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Research article

Perov-fixed point theorems on a metric space equipped with ordered theoretic relation

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Abstract: In this paper, we introduce a few new generalizations of the classical Perov-fixed point theorem for single-valued and multi-valued mappings in a complete generalized metric space endowed with a binary relation. We have furnished our work with examples to show that several metrical-fixed point theorems can be obtained from an arbitrary binary relation.

Keywords: binary relation; Perov-fixed point; vector-valued metric space
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1. Introduction

The classical Banach fixed point theorem [1] is a very useful and effective tool of functional analysis. This classical result has applications not only in different branches of Mathematics, such as ordinary differential equations, partial differential equations, integral equations, optimization, and variational analysis, but used as an effective tool in other subjects, such as economics, game theory, and biology as well. This theorem provides the existence and uniqueness of the fixed point of a self-map satisfies the contraction condition defined on a complete metric space. There are many extensions of the Banach fixed point theorem, like Ciric theorem, Caristi theorem, Boyd-Wong theorem, and Browder-Kirk theorem. One of the most interesting extensions of this classical result was made by Perov [2]. In his work, Perov extended the Banach contraction principle to mappings defined on product spaces. Many contributions related to fixed point theory in that context have been made by different researchers, for example, Abbas et al. [3] investigated fixed points of Perov type contractive mappings on a set endowed with a graphic structure, Filip and Petrusel [4] explored fixed point theorems on a set endowed with vector-valued metric. Many other researchers also showed keen interest in this result, namely, Cretkovic and Rakocevic [5], Altun and Olgun [6], Ilic et al. [7], Vetro and Radenovic [8], discussed this result under various circumstances to get fixed points.

Alam and Imdad [9] extended the Banach fixed point theorem to a complete metric space endowed with a binary relation and discussed a more generalized way to get fixed points. In this work, we extend the Perov-fixed point theorem for single-valued and multi-valued mappings in the framework of a generalized metric space equipped with a binary relation. Here, the contractive inequality is considerably weaker than the classical contractive inequality because it is needed that the contractive inequality holds only on those elements which are related to each other under the respect binary relation instead of the entire space.

2. Preliminaries

In this section, we recollect the material required to prove our results. Here, by $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Q}$ and $\mathbb{R}$, we mean the set of all natural numbers, whole numbers, rational numbers and the set of all real numbers, respectively. Let X be a non-empty set, $\mathbb{R}_+$ be the set of all non-negative real numbers and $M(IR)$ be the set consisting of all $m \times 1$ real matrices. Let $\underline{u}, \underline{v} \in M(IR)$, then these elements are of the form $\underline{u} = (u_1, u_2, \ldots, u_m)^T$ and $\underline{v} = (v_1, v_2, \ldots, v_m)^T$, where T denotes the transpose of a matrix. Then by $\underline{u} \leq \underline{v}$ (respectively $\underline{u} < \underline{v}$), we mean that $u_i \leq v_i$ (respectively $u_i < v_i$), $\forall\ i = 1, 2, \cdots, m$.

The concept of generalized metric space, also known as vector-valued metric space, was discussed by Perov in the following way.

Definition 2.1. *[2] Let X be a non-empty set and let $\rho : X \times X \to M(IR)$ be a map such that for all $a, b, c \in X$ the following axioms hold.*

(M_1) $\rho(a, b) \geq \overline{0}$,

(M_2) $\rho(a, b) = \overline{0} \Leftrightarrow a = b$,

(M_3) $\rho(a, b) = \rho(b, a)$,

(M_4) $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$,

where $\overline{0}$ is the zero matrix of order $m \times 1$. Then the map ρ is called a vector-valued metric on X and the pair (X, ρ) is called a generalized metric space or vector-valued metric space.

The convergence and Cauchyness of a sequence in a generalized metric space are analogous to those in a metric space.

Definition 2.2. *[2] Let (X, ρ) be a generalized metric space and let $\{a_n\}$ be a sequence in X. Then, we say that*

(1) $\{a_n\}$ is convergent to $a \in X$ if $\lim_{n\to\infty} \rho(a_n, a) = \overline{0}$.

(2) $\{a_n\}$ is a Cauchy sequence if $\lim_{n,m\to\infty} \rho(a_n, a_m) = \overline{0}$.

Note that a generalized metric space (X, ρ) is said to be complete if each Cauchy sequence in it is also convergent.

Throughout the paper, we denote the set of all square matrices of order m with non-negative entries by $\mathbb{M}_m(\mathbb{R}_+)$, the zero matrix of order $m \times m$ by $\overline{0}_m$, and the identity matrix by I_m. Note that for any $A \in \mathbb{M}_m(\mathbb{R}_+)$, we have $A^0 = I_m$.

The concept of a matrix converges to zero is elaborated through the following definition and example.

Definition 2.3. *[2] Let $A \in \mathbb{M}_m(\mathbb{R}_+)$, then A is said to be a matrix convergent to a zero matrix, if $A^n \to \overline{0}_m$ as $n \to \infty$.*

Example 2.4. *[4] The given below matrices are convergent to a zero matrix.*
$A := \begin{pmatrix} x & x \\ y & y \end{pmatrix}$, *where $x, y \in \mathbb{R}_+$ and $x + y < 1$;*
$B := \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$, *where $x, y, z \in \mathbb{R}_+$ and $\max\{x, z\} < 1$.*

From Petrusel-Filip [4], we discuss other equivalent conditions of matrices converge to zero.

Theorem 2.5. *[4] Let $A \in \mathbb{M}_m(\mathbb{R}_+)$. Then the following conditions are equivalent.*

(a) A is convergent to zero.

(b) The set of all eigenvalues of A is contained in the open unit disc, that is, $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $\det(A - \lambda I_m) = 0$.

(c) The matrix $I_m - A$ is nonsingular (i.e. det$(I_m - A) \neq 0$) and

$$(I_m - A)^{-1} = I_m + A + \cdots + A^n + \cdots .$$

(d) $A^n \underline{u} \to \overline{0}$ as $n \to \infty$, for all $\underline{u} \in M(IR)$.

(e) $A^n \to \overline{0}$ as $n \to \infty$.

The following is another example of a matrix convergent to a zero matrix.

Example 2.6. *If each $a_1, a_2, \ldots, a_m$ is less than 1, then*

$$A := \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_m \end{pmatrix}_{m\times m}$$

in $\mathbb{M}_m(\mathbb{R}_+)$ is convergent to zero.

We are now going to discuss some concepts regarding the binary relation.

Definition 2.7. *[10] Let X be a non-empty set. Then the Cartesian product on X is defined as:*

$$X^2 = X \times X = \{(a, b) : a, b \in X\}.$$

Any subset $\mathcal{R}$ of X^2 is called a binary relation on X.

Note that for every pair $(a, b) \in X^2$, one of the following two situations must hold:

(1) $(a, b) \in \mathcal{R}$, which means that a is $\mathcal{R}$-related to b or a relates to b under $\mathcal{R}$. We also write $(a, b) \in \mathcal{R}$ as $a\mathcal{R}b$.

(2) $(a, b) \notin \mathcal{R}$, which means that a is not $\mathcal{R}$-related to b or a doesn't relates to b under $\mathcal{R}$. We also write $(a, b) \notin \mathcal{R}$ as $a \not\mathcal{R} b$.

As X^2 and ϕ are two trivial subsets of X^2, we call them trivial binary relations on X. The binary relations X^2 and ϕ are called the universal relation (or the full relation) and empty relation, respectively. There is another important and useful binary relation on X named as the equality relation or identity relation (or the diagonal relation) and is defined as

$$\mathrm{dia}(X) = \Delta_X := \{(a, a) : a \in X\}.$$

Throughout the paper, we denote $\mathcal{R}$ for a non-empty binary relation, but for our convenience, we write it as "binary relation" instead of writing "non-empty binary relation".

Alam and Imdad [9] presented the notion of $\mathcal{R}$-Comparative in the following way.

Definition 2.8. *[9] Let X be a non-empty set and $\mathcal{R}$ be a binary relation defined on X. Then we say that any two elements $a, b \in X$ are $\mathcal{R}$-Comparative, if either $(a, b) \in \mathcal{R}$ or $(b, a) \in \mathcal{R}$.*
If $a, b \in X$ are $\mathcal{R}$-Comparative, then we denote them by $[a, b] \in \mathcal{R}$.

A binary relation is classified into several types through the appropriate conditions. A few types of binary relations are defined in the following definitions.

Definition 2.9. *[10] Let X be a non-empty set and $\mathcal{R}$ be a binary relation on X. Then $\mathcal{R}$ is said to be:*

(1) Reflexive if $(a, a) \in \mathcal{R}, \forall a \in X$.

(2) Irreflexive if $(a, a) \notin \mathcal{R}$, $\forall a \in X$.

(3) Non-reflexive if $(a, a) \notin \mathcal{R}$ for some $a \in X$.

(4) Non-irreflexive if $(a, a) \in \mathcal{R}$ for some $a \in X$.

(5) Symmetric if $(a, b) \in \mathcal{R}$ implies $(b, a) \in \mathcal{R}$, $\forall a, b \in X$.

(6) Anti-symmetric if $(a, b) \in \mathcal{R}$ and $(b, a) \in \mathcal{R}$ implies $a = b$, $\forall a, b \in X$.

(7) Transitive if $(a, b) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}$ implies $(a, c) \in \mathcal{R}$, $\forall a, b, c \in X$.

(8) Complete or connected if $[a, b] \in \mathcal{R}$, $\forall a, b \in X$.

(9) Weakly complete or weakly connected if $[a, b] \in \mathcal{R}$ or $a = b$, $\forall a, b \in X$.

Definition 2.10. *[10] Let X be a non-empty set and $\mathcal{R}$ be a binary relation defined on X. Then the binary relation $\mathcal{R}$ is called:*

(1) Sharp order or strict order if $\mathcal{R}$ is irreflexive and transitive.

(2) Near order if $\mathcal{R}$ is anti-symmetric and transitive.

(3) Pre-order or quasi order if $\mathcal{R}$ is reflexive and transitive.

(4) Partial order if $\mathcal{R}$ is reflexive, anti-symmetric and transitive.

(5) Pseudo order if $\mathcal{R}$ is reflexive and anti-symmetric.

(6) Equivalence order if $\mathcal{R}$ is reflexive, symmetric and transitive.

(7) Simple order if $\mathcal{R}$ is weakly complete strict order.

(8) Weak order if $\mathcal{R}$ is complete pre-order.

(9) Total order, Chain or Linear order if $\mathcal{R}$ is complete partial order.

(10) Tolerance if $\mathcal{R}$ is reflexive and symmetric.

The following well-known proposition states that every universal relation is a complete equivalence relation.

Proposition 2.11. *[10] Let $\mathcal{R}$ be the full binary relation (universal relation) defined on a non-empty set X. Then $\mathcal{R}$ is a complete equivalence relation.*

The following proposition is an extended form of Proposition 2.3 presented in [9].

Proposition 2.12. *Let (X, ρ) be a generalized metric space endowed with a binary relation $\mathcal{R}$, and let $T : X \to X$ be a mapping. Also assume that $A \in \mathbb{M}_m(\mathbb{R}_+)$ be a matrix convergent to zero, then the following contractive conditions are equivalent:*

(1) $\rho(Ta, Tb) \leq A\rho(a, b), \forall a, b \in X$ with $(a, b) \in \mathcal{R}$,

(2) $\rho(Ta, Tb) \leq A\rho(a, b), \forall a, b \in X$ with $[a, b] \in \mathcal{R}$.

Proof. It is trivial that if (2) holds, then (1) exists. Now, we show that the existence of (1) implies the existence of (2).

Suppose (1) holds and $a, b \in X$ with $[a, b] \in \mathcal{R}$, that is either $(a, b) \in \mathcal{R}$ or $(b, a) \in \mathcal{R}$.

If $(a, b) \in \mathcal{R}$, then (2) holds directly from (1).

If $(b, a) \in \mathcal{R}$, using M_3, we get

$$\rho(Ta, Tb) = \rho(Tb, Ta) \leq A\rho(b, a) = A\rho(a, b).$$

This implies that (2) holds. □

Alam and Imdad [9] stated the notion of $\mathcal{R}$-preserving sequence as follows.

Definition 2.13. *[9] Let $\mathcal{R}$ be a binary relation on a non-empty set X. A sequence $(a_n) \subseteq X$ is called $\mathcal{R}$-preserving if*

$$(a_n, a_{n+1}) \in \mathcal{R}, \forall n \in \mathbb{W}.$$

The concept of d-self closedness for an arbitrary binary relation defined on a metric space (X, d) was presented by Alam and Imdad [9].

Definition 2.14. *Let (X, ρ) be a generalized metric space endowed with a binary relation $\mathcal{R}$. Then $\mathcal{R}$ is said to be ρ-self-closed, if for each $\mathcal{R}$-preserving sequence (a_n) with limit point $a \in X$, there exists a subsequence (a_{n_k}) of (a_n) with $[a_{n_k}, a] \in \mathcal{R}$, $\forall k \in \mathbb{N}$.*

The above definition is elaborated with the following example.

Example 2.15. *Let $X = \mathbb{R}$ equipped with a generalized metric ρ defined by*

$$\rho(a, b) = \begin{bmatrix} |a - b| \\ |a - b| \end{bmatrix}$$

and a binary relation $\mathcal{R}$ defined by $\mathcal{R} = \{(a, b) : a, b \geq 0\}$. Consider an arbitrary $\mathcal{R}$-preserving sequence (a_n) with limit point $a \in X$, that is $\lim_{n\to\infty} \rho(a_n, a) = \overline{0}$ and $a_n \mathcal{R} a_{n+1}$ for each $n \in \mathbb{N}$. By $\mathcal{R}$-preserving nature of (a_n), we get $a_n, a_{n+1} \geq 0$ for each $n \in \mathbb{N}$. Also, the fact $\lim_{n\to\infty} \rho(a_n, a) = \overline{0}$ implies $a \geq 0$, since $a_n \geq 0$ for each $n \in \mathbb{N}$. Thus, $a_n, a \geq 0$ for each $n \in \mathbb{N}$, that is $a_n \mathcal{R} a$ for each $n \in \mathbb{N}$. Hence, we say that there exists a subsequence (a_{n_k}) of (a_n) with $[a_{n_k}, a] \in \mathcal{R}$ $\forall k \in \mathbb{N}$.

Alam and Imdad [9] stated the notions of T-closed binary relation and $\mathcal{R}$-directed set as follows.

Definition 2.16. *[9] Let X be a non-empty set and $T : X \to X$ be a mapping. Then a binary relation $\mathcal{R}$ is called T-closed, if for each $a, b \in X$*

$$(a, b) \in \mathcal{R} \Rightarrow (Ta, Tb) \in \mathcal{R}.$$

Definition 2.17. *[9] Let X be a non-empty set and $\mathcal{R}$ be a binary relation on X. A subset E of X is said to be $\mathcal{R}$-directed if for every $a, b \in E$, there exists $c \in X$ such that $(a, c) \in \mathcal{R}$ as well as $(b, c) \in \mathcal{R}$.*

The following definition presents the notion of a path between two points of a set endowed with a binary relation.

Definition 2.18. *Let $\mathcal{R}$ be a binary relation on a non-empty set X. For $a, b \in X$, a finite sequence $\{c_0, c_1, \ldots, c_l\} \subseteq X$ is called a path from a to b of length l in $\mathcal{R}$, where $l \in \mathbb{N}$, if:*
(1) $c_0 = a$ and $c_l = b$,
(2) $[c_i, c_{i+1}] \in \mathcal{R}\ \forall i = 0, 1, 2, \cdots, l-1$.

Note that every path of length l contains $l + 1$ elements of X, but not necessarily distinct.

3. Main results

Before moving toward the results of this section, we denote the set of all fixed points of T in X by $fix_X(T)$ and the family of all finite paths from a to b (where $a, b \in X$) in $\mathcal{R}$ by $C_{\mathcal{R}}(a, b)$.

We are now going to state and prove our first result that is a generalized form of the result presented by Perov [2].

Theorem 3.1. *Let (X, ρ) be a complete vector-valued metric space endowed with a binary relation $\mathcal{R}$ and $T : X \to X$ be a mapping. Suppose that:*

(i) There exists $a \in X$ such that $(a, Ta) \in \mathcal{R}$;

(ii) $\mathcal{R}$ is T-closed, that is, for each $a, b \in X$ with $(a, b) \in \mathcal{R}$, we have $(Ta, Tb) \in \mathcal{R}$;

(iii) Either T is continuous or $\mathcal{R}$ is ρ-self-closed;

(iv) There exists a matrix $A \in \mathbb{M}_m(\mathbb{R}_+)$ convergent to zero such that

$$\rho(Ta, Tb) \leq A\rho(a, b), \quad \forall a, b \in X \quad with \quad (a, b) \in \mathcal{R}.$$

Then T has a fixed point.

(v) Further, if $C_{\mathcal{R}}(a, b) \neq \phi, \forall a, b \in X$, then T has a unique fixed point.

Proof. Using hypothesis (i), we get an element $a_0 \in X$ with $(a_0, Ta_0) \in \mathcal{R}$. Define a Picard iterative sequence (a_n) in the following way:

$$a_0, \quad a_1 = Ta_0, \quad a_2 = Ta_1 = T^2 a_0, \quad \cdots, \quad a_n = T^n a_0, \cdots.$$

As $(a_0, a_1) \in \mathcal{R}$, thus, using hypothesis (ii) we reach the fact that

$$(a_n, a_{n+1}) \in \mathcal{R} \quad \forall n \in \mathbb{W}. \tag{3.1}$$

That is, the sequence (a_n) is $\mathcal{R}$-preserving. By using (iv) and (3.1), we get

$$\rho(a_{n+1},a_{n+2}) \leq A\rho(a_n,a_{n+1}) \leq A^2\rho(a_{n-1},a_n)\cdots \leq A^{n+1}\rho(a_0,a_1)\forall n \in \mathbb{W}.$$

From the above and triangular inequality, for all $n,m \in \mathbb{W}$ with $m > n$ and $m \geq 2$, we get

$$\begin{aligned}\rho(a_{n+1},a_{n+m}) &\leq \rho(a_{n+1},a_{n+2}) + \rho(a_{n+2},a_{n+3}) + \cdots + \rho(a_{n+m-1},a_{n+m})\\ &\leq (A^{n+1} + A^{n+2} + \cdots + A^{n+m-1})\rho(a_0,a_1)\\ &= A^{n+1}(I_m + A + A^2 + \cdots + A^{m-2})\rho(a_0,Ta_0)\\ &\leq A^{n+1}(I_m + A + A^2 + \cdots)\rho(a_0,Ta_0)\\ &= A^{n+1}(I_m - A)^{-1}\rho(a_0,Ta_0) \to \overline{0} \quad as \quad n \to \infty,\end{aligned}$$

which implies that the sequence (a_n) is a Cauchy in X. As (X,ρ) is a complete space, there exists $a \in X$ such that

$$a_n \xrightarrow{\rho} a \quad or \lim_{n\to\infty} a_n = a.$$

Now, in lieu of (i), if T is continuous then we have $T(a_n) \xrightarrow{\rho} T(a)$, that is,

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} T(a_n) = T(a).$$

Hence, T has a fixed point.

On the other hand, if $\mathcal{R}$ is ρ-self-closed then for the above defined $\mathcal{R}$-preserving (a_n) (with $\lim_{n\to\infty} a_n = a$), there exists a subsequence (a_{n_k}) of (a_n) with $[a_{n_k},a] \in \mathcal{R}\ \forall k \in \mathbb{N}$. Using this fact in lieu of (iv), we get

$$\rho(a_{n_k+1},Ta) = \rho(Ta_{n_k},Ta) \leq A\rho(a_{n_k},a) \to \overline{0} \quad as \quad k \to \infty.$$

This implies that

$$\lim_{k\to\infty} a_{n_k+1} = T(a).$$

But

$$T(a) = \lim_{k\to\infty} a_{n_k+1} = \lim_{k\to\infty} a_{n_k} = \lim_{n\to\infty} a_n = a.$$

Hence, a is a fixed point of T.

To prove the uniqueness, suppose that (v) holds and $a,b \in fix_X(T)$, i.e., $T(a) = a, \quad T(b) = b$.

By assumption (v), there exists a finite path from a to b (say $\{c_0,c_1,\ldots,c_l\}$) of length l in $\mathcal{R}$ such that

$$c_0 = a, \quad c_l = b, \quad [c_i,c_{i+1}] \in \mathcal{R}, \quad \forall i = 0,1,\cdots,l-1. \tag{3.2}$$

Since $\mathcal{R}$ is T-closed, thus, we get

$$[T^n(c_i),T^n(c_{i+1})] \in \mathcal{R}, \quad \forall i = 0,1,\cdots,l-1, \quad and \quad \forall n \in \mathbb{N}. \tag{3.3}$$

By the triangular inequality, (3.3) and assumption (iv), we reach to

$$\begin{aligned}\rho(a,b) = \rho(T^n(c_0),T^n(c_l)) &\leq \rho(T^n(c_0),T^n(c_1)) + \rho(T^n(c_1),T^n(c_2)) + \cdots + \rho(T^n(c_{l-1}),T^n(c_l))\\ &\leq A\rho(T^{n-1}(c_0),T^{n-1}(c_1)) + A\rho(T^{n-1}(c_1),T^{n-1}(c_2)) + \cdots + A\rho(T^{n-1}(c_{l-1}),T^{n-1}(c_l))\\ &\cdots \leq A^n\rho(c_0,c_1) + A^n\rho(c_1,c_2) + \cdots + A^n\rho(c_{l-1},c_l) \quad \to \overline{0} \quad as \quad n \to \infty.\end{aligned}$$

That is $\rho(a,b) = \overline{0} \Rightarrow a = b$. Hence, a is the unique fixed point of T. □

Remark 3.2. *Note that if $\mathcal{R}$ is a complete order or X is a $\mathcal{R}$-directed set, then $C_{\mathcal{R}}(a,b) \neq \phi$ $\forall a,b \in X$.*

If $\mathcal{R}$ is complete order, then each $a,b \in X$ is $\mathcal{R}$-Comparative, i.e., $[a,b] \in \mathcal{R}$ $\forall a,b \in X$, which implies that $\{a,b\}$ is a path from a to b of length 1 in $\mathcal{R}$. Hence, $C_{\mathcal{R}}(a,b) \neq \phi$ $\forall a,b \in X$.

If X is $\mathcal{R}$-directed set, then for each $a,b \in X$, there exists $c \in X$ such that $(a,c) \in \mathcal{R}$ as well as $(b,c) \in \mathcal{R}$. This shows that for each $a,b \in X$, we have a path $\{a,c,b\}$ from a to b of length 2 in $\mathcal{R}$. Hence, $C_{\mathcal{R}}(a,b) \neq \phi$ $\forall a,b \in X$.

By considering the above remark and Theorem 3.1, one can conclude the following corollary.

Corollary 3.3. *If the hypotheses (i)–(iv) of Theorem 3.1 are true with either $\mathcal{R}$ as a complete order or X as a $\mathcal{R}$-directed set. Then T has a unique fixed point.*

Theorem 3.1 deduces to several well-known fixed point theorems of the existing literature by considering the following special cases.

(1) Under the full binary relation (i.e. $\mathcal{R} = X^2$), Theorem 3.1 yields the Perov-fixed point theorem [2].

(2) By taking $m = 1$ and the universal relation, Theorem 3.1 yields the Banach contraction principle [1].

(3) By setting $m = 1$ and $\mathcal{R}$ as the partial order $\preceq$ in Theorem 3.1, we get the Theorems 2.1–2.3 of Nieto and Rodriguez-Lopez [11]. Here, the property "$\preceq$ is T-closed" is equivalent to the increasing property of T.

(4) Putting $m = 1$ and $\mathcal{R} = \succeq$, the dual binary relation associated with a partial order $\preceq$ in Theorem 3.1, we obtain Theorems 2.4 and 2.5 of Nieto and Rodriguez-Lopez [11].

(5) On setting $\mathcal{R}$ by pre-order $\preceq$ with $m = 1$ in Theorem 3.1, we have Theorem 1 of Turinici [12].

(6) By choosing $\mathcal{R}$ as the transitive binary relation $\preceq$ in Theorem 3.1 with $m = 1$, we can obtain Theorems 2.2 and 2.4 of Ben-El-Mechaiekh [13].

(7) Particularizing $m = 1$ and choosing $\mathcal{R}$ by the tolerance relation $\prec\succ$ associated with a partial order $\preceq$ in Theorem 3.1, we can obtain Theorem 2.1 of Turinici [14] and Theorem 2.1 of Turinici [15].

To elaborate Theorem 3.1, we provide the following examples.

Example 3.4. *Let $X = \mathbb{R}$ and $\rho(a,b) = \begin{bmatrix} |a-b| \\ \frac{|a-b|}{2} \end{bmatrix}$, then clearly (X,ρ) is complete vector-valued metric space. Define a binary relation $\mathcal{R}$ and a self mapping $T : X \to X$ as follows:*

$$\begin{aligned} \mathcal{R} &= \left\{(a,b) \in X^2 : a,b \in \mathbb{Q}\right\}, \\ T(a) &= \tfrac{a}{2} + 4 \ \forall a \in X. \end{aligned}$$

As for each $a \in \mathbb{Q}$, we get $T(a) \in \mathbb{Q}$, thus $(a,Ta) \in \mathcal{R}$. Also, $(a,b) \in \mathcal{R}$ implies that $(Ta,Tb) \in \mathcal{R}$ (i.e., $\mathcal{R}$ is T-closed). Further, T is continuous map. Furthermore, for each $(a,b) \in \mathcal{R}$, we get

$$\rho(Ta,Tb) = \rho(\frac{a}{2}+4, \frac{b}{2}+4) = \begin{bmatrix} \frac{|a-b|}{2} \\ \frac{|a-b|}{4} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} |a-b| \\ \frac{|a-b|}{2} \end{bmatrix} < \begin{bmatrix} 3/4 & 0 \\ 0 & 3/4 \end{bmatrix} \rho(a,b).$$

It is clear by Example 2.4 that the matrix $\begin{bmatrix} 3/4 & 0 \\ 0 & 3/4 \end{bmatrix}$ is a convergent to zero matrix. Thus, Theorem 3.1 ensures that T has a fixed point in X.

Example 3.5. *Let $X = [0,2]$ with $\rho(a,b) = \left(|a-b|, \quad \frac{|a-b|}{2}\right)^T$. Clearly, (X,ρ) is a complete vector-valued metric space. Define a binary relation on X by*

$$\mathcal{R} = \{(0,0),(0,1),(1,0),(1,1),(0,2)\}.$$

Consider a mapping $T : X \longrightarrow X$ by

$$T(a) = \begin{cases} 0, & 0 \le a \le 1. \\ 1, & 1 < a \le 2. \end{cases}$$

Clearly, T is not continuous but it is not difficult to check that $\mathcal{R}$ is ρ-self-closed. Take an arbitrary $\mathcal{R}$-preserving sequence (a_n) such that $a_n \xrightarrow{\rho} a$ and $(a_n, a_{n+1}) \in \mathcal{R}$ $\forall n \in \mathbb{W}$ this implies that $a_n, a \in \{0,1\}$ $\forall n \in \mathbb{W}$, thus, we have $[a_n, a] \in \mathcal{R}$. Hence, $\mathcal{R}$ is ρ-self-closed. Also, $\mathcal{R}$ is T-closed and for $1 \in X$, we have $(1, T(1)) = (1,0) \in \mathcal{R}$. Further, hypothesis (iv) of Theorem 3.1 also holds with $A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$. Hence, T has a fixed point (which is $a = 0$).

We have defined the Perov-fixed point theorem endowed with an arbitrary binary relation for single-valued mappings. We are now moving to extend this result to multi-valued mappings.

Theorem 3.6. *Let (X,ρ) be a complete vector-valued metric space endowed with a binary relation $\mathcal{R}$ and $T : X \to CL(X)$ be a multi-valued mapping. Suppose that:*

(i) There exists $a \in X$ such that $(a,u) \in \mathcal{R}$ for some $u \in Ta$;

(ii) $\mathcal{R}$ is T-closed, that is, for each $a, b \in X$ with $(a,b) \in \mathcal{R}$, we have $(u,v) \in \mathcal{R}$ for all $u \in Ta$ and $v \in Tb$;

(iii) $\mathcal{R}$ is strongly ρ-self-closed, that is, for each sequence (a_n) in X with $(a_n, a_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$ and $a_n \xrightarrow{\rho} a$, we have $(a_n, a) \in \mathcal{R}$ for all $n \ge k$.

(iv) For each $(a,b) \in \mathcal{R}$ and $u \in Ta$, there exists an element $v \in Tb$ such that

$$\rho(u,v) \le A\rho(a,b)$$

where $A \in \mathbb{M}_m(\mathbb{R}_+)$ is a matrix convergent to zero.

Then T has a fixed point.

Proof. By axiom (i), we have elements $a_0 \in X$ and $a_1 \in Ta_0$ with $(a_0, a_1) \in \mathcal{R}$. Also, by (iv), for $(a_0, a_1) \in \mathcal{R}$ and $a_1 \in Ta_0$, there exists $a_2 \in Ta_1$ such that

$$\rho(a_1, a_2) \le A\rho(a_0, a_1).$$

Since $\mathcal{R}$ is T-closed, by (ii), thus we have $(a_1, a_2) \in \mathcal{R}$. Again, by (iv) and (ii), for $(a_1, a_2) \in \mathcal{R}$ and $a_2 \in Ta_1$, $\exists a_3 \in Ta_2$ such that

$$\rho(a_2, a_3) \le A\rho(a_1, a_2) \le A^2\rho(a_0, a_1)$$

and $(a_2, a_3) \in \mathcal{R}$. Thus, by continuing this, we get a sequence (a_n) in X defined by $a_n \in Ta_{n-1}$ such that

$$\rho(a_n, a_{n+1}) \leq A^n \rho(a_0, a_1) \quad \forall n \in \mathbb{N} \tag{3.4}$$

and $(a_n, a_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$. This shows that (a_n) is an $\mathcal{R}$-preserving sequence. Now, we show that this sequence (a_n) is a Cauchy sequence. For this consider arbitrary $m, n \in \mathbb{N}$ with $m > n$. Also, by using triangular inequality and (3.4), we have

$$\begin{aligned}
\rho(a_n, a_m) &\leq \rho(a_n, a_{n+1}) + \rho(a_{n+1}, a_{n+2}) + \cdots + \rho(a_{m-1}, a_m) \\
&\leq (A^n + A^{n+1} + \cdots + A^{m-1})\rho(a_0, a_1) \\
&= A^n(I_m + A + A^2 + \cdots + A^{m-n-1})\rho(a_0, a_1) \\
&\leq A^n(I_m + A + A^2 + \cdots)\rho(a_0, a_1) \\
&= A^n(I_m - A)^{-1}\rho(a_0, a_1) \to \overline{0} \quad as \quad n \to \infty.
\end{aligned}$$

Hence, (a_n) is a Cauchy sequence. As X is complete, there exists $a \in X$ such that $a_n \xrightarrow{\rho} a$. Since $\mathcal{R}$ is strongly ρ-self-closed, by (iii), we get $(a_n, a) \in \mathcal{R}$ $\forall n \geq k$, for some natural k. By using (iv), for each $(a_n, a) \in \mathcal{R}$, and $a_{n+1} \in Ta_n$, there exists $a_n^* \in Ta$ such that

$$\rho(a_{n+1}, Ta) \leq \rho(a_{n+1}, a_n^*) \leq A\rho(a_n, a) \; \forall n \geq k.$$

By using the above inequality and the triangular inequality, we get

$$\begin{aligned}
\rho(a, Ta) &\leq \rho(a, a_{n+1}) + \rho(a_{n+1}, Ta) \\
&\leq \rho(a, a_{n+1}) + A\rho(a_n, a) \to \overline{0} \quad as \quad n \to \infty.
\end{aligned}$$

Hence, $a \in Ta$, that is T has a fixed point. □

In the following example, the existence of a fixed point of a multi-valued mapping will be discussed.

Example 3.7. *Let $X = \mathbb{R}$ with $\rho(a, b) = \begin{bmatrix} |a-b| \\ |a-b| \end{bmatrix}$. Clearly, (X, ρ) is a complete vector-valued metric space. Define a binary relation on X by*

$$\mathcal{R} = \{(n, m) : n, m \in \mathbb{W}\}.$$

Define a map $T : X \longrightarrow CL(X)$ by

$$T(a) = \begin{cases} \{a\}, & a < 0 \\ \{0, 1\}, & a \geq 0. \end{cases}$$

It is easy to check that the axioms of Theorem 3.6 hold. Hence, T has a fixed point.

The following result is an extended version of the above result.

Theorem 3.8. *Let (X, ρ) be a complete vector-valued metric space with an arbitrary binary relation $\mathcal{R}$ and let $T : X \to CL(X)$ be a multi-valued map. Suppose that:*

(i) There exists $a \in X$ such that $(a, u) \in \mathcal{R}$ for some $u \in Ta$;

(ii) $\mathcal{R}$ *is* T*-closed, that is, for each* $a, b \in X$ *with* $(a, b) \in \mathcal{R}$*, we have* $(u, v) \in \mathcal{R}$ *for all* $u \in Ta$ *and* $v \in Tb$*;*

(iii) $\mathcal{R}$ *is strongly* ρ*-self-closed, that is, for each sequence* (a_n) *in* X *with* $(a_n, a_{n+1}) \in \mathcal{R}$ *for all* $n \in \mathbb{N}$ *and* $a_n \xrightarrow{\rho} a$*, we have* $(a_n, a) \in \mathcal{R}$ *for all* $n \geq k$*.*

(iv) For each $(a, b) \in \mathcal{R}$ *and* $u \in Ta$*, there exists an element* $v \in Tb$ *such that*

$$\rho(u, v) \leq A\rho(a, b) + B\rho(b, u) \tag{3.5}$$

where $A, B \in \mathbb{M}_m(\mathbb{R}_+)$ *and* A *is a matrix convergent to zero.*

Then T *has a fixed point.*

Proof. Using hypothesis (i), we can say that $a_0 \in X$ and $a_1 \in Ta_0$ such that $(a_0, a_1) \in \mathcal{R}$. Also, by (iv), for $(a_0, a_1) \in \mathcal{R}$ and $a_1 \in Ta_0$, there exists $a_2 \in Ta_1$ such that

$$\rho(a_1, a_2) \leq A\rho(a_0, a_1) + B\rho(a_1, a_1) = A\rho(a_0, a_1).$$

As $\mathcal{R}$ is T-closed, by (ii), we get $(a_1, a_2) \in \mathcal{R}$. Again, by (iv) and (ii), for $(a_1, a_2) \in \mathcal{R}$ and $a_2 \in Ta_1$, $\exists a_3 \in Ta_2$ such that

$$\rho(a_2, a_3) \leq A\rho(a_1, a_2) + B\rho(a_2, a_2) = A\rho(a_1, a_2) \leq A^2\rho(a_0, a_1).$$

and $(a_2, a_3) \in \mathcal{R}$. By repeating these steps, we get a sequence (a_n) in X defined by $a_n \in Ta_{n-1}$ such that $(a_n, a_{n+1}) \in \mathcal{R}$ and

$$\rho(a_n, a_{n+1}) \leq A^n\rho(a_0, a_1) \ \forall n \in \mathbb{N}. \tag{3.6}$$

By considering the above inequality and the proof of Theorem 3.6, we say that (a_n) is a Cauchy sequence in X and there exists $a \in X$ such that $a_n \xrightarrow{\rho} a$. As $\mathcal{R}$ is strongly ρ-self-closed, we have $(a_n, a) \in \mathcal{R} \ \forall n \geq k$, for some natural k. By using (ii), for each $(a_n, a) \in \mathcal{R}$, and $a_{n+1} \in Ta_n$, there exists $a_n^* \in Ta$ such that

$$\rho(a_{n+1}, Ta) \leq \rho(a_{n+1}, a_n^*) \leq A\rho(a_n, a) + B\rho(a, a_{n+1}). \tag{3.7}$$

By letting $n \to \infty$, we get $\rho(a, Ta) = \overline{0}$. Hence $a \in Ta$, which shows that T has a fixed point. □

4. Application

In this section, we will discuss the existence of a solution for the system of integral equations given below:

$$\begin{aligned} p(t) &= g(t) + \lambda \int_a^b f_1(t, s, p(s), q(s))ds \\ q(t) &= g(t) + \lambda \int_a^b f_2(t, s, p(s), q(s))ds, \end{aligned} \tag{4.1}$$

for each $t \in J = [a, b]$, where $\lambda > 0$ and $g\colon J \to \mathbb{R}$, $f_j\colon J \times J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$, for each $j = 1, 2$, are continuous functions. Here, $(C[a, b], \mathbb{R})$ presents the collection of all continuous real-valued functions defined on $[a, b]$.

The following theorem ensures the existence of a solution to the system (4.1).

Theorem 4.1. *Let $X = (C[a,b], \mathbb{R})$. Consider the operator $T_i\colon X \times X \to X$ defined by*

$$T_j(p(t), q(t)) = g(t) + \lambda \int_a^b f_j(t, s, p(s), q(s))ds$$

where $g\colon J \to \mathbb{R}$, $f_j\colon J \times J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$, for each $j = 1, 2$, are continuous functions and $\lambda > 0$. Assume that, for each $t, s \in [a,b]$ and $p, q, w, x \in X$ with $p \geq g$, $q \geq g$, $w \geq g$ and $x \geq g$, the following inequality exists

$$|f_j(t, s, p(s), q(s)) - f_j(t, s, w(s), x(s))| \leq c_{j1} \max_{s\in J} |p(s) - w(s)| + c_{j2} \max_{s\in J} |q(s) - x(s)| \text{ for } j = 1, 2.$$

Also, assume that the matrix $A = (\lambda b - \lambda a)\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ converges to zero matrix. Then, the system (4.1) *possesses a solution.*

Proof. Using the given hypothesis, for each $t, s \in [a,b]$ and $p, q, w, x \in X$ with $p \geq g$, $q \geq g$, $w \geq g$ and $x \geq g$, we can obtain the following inequality:

$$\begin{aligned} |T_j(p(t), q(t)) - T_j(w(t), x(t))| &\leq \lambda \int_a^b |f_j(t, s, p(s), q(s)) - f_j(t, s, w(s), x(s))|ds \\ &\leq \lambda \int_a^b [c_{j1} \max_{s\in J} |p(s) - w(s)| + c_{j2} \max_{s\in J} |q(s) - x(s)|]ds \\ &\leq \lambda(b - a)[c_{j1} \max_{s\in J} |p(s) - w(s)| + c_{j2} \max_{s\in J} |q(s) - x(s)|], \end{aligned}$$

for each $j = 1, 2$.

Consider an operator $T\colon X \times X \to X \times X$ defined by

$$T(\overline{p}) = T(p_1, p_2) = (T_1(p_1, p_2), T_2(p_1, p_2)) \text{ for each } \overline{p} = (p_1, p_2) \in X \times X,$$

and consider a vector-valued metric $\rho\colon X^2 \times X^2 \to M(IR)$ defined by

$$\rho(\overline{p}(t), \overline{q}(t)) = \begin{bmatrix} \max_{t\in J} |p_1(t) - q_1(t)| \\ \max_{t\in J} |p_2(t) - q_2(t)| \end{bmatrix}$$

for each $\overline{p} = (p_1, p_2), \overline{q} = (q_1, q_2) \in X^2$. Also, define a binary relation $\mathcal{R}$ on $X \times X$ as

$$\mathcal{R} = \{(\overline{p}, \overline{q}) : \overline{p} = (p_1, p_2) \geq (g, g) = \overline{g}, \overline{q} = (q_1, q_2) \geq (g, g) = \overline{g}\}$$

where $(a_1, a_2) \geq (b_1, b_2)$ means $a_1 \geq b_1$, $a_2 \geq b_2$, and g is the function given in (4.1). Now, for the above defined operator, vector-valued metric space and the binary relation, we can obtain the following inequality:

$$\rho(T\overline{p}, T\overline{q}) \leq A\rho(\overline{p}, \overline{q}) \ \forall \overline{p}, \overline{q} \in X \times X \text{ with } (\overline{p}, \overline{q}) \in \mathcal{R}.$$

Note that for the function g we have $T_j(g(t), g(t)) \geq g(t)$ $\forall t \in [a,b]$ for $j = 1, 2$, thus, we say that $(\overline{g}, T(\overline{g})) \in \mathcal{R}$, where $\overline{g} = (g, g)$. It is now easy to see that the other axioms of Theorem 3.1 also exist. Hence, T has a fixed point, that is, $\overline{p} \in X \times X$ such that $T\overline{p} = T(p_1, p_2) = (T_1(p_1, p_2), T_2(p_1, p_2)) = (p_1, p_2)$. This implies that $p_1 = T_1(p_1, p_2)$ and $p_2 = T_2(p_1, p_2)$. Therefore, we conclude that the system (4.1) possesses a solution. □

5. Conclusions

In this manuscript, we have presented a new modification of the classical Perov-fixed point theorem for single-valued and multi-valued mappings in a complete generalized metric space endowed with a binary relation, which under trivial relation (universal) reduces to Perov-fixed point theorems. During this process, we have made a detailed discussion that, instead of choosing several types of binary relations, such as partial order, pre-order, connected, weakly connected, transitive relation, tolerance, strict order, symmetrical closure, etc. which are used by numerous authors, we can choose binary relations to obtain several well-known metrical-fixed point theorems.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.*, **3** (1922), 133–181.
2. A. I. Perov, On the Cauchy problem for a system of ordinary differential equations, *Pviblizhen. Met. Reshen. Differ. Uvavn.*, **2** (1964), 115–134.
3. M. Abbas, V. Rakocevic, A. Iqbal, Fixed points of Perov type contractive mappings on the set endowed with a graphic structure, *RACSAM*, **112** (2018), 209–228. https://doi.org/10.1007/s13398-016-0373-4
4. A. D. Flip, A. Petrusel, Fixed point theorems on space endowed with vector valued metrics, *Fixed Point Theory Appl.*, **2010** (2010), 281381. https://doi.org/10.1155/2010/281381
5. M. Cvetković, V. Rakocević, Extensions of Perov theorem, *Carpathian J. Math.*, **31** (2015), 181–188. https://doi.org/10.37193/CJM.2015.02.05
6. I. Altun, M. Olgun, Fixed point results for Perov type F-contractions and an application, *J. Fixed Point Theory Appl.*, **22** (2020), 46. https://doi.org/10.1007/s11784-020-00779-4
7. D. Ilić, M. Cvetković, L. Gajić, V. Rakočević, Fixed points of sequence of Ćirić generalized contractions of Perov type, *Mediterr. J. Math.*, **13** (2016), 3921–3937. https://doi.org/10.1007/s00009-016-0724-6
8. F. Vetro, S. Radenović, Some results of Perov type in rectangular cone metric spaces, *J. Fixed Point Theory Appl.*, **20** (2018), 41. https://doi.org/10.1007/s11784-018-0520-y
9. A. Alam, M. Imdad, Relation-theoretic contraction principle, *J. Fixed Point Theory Appl.*, **17** (2015), 693–702. https://doi.org/10.1007/s11784-015-0247-y

10. S. Lipschutz, *Schaum's outline of theory and problems of set theory and related topics*, New York: McGraw-Hill, 1964.

11. J. J. Nieto, R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order*, **22** (2005), 223–239. https://doi.org/10.1007/s11083-005-9018-5

12. M. Turinici, Linear contractions in product ordered metric spaces, *Ann. Univ. Ferrara*, **59** (2013), 187–198. https://doi.org/10.1007/s11565-012-0164-6

13. H. Ben-El-Mechaiekh, The Ran-Reurings fixed point theorem without partial order: A simple proof, *J. Fixed Point Theory Appl.*, **16** (2014), 373–383. https://doi.org/10.1007/s11784-015-0218-3

14. M. Turinici, Ran-Reurings fixed point results in ordered metric spaces, *Libertas Math.*, **31** (2011), 49–55.

15. M. Turinici, Nieto-Lopez theorems in ordered metric spaces, *arXiv*, 2011. https://doi.org/10.48550/arXiv.1105.2401

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