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Research article

Approximate analytical solution of time-fractional vibration equation via reliable numerical algorithm

M. Mossa Al-Sawalha[1], Azzh Saad Alshehry[2], Kamsing Nonlaopon[3,*], Rasool Shah[4] and Osama Y. Ababneh[5]

[1] Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia
[2] Department of Mathematical Sciences, Faculty of Sciences, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
[3] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
[4] Department of Mathematics, Abdul Wali khan university, Mardan 23200, Pakistan
[5] Department of Mathematics, Faculty of Science, Zarqa University, Zarqa 13110, Jordan

* **Correspondence:** Email: nkamsi@kku.ac.th; Tel:+66866421582.

Abstract: With effective techniques like the homotopy perturbation approach and the Adomian decomposition method via the Yang transform, the time-fractional vibration equation's solution is found for large membranes. In Caputo's sense, the fractional derivative is taken. Numerical experiments with various initial conditions are carried out through a few test examples. The findings are described using various wave velocity values. The outcomes demonstrate the competence and reliability of this analytical framework. Figures are used to discuss the solution of the fractional vibration equation using the suggested strategies for different orders of memory-dependent derivative. The suggested approaches reduce computation size and time even when the accurate solution of a nonlinear differential equation is unknown. It is helpful for both small and large parameters. The results show that the suggested techniques are trustworthy, accurate, appealing and effective strategies.

Keywords: homotopy perturbation method; Adomian decomposition method; Yang transform; fractional vibration equation; Caputo operator
Mathematics Subject Classification: 33B15, 34A34, 35A20, 35A22, 44A10

1. Introduction

The study of fractional order derivatives and integrations is known as fractional calculus (FC). When L'Hospital questioned Leibniz in 1965 about the fractional order derivative, he first proposed the concept of FC. The theory of FC was initially given as an apparent paradox, but as time passed, it grew in popularity as a topic of study. Due to FC's wide range of uses in several fields of study, many mathematicians were attracted to it. Fractional order derivatives describe different real-world processes and materials in terms of their inherited and memory-related characteristics [1]. The analysis of fractional differential equations (FDEs) [2–5] in engineering and science like as biology [6], viscoelasticity [7], signal processing [8], chemical engineering [9], modelling of diseases [10], seismic wave propagation [11] etc. is a growing field of interest for the researchers.

Both partial differential equations (PDEs) and ordinary differential equations (ODEs) are frequently employed in science and technology to express physical processes mathematically. The mathematical representation of physical processes makes them freely understandable and easy to study. It also conveniently explains the entire situation of the occurrences. Fractional differential equations have mitigated this problem and, in comparison, provide the best and acceptable modeling of the presented challenges [12–14]. Initially, these phenomena were not only modeled accurately by employing integer-order differential equations.

Fractional order ODEs and PDEs have many uses in applied sciences and are more accurate at describing some phenomena than non-fractional order ODEs and PDEs. The fractional-order ODEs and PDEs are non-local and imply that a system's future state depends on its present state and past states. Therefore, the integration and fractional derivatives have a wide range of applications, including the nonlinear oscillation of earth quack is molded with fractional-order derivatives [15], fractional diabetes model [16], chaos theory [17], optics [18], fractional order Covid-19 Model [19], effect of fractional order on ferromagnetic fluid [20], fractional model of cancer chemotherapy [21], electrodynamics [22], signal processing phenomena [23], fractional model for tuberculosis [24], fractional model for the dynamics of Hepatitis B Virus [25], fractional-order pine wilt disease model [26], and some others references therein [27–31].

Science and engineering are very interested in and need to study massive membrane vibration. Significant elements of acoustics and music are membranes. Membranes are also a part of speakers, microphones, and other technology. Membranes can also be used to study the dynamics and propagation of waves in two dimensions. Many human tissues are treated as membranes in bioengineering. The eardrum's vibrational properties aid in understanding hearing. Understanding membrane vibration behavior is necessary for designing assistive technology for the deaf. The common vibration model is provided by

$$\frac{1}{\ell^2}\frac{\partial^2 \mathbb{V}(\mathbf{w},\varsigma)}{\partial \varsigma^2} = \frac{\partial^2 \mathbb{V}(\mathbf{w},\varsigma)}{\partial \mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial \mathbb{V}(\mathbf{w},\varsigma)}{\partial \mathbf{w}} \quad \mathbf{w} \geq 0, \varsigma \geq 0, \tag{1.1}$$

where $\mathbf{w}$ is the spatial domain, ς is the time variable, ℓ is the wave velocity of free vibration and component $\mathbb{V}$ is the displacement of the particle at the position $\mathbf{w}$ at the time instant ς.

In this paper, we will consider a more general form of vibration equation (VE) by replacing integer order time derivative with fractional order Liouville-Caputo derivative as follows:

$$\frac{1}{\ell^2}\frac{\partial^{\wp} \mathbb{V}(\mathbf{w},\varsigma)}{\partial \varsigma^{\wp}} = \frac{\partial^2 \mathbb{V}(\mathbf{w},\varsigma)}{\partial \mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial \mathbb{V}(\mathbf{w},\varsigma)}{\partial \mathbf{w}} \tag{1.2}$$

with initial source

$$\mathbb{V}(\mathbf{w},0) = \mathbf{w}^2, \quad \frac{\partial}{\partial \varsigma}\mathbb{V}(\mathbf{w},0) = \ell \mathbf{w}.$$

Numerous analytical techniques, such as the modified decomposition approach [32] and the homotopy perturbation method [33], are available to solve vibration equations of fractional and integer order vibration equations. For the solution of the fractional vibration equation (FVE), Das and Gupta provided an analytical method based on the homotopy analysis method in [34]. Modified decomposition and variational iteration techniques are employed in [35] to address this issue. Recently, Srivastava et al. [36] attempted to achieve an analytical solution to VE using the q-homotopy analysis transform technique and the Laplace decomposition technique. In the current study, we employ the Adomian decomposition method and homotopy perturbation method along with the Yang transform (YT) called homotopy perturbation transform method (HPTM) and Yang transform decomposition method (YTDM).

The rest of the paper is organized as follows: Introduction is included in Section 1. The fundamental definitions of FC, the Yang transform and its properties are given in Section 2. The notion of HPTM is presented in Section 3, whereas the idea of YTDM is presented in Section 4. Its application to the fractional vibration equation is demonstrated in Section 5. We summarise the conclusion in Section 6.

2. Preliminaries

In this section, we give some basic notions about fractional calculus, Yang transform and Yang transform of fractional derivatives, which are used further in this paper.

Definition 2.1. The Caputo fractional-order derivative as

$$D_{\varsigma}^{\wp}\mathbb{V}(\mathbf{w},\varsigma) = \frac{1}{\Gamma(k-\wp)}\int_0^{\varsigma}(\varsigma-\wp)^{k-\wp-1}\mathbb{V}^{(k)}(\mathbf{w},\wp)d\wp, \quad k-1<\wp\leq k, \quad k\in\mathbb{N}. \tag{2.1}$$

Definition 2.2. The function $\mathbb{V}(\varsigma)$ having Yang transform is defined as

$$Y(\mathbb{V}(\varsigma)) = M(u) = \int_0^{\infty} e^{\frac{-\varsigma}{u}}\mathbb{V}(\varsigma)d\varsigma, \quad \varsigma>0, \ \ u\in(-\varsigma_1,\varsigma_2), \tag{2.2}$$

considering inverse Yang transform as

$$Y^{-1}\{M(u)\} = \mathbb{V}(\varsigma). \tag{2.3}$$

Definition 2.3. The Yang transform of the function having nth derivatives is defined as

$$Y(\mathbb{V}^{n}(\varsigma)) = \frac{M(u)}{u^{n}} - \sum_{k=0}^{n-1}\frac{\mathbb{V}^{k}(0)}{u^{n-k-1}} \tag{2.4}$$

for all $n\in\mathbb{N}$.

Definition 2.4. The Yang transform of the function having fractional derivatives is defined as

$$Y(\mathbb{V}^{\wp}(\varsigma)) = \frac{M(u)}{u^{\wp}} - \sum_{k=0}^{n-1}\frac{\mathbb{V}^{k}(0)}{u^{\wp-(k+1)}}, \quad 0<\wp\leq n. \tag{2.5}$$

3. Implementation of HPTM

To illustrate the basic idea of this method, we consider the general fractional nonlinear partial differential equations of the form

$$D_{\varsigma}^{\wp}\mathbb{V}(\mathbf{w},\varsigma) = \mathcal{P}_1[\mathbf{w}]\mathbb{V}(\mathbf{w},\varsigma) + \mathcal{Q}_1[\mathbf{w}]\mathbb{V}(\mathbf{w},\varsigma), \quad 0 < \wp \leq 2 \tag{3.1}$$

with the initial conditions

$$\mathbb{V}(\mathbf{w},0) = \xi(\mathbf{w}), \quad \frac{\partial}{\partial\varsigma}\mathbb{V}(\mathbf{w},0) = \zeta(\mathbf{w}),$$

where $D_{\varsigma}^{\wp} = \frac{\partial^{\wp}}{\partial\varsigma^{\wp}}$ is the Caputo operator, $\mathcal{P}_1[\mathbf{w}], \mathcal{Q}_1[\mathbf{w}]$ are linear, nonlinear differential operators.

By taking YT and by applying its differentiation property, we have

$$Y\left(D_{\varsigma}^{\wp}\mathbb{V}(\mathbf{w},\varsigma)\right) = Y\left(\mathcal{P}_1[\mathbf{w}]\mathbb{V}(\mathbf{w},\varsigma) + \mathcal{Q}_1[\mathbf{w}]\mathbb{V}(\mathbf{w},\varsigma)\right), \tag{3.2}$$

and

$$\frac{1}{u^{\wp}}\left\{M(u) - u\mathbb{V}(0) - u^2\mathbb{V}^{'}(0)\right\} = Y\left(\mathcal{P}_1[\mathbf{w}]\mathbb{V}(\mathbf{w},\varsigma) + \mathcal{Q}_1[\mathbf{w}]\mathbb{V}(\mathbf{w},\varsigma)\right). \tag{3.3}$$

After simplification, we have

$$M(u) = u\mathbb{V}(0) + u^2\mathbb{V}^{'}(0) + u^{\wp}Y\left(\mathcal{P}_1[\mathbf{w}]\mathbb{V}(\mathbf{w},\varsigma) + \mathcal{Q}_1[\mathbf{w}]\mathbb{V}(\mathbf{w},\varsigma)\right). \tag{3.4}$$

Using inverse YT both sides, we get

$$\mathbb{V}(\mathbf{w},\varsigma) = \mathbb{V}(0) + \mathbb{V}^{'}(0) + Y^{-1}\left\{u^{\wp}Y\left(\mathcal{P}_1[\mathbf{w}]\mathbb{V}(\mathbf{w},\varsigma) + \mathcal{Q}_1[\mathbf{w}]\mathbb{V}(\mathbf{w},\varsigma)\right)\right\}. \tag{3.5}$$

On utilizing the HPM

$$\mathbb{V}(\mathbf{w},\varsigma) = \sum_{k=0}^{\infty}\epsilon^k\mathbb{V}_k(\mathbf{w},\varsigma), \tag{3.6}$$

with perturbation parameter $\epsilon \in [0,1]$.

The nonlinear operator is

$$\mathcal{Q}_1[\mathbf{w}]\mathbb{V}(\mathbf{w},\varsigma) = \sum_{k=0}^{\infty}\epsilon^k H_n(\mathbb{V}), \tag{3.7}$$

and He's polynomials $H_k(\mathbb{V})$ are stated as

$$H_n\left(\mathbb{V}_0,\mathbb{V}_1,\ldots,\mathbb{V}_n\right) = \frac{1}{\Gamma(n+1)}D_{\epsilon}^{k}\left[\mathcal{Q}_1\left(\sum_{k=0}^{\infty}\epsilon^i\mathbb{V}_i\right)\right]_{\epsilon=0}, \tag{3.8}$$

where $D_{\epsilon}^{k} = \frac{\partial^k}{\partial\epsilon^k}$.

On substituting (3.7) and (3.8) in (3.5), we obtain

$$\sum_{k=0}^{\infty}\epsilon^k\mathbb{V}_k(\mathbf{w},\varsigma) = \mathbb{V}(0) + \mathbb{V}^{'}(0) + \epsilon\left(Y^{-1}\left\{u^{\wp}Y\left(\mathcal{P}_1\sum_{k=0}^{\infty}\epsilon^k\mathbb{V}_k(\mathbf{w},\varsigma) + \sum_{k=0}^{\infty}\epsilon^k H_k(\mathbb{V})\right)\right\}\right). \tag{3.9}$$

On comparing the ϵ coefficients, we have

$$
\begin{aligned}
\epsilon^0 &: \mathbb{V}_0(\mathbf{w},\varsigma) = \mathbb{V}(0) + \mathbb{V}'(0),\\
\epsilon^1 &: \mathbb{V}_1(\mathbf{w},\varsigma) = Y^{-1}\left\{u^{\wp} Y(\mathcal{P}_1[\mathbf{w}]\mathbb{V}_0(\mathbf{w},\varsigma) + H_0(\mathbb{V}))\right\},\\
\epsilon^2 &: \mathbb{V}_2(\mathbf{w},\varsigma) = Y^{-1}\left\{u^{\wp} Y(\mathcal{P}_1[\mathbf{w}]\mathbb{V}_1(\mathbf{w},\varsigma) + H_1(\mathbb{V}))\right\},\\
&\vdots\\
\epsilon^k &: \mathbb{V}_k(\mathbf{w},\varsigma) = Y^{-1}\left\{u^{\wp} Y(\mathcal{P}_1[\mathbf{w}]\mathbb{V}_{k-1}(\mathbf{w},\varsigma) + H_{k-1}(\mathbb{V}))\right\}
\end{aligned}
\tag{3.10}
$$

for $k \in \mathbb{N}$.

Finally, the approximate solution is calculated as

$$
\mathbb{V}(\mathbf{w},\varsigma) = \lim_{M\to\infty} \sum_{k=1}^{M} \mathbb{V}_k(\mathbf{w},\varsigma). \tag{3.11}
$$

4. Implementation of YTDM

To illustrate the basic idea of this method, we consider the general fractional nonlinear partial differential equations of the form

$$
D_{\varsigma}^{\wp}\mathbb{V}(\mathbf{w},\varsigma) = \mathcal{P}_1(\mathbf{w},\varsigma) + Q_1(\mathbf{w},\varsigma), \quad 0 < \wp \le 1 \tag{4.1}
$$

with the initial conditions

$$
\mathbb{V}(\mathbf{w},0) = \xi(\mathbf{w}), \quad \frac{\partial}{\partial\varsigma}\mathbb{V}(\mathbf{w},0) = \zeta(\mathbf{w}),
$$

where $D_{\varsigma}^{\wp} = \frac{\partial^{\wp}}{\partial\varsigma^{\wp}}$ is the Caputo operator, $\mathcal{P}_1, Q_1$ are linear, nonlinear differential operators.

By taking YT and by applying its differentiation property, we have

$$
\begin{aligned}
Y\left(D_{\varsigma}^{\wp}\mathbb{V}(\mathbf{w},\varsigma)\right) &= Y\left(\mathcal{P}_1(\mathbf{w},\varsigma) + Q_1(\mathbf{w},\varsigma)\right),\\
\frac{1}{u^{\wp}}\left\{M(u) - u\mathbb{V}(0) - u^2\mathbb{V}'(0)\right\} &= Y\left(\mathcal{P}_1(\mathbf{w},\varsigma) + Q_1(\mathbf{w},\varsigma)\right).
\end{aligned}
\tag{4.2}
$$

After simplification, we have

$$
M(u) = u\mathbb{V}(0) + u^2\mathbb{V}'(0) + u^{\wp} Y\left(\mathcal{P}_1(\mathbf{w},\varsigma) + Q_1(\mathbf{w},\varsigma)\right). \tag{4.3}
$$

Using inverse YT both sides, we get

$$
\mathbb{V}(\mathbf{w},\varsigma) = \mathbb{V}(0) + \mathbb{V}'(0) + Y^{-1}\left\{u^{\wp} Y\left(\mathcal{P}_1(\mathbf{w},\varsigma) + Q_1(\mathbf{w},\varsigma)\right)\right\}. \tag{4.4}
$$

The YTDM series form solution for $\mathbb{V}(\mathbf{w},\varsigma)$ is determined as

$$
\mathbb{V}(\mathbf{w},\varsigma) = \sum_{m=0}^{\infty} \mathbb{V}_m(\mathbf{w},\varsigma), \tag{4.5}
$$

and the nonlinear terms Q_1 decomposition are defined as

$$
Q_1(\mathbf{w},\varsigma) = \sum_{m=0}^{\infty} \mathcal{A}_m, \tag{4.6}
$$

where

$$\mathcal{A}_m = \frac{1}{m!}\left[\frac{\partial^m}{\partial \ell^m}\left\{Q_1\left(\sum_{k=0}^{\infty}\ell^k \mathbf{w}_k, \sum_{k=0}^{\infty}\ell^k \varsigma_k\right)\right\}\right]_{\ell=0}.$$

On putting (4.5) and (4.6) into (4.4), we get

$$\sum_{m=0}^{\infty}\mathbb{V}_m(\mathbf{w},\varsigma) = \mathbb{V}(0) + \mathbb{V}'(0) + Y^{-1}\left\{u^{\wp}\left[Y\left(\mathcal{P}_1\left(\sum_{m=0}^{\infty}\mathbf{w}_m, \sum_{m=0}^{\infty}\varsigma_m\right) + \sum_{m=0}^{\infty}\mathcal{A}_m\right)\right]\right\}. \tag{4.7}$$

Thus, the given terms are obtained:

$$\begin{aligned}
\mathbb{V}_0(\mathbf{w},\varsigma) &= \mathbb{V}(0) + \varsigma\mathbb{V}'(0),\\
\mathbb{V}_1(\mathbf{w},\varsigma) &= Y^{-1}\left\{u^{\wp}Y\left(\mathcal{P}_1(\mathbf{w}_0,\varsigma_0) + \mathcal{A}_0\right)\right\}.
\end{aligned}$$

Generally for $m \geq 1$, we can write

$$\mathbb{V}_{m+1}(\mathbf{w},\varsigma) = Y^{-1}\left\{u^{\wp}Y\left(\mathcal{P}_1(\mathbf{w}_m,\varsigma_m) + \mathcal{A}_m\right)\right\}.$$

5. Examples

In this section, we implemented HPTM and YTDM to solve different fractional-order vibration equations.

Example 1. Consider the fractional-order vibration equation

$$\frac{1}{\ell^2}\frac{\partial^{\wp}\mathbb{V}(\mathbf{w},\varsigma)}{\partial\varsigma^{\wp}} = \frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}} \tag{5.1}$$

with the initial conditions

$$\mathbb{V}(\mathbf{w},0) = \mathbf{w}^2, \quad \frac{\partial}{\partial\varsigma}\mathbb{V}(\mathbf{w},0) = \ell\mathbf{w}.$$

By taking YT, we get

$$Y\left(\frac{\partial^{\wp}\mathbb{V}}{\partial\varsigma^{\wp}}\right) = Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right), \tag{5.2}$$

and by applying its differentiation property, we have

$$\frac{1}{u^{\wp}}\left\{M(u) - u\mathbb{V}(0) - u^2\mathbb{V}'(0)\right\} = Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right). \tag{5.3}$$

Thus, we obtain

$$M(u) = u\mathbb{V}(0) + u^2\mathbb{V}'(0) + u^{\wp}Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right). \tag{5.4}$$

Using inverse YT both sides, we get

$$\begin{aligned}
\mathbb{V}(\mathbf{w},\varsigma) &= Y^{-1}[u\mathbb{V}(0) + u^2\mathbb{V}'(0)] + Y^{-1}\left\{u^{\wp}\left[Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right)\right]\right\},\\
\mathbb{V}(\mathbf{w},\varsigma) &= (\mathbf{w}^2 + \ell\mathbf{w}\varsigma) + Y^{-1}\left\{u^{\wp}\left[Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right)\right]\right\}.
\end{aligned} \tag{5.5}$$

On utilizing the HPM, we get

$$\sum_{k=0}^{\infty}\epsilon^k\mathbb{V}_k(\mathbf{w},\varsigma)=(\mathbf{w}^2+\ell\mathbf{w}\varsigma)+\epsilon\left(Y^{-1}\left\{u^{\wp}Y\left(\ell^2\left(\sum_{k=0}^{\infty}\epsilon^k\mathbb{V}_k(\mathbf{w},\varsigma)\right)_{\mathbf{ww}}+\frac{\ell^2}{\mathbf{w}}\left(\sum_{k=0}^{\infty}\epsilon^k\mathbb{V}_k(\mathbf{w},\varsigma)\right)_{\mathbf{w}}\right)\right\}\right). \quad (5.6)$$

On comparing the ϵ coefficients, we have

$$\begin{aligned}
&\epsilon^0:\mathbb{V}_0(\mathbf{w},\varsigma)=(\mathbf{w}^2+\ell\mathbf{w}\varsigma),\\
&\epsilon^1:\mathbb{V}_1(\mathbf{w},\varsigma)=Y^{-1}\left\{u^{\wp}Y\left(\ell^2\frac{\partial^2\mathbb{V}_0(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{\ell^2}{\mathbf{w}}\frac{\partial\mathbb{V}_0(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right\}=\frac{\ell^2\varsigma^{\wp}(\ell\varsigma+4\mathbf{w}(1+\wp))}{\mathbf{w}\Gamma(2+\wp)},\\
&\epsilon^2:\mathbb{V}_2(\mathbf{w},\varsigma)=Y^{-1}\left\{u^{\wp}Y\left(\ell^2\frac{\partial^2\mathbb{V}_1(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{\ell^2}{\mathbf{w}}\frac{\partial\mathbb{V}_1(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right\}=\frac{\ell^5\varsigma^{1+2\wp}}{\mathbf{w}^3\Gamma(2+2\wp)},\\
&\epsilon^3:\mathbb{V}_3(\mathbf{w},\varsigma)=Y^{-1}\left\{u^{\wp}Y\left(\ell^2\frac{\partial^2\mathbb{V}_2(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{\ell^2}{\mathbf{w}}\frac{\partial\mathbb{V}_2(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right\}=\frac{9\ell^7\varsigma^{1+3\wp}}{\mathbf{w}^5\Gamma(2+3\wp)},\\
&\qquad\vdots
\end{aligned}$$

The HPTM solution is

$$\mathbb{V}(\mathbf{w},\varsigma)=\mathbb{V}_0(\mathbf{w},\varsigma)+\mathbb{V}_1(\mathbf{w},\varsigma)+\mathbb{V}_2(\mathbf{w},\varsigma)+\mathbb{V}_3(\mathbf{w},\varsigma)+\cdots,$$

or

$$\mathbb{V}(\mathbf{w},\varsigma)=(\mathbf{w}^2+\ell\mathbf{w}\varsigma)+\frac{\ell^2\varsigma^{\wp}(\ell\varsigma+4\mathbf{w}(1+\wp))}{\mathbf{w}\Gamma(2+\wp)}+\frac{\ell^5\varsigma^{1+2\wp}}{\mathbf{w}^3\Gamma(2+2\wp)}+\frac{9\ell^7\varsigma^{1+3\wp}}{\mathbf{w}^5\Gamma(2+3\wp)}+\cdots.$$

By taking YT, we get

$$Y\left(\frac{\partial^{\wp}\mathbb{V}}{\partial\varsigma^{\wp}}\right)=Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right), \quad (5.7)$$

and by applying its differentiation property, we have

$$\frac{1}{u^{\wp}}\{M(u)-u\mathbb{V}(0)-u^2\mathbb{V}'(0)\}=Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right). \quad (5.8)$$

Thus, we obtain

$$M(u)=u\mathbb{V}(0)+u^2\mathbb{V}'(0)+u^{\wp}Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right). \quad (5.9)$$

Using inverse YT both sides, we get

$$\begin{aligned}
\mathbb{V}(\mathbf{w},\varsigma)&=Y^{-1}[u\mathbb{V}(0)+u^2\mathbb{V}'(0)]+Y^{-1}\left\{u^{\wp}\left[Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right)\right]\right\},\\
\mathbb{V}(\mathbf{w},\varsigma)&=(\mathbf{w}^2+\ell\mathbf{w}\varsigma)+Y^{-1}\left\{u^{\wp}\left[Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right)\right]\right\}.
\end{aligned} \quad (5.10)$$

The YTDM series form solution for $\mathbb{V}(\mathbf{w},\varsigma)$ is determined as

$$\mathbb{V}(\mathbf{w},\varsigma)=\sum_{m=0}^{\infty}\mathbb{V}_m(\mathbf{w},\varsigma). \quad (5.11)$$

Then, we have

$$\sum_{m=0}^{\infty} \mathbb{V}_m(\mathbf{w}, \varsigma) = (\mathbf{w}^2 + \ell\mathbf{w}\varsigma) + Y^{-1}\left\{u^{\wp}\left[Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w}, \varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w}, \varsigma)}{\partial\mathbf{w}}\right)\right)\right]\right\}. \tag{5.12}$$

Thus, by comparing both sides, we obtain

$$\mathbb{V}_0(\mathbf{w}, \varsigma) = (\mathbf{w}^2 + \ell\mathbf{w}\varsigma).$$

On $m = 0$, we have

$$\mathbb{V}_1(\mathbf{w}, \varsigma) = \frac{\ell^2\varsigma^{\wp}(\ell\varsigma + 4\mathbf{w}(1 + \wp))}{\mathbf{w}\Gamma(2 + \wp)}.$$

On $m = 1$, we have

$$\mathbb{V}_2(\mathbf{w}, \varsigma) = \frac{\ell^5\varsigma^{1+2\wp}}{\mathbf{w}^3\Gamma(2 + 2\wp)}.$$

On $m = 2$, we have

$$\mathbb{V}_3(\mathbf{w}, \varsigma) = \frac{9\ell^7\varsigma^{1+3\wp}}{\mathbf{w}^5\Gamma(2 + 3\wp)}.$$

The YTDM solution is

$$\mathbb{V}(\mathbf{w}, \varsigma) = \sum_{m=0}^{\infty} \mathbb{V}_m(\mathbf{w}, \varsigma) = \mathbb{V}_0(\mathbf{w}, \varsigma) + \mathbb{V}_1(\mathbf{w}, \varsigma) + \mathbb{V}_2(\mathbf{w}, \varsigma) + \mathbb{V}_3(\mathbf{w}, \varsigma) + \cdots,$$

or

$$\mathbb{V}(\mathbf{w}, \varsigma) = (\mathbf{w}^2 + \ell\mathbf{w}\varsigma) + \frac{\ell^2\varsigma^{\wp}(\ell\varsigma + 4\mathbf{w}(1 + \wp))}{\mathbf{w}\Gamma(2 + \wp)} + \frac{\ell^5\varsigma^{1+2\wp}}{\mathbf{w}^3\Gamma(2 + 2\wp)} + \frac{9\ell^7\varsigma^{1+3\wp}}{\mathbf{w}^5\Gamma(2 + 3\wp)} + \cdots.$$

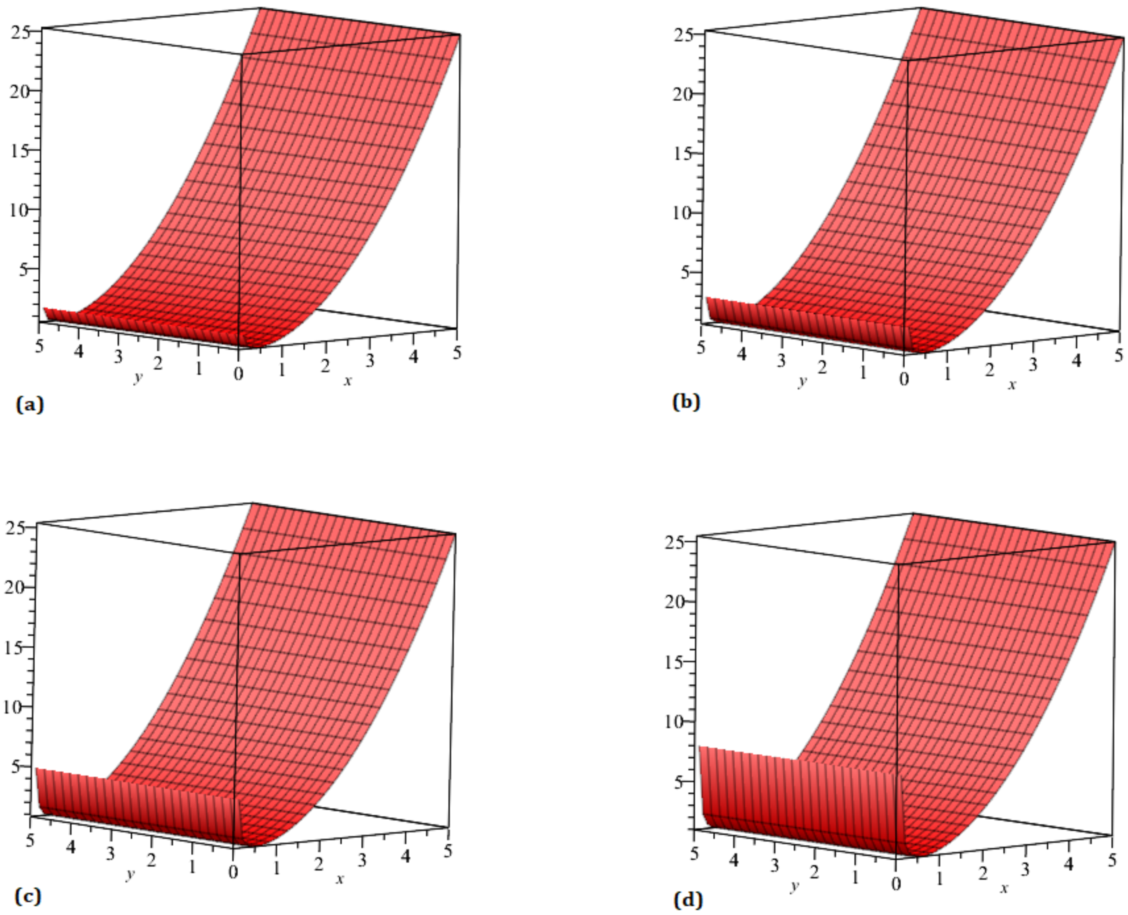

Figure 1. The behavior of $\mathbb{V}(\mathbf{w}, \varsigma)$ with respect to $\mathbf{w}$ and ς for different values $\wp$ within the domain $0 \leq \mathbf{w}, \varsigma \leq 5$. (a) Behavior of the $\mathbb{V}(\mathbf{w}, \varsigma)$ solution, when $\wp = 1.25$. (b) Behavior of the $\mathbb{V}(\mathbf{w}, \varsigma)$ solution, when $\wp = 1.50$. (c) Behavior of the $\mathbb{V}(\mathbf{w}, \varsigma)$ solution, when $\wp = 1.75$. (d) Behavior of the $\mathbb{V}(\mathbf{w}, \varsigma)$ solution, when $\wp = 2$.

Table 1. Suggested methods solution at different values of $\wp$ and $\ell = 5$ of Example 1.

$\varsigma = 0.001$	HPTM solution	YTDM solution	Solution at	Solution at	Solution at
$\mathbf{w}$	$\wp = 2$	$\wp = 2$	$\wp = 1.9$	$\wp = 1.8$	$\wp = 1.7$
0.5	0.25250833	0.25250833	0.25251882	0.25254240	0.25259522
1.0	1.00500416	1.00500416	1.00500941	1.00502120	1.00504761
1.5	2.25750277	2.25750277	2.25750627	2.25751413	2.25753174
2.0	4.01000208	4.01000208	4.01000470	4.01001060	4.01002380
2.5	6.26250166	6.26250166	6.26250376	6.26250848	6.26251904
3.0	9.01500138	9.01500138	9.01500313	9.01500706	9.01501587
3.5	12.26750119	12.26750119	12.26750269	12.26750606	12.26751360
4.0	16.02000104	16.02000104	16.02000235	16.02000530	16.02001190
4.5	20.27250093	20.27250093	20.27250209	20.27250471	20.27251058
5.0	25.02500083	25.02500083	25.02500188	25.02500424	25.02500952

Example 2. Consider the fractional-order vibration equation

$$\frac{1}{\ell^2}\frac{\partial^{\wp}\mathbb{V}(\mathbf{w},\varsigma)}{\partial\varsigma^{\wp}} = \frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}} \tag{5.13}$$

with the initial conditions

$$\mathbb{V}(\mathbf{w},0) = \mathbf{w}, \quad \frac{\partial}{\partial\varsigma}\mathbb{V}(\mathbf{w},0) = \ell\mathbf{w}.$$

By taking YT, we get

$$Y\left(\frac{\partial^{\wp}\mathbb{V}}{\partial\varsigma^{\wp}}\right) = Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right), \tag{5.14}$$

and by applying its differentiation property, we have

$$\frac{1}{u^{\wp}}\left\{M(u) - u\mathbb{V}(0) - u^2\mathbb{V}'(0)\right\} = Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right). \tag{5.15}$$

Then, we obtain

$$M(u) = u\mathbb{V}(0) + u^2\mathbb{V}'(0) + u^{\wp}Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right). \tag{5.16}$$

Using inverse YT both sides, we get

$$\mathbb{V}(\mathbf{w},\varsigma) = Y^{-1}\left\{u\mathbb{V}(0) + u^2\mathbb{V}'(0)\right\} + Y^{-1}\left\{u^{\wp}\left[Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right)\right]\right\}, \tag{5.17}$$

or

$$\mathbb{V}(\mathbf{w},\varsigma) = (\mathbf{w} + \ell\mathbf{w}\varsigma) + Y^{-1}\left\{u^{\wp}\left[Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right)\right]\right\}. \tag{5.18}$$

On utilizing the HPM, we get

$$\sum_{k=0}^{\infty}\epsilon^k\mathbb{V}_k(\mathbf{w},\varsigma) = (\mathbf{w} + \ell\mathbf{w}\varsigma) + \epsilon\left(Y^{-1}\left\{u^{\wp}Y\left(\ell^2\left(\sum_{k=0}^{\infty}\epsilon^k\mathbb{V}_k(\mathbf{w},\varsigma)\right)_{\mathbf{ww}} + \frac{\ell^2}{\mathbf{w}}\left(\sum_{k=0}^{\infty}\epsilon^k\mathbb{V}_k(\mathbf{w},\varsigma)\right)_{\mathbf{w}}\right)\right\}\right). \tag{5.19}$$

On comparing the ϵ coefficients, we have

$$\begin{aligned}
\epsilon^0 &: \mathbb{V}_0(\mathbf{w},\varsigma) = (\mathbf{w} + \ell\mathbf{w}\varsigma),\\
\epsilon^1 &: \mathbb{V}_1(\mathbf{w},\varsigma) = Y^{-1}\left\{u^{\wp}Y\left(\ell^2\frac{\partial^2\mathbb{V}_0(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{\ell^2}{\mathbf{w}}\frac{\partial\mathbb{V}_0(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right\} = \frac{\ell^2\varsigma^{\wp}(1+\ell\varsigma+\wp)}{\mathbf{w}\Gamma(2+\wp)},\\
\epsilon^2 &: \mathbb{V}_2(\mathbf{w},\varsigma) = Y^{-1}\left\{u^{\wp}Y\left(\ell^2\frac{\partial^2\mathbb{V}_1(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{\ell^2}{\mathbf{w}}\frac{\partial\mathbb{V}_1(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right\} = \frac{\ell^2\varsigma^{2\wp}(1+\ell\varsigma+2\wp)}{\mathbf{w}^3\Gamma(2+2\wp)},\\
\epsilon^3 &: \mathbb{V}_3(\mathbf{w},\varsigma) = Y^{-1}\left\{u^{\wp}Y\left(\ell^2\frac{\partial^2\mathbb{V}_2(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{\ell^2}{\mathbf{w}}\frac{\partial\mathbb{V}_2(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right\} = \frac{9\ell^6\varsigma^{3\wp}(1+\ell\varsigma+3\wp)}{\mathbf{w}^5\Gamma(2+3\wp)},\\
&\vdots
\end{aligned}$$

The HPTM solution is

$$\mathbb{V}(\mathbf{w},\varsigma)=\mathbb{V}_0(\mathbf{w},\varsigma)+\mathbb{V}_1(\mathbf{w},\varsigma)+\mathbb{V}_2(\mathbf{w},\varsigma)+\mathbb{V}_3(\mathbf{w},\varsigma)+\cdots,$$

or

$$\mathbb{V}(\mathbf{w},\varsigma)=\frac{\ell^2\varsigma^{\wp}(1+\ell\varsigma+\wp)}{\mathbf{w}\Gamma(2+\wp)}+\frac{\ell^2\varsigma^{2\wp}(1+\ell\varsigma+2\wp)}{\mathbf{w}^3\Gamma(2+2\wp)}+\frac{9\ell^6\varsigma^{3\wp}(1+\ell\varsigma+3\wp)}{\mathbf{w}^5\Gamma(2+3\wp)}+\cdots.$$

By taking YT, we get

$$Y\left(\frac{\partial^{\wp}\mathbb{V}}{\partial\varsigma^{\wp}}\right)=Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right),\tag{5.20}$$

and by applying its differentiation property, we have

$$\frac{1}{u^{\wp}}\left\{M(u)-u\mathbb{V}(0)-u^2\mathbb{V}'(0)\right\}=Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right).\tag{5.21}$$

Thus, we obtain

$$M(u)=u\mathbb{V}(0)+u^2\mathbb{V}'(0)+u^{\wp}Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right).\tag{5.22}$$

Using inverse YT both sides, we get

$$\mathbb{V}(\mathbf{w},\varsigma)=Y^{-1}\left\{u\mathbb{V}(0)+u^2\mathbb{V}'(0)\right\}+Y^{-1}\left\{u^{\wp}\left[Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right)\right]\right\},\tag{5.23}$$

or

$$\mathbb{V}(\mathbf{w},\varsigma)=(\mathbf{w}+\ell\mathbf{w}\varsigma)+Y^{-1}\left\{u^{\wp}\left[Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right)\right]\right\}.\tag{5.24}$$

The YTDM series form solution for $\mathbb{V}(\mathbf{w},\varsigma)$ is determined as

$$\mathbb{V}(\mathbf{w},\varsigma)=\sum_{m=0}^{\infty}\mathbb{V}_m(\mathbf{w},\varsigma).\tag{5.25}$$

Then, we obtain

$$\sum_{m=0}^{\infty}\mathbb{V}_m(\mathbf{w},\varsigma)=(\mathbf{w}+\ell\mathbf{w}\varsigma)+Y^{-1}\left\{u^{\wp}\left[Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right)\right]\right\}.\tag{5.26}$$

Thus by comparing both sides, we obtain

$$\mathbb{V}_0(\mathbf{w},\varsigma)=(\mathbf{w}+\ell\mathbf{w}\varsigma).$$

On $m=0$, we have

$$\mathbb{V}_1(\mathbf{w},\varsigma)=\frac{\ell^2\varsigma^{\wp}(1+\ell\varsigma+\wp)}{\mathbf{w}\Gamma(2+\wp)}.$$

On $m=1$, we have

$$\mathbb{V}_2(\mathbf{w},\varsigma)=\frac{\ell^2\varsigma^{2\wp}(1+\ell\varsigma+2\wp)}{\mathbf{w}^3\Gamma(2+2\wp)}.$$

On $m = 2$, we have

$$\mathbb{V}_3(\mathbf{w}, \varsigma) = \frac{9\ell^6\varsigma^{3\wp}(1 + \ell\varsigma + 3\wp)}{\mathbf{w}^5\Gamma(2 + 3\wp)}.$$

The YTDM solution is

$$\mathbb{V}(\mathbf{w}, \varsigma) = \sum_{m=0}^{\infty} \mathbb{V}_m(\mathbf{w}, \varsigma) = \mathbb{V}_0(\mathbf{w}, \varsigma) + \mathbb{V}_1(\mathbf{w}, \varsigma) + \mathbb{V}_2(\mathbf{w}, \varsigma) + \mathbb{V}_3(\mathbf{w}, \varsigma) + \cdots,$$

or

$$\mathbb{V}(\mathbf{w}, \varsigma) = \frac{\ell^2\varsigma^{\wp}(1 + \ell\varsigma + \wp)}{\mathbf{w}\Gamma(2 + \wp)} + \frac{\ell^2\varsigma^{2\wp}(1 + \ell\varsigma + 2\wp)}{\mathbf{w}^3\Gamma(2 + 2\wp)} + \frac{9\ell^6\varsigma^{3\wp}(1 + \ell\varsigma + 3\wp)}{\mathbf{w}^5\Gamma(2 + 3\wp)} + \cdots.$$

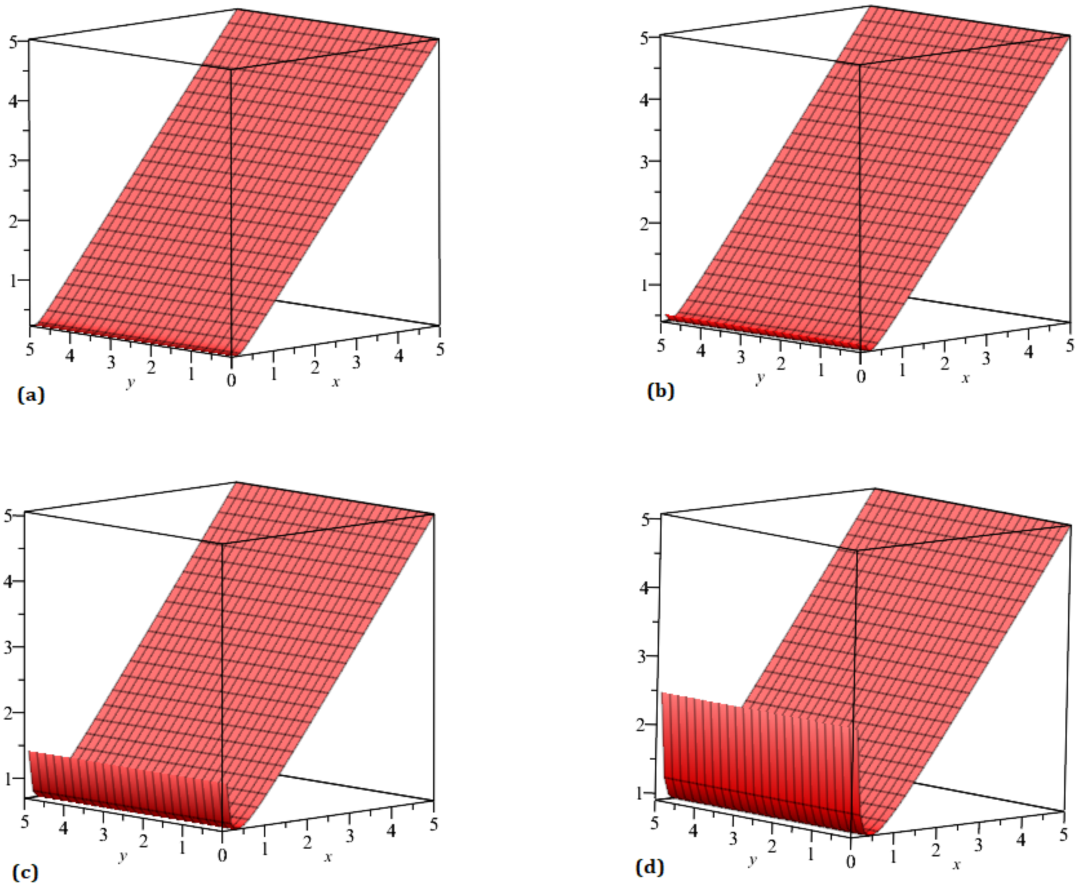

Figure 2. The behavior of $\mathbb{V}(\mathbf{w}, \varsigma)$ with respect to $\mathbf{w}$ and ς for different values $\wp$ within the domain $0 \leq \mathbf{w}, \varsigma \leq 5$. (a) Behavior of the $\mathbb{V}(\mathbf{w}, \varsigma)$ solution, when $\wp = 1.25$. (b) Behavior of the $\mathbb{V}(\mathbf{w}, \varsigma)$ solution, when $\wp = 1.50$. (c) Behavior of the $\mathbb{V}(\mathbf{w}, \varsigma)$ solution, when $\wp = 1.75$. (d) Behavior of the $\mathbb{V}(\mathbf{w}, \varsigma)$ solution, when $\wp = 2$.

Table 2. Suggested methods solution at different values of $\wp$ and $\ell = 5$ of Problem 2.

$\varsigma = 0.001$	HPTM solution	YTDM solution	Solution at	Solution at	Solution at
w	$\wp = 2$	$\wp = 2$	$\wp = 1.9$	$\wp = 1.8$	$\wp = 1.7$
0.5	0.50250833	0.50250833	0.50251882	0.50254240	0.50259522
1.0	1.00500416	1.00500416	1.00500941	1.00502120	1.00504761
1.5	1.50750277	1.50750277	1.50750627	1.50751413	1.50753174
2.0	2.01000208	2.01000208	2.01000470	2.01001060	2.01002380
2.5	2.51250166	2.51250166	2.51250376	2.51250848	2.51251904
3.0	3.01500138	3.01500138	3.01500313	3.01500706	3.01501587
3.5	3.51750119	3.51750119	3.51750268	3.51750605	3.51751360
4.0	4.02000104	4.02000104	4.02000235	4.02000530	4.02001190
4.5	4.52250092	4.52250092	4.52250209	4.52250471	4.52251058
5.0	5.02500083	5.02500083	5.02500188	5.02500424	5.02500952

Example 3. Consider the fractional-order vibration equation

$$\frac{1}{\ell^2}\frac{\partial^{\wp}\mathbb{V}(\mathbf{w},\varsigma)}{\partial\varsigma^{\wp}} = \frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}} \tag{5.27}$$

with the initial conditions

$$\mathbb{V}(\mathbf{w},0) = \sqrt{\mathbf{w}}, \quad \frac{\partial}{\partial\varsigma}\mathbb{V}(\mathbf{w},0) = \frac{\ell}{\sqrt{\mathbf{w}}}.$$

By taking YT, we get

$$Y\left(\frac{\partial^{\wp}\mathbb{V}}{\partial\varsigma^{\wp}}\right) = Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right), \tag{5.28}$$

and by applying its differentiation property, we have

$$\frac{1}{u^{\wp}}\left\{M(u) - u\mathbb{V}(0) - u^2\mathbb{V}'(0)\right\} = Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right). \tag{5.29}$$

Then, we obtain

$$M(u) = u\mathbb{V}(0) + u^2\mathbb{V}'(0) + u^{\wp}Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right). \tag{5.30}$$

Using inverse YT both sides, we get

$$\mathbb{V}(\mathbf{w},\varsigma) = Y^{-1}\left\{u\mathbb{V}(0) + u^2\mathbb{V}'(0)\right\} + Y^{-1}\left\{u^{\wp}\left[Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right)\right]\right\}. \tag{5.31}$$

Then, we obtain

$$\mathbb{V}(\mathbf{w},\varsigma) = \left(\sqrt{\mathbf{w}} + \frac{\ell\varsigma}{\sqrt{\mathbf{w}}}\right) + Y^{-1}\left\{u^{\wp}\left[Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2} + \frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right)\right]\right\}. \tag{5.32}$$

On utilizing the HPM, we get

$$\sum_{k=0}^{\infty}\epsilon^k\mathbb{V}_k(\mathbf{w},\varsigma)=\left(\sqrt{\mathbf{w}}+\frac{\ell\varsigma}{\sqrt{\mathbf{w}}}\right)+\epsilon\left(Y^{-1}\left\{u^{\wp}Y\left(\ell^2\left(\sum_{k=0}^{\infty}\epsilon^k\mathbb{V}_k(\mathbf{w},\varsigma)\right)_{\mathbf{ww}}+\frac{\ell^2}{\mathbf{w}}\left(\sum_{k=0}^{\infty}\epsilon^k\mathbb{V}_k(\mathbf{w},\varsigma)\right)_{\mathbf{w}}\right)\right\}\right). \tag{5.33}$$

On comparing the ϵ coefficients, we have

$$\begin{aligned}
\epsilon^0 &: \mathbb{V}_0(\mathbf{w},\varsigma)=\left(\sqrt{\mathbf{w}}+\frac{\ell\varsigma}{\sqrt{\mathbf{w}}}\right),\\
\epsilon^1 &: \mathbb{V}_1(\mathbf{w},\varsigma)=Y^{-1}\left\{u^{\wp}Y\left(\ell^2\frac{\partial^2\mathbb{V}_0(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{\ell^2}{\mathbf{w}}\frac{\partial\mathbb{V}_0(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right\}=\frac{\ell^2\varsigma^{\wp}(\mathbf{w}+\ell\varsigma+\mathbf{w}\wp)}{4\mathbf{w}^{\frac{5}{2}}\Gamma(2+\wp)},\\
\epsilon^2 &: \mathbb{V}_2(\mathbf{w},\varsigma)=Y^{-1}\left\{u^{\wp}Y\left(\ell^2\frac{\partial^2\mathbb{V}_1(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{\ell^2}{\mathbf{w}}\frac{\partial\mathbb{V}_1(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right\}=\frac{\ell^4\varsigma^{2\wp}\left\{25\ell\varsigma+9\mathbf{w}(1+2\wp)\right\}}{16\mathbf{w}^{\frac{9}{2}}\Gamma(2+2\wp)},\\
\epsilon^3 &: \mathbb{V}_3(\mathbf{w},\varsigma)=Y^{-1}\left\{u^{\wp}Y\left(\ell^2\frac{\partial^2\mathbb{V}_2(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{\ell^2}{\mathbf{w}}\frac{\partial\mathbb{V}_2(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right\}=\frac{9\ell^6\varsigma^{3\wp}\{225\ell\varsigma+49\mathbf{w}(1+3\wp)\}}{64\mathbf{w}^{\frac{13}{2}}\Gamma(2+3\wp)},\\
&\quad\vdots
\end{aligned}$$

The HPTM solution is

$$\mathbb{V}(\mathbf{w},\varsigma)=\mathbb{V}_0(\mathbf{w},\varsigma)+\mathbb{V}_1(\mathbf{w},\varsigma)+\mathbb{V}_2(\mathbf{w},\varsigma)+\mathbb{V}_3(\mathbf{w},\varsigma)+\cdots,$$

or

$$\mathbb{V}(\mathbf{w},\varsigma)=\left(\sqrt{\mathbf{w}}+\frac{\ell\varsigma}{\sqrt{\mathbf{w}}}\right)+\frac{\ell^2\varsigma^{\wp}(\mathbf{w}+\ell\varsigma+\mathbf{w}\wp)}{4\mathbf{w}^{\frac{5}{2}}\Gamma(2+\wp)}+\frac{\ell^4\varsigma^{2\wp}\{25\ell\varsigma+9\mathbf{w}(1+2\wp)\}}{16\mathbf{w}^{\frac{9}{2}}\Gamma(2+2\wp)}+\frac{9\ell^6\varsigma^{3\wp}\{225\ell\varsigma+49\mathbf{w}(1+3\wp)\}}{64\mathbf{w}^{\frac{13}{2}}\Gamma(2+3\wp)}+\cdots.$$

By taking YT, we get

$$Y\left(\frac{\partial^{\wp}\mathbb{V}}{\partial\varsigma^{\wp}}\right)=Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right), \tag{5.34}$$

and by applying its differentiation property, we have

$$\frac{1}{u^{\wp}}\left\{M(u)-u\mathbb{V}(0)-u^2\mathbb{V}'(0)\right\}=Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right). \tag{5.35}$$

Then, we obtain

$$M(u)=u\mathbb{V}(0)+u^2\mathbb{V}'(0)+u^{\wp}Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right). \tag{5.36}$$

Using inverse YT both sides, we get

$$\mathbb{V}(\mathbf{w},\varsigma)=Y^{-1}\left\{u\mathbb{V}(0)+u^2\mathbb{V}'(0)\right\}+Y^{-1}\left\{u^{\wp}\left[Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right)\right]\right\}, \tag{5.37}$$

or

$$\mathbb{V}(\mathbf{w},\varsigma)=\left(\sqrt{\mathbf{w}}+\frac{\ell\varsigma}{\sqrt{\mathbf{w}}}\right)+Y^{-1}\left\{u^{\wp}\left[Y\left(\ell^2\left(\frac{\partial^2\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}^2}+\frac{1}{\mathbf{w}}\frac{\partial\mathbb{V}(\mathbf{w},\varsigma)}{\partial\mathbf{w}}\right)\right)\right]\right\}. \tag{5.38}$$

The YTDM series form solution for $\mathbb{V}(\mathbf{w}, \varsigma)$ is determined as

$$\mathbb{V}(\mathbf{w}, \varsigma) = \sum_{m=0}^{\infty} \mathbb{V}_m(\mathbf{w}, \varsigma). \tag{5.39}$$

Thus, we obtain

$$\sum_{m=0}^{\infty} \mathbb{V}_m(\mathbf{w}, \varsigma) = \left(\sqrt{\mathbf{w}} + \frac{\ell\varsigma}{\sqrt{\mathbf{w}}} \right) + Y^{-1} \left\{ u^{\wp} \left[Y \left(\ell^2 \left(\frac{\partial^2 \mathbb{V}(\mathbf{w}, \varsigma)}{\partial \mathbf{w}^2} + \frac{1}{\mathbf{w}} \frac{\partial \mathbb{V}(\mathbf{w}, \varsigma)}{\partial \mathbf{w}} \right) \right) \right] \right\}. \tag{5.40}$$

Thus by comparing both sides, we obtain

$$\mathbb{V}_0(\mathbf{w}, \varsigma) = \left(\sqrt{\mathbf{w}} + \frac{\ell\varsigma}{\sqrt{\mathbf{w}}} \right).$$

On $m = 0$, we have

$$\mathbb{V}_1(\mathbf{w}, \varsigma) = \frac{\ell^2 \varsigma^{\wp} (\mathbf{w} + \ell\varsigma + \mathbf{w}\wp)}{4\mathbf{w}^{\frac{5}{2}} \Gamma(2 + \wp)}.$$

On $m = 1$, we have

$$\mathbb{V}_2(\mathbf{w}, \varsigma) = \frac{\ell^4 \varsigma^{2\wp} \{25\ell\varsigma + 9\mathbf{w}(1 + 2\wp)\}}{16\mathbf{w}^{\frac{9}{2}} \Gamma(2 + 2\wp)}.$$

On $m = 2$, we have

$$\mathbb{V}_3(\mathbf{w}, \varsigma) = \frac{9\ell^6 \varsigma^{3\wp} \{225\ell\varsigma + 49\mathbf{w}(1 + 3\wp)\}}{64\mathbf{w}^{\frac{13}{2}} \Gamma(2 + 3\wp)}.$$

The YTDM solution is

$$\mathbb{V}(\mathbf{w}, \varsigma) = \sum_{m=0}^{\infty} \mathbb{V}_m(\mathbf{w}, \varsigma) = \mathbb{V}_0(\mathbf{w}, \varsigma) + \mathbb{V}_1(\mathbf{w}, \varsigma) + \mathbb{V}_2(\mathbf{w}, \varsigma) + \mathbb{V}_3(\mathbf{w}, \varsigma) + \cdots,$$

or

$$\mathbb{V}(\mathbf{w}, \varsigma) = (\sqrt{\mathbf{w}} + \frac{\ell\varsigma}{\sqrt{\mathbf{w}}}) + \frac{\ell^2 \varsigma^{\wp} (\mathbf{w} + \ell\varsigma + \mathbf{w}\wp)}{4\mathbf{w}^{\frac{5}{2}} \Gamma(2 + \wp)} + \frac{\ell^4 \varsigma^{2\wp} \{25\ell\varsigma + 9\mathbf{w}(1 + 2\wp)\}}{16\mathbf{w}^{\frac{9}{2}} \Gamma(2 + 2\wp)} + \frac{9\ell^6 \varsigma^{3\wp} \{225\ell\varsigma + 49\mathbf{w}(1 + 3\wp)\}}{64\mathbf{w}^{\frac{13}{2}} \Gamma(2 + 3\wp)} + \cdots.$$

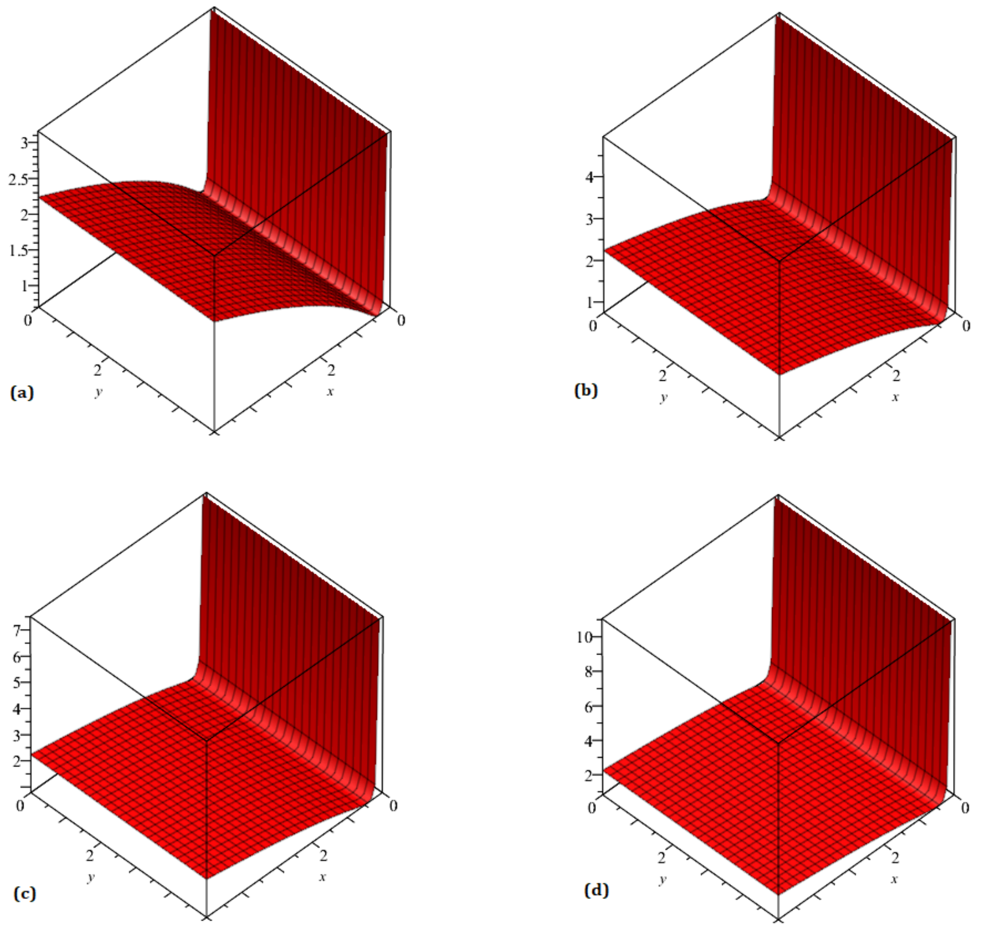

Figure 3. The behavior of $\mathbb{V}(\mathbf{w}, \varsigma)$ with respect to $\mathbf{w}$ and ς for different values $\wp$ within the domain $0 \leq \mathbf{w}, \varsigma \leq 5$. (a) Behavior of the $\mathbb{V}(\mathbf{w}, \varsigma)$ solution, when $\wp = 1.25$. (b) Behavior of the $\mathbb{V}(\mathbf{w}, \varsigma)$ solution, when $\wp = 1.50$. (c) Behavior of the $\mathbb{V}(\mathbf{w}, \varsigma)$ solution, when $\wp = 1.75$. (d) Behavior of the $\mathbb{V}(\mathbf{w}, \varsigma)$ solution, when $\wp = 2$.

Table 3. Suggested methods solution at different values of $\wp$ and $\ell = 5$ of Problem 3.

$\varsigma = 0.001$	HPTM solution	YTDM solution	Solution at	Solution at	Solution at
$\mathbf{w}$	$\wp = 2$	$\wp = 2$	$\wp = 1.9$	$\wp = 1.8$	$\wp = 1.7$
0.5	0.71418374	0.71418374	0.71419116	0.71420783	0.71424518
1.0	1.00500104	1.00500104	1.00500235	1.00500530	1.00501190
1.5	1.22882773	1.22882773	1.22882820	1.22882927	1.22883167
2.0	1.41774928	1.41774928	1.41774951	1.41775003	1.41775120
2.5	1.58430121	1.58430121	1.58430134	1.58430164	1.58430231
3.0	1.73493762	1.73493762	1.73493771	1.73493789	1.73493832
3.5	1.87350135	1.87350135	1.87350140	1.87350153	1.87350182
4.0	2.00250003	2.00250003	2.00250007	2.00250016	2.00250037
4.5	2.12367739	2.12367739	2.12367742	2.12367749	2.12367764
5.0	2.23830406	2.23830406	2.23830408	2.23830414	2.23830425

6. Conclusions

In this innovative research, the time fractional vibration equation is effectively examined using the HPTM and YTDM. The results show that the generated results are reliable and the found solution is convergent. These strategies are capable of lowering the time and the size of computation. Using it is simpler for both small and large parameters. The derived solutions are positive and bounded. Additionally, the suggested methods make it simple to manage the non-linear term. It is encouraging to see that both approaches effectively function even when the actual solution is unknown. Thus, these schemes are quite exact, fair, systematic, and effective. Numerous fractional order mathematical models of physics, biology and society can be studied and their solutions may be obtained.

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Conflict of interest

The authors declare that they have no competing interests.

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