqrt{1+s^2}}, \frac{2s}{2+s^2} - \frac{u}{\sqrt{1+s^2}}, \frac{s^2}{2+s^2}\right),$$

$$\varphi_{T}^{D_1} = \left(\frac{s}{\sqrt{1+s^2}} + \frac{2u}{2+s^2}, -\frac{1}{\sqrt{1+s^2}} + \frac{2su}{2+s^2}, \frac{s^2u}{2+s^2}\right).$$

See Figure 2.



(a) $\varphi_{D_1}^{T}$ surface **(b)** $\varphi_{T}^{D_1}$ surface **(c)** $\varphi_{D_1}^{T}$ and $\varphi_{T}^{D_1}$ surfaces

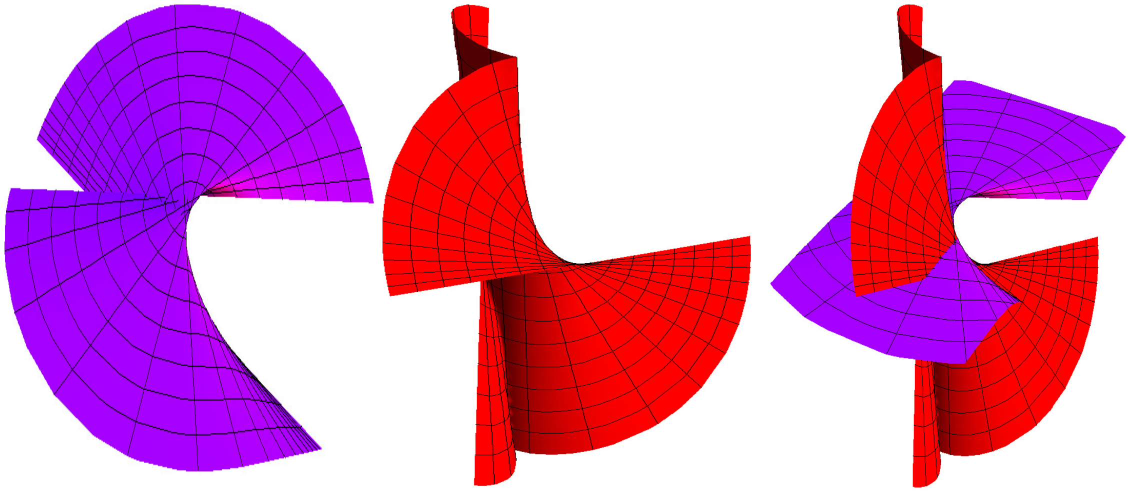
Figure 2. TD_1-partner ruled surfaces with $s \in [-2, 5]$ and $u \in [-5, 5]$.

(3) Thus, we have the parametric forms for D_2D_1-partner ruled surfaces as the following:

$$\varphi_{D_1}^{D_2} = \left(\frac{-s^2+s\,u\left(2+s^2\right)}{\sqrt{1+s^2}\left(2+s^2\right)}, \frac{-2u-s^2(s+u)}{\sqrt{1+s^2}\left(2+s^2\right)}, \frac{2\sqrt{1+s^2}}{2+s^2}\right),$$

$$\varphi_{D_2}^{D_1} = \left(\frac{s\left(2+s^2-su\right)}{\sqrt{1+s^2}\left(2+s^2\right)}, \frac{-2-s^2(1+su)}{\sqrt{1+s^2}\left(2+s^2\right)}, \frac{2\sqrt{1+s^2}u}{2+s^2}\right).$$

See Figure 3.



(a) $\varphi_{D_1}^{D_2}$ surface **(b)** $\varphi_{D_2}^{D_1}$ surface **(c)** $\varphi_{D_1}^{D_2}$ and $\varphi_{D_2}^{D_1}$ surfaces

Figure 3. D_2D_1-partner ruled surfaces with $s \in [-2, 5]$ and $u \in [-5, 5]$.

Example 3.2. *Let us consider a helical polynomial curve parameterized as* $\beta(s) = \left(s^3, s^4, s^5\right)$*.Then the Flc frame elements of* β *are given by*

$$T(s) = \left(\frac{3}{\sqrt{25s^4+16s^2+9,}}, \frac{4s}{\sqrt{25s^4+16s^2+9}}, \frac{5s^2}{\sqrt{25s^4+16s^2+9}}\right),$$

$$D_1(s) = \left(\frac{4s}{\sqrt{16s^2+9}}, -\frac{3}{\sqrt{16s^2+9}}, 0\right),$$

$$D_2(s) = \left(-\frac{15s^2}{\sqrt{16s^2+9}\sqrt{25s^4+16s^2+9}}, -\frac{20s^3}{\sqrt{16s^2+9}\sqrt{25s^4+16s^2+9}}, \frac{\sqrt{16s^2+9}}{\sqrt{25s^4+16s^2+9}}\right),$$

and the corresponding curvatures according to Flc frame are as the following:

$$d_1(s) = \frac{10(8s^2+9)}{s(25s^4+16s^2+9)^{\frac{3}{2}}\sqrt{16s^2+9}},$$

$$d_2(s) = -\frac{12}{s^2(25s^4+16s^2+9)\sqrt{16s^2+9}},$$

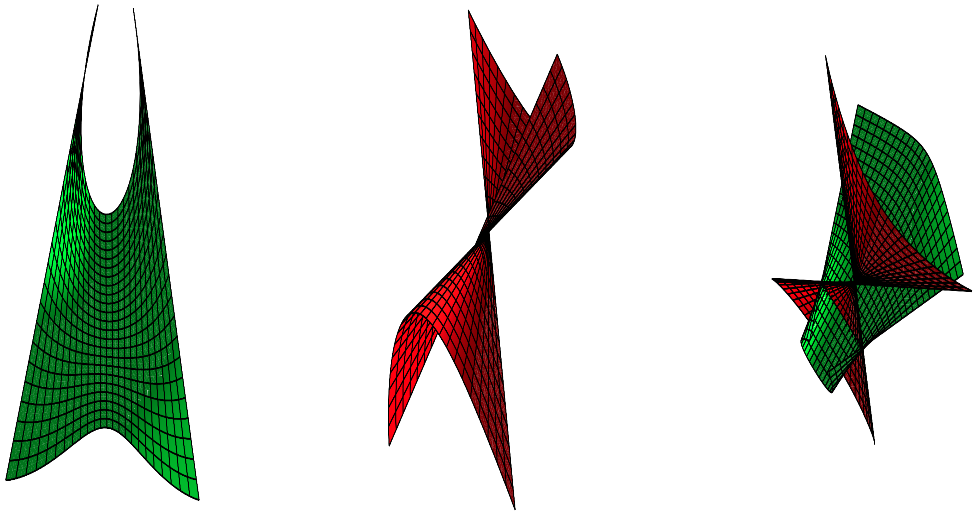
$$d_3(s) = \frac{60}{(25s^4+16s^2+9)(16s^2+9)}.$$

(1) Thus, we have the parametric forms for TD_2*-partner ruled surfaces as follows:*

$$\chi_{D_2}^{T} = \begin{pmatrix} \frac{3}{\sqrt{25s^4+16s^2+9}} - \frac{15us^2}{\sqrt{16s^2+9}\sqrt{25s^4+16s^2+9}}, \\ \frac{4s}{\sqrt{25s^4+16s^2+9}} - \frac{20us^3}{\sqrt{16s^2+9}\sqrt{25s^4+16s^2+9}}, \\ \frac{5s^2}{\sqrt{25s^4+16s^2+9}} + \frac{u\sqrt{16s^2+9}}{\sqrt{25s^4+16s^2+9}}, \end{pmatrix}$$

$$\chi_T^{D_2}=\begin{pmatrix} -\frac{15s^2}{\sqrt{16s^2+9}\sqrt{25s^4+16s^2+9}}+\frac{3u}{\sqrt{25s^4+16s^2+9}}, \\ -\frac{20s^3}{\sqrt{16s^2+9}\sqrt{25s^4+16s^2+9}}+\frac{4us}{\sqrt{25s^4+16s^2+9}}, \\ \frac{\sqrt{16s^2+9}}{\sqrt{25s^4+16s^2+9}}+\frac{5us^2}{\sqrt{25s^4+16s^2+9}}. \end{pmatrix}$$

See Figure 4.



(a) $\chi_{D_2}^T$ surface **(b)** $\chi_T^{D_2}$ surface **(c)** $\chi_{D_2}^T$ and $\chi_T^{D_2}$ surfaces

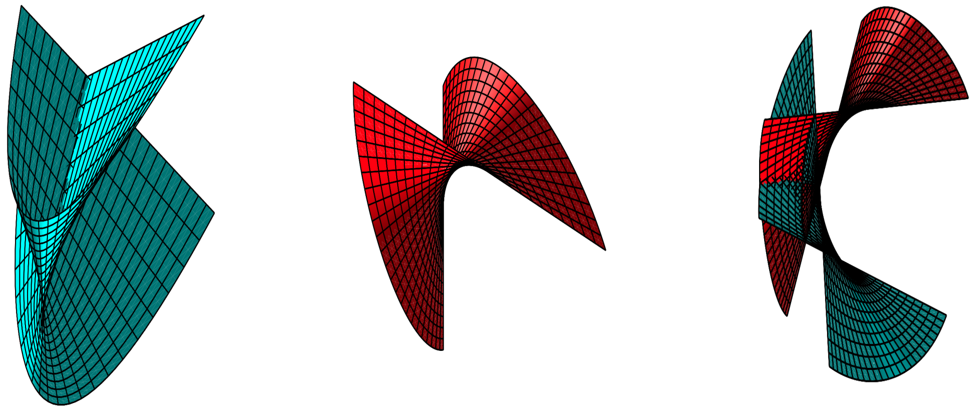
Figure 4. TD_2-partner ruled surfaces with $s \in [-0.5, 0.5]$ and $u \in [-2, 2]$.

(2) Thus, we have the parametric forms for TD_1-partner ruled surfaces as follows:

$$\chi_{D_1}^{T}=\begin{pmatrix} \frac{3}{\sqrt{25s^4+16s^2+9}}+\frac{4us}{\sqrt{16s^2+9}}, \\ \frac{4s}{\sqrt{25s^4+16s^2+9}}-\frac{3u}{\sqrt{16s^2+9}}, \\ \frac{5s^2}{\sqrt{25s^4+16s^2+9}}, \end{pmatrix}$$

$$\chi_T^{D_1}=\begin{pmatrix} \frac{4s}{\sqrt{16s^2+9}}+\frac{3u}{\sqrt{25s^4+16s^2+9}}, \\ -\frac{3}{\sqrt{16s^2+9}}+\frac{4us}{\sqrt{25s^4+16s^2+9}} \\ \frac{5us^2}{\sqrt{25s^4+16s^2+9}}. \end{pmatrix}$$

See Figure 5.



(a) $\chi_{D_1}^{T}$ surface **(b)** $\chi_{T}^{D_1}$ surface **(c)** $\chi_{D_1}^{T}$ and $\chi_{T}^{D_1}$ surfaces

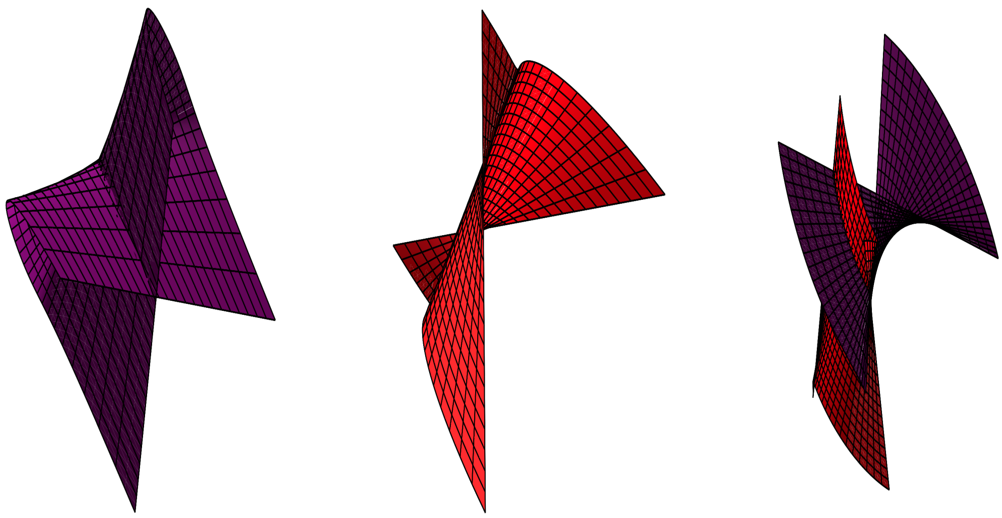
Figure 5. TD_1-partner ruled surfaces with $s \in [-0.5, 0.5]$ and $u \in [-2, 2]$.

(3) Thus, we have the parametric forms for D_2D_1-partner ruled surfaces as the following:

$$\chi_{D_1}^{D_2} = \left(\begin{array}{l} -\frac{15s^2}{\sqrt{16s^2+9}\sqrt{25s^4+16s^2+9}} + \frac{4us}{\sqrt{16s^2+9}}, \\ -\frac{20s^3}{\sqrt{16s^2+9}\sqrt{25s^4+16s^2+9}} - \frac{3u}{\sqrt{16s^2+9}}, \\ \frac{\sqrt{16s^2+9}}{\sqrt{25s^4+16s^2+9}}, \end{array} \right)$$

$$\chi_{D_2}^{D_1} = \left(\begin{array}{l} \frac{4s}{\sqrt{16s^2+9}} - \frac{15us^2}{\sqrt{16s^2+9}\sqrt{25s^4+16s^2+9}}, \\ -\frac{3}{\sqrt{16s^2+9}} - \frac{20us^3}{\sqrt{16s^2+9}\sqrt{25s^4+16s^2+9}}, \\ \frac{u\sqrt{16s^2+9}}{\sqrt{25s^4+16s^2+9}}. \end{array} \right)$$

See Figure 6.



(a) $\chi_{D_1}^{D_2}$ surface (b) $\chi_{D_2}^{D_1}$ surface (c) $\chi_{D_1}^{D_2}$ and $\chi_{D_2}^{D_1}$ surfaces

Figure 6. D_2D_1-partner ruled surfaces with $s \in [-0.5, 0.5]$ and $u \in [-2, 2]$.

4. Conclusions

In this paper, the invariants of partner ruled surfaces formed by tangent, normal-like and binormal-like vector fields of a polynomial space curve simultaneously are presented. Also, some characterizations of the parameter curves are examined. Examples of these surfaces are given and their graphics are drawn using the MATLAB R2021b program.

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Conflict of interest

The authors declare no conflict of interest.

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