Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(10): 17967–17988.
DOI: 10.3934/math.2022990
Received: 28 March 2022
Revised: 14 July 2022
Accepted: 18 July 2022
Published: 05 August 2022

Research article

Existence of fixed points of generalized set-valued F-contractions of b-metric spaces

Basit Ali [1,*], **Hafiza Aqsa Butt** [1,2] **and Manuel De la Sen** [3]

[1] Department of Mathematics, School of Science, University of Management and Technology, C-II, Johar Town, Lahore 54770, Pakistan

[2] Department of Mathematics, The Sahara College Narowal, Narowal, Punjab, Pakistan

[3] Department of Electricity and Electronics, Institute of Research and Development of Processes, Faculty of Science and Technology, University of the Basque Country (UPV/EHU), Campus of Leioa, 48940 Leioa, Bizkaia, Spain

* **Correspondence:** Email: basit.aa@gmail.com; Tel: +923334710429.

Abstract: This paper deals with the existence of non-empty fixed point sets of newly introduced generalized set-valued F-contractions of b-metric spaces. Some illustrative examples show that the new results in this paper generalize properly, unify and extend some related results in the existing literature. Moreover, we extract some important consequences of the results in b-metric spaces. Particularly, by setting b-metric constant equal to one, we obtain some specific cases showing notable enhancement of existing results yet in metric spaces.

Keywords: b-metric; fixed points; set-valued mapping; F-contractions; almost F-contractions
Mathematics Subject Classification: Primary 47H10, 47H04, 54E99, Secondary 54H25

1. Introduction and preliminaries

Axioms of metric have been modified to get more general distance functions (compare [8]). Among the generalizations of metric, b-metric was initially considered by Bakhtin [11], Czerwik [14–16] and Berinde [13] to generalize the well known Banach contraction principle (shortly as BCP) [12]. Due to the useful applications of BCP, it has been attempted successfully by a long list of researchers to generalize in various directions. Wardowski [26] set up a contraction termed as F-contraction and obtained a generalization of BCP. After that, several authors have established different versions of F-contractions to generalize the results of Wardowski, for instance, see [1, 2, 4–7, 10, 19, 20] and references therein. For self mappings of metric spaces, Proinov [21] proved that some results including Wardowski's result are equivalent to a special case of a well-known fixed point theorem of Skof [23].

Abbas et al. [3] obtained some coincidence point results for generalized set-valued (f, L)-almost F-contractions of metric spaces along with some applications. Recently, Karapınar et al. [17] provided a survey on F-contractions in which a collection of various results of F-contractions are given.

Miculescu [18] introduced a sufficient condition for a sequence in a b-metric space (shortly as b-MS) to be Cauchy and proved some results involving set-valued contractions of a b-MS. After this, Suzuki [25] provided a sufficient condition (weaker than the one given by Miculescu) for a sequence to be Cauchy in a b-MS and proved some fixed point theorems for set-valued F-contractions.

We present new generalized set-valued F-contractions of a b-MS and extend results given in [3,25] and in some references therein. We provide with some examples to substantiate the main results and to proclaim that the results in this paper are proper generalizations of some existing results in the literature. We start by fixing some notations to be used in the sequel. The letters $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_+$ represent the set of positive integers, real numbers, non-negative and positive real numbers, respectively and X a non-empty set. Now we give some preliminary notions.

Definition 1.1. *[14] Let $d : X \times X \to \mathbb{R}^+$ be a function and $s \geq 1$ a real number. Then (X, d) is termed as b-MS if d satisfies*

(1) $d(r, w) = 0 \Leftrightarrow r = w$,
(2) $d(r, w) = d(w, r)$,
(3) $d(r, z) \leq sd(r, w) + sd(w, z)$,

for all $r, w, z \in X$, where s is a b-metric constant. For $s = 1$, d is a metric.

Throughout this article, s represent b-metric constant unless otherwise stated. Now consider the following example.

Example 1.1. [11, 13] Consider

$$l_p = \left\{ \left\{r_j\right\} : \left\{r_j\right\} \subset \mathbb{R}, \text{ and } \sum_{j=1}^{\infty} \left|r_j\right|^p < \infty, \ 0 < p < 1 \right\}.$$

For all $r = \left\{r_j\right\}$ and $w = \left\{w_j\right\}$ in l_p, the mapping $d : l_p \times l_p \longrightarrow \mathbb{R}$ defined as

$$d(r, w) = \left[\sum_{j=1}^{\infty} \left|r_j - w_j\right|^p \right]^{\frac{1}{p}},$$

is a b-metric on l_p for $s = 2^{\frac{1}{p}}$ as

$$d(r, z) \leq 2^{\frac{1}{p}} \left(d(r, w) + d(w, z)\right),$$

for all r, z, w in l_p.

Let (X, d) be a b-MS. A sequence $\{r_j\}$ in (X, d) is Cauchy if for any given $\epsilon > 0$, there is a $J_\epsilon \in \mathbb{N}$ so that $d(r_j, r_m) < \epsilon$ for all $m, j \geq J_\epsilon$, or equivalently

$$\lim_{j\to\infty} d(r_{j+p}, r_j) = 0,$$

for all $p \in \mathbb{N}$. A sequence $\{r_j\}$ in (X, d) is convergent if for any given $\epsilon > 0$, there is $J_\epsilon \in \mathbb{N}$ and an r in X so that $d(r_j, r) < \epsilon$ for all $j \geq J_\epsilon$, or equivalently

$$\lim_{j\to\infty} d(r_j, r) = 0,$$

and we write $r_j \to r$ as $j \to \infty$.

Further, a subset $E \subseteq X$ is closed if for every sequence $\{r_j\}$ in E and $r_j \to r$ as $j \to \infty$, implies $r \in E$ and $E \subseteq X$ is bounded if

$$\sup_{z,w\in E} d(z, w)$$

is finite. A b-MS (X, d) is complete if every Cauchy sequence in X converges. An et al. [9] explored some topological aspects of b-MS (X, d) and asserted that d is not necessarily continuous in both arguments. However, if d is continuous in one variable then it is continuous in the other variable as well. Moreover, the subset

$$B_\epsilon(r_0) = \{r \in X : d(r_0, r) < \epsilon\},$$

in (X, d) is not an open set (in general) but if d is continuous in one variable then $B_\epsilon(r_0)$ is open in X. Throughout in this paper b-metric d is continuous.

Let (X, d) be a b-MS and $C_B(X)$ and $P(X)$ the set of non-empty, closed, bounded, and the set of non-empty subsets of X, respectively. For $E, G \in C_B(X)$, the mapping $H : C_B(X) \times C_B(X) \to \mathbb{R}^+$ defined as

$$H(E, G) = \max \{\delta(E, G), \delta(G, E)\},$$

is Hausdorff metric on $C_B(X)$ generated by d, where

$$\delta(E, G) = \sup_{r\in E} d(r, G) \ \text{ and } \ d(r, G) = \inf_{w\in G} d(r, w).$$

The following lemma provides important tools in connection with a b-MS.

Lemma 1.1. *[14–16, 22] For a b-MS (X, d), $r, w \in X$ and $E, G \in C_B(X)$, the following statements hold:*

(1) $(C_B(X), H)$ is a b-MS.
(2) For all $r \in E$, $d(r, G) \leq H(E, G)$.
(3) For all r, w in X, $d(r, E) \leq sd(r, w) + sd(w, E)$.
(4) For $k > 1$ and $c \in E$, there is a $w \in G$ so that $d(c, w) \leq kH(E, G)$.
(5) For every $k > 0$ and $c \in E$, there is a $w \in G$ so that $d(c, w) \leq H(E, G) + k$.
(6) $c \in \overline{E} = E$ if and only if $d(c, E) = 0$, where $\overline{E}$ is the closure of E in (X, d).
(7) For any sequence $\{r_j\}$ in X,

$$d(r_0, r_j) \leq sd(r_0, r_1) + s^2 d(r_1, r_2) + \cdots + s^{j-1}\left(d(r_{j-2}, r_{j-1}) + d(r_{j-1}, r_j)\right).$$

Now consider a mapping $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ that satisfies:

(A_1) If $\lambda_1 < \lambda_2$, then $F(\lambda_1) < F(\lambda_2)$, for all $\lambda_1, \lambda_2 \in \mathbb{R}_+$;

(A_2) For each sequence $\{\lambda_j\}$, $\lim_{j\to\infty} \lambda_j = 0$ if and only if $\lim_{j\to\infty} F(\lambda_j) = -\infty$;

(A_3) There is $k \in (0,1)$ such that $\lim_{\lambda\to 0^+} \lambda^k F(\lambda) = 0$;

(A_4) $F(\inf E) = \inf F(E)$ for all $E \subset (0,\infty)$ and $\inf E \in (0,\infty)$;

(A_5) F is upper semicontinuous;

(A_6) F is continuous.

Note that under (A_1), (A_4) and (A_5) are equivalent (compare [25]). Wardowski [26] initiated the idea of F-contraction.

Definition 1.2. *[26] Let (X, d) be a metric space. A mapping $f : X \to X$ is F-contraction if*

$$d(fr, fw) > 0 \textit{ implies } \tau + F(d(fr, fw)) \leq F(d(r, w)),$$

for all $r, w \in X$ and for some $\tau > 0$, where $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfies A_1–A_3.

Further, they obtained the existence and uniqueness of fixed point of the $F-$contraction f of a complete metric space (X, d). Throughout this paper, for mappings $f : X \to X$ and $S : X \to C_B(X)$, we use the notations $F^{ix}(f)$ and $F^{ix}(S)$ for the set of fixed points of f and S, respectively.

Remark 1.1. *For different choices of the function F in Definition 1.2, one can get different contractions, for instance if $F(\kappa) = \ln(\kappa)$ for $\kappa > 0$, then the mapping f becomes Banach contraction (compare [26] for details).*

Definition 1.3. *[25] Let a sequence $\{a_j\}$ be in $\mathbb{R}^+$ and $\{b_j\}$ a sequence in $\mathbb{R}_+$. If there is a real number $C > 0$ such that*

$$a_j \leq Cb_j,$$

for all $j \in \mathbb{N}$, then we say

$$\{a_j\} \in O(b_j).$$

Lemma 1.2. *[24, 25] Let $\{r_j\}$ be a sequence in a b-MS (X, d). If for $\beta > 1 + \log_2 s$,*

$$\left\{d\left(r_j, r_{j+1}\right)\right\} \in \bigcup \left\{O\left(j^{-\beta}\right)\right\}$$

holds. Then $\{r_j\}$ is a Cauchy sequence.

Lemma 1.3. *[25] Let $\left\{t_j\right\}$ be a sequence in $\mathbb{R}_+$. If there is a function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$, a real number $c \in (0,1)$ and $\tau \in \mathbb{R}_+$ satisfying (A_2), (A_3) and*

$$j\tau + F\left(t_{j+1}\right) \leq F(t_1),$$

then

$$\left\{t_j\right\} \in O\left(j^{-\frac{1}{c}}\right)$$

holds.

Theorem 1.1. *[25] Let (X, d) be a complete b-MS and $S : X \longrightarrow C_B(X)$ a mapping. Suppose there is a function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$,*

$$c \in \left(0, \frac{1}{1 + \log_2 s}\right),$$

and $\tau \in \mathbb{R}_+$ satisfying (A_2), (A_3), and for any $r, w \in X$ and $\eta \in Sr$, there is $\mu \in Sw$ such that either $\eta = \mu$ or

$$\tau + F\left(d\left(\eta, \mu\right)\right) \leq F\left(d\left(r, w\right)\right)$$

holds. Then $F^{ix}(S)$ is non-empty.

Theorem 1.2. *[25] Let (X, d) be a complete b-MS and a mapping $S : X \longrightarrow C_B(X)$. Suppose there is a function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$,*

$$c \in \left(0, \frac{1}{1 + \log_2 s}\right),$$

and $\tau \in \mathbb{R}_+$ satisfying (A_1)–(A_3), (A_5), and for any $r, w \in X$ with $Sr \neq Sw$,

$$\tau + F\left(H\left(Sr, Sw\right)\right) \leq F\left(d\left(r, w\right)\right)$$

holds. Then $F^{ix}(S)$ is non-empty.

Let $f : X \rightarrow X$ be single valued mapping and $S : X \rightarrow C_B(X)$ a set-valued mapping. For $r, w \in X$, $\eta \in Sr$ and $\mu \in Sw$, we use the following notations in the sequel.

$$\begin{aligned} M_1\left(r, w\right) &= \max\left\{d\left(r, w\right), d\left(r, \eta\right), d\left(w, \eta\right), \left(d\left(r, \eta\right) d\left(w, \eta\right)\right)\right\}, \\ N_1\left(r, w\right) &= \min\left\{d\left(r, \eta\right), d\left(w, \mu\right), d\left(r, \mu\right), d\left(w, \eta\right)\right\}, \end{aligned}$$

and

$$\begin{aligned} M_2\left(r, w\right) &= \max\left\{d\left(r, w\right), d\left(r, Tr\right), d\left(w, Tr\right), \left(d\left(r, Tr\right) d\left(w, Tr\right)\right)\right\}, \\ M_3\left(r, w\right) &= \max\left\{\begin{array}{l} d\left(r, w\right), d\left(r, Tr\right), d\left(w, Tw\right), d\left(r, Tr\right) d\left(w, Tr\right), d\left(r, Tw\right) d\left(w, Tr\right) \\ \dfrac{d\left(r, Tw\right) + d\left(w, Tr\right)}{2s}, \left[\dfrac{d\left(r, Tw\right) + d\left(w, Tr\right)}{s\left(1 + d\left(r, Tr\right) + d\left(w, Tw\right)\right)}\right] d\left(r, w\right) \end{array}\right\}, \\ M_4\left(r, w\right) &= \max\left\{\begin{array}{l} d\left(r, w\right), d\left(r, Tr\right), d\left(w, Tw\right), d\left(r, Tr\right) d\left(w, Tr\right), d\left(r, Tw\right) d\left(w, Tr\right) \\ \dfrac{d\left(r, Tw\right) + d\left(w, Tr\right)}{2}, \left[\dfrac{d\left(r, Tw\right) + d\left(w, Tr\right)}{1 + d\left(r, Tr\right) + d\left(w, Tw\right)}\right] d\left(r, w\right) \end{array}\right\}, \\ M_5\left(r, w\right) &= \max\left\{d\left(r, w\right), d\left(r, Tr\right), d\left(w, Tw\right), \frac{d\left(r, Tw\right) + d\left(w, Tr\right)}{2}\right\}, \\ N_2\left(r, w\right) &= \min\left\{d\left(r, Tr\right), d\left(w, Tw\right), d\left(r, Tw\right), d\left(w, Tr\right)\right\}, \end{aligned}$$

for $T \in \{f, S\}$.

2. Fixed points of generalized set-valued F-contractions of a b-metric space

We start with the following theorem.

Theorem 2.1. *Let (X, d) be a complete b-MS and $S : X \longrightarrow C_B(X)$. Suppose there is a function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$,*

$$c \in \left(0, \frac{1}{1 + \log_2 s}\right),$$

and $\tau \in \mathbb{R}_+$ satisfying (A_2) and (A_3), and for any $r, w \in X$ and $\eta \in S r$, there is $\mu \in S w$ such that $\eta = \mu$ or

$$\tau + F(d(\eta, \mu)) \leq F(M_1(r, w) + LN_1(r, w))$$

holds for some $L \geq 0$. Then $F^{ix}(S)$ is non-empty.

Proof. On contrary, consider that $F^{ix}(S)$ is empty. Fix $r_1 \in X$ and $r_2 \in S r_1$. From our assumption $r_1 \neq r_2$ and $r_2 \notin S r_2$. We can pick $r_3 \in S r_2$. As $r_2 \neq r_3$, so

$$\begin{aligned}
\tau + F(d(r_2, r_3)) &\leq F(M_1(r_1, r_2) + LN_1(r_1, r_2)) \\
&\leq F\left(\begin{array}{l} \max\left\{ \begin{array}{l} d(r_1, r_2), d(r_1, r_2), d(r_2, r_2), \\ (d(r_1, r_2)\, d(r_2, r_2)) \end{array} \right\} + \\ L \min\{d(r_1, r_2), d(r_2, r_3), d(r_1, r_3), d(r_2, r_2)\} \end{array} \right) \\
&= F(d(r_1, r_2)).
\end{aligned}$$

Similarly, we can choose a sequence $\{r_j\}$ in X for any $j \in \mathbb{N}$ satisfying $r_{j+1} \in S r_j$ and

$$\begin{aligned}
&\tau + F\left(d\left(r_{j+1}, r_{j+2}\right)\right) \\
&\leq F\left(M_1\left(r_j, r_{j+1}\right) + LN_1\left(r_j, r_{j+1}\right)\right) \\
&= F\left(\begin{array}{l} \max\left\{ \begin{array}{l} d\left(r_j, r_{j+1}\right), d\left(r_j, r_{j+1}\right), d\left(r_{j+1}, r_{j+1}\right), \\ \left(d\left(r_j, r_{j+1}\right) d\left(r_{j+1}, r_{j+1}\right)\right) \end{array} \right\} + \\ L \min\left\{d\left(r_j, r_{j+1}\right), d\left(r_{j+1}, r_{j+2}\right), d\left(r_j, r_{j+2}\right), d\left(r_{j+1}, r_{j+1}\right)\right\} \end{array} \right) \\
&= F\left(d\left(r_j, r_{j+1}\right)\right).
\end{aligned}$$

That is

$$\begin{aligned}
F\left(d\left(r_{j+1}, r_{j+2}\right)\right) &\leq F\left(d\left(r_j, r_{j+1}\right)\right) - \tau \leq F\left(d\left(r_{j-1}, r_j\right)\right) - 2\tau \\
&\leq F\left(d\left(r_{j-2}, r_{j-1}\right)\right) - 3\tau \leq \ldots \leq F(d(r_1, r_2)) - j\tau,
\end{aligned}$$

implies

$$j\tau + F\left(d\left(r_{j+1}, r_{j+2}\right)\right) \le F\left(d\left(r_1, r_2\right)\right).$$

Now, using Lemma 1.3,

$$\left\{d\left(r_j, r_{j+1}\right)\right\} \in O\left(j^{-\frac{1}{c}}\right)$$

holds. As $c \in \left(0, \frac{1}{1+\log_2 s}\right)$, so

$$\frac{1}{c} \in \left(1 + \log_2 s, +\infty\right).$$

Hence, by Lemma 1.2, $\{r_j\}$ is a Cauchy sequence. Because X is complete, $\{r_j\}$ converges to a $z \in X$, there is $z_j \in Sz$ such that either $z_j = r_{j+1}$ or

$$\tau + F\left(d\left(r_{j+1}, z_j\right)\right) \le F\left(M_1\left(r_j, z\right) + LN_1\left(r_j, z\right)\right)$$

holds. Let $\{j\}$ be a sequence in $\mathbb{N}$ and $\{f(j)\}$ be an arbitrary subsequence of $\{j\}$. Here, two cases arise:

(1) $\#\left\{j \in \mathbb{N} : z_{f(j)} = r_{f(j)+1}\right\} = \infty$,
(2) $\#\left\{j \in \mathbb{N} : z_{f(j)} = r_{f(j)+1}\right\} < \infty$,

where $\#A$ denotes the cardinality of the set A.

Case 1: Let $\{g(j)\}$ be a subsequence of $\{j\}$ in $\mathbb{N}$ which satisfy $z_{f\circ g(j)} = r_{f\circ g(j)+1}$. Since $r_j \to z$ as $j \to \infty$,

$$z_{f\circ g(j)} = r_{f\circ g(j)+1} \to z,$$

as $j \to \infty$. Hence,

$$\lim_{j\to\infty} d\left(z_{fog(j)}, z\right) = 0.$$

Case 2: Let $\{g(j)\}$ be a subsequenceof $\{j\}$ in $\mathbb{N}$ such that $g(j) \notin \left\{j \in \mathbb{N} : z_{f(j)} = r_{f(j)+1}\right\}$. This implies

$$\begin{aligned}
&\tau + F\left(d\left(r_{fog(j)+1}, z_{fog(j)}\right)\right) \\
&\le F\left(M_1\left(r_{fog(j)}, z\right) + LN_1\left(r_{fog(j)}, z\right)\right) \\
&= F\left(\begin{array}{l} \max\left\{\begin{array}{l} d\left(r_{fog(j)}, z\right), d\left(r_{fog(j)}, r_{fog(j)+1}\right), d\left(z, r_{fog(j)+1}\right), \\ \left(d\left(r_{fog(j)}, r_{fog(j)+1}\right) d\left(z, r_{fog(j)+1}\right)\right) \end{array}\right\} + \\ L\min\left\{d\left(r_{fog(j)}, r_{fog(j)+1}\right), d\left(z, z_{fog(j)}\right), d\left(r_{fog(j)}, z_{fog(j)}\right), d\left(z, r_{fog(j)+1}\right)\right\} \end{array}\right) \\
&\le F\left(\begin{array}{l} \max\left\{\begin{array}{l} d\left(r_{fog(j)}, z\right), sd\left(r_{fog(j)}, z\right) + sd\left(z, r_{fog(j)+1}\right), \\ d\left(z, r_{fog(j)+1}\right), \left(d\left(r_{fog(j)}, r_{fog(j)+1}\right) d\left(z, r_{fog(j)+1}\right)\right) \end{array}\right\} + \\ L\min\left\{d\left(r_{fog(j)}, r_{fog(j)+1}\right), d\left(z, z_{fog(j)}\right), d\left(r_{fog(j)}, z_{fog(j)}\right), d\left(z, r_{fog(j)+1}\right)\right\} \end{array}\right).
\end{aligned}$$

That is

$$\tau + F\left(d\left(r_{fog(j)+1}, z_{fog(j)}\right)\right) \leq F(u_j), \tag{2.1}$$

where

$$u_j = \max \left\{ \begin{array}{c} d\left(r_{fog(j)}, z\right), sd\left(r_{fog(j)}, z\right) + sd\left(z, r_{fog(j)+1}\right), \\ d\left(z, r_{fog(j)+1}\right), \left(d\left(r_{fog(j)}, r_{fog(j)+1}\right) d\left(z, r_{fog(j)+1}\right)\right) \end{array} \right\}$$
$$+L \min\left\{d\left(r_{fog(j)}, r_{fog(j)+1}\right), d\left(z, z_{fog(j)}\right), d\left(r_{fog(j)}, z_{fog(j)}\right), d\left(z, r_{fog(j)+1}\right)\right\}.$$

Since $r_j \to z$ as $j \to \infty$, therefore, $u_j \to 0$ as $j \to \infty$. Hence, by (A$_2$), we get

$$\lim_{j\to\infty} F(u_j) = -\infty. \tag{2.2}$$

From (2.1) and (2.2), we obtain

$$\tau + \lim_{j\to\infty} F\left(d\left(r_{fog(j)+1}, z_{fog(j)}\right)\right) \leq -\infty.$$

Again by (A$_2$), we get

$$\lim_{j\to\infty} d\left(r_{fog(j)+1}, z_{fog(j)}\right) = 0.$$

Hence,

$$\lim_{j\to\infty} d\left(z_{fog(j)}, z\right) \leq \lim_{j\to\infty} s(d\left(z_{fog(j)}, r_{fog(j)+1}\right) + d\left(r_{fog(j)+1}, z\right)) = 0.$$

Consequently, both cases imply

$$\lim_{j\to\infty} d\left(z_{fog(j)}, z\right) = 0.$$

As f was taken to be arbitrary, so

$$\lim_{j\to\infty} d\left(z_j, z\right) = 0.$$

As Sz is closed, we get $z \in Sz$, a contradiction. Thus $F^{ix}(S)$ is non-empty. □

Consider an example to illustrate the Theorem 2.1 and to show that it is a proper generalization of some results in the literature.

Example 2.1. Let $X = \{1, 2, 3\}$ be a set and $d : X \times X \to \mathbb{R}^+$ a mapping defined as

$$\begin{aligned} d(1,2) &= d(2,1) = 3,\ d(1,3) = d(3,1) = 1.5, \\ d(2,3) &= d(3,2) = 5,\ d(r,r) = 0,\ \text{for all } r \in X, \\ \text{and }\ d(r,w) &= d(w,r),\ \text{for all } r, w \in X. \end{aligned}$$

As

$$d(2,3) = 5 \nleq d(2,1) + d(1,3) = 4.5,$$

so d is not a metric on X but for $s = 1.12$, d is a complete b-metric. Define $S : X \to C_B(X)$ and $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ as

$$Sr = \begin{cases} \{1,2\}, & \text{if } r = 3, \\ \{1\}, & \text{if } r = 1, 2, \end{cases} \quad \text{and } F(r) = \ln(r).$$

Note that

$$\frac{1}{1+\log_2 1.12} \approx 0.859 > 0.$$

If $r = 3$ and $w = 1$, then $Sr = \{1,2\}$, for $\eta = 1 \in S3$, there is $\mu = 1 \in S1$ such that $\eta = \mu$. For $\eta = 2$, there is $\mu = 1$ such that

$$\begin{aligned}
& F(d(2,1)) = \ln(3) \approx 1.099, \text{ and} \\
& F(M_1(3,1) + LN_1(3,1)) \\
= & F\left(\begin{array}{l} \max\left\{\begin{array}{l} d(3,1), d(3,2), d(1,2), \\ (d(3,2)d(1,2)) \end{array}\right\} + \\ +L\min\{d(3,2), d(1,1), d(3,1), d(1,2)\} \end{array}\right) \\
= & F(\max\{1.5, 5, 3, (5)(3)\} + L\min\{5, 0, 1.5, 3\}) \\
= & \ln(15+0) = \ln(15) \approx 2.708.
\end{aligned}$$

If $r = 3$ and $w = 2$, then $Sr = \{1,2\}$, for $\eta = 1$, there exists $\mu = 1 \in S2$ such that $\eta = \mu$. For $\eta = 2$, there is $\mu = 1 \in S2$ such that

$$\begin{aligned}
& F(d(2,1)) = \ln(3) \approx 1.099 \text{ and} \\
& F(M_1(3,2) + LN_1(3,2)) \\
= & F\left(\begin{array}{l} \max\{d(3,2), d(3,2), d(2,2), (d(3,2)d(2,2))\} + \\ L\min\{d(3,2), d(2,1), d(3,1), d(2,2)\} \end{array}\right) \\
= & F(\max\{5, 5, 0, 0\} + L\min\{5, 3, 1.5, 0\}) \\
= & \ln(5+0) = \ln(5) \approx 1.609.
\end{aligned}$$

Hence, for any $\tau \in (0, 0.51)$, $r, w \in X$ and $\eta \in Sr$, there exists $\mu \in Sw$ such that either $\eta = \mu$ or

$$\tau + F(d(\eta,\mu)) \leq F(M_1(r,w) + LN_1(r,w))$$

holds for all $r, w \in X$ and for any $L \geq 0$. Hence, all the assumptions of Theorem 2.1 are met and $1 \in S(1)$.

Remark 2.1. *In the above example, if* $r = 3$ *and* $w = 1$*, then* $Sr = \{1,2\}$*, for* $\eta = 2$*, there does not exist* $\mu \in S1$ *such that either* $\eta = \mu$ *or*

$$\tau + F(d(\eta,\mu)) \leq Fd(r,w),$$

for any $\tau > 0$*, because for* $\eta = 2$*, and for all* $\mu \in S1 = \{1\}$*, we have*

$$F(d(2,1)) = \ln(3) = 1.099 \nleq 0.405 \approx \ln(1.5) = F(d(3,1)),$$

that is Theorem 1.1 is not applicable in this example. Hence, Theorem 2.1 is a proper extension of Theorem 1.1.

In the following, we obtain some corollaries of Theorem 2.1.

Corollary 2.1. *Let (X,d) be a complete b-MS and $S : X \longrightarrow C_B(X)$ a mapping. Suppose there is a function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$,*

$$c \in \left(0, \frac{1}{1+\log_2 s}\right),$$

$\tau \in \mathbb{R}_+$ satisfying (A_2) and (A_3), and for any $r, w \in X$ and $\eta \in Sr$, there is $\mu \in Sw$ such that either $\eta = \mu$ or

$$\tau + F(d(\eta,\mu)) \leq F(M_1(r,w))$$

holds. Then $F^{ix}(S)$ is non-empty.

Corollary 2.2. *Let (X,d) be a complete b-MS and $S : X \longrightarrow C_B(X)$ a mapping. Suppose there is a function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$,*

$$c \in \left(0, \frac{1}{1+\log_2 s}\right),$$

$\tau \in \mathbb{R}_+$ satisfying A_2, A_3, and for any $r, w \in X$ and $\eta \in Sr$, there is $\mu \in Sw$ such that either $\eta = \mu$ or

$$\tau + F(d(\eta,\mu)) \leq F(\max\{d(r,w), d(r,\eta), d(w,\eta)\})$$

holds. Then $F^{ix}(S)$ is non-empty.

Following result is the corollary of Theorem 2.1 for single valued mapping.

Corollary 2.3. *Let (X,d) be a complete b-MS and a mapping $f : X \longrightarrow X$. Suppose there is a function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$,*

$$c \in \left(0, \frac{1}{1+\log_2 s}\right),$$

$\tau \in \mathbb{R}_+$ satisfying (A_2) and (A_3), and for any $r, w \in X$ and either $fr = fw$ or

$$\tau + F(d(fr,fw)) \leq F(M_2(r,w) + LN_2(r,w))$$

holds for some $L \geq 0$. Then $F^{ix}(f)$ is singleton.

Proof. From Theorem 2.1, $F^{ix}(f)$ is non-empty. To check the uniqueness, assume κ and ϖ be fixed points of f with $\kappa \neq \varpi$, that is $f\kappa \neq f\varpi$. Hence, from given condition, we get

$$\begin{aligned}
&\tau + F(d(\kappa,\varpi)) = \tau + F(d(f\kappa, f\varpi)) \\
&\leq F(M_2(\kappa,\varpi) + LN_2(\kappa,\varpi)) \\
&= F\left(\begin{array}{l} \max\{d(\kappa,\varpi), d(\kappa,f\kappa), d(\varpi,f\kappa), (d(\kappa,f\kappa)\, d(\varpi,f\kappa))\} \\ + \min\{d(\kappa,f\kappa), d(\varpi,f\varpi), d(\kappa,f\varpi), d(\varpi,f\kappa)\} \end{array}\right) \\
&= F\left(\begin{array}{l} \max\left\{\begin{array}{l} d(\kappa,\varpi), d(\kappa,\kappa), d(\varpi,\kappa), \\ (d(\kappa,\kappa)\, d(\varpi,\kappa)) \end{array}\right\} \\ + \min\{d(\kappa,\kappa), d(\varpi,\varpi), d(\kappa,\varpi), d(\varpi,\kappa)\} \end{array}\right) \\
&= F(\max\{d(\kappa,\varpi), 0, d(\varpi,\kappa), 0\} + \min\{0, 0, d(\kappa,\varpi), d(\varpi,\kappa)\}) \\
&= F(d(\kappa,\varpi)),
\end{aligned}$$

implies $\tau \leq 0$, a contradiction. Hence, $F^{ix}(f)$ is singleton. ◻

The following result is an extension of a result given in [25, Theorem 14].

Theorem 2.2. *Let (X, d) be a complete b-MS and $S : X \longrightarrow C_B(X)$ a mapping. Suppose there is a function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$,*

$$c \in \left(0, \frac{1}{1 + \log_2 s}\right),$$

and $\tau \in \mathbb{R}_+$ satisfying (A_1)–(A_3) and (A_5), and for any $r, w \in X$, $r \neq w$ with $Sr \neq Sw$,

$$\tau + F\left(H\left(Sr, Sw\right)\right) \leq F\left(M_2\left(r, w\right) + LN_2\left(r, w\right)\right)$$

holds for some $L \geq 0$. Then $F^{ix}(S)$ is non-empty.

Proof. Let $\alpha \in Sr$ where $r, w \in X$. We have following two cases:

(1) $d\left(\alpha, Sw\right) = 0$,
(2) $d\left(\alpha, Sw\right) > 0$.

If $d\left(\alpha, Sw\right) = 0$ then $\alpha \in Sw$, because Sw is closed. In the second case $Sr \neq Sw$. So

$$\tau + F\left(H\left(Sr, Sw\right)\right) \leq F\left(M_2\left(r, w\right) + LN_2\left(r, w\right)\right)$$

holds as given. As

$$d\left(\alpha, Sw\right) \leq H\left(Sr, Sw\right),$$

so using (A_1) we get

$$\begin{aligned} & \tau + F(d\left(\alpha, Sw\right)) \\ \leq\ & \tau + F\left(H\left(Sr, Sw\right)\right) \\ \leq\ & F\left(M_2\left(r, w\right) + LN_2\left(r, w\right)\right). \end{aligned}$$

From (A_5),

$$\begin{aligned} & \inf\left\{F\left(d\left(\alpha, \gamma\right)\right) : \gamma \in Sw\right\} \\ =\ & F\left(d\left(\alpha, Sw\right)\right) \\ \leq\ & F\left(M_2\left(r, w\right) + LN_2\left(r, w\right)\right) - \tau \\ <\ & F\left(M_2\left(r, w\right) + LN_2\left(r, w\right)\right) - \frac{\tau}{2}. \end{aligned}$$

So, we can pick $\beta \in Sw$ fulfilling

$$\frac{\tau}{2} + F\left(d\left(\alpha, \beta\right)\right) \leq F\left(M_2\left(r, w\right) + LN_2\left(r, w\right)\right).$$

If we replace τ with $\dfrac{\tau}{2}$ in Theorem 2.1, then we get the desired result. □

Here, we obtain some corollaries of Theorem 2.2.

Corollary 2.4. *Let (X, d) be a complete b-MS and a mapping $S : X \longrightarrow C_B(X)$. Suppose there is a function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$,*

$$c \in \left(0, \frac{1}{1 + \log_2 s}\right),$$

and $\tau \in \mathbb{R}_+$ satisfying (A_1)–(A_3) and (A_5), and for any $r, w \in X$, $r \neq w$ with $Sr \neq Sw$,

$$\tau + F(H(Sr, Sw)) \leq F(M_2(r, w))$$

holds. Then $F^{ix}(S)$ is non-empty.

Corollary 2.5. *Let (X, d) be a complete b-MS and a mapping $S : X \longrightarrow C_B(X)$. Suppose there is a function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$,*

$$c \in \left(0, \frac{1}{1 + \log_2 s}\right),$$

and $\tau \in \mathbb{R}_+$ satisfying (A_1)–(A_3) and (A_5), and for any $r, w \in X$, $r \neq w$ with $Sr \neq Sw$,

$$\tau + F(H(Sr, Sw)) \leq F(\max\{d(r, w), d(r, Sr), d(w, Sr)\})$$

holds. Then $F^{ix}(S)$ is non-empty.

In the following, we obtain another result for a new set-valued F-contractions of a b-MS.

Theorem 2.3. *Let (X, d) be a complete b-MS and $S : X \to C_B(X)$ a set-valued mapping. Suppose there is a function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$,*

$$c \in \left(0, \frac{1}{1 + \log_2 s}\right),$$

$\tau \in \mathbb{R}_+$ satisfying (A_1)–(A_3), (A_6) and

$$2\tau + F(H(Sr, Sw)) \leq F(M_3(r, w) + LN_2(r, w)), \tag{2.3}$$

for all $r, w \in X$ with $Sr \neq Sw$ and for some $L \geq 0$. Then $F^{ix}(S)$ is non-empty.

Proof. Let $r_0 \in X$ and $r_{j+1} \in Sr_j$ for all $j \in \mathbb{N}$. If $r_j = r_{j+1}$ for some $j \in \mathbb{N}$, then $r_j \in Sr_j$ and there is nothing to prove further. Now suppose $r_j \neq r_{j+1}$ for all $j \in \mathbb{N}$. As F is right continuous at $H\left(Sr_j, Sr_{j+1}\right)$ for each $j \in \mathbb{N}$, so there is a $k > 1$ such that

$$F\left(kH\left(Sr_j, Sr_{j+1}\right)\right) < F\left(H\left(Sr_j, Sr_{j+1}\right)\right) + \tau. \tag{2.4}$$

Moreover, there exists $r_{j+2} \in Sr_{j+1}$ such that

$$d\left(r_{j+1}, r_{j+2}\right) \leq kH\left(Sr_j, Sr_{j+1}\right). \tag{2.5}$$

Now, from (2.3)–(2.5) and (A_1) we get

$$
\begin{aligned}
&F\left(d\left(r_{j+1},r_{j+2}\right)\right)\leq F\left(kH\left(Sr_j,Sr_{j+1}\right)\right)\\
&<F\left(H\left(Sr_j,Sr_{j+1}\right)\right)+\tau\leq F\left(M_3\left(r_j,r_{j+1}\right)+LN_2\left(r_j,r_{j+1}\right)\right)-2\tau+\tau\\
&=F\left(\begin{array}{l}\max\left\{\begin{array}{l}d\left(r_j,r_{j+1}\right),d\left(r_j,Sr_j\right),d\left(r_{j+1},Sr_{j+1}\right),d\left(r_j,Sr_j\right)d\left(r_{j+1},Sr_j\right),\\ d\left(r_j,Sr_{j+1}\right)d\left(r_{j+1},Sr_j\right),\dfrac{d\left(r_j,Sr_{j+1}\right)+d\left(r_{j+1},Sr_j\right)}{2s},\\ \left[\dfrac{d\left(r_j,Sr_{j+1}\right)+d\left(r_{j+1},Sr_j\right)}{s\left(1+d\left(r_j,Sr_j\right)+d\left(r_{j+1},Sr_{j+1}\right)\right)}\right]d\left(r_j,r_{j+1}\right)\end{array}\right\}\\ +L\min\left\{d\left(r_j,Sr_j\right),d\left(r_{j+1},Sr_{j+1}\right),d\left(r_j,Sr_{j+1}\right),d\left(r_{j+1},Sr_j\right)\right\}\end{array}\right)-\tau\\
&\leq F\left(\begin{array}{l}\max\left\{\begin{array}{l}d\left(r_j,r_{j+1}\right),d\left(r_j,r_{j+1}\right),d\left(r_{j+1},r_{j+2}\right),d\left(r_j,r_{j+1}\right)d\left(r_{j+1},r_{j+1}\right),\\ d\left(r_j,r_{j+2}\right)d\left(r_{j+1},r_{j+1}\right),\dfrac{d\left(r_j,r_{j+2}\right)+d\left(r_{j+1},r_{j+1}\right)}{2s},\\ \left[\dfrac{d\left(r_j,r_{j+2}\right)+d\left(r_{j+1},r_{j+1}\right)}{s\left(1+d\left(r_j,r_{j+1}\right)+d\left(r_{j+1},r_{j+2}\right)\right)}\right]d\left(r_j,r_{j+1}\right)\end{array}\right\}\\ +L\min\left\{d\left(r_j,Sr_j\right),d\left(r_{j+1},r_{j+2}\right),d\left(r_j,r_{j+2}\right),d\left(r_{j+1},r_{j+1}\right)\right\}\end{array}\right)-\tau\\
&=F\left(\max\left\{\begin{array}{l}d\left(r_j,r_{j+1}\right),d\left(r_{j+1},r_{j+2}\right),\dfrac{sd\left(r_j,r_{j+1}\right)+sd\left(r_{j+1},r_{j+2}\right)}{2s},\\ \left[\dfrac{sd\left(r_j,r_{j+1}\right)+sd\left(r_{j+1},r_{j+2}\right)}{s\left(1+d\left(r_j,r_{j+1}\right)+d\left(r_{j+1},r_{j+2}\right)\right)}\right]d\left(r_j,r_{j+1}\right)\end{array}\right\}\right)-\tau\\
&\leq F\left(\max\left\{d\left(r_j,r_{j+1}\right),d\left(r_{j+1},r_{j+2}\right)\right\}\right)-\tau.
\end{aligned}
$$

If

$$\max\left\{d\left(r_j,r_{j+1}\right),d\left(r_{j+1},r_{j+2}\right)\right\}=d\left(r_{j+1},r_{j+2}\right),$$

then

$$\tau+F\left(d\left(r_{j+1},r_{j+2}\right)\right)\leq F\left(d\left(r_{j+1},r_{j+2}\right)\right),$$

implies $\tau\leq 0$, a contradiction. So

$$\tau+F\left(d\left(r_{j+1},r_{j+2}\right)\right)\leq F\left(d\left(r_j,r_{j+1}\right)\right),$$

which further implies

$$\begin{aligned} F\left(d\left(r_{j+1}, r_{j+2}\right)\right) &\leq F\left(d\left(r_j, r_{j+1}\right)\right) - \tau \leq F\left(d\left(r_{j-1}, r_j\right)\right) - 2\tau \\ &\leq F\left(d\left(r_{j-2}, r_{j-1}\right)\right) - 3\tau \leq \cdots \leq F\left(d\left(r_1, r_2\right)\right) - j\tau, \end{aligned}$$

we get

$$j\tau + F\left(d\left(r_{j+1}, r_{j+2}\right)\right) \leq F\left(d\left(r_1, r_2\right)\right),$$

for any $j \in \mathbb{N}$. By Lemma 1.3,

$$\left\{d\left(r_j, r_{j+1}\right)\right\} \in O\left(j^{-\frac{1}{c}}\right)$$

holds. Since

$$\frac{1}{c} \in \left(1 + \log_2 s, \infty\right),$$

by Lemma 1.2, $\{r_j\}$ is a Cauchy sequence. Since X is complete, $\{r_j\}$ converges to some $w \in X$.

$$\lim_{j\to\infty} r_j = w.$$

Now we will show that $w \in Sw$. On contrary assume that $w \notin Sw$, that is $d(w, Sw) > 0$. As

$$d\left(r_{j+1}, Sw\right) \leq H\left(Sr_j, Sw\right).$$

By (A_1), we get

$$\begin{aligned} & 2\tau + F\left(d\left(r_{j+1}, Sw\right)\right) \\ & \leq 2\tau + F\left(H\left(Sr_j, Sw\right)\right) \leq F\left(M_3\left(r_j, w\right) + LN_2\left(r_j, w\right)\right) \\ & = F\left(\begin{array}{l} \max\left\{\begin{array}{l} d\left(r_j, w\right), d\left(r_j, Sr_j\right), d\left(w, Sw\right), d\left(r_j, Sr_j\right) d\left(w, Sr_j\right), d\left(r_j, Sw\right) d\left(w, Sr_j\right), \\ \dfrac{d\left(r_j, Sw\right) + d\left(w, Sr_j\right)}{2s}, \left[\dfrac{d\left(r_j, Sw\right) + d\left(w, Sr_j\right)}{s\left(1 + d\left(r_j, Sr_j\right) + d\left(w, Sw\right)\right)}\right] d\left(r_j, w\right) \end{array}\right\} \\ + L\min\left\{d\left(r_j, Sr_j\right), d\left(w, Sw\right), d\left(r_j, Sw\right), d\left(w, Sr_j\right)\right\} \end{array}\right) \\ & \leq F\left(\begin{array}{l} \max\left\{\begin{array}{l} d\left(r_j, w\right), d\left(r_j, r_{j+1}\right), d\left(w, Sw\right), d\left(r_j, r_{j+1}\right) d\left(w, r_{j+1}\right), d\left(r_j, Sw\right) d\left(w, r_{j+1}\right), \\ \dfrac{d\left(r_j, Sw\right) + d\left(w, r_{j+1}\right)}{2s}, \left[\dfrac{d\left(r_j, Sw\right) + d\left(w, r_{j+1}\right)}{s\left(1 + d\left(r_j, r_{j+1}\right) + d\left(w, Sw\right)\right)}\right] d\left(r_j, w\right) \end{array}\right\} \\ + L\min\left\{d\left(r_j, r_{j+1}\right), d\left(w, Sw\right), d\left(r_j, Sw\right), d\left(w, r_{j+1}\right)\right\} \end{array}\right). \end{aligned}$$

On taking limit as j tends to ∞ and by the continuity of F, we get

$$2\tau + F\left(d\left(w, Sw\right)\right) \leq F\left(d\left(w, Sw\right)\right),$$

implies $2\tau \leq 0$, a contradiction. Hence, S has a fixed point. □

Here is an example to explain the Theorem 2.3.

Example 2.2. *Let* $X = \{a, b, c, \rho, e\}$ *and set a mapping* $d : X \times X \to \mathbb{R}^+$ *by*

$$\begin{aligned} d(a,b) &= d(a,c) = 3,\ d(b,e) = d(c,\rho) = d(c,e) = 9, \\ d(a,\rho) &= d(a,e) = 12,\ d(b,\rho) = 8, d(b,c) = 6,\ d(\rho,e) = 2, \\ d(r,r) &= 0 \text{ for all } r \in X \text{ and } d(r,w) = d(w,r) \text{ for all } r,w \in X. \end{aligned}$$

As

$$d(a,\rho) = 12 \nleq d(a,b) + d(b,\rho) = 11,$$

so d *is not a metric on* X*. For any* $s \geq \frac{12}{11}$*,* d *is a complete b-metric. Set* $L = 2$ *and define a mapping* $S : X \to C_B(X)$ *as*

$$Sr = \begin{cases} \{a\}, & \text{if } r = a, b, \\ \{b\}, & \text{if } r = c, \rho, \\ \{c, \rho\}, & \text{if } r = e. \end{cases}$$

For $r \in \{a, b\}$ *and* $w \in \{c, \rho, e\}$*, there are following cases:*

If $r = a$*,* $w = c$*, then*

$$\begin{aligned} & F(H(Sa, Sc)) = F(d(a,b)) = \ln(3) \approx 1.0986 \text{ and} \\ & F(M_3(a,c) + LN_2(a,c)) \\ = {} & F\left(\begin{array}{l} \max\left\{ \begin{array}{l} d(a,c), d(a,Sa), d(c,Sc), d(a,Sa)d(c,Sa), d(a,Sc)d(c,Sa), \\ \frac{d(a,Sc)+d(c,Sa)}{2s}, \left[\frac{d(a,Sc)+d(c,Sa)}{s(1+d(a,Sa)+d(c,Sc))}\right] d(a,c) \end{array} \right\} \\ +L\min\{d(a,Sa), d(c,Sc), d(a,Sc), d(c,Sa)\} \end{array} \right) \\ = {} & F\left(\max\left\{3, 0, 6, 0, 9, \frac{66}{24}, \left(\frac{66}{84}\right)3\right\} + L\min\{0, 6, 3, 3\}\right) \\ = {} & F(\max\{3, 0, 6, 0, 9, 2.75, 2.36\} + L(0)) \\ = {} & F(9) = \ln(9) \approx 2.197. \end{aligned}$$

If $r = a$*,* $w = \rho$*, then*

$$\begin{aligned} & F(H(Sa, S\rho)) = F(d(a,b)) = \ln(3) \approx 1.0986 \text{ and} \\ & F(M_3(a,\rho) + LN_2(a,\rho)) \\ = {} & F\left(\begin{array}{l} \max\left\{ \begin{array}{l} d(a,\rho), d(a,Sa), d(\rho,S\rho), d(a,Sa)d(\rho,Sa), d(a,S\rho)d(\rho,Sa), \\ \frac{d(a,S\rho)+d(\rho,Sa)}{2s}, \left[\frac{d(a,S\rho)+d(\rho,Sa)}{s(1+d(a,Sa)+d(\rho,S\rho))}\right] d(a,\rho) \end{array} \right\} \\ +L\min\{d(a,Sa), d(\rho,S\rho), d(a,S\rho), d(\rho,Sa)\} \end{array} \right) \\ = {} & F\left(\max\left\{12, 0, 8, 0, 36, \frac{165}{24}, \left[\frac{165}{108}\right] 12\right\} + L\min\{0, 8, 3, 12\}\right) \\ = {} & F(\max\{12, 0, 8, 0, 36, 6.875, 18.33\} + L(0)) \\ = {} & F(36) = \ln(36) \approx 3.5835. \end{aligned}$$

If $r = a$, $w = e$, then

$$F(H(Sa, Se)) = F(H(a, \{c, \rho\})) = F(12) = \ln(12) \approx 2.4849 \text{ and}$$
$$F(M_3(a,e) + LN_2(a,e))$$
$$= F\left(\begin{array}{l} \max\left\{\begin{array}{l} d(a,e), d(a,Sa), d(e,Se), d(a,Sa)\,d(e,Sa), d(a,Se)\,d(e,Sa), \\ \dfrac{d(a,Se)+d(e,Sa)}{2s}, \left[\dfrac{d(a,Se)+d(e,Sa)}{s(1+d(a,Sa)+d(e,Se))}\right]d(a,e) \end{array}\right\} \\ +L\min\{d(a,Sa), d(e,Se), d(a,Se), d(e,Sa)\} \end{array}\right)$$
$$= F\left(\max\left\{12, 0, 9, 0, 12(12), \frac{264}{24}, \left[\frac{264}{120}\right]12\right\} + L\min\{0, 9, 12, 12\}\right)$$
$$= F(\max\{12, 0, 9, 0, 144, 11, 26.4\} + L(0))$$
$$= F(144) = \ln(144) \approx 4.9698.$$

If $r = b$, $w = c$, then

$$F(H(Sb, Sc)) = F(d(a,b)) = \ln(3) \approx 1.0986 \text{ and}$$
$$F(M_3(b,c) + LN_2(b,c))$$
$$= F\left(\begin{array}{l} \max\left\{\begin{array}{l} d(b,c), d(b,Sb), d(c,Sc), d(b,Sb)\,d(c,Sb), d(b,Sc)\,d(c,Sb), \\ \dfrac{d(b,Sc)+d(c,Sb)}{2s}, \left[\dfrac{d(b,Sc)+d(c,Sb)}{s(1+d(b,Sb)+d(c,Sc))}\right]d(b,c) \end{array}\right\} \\ +L\min\{d(b,Sb), d(c,Sc), d(b,Sc), d(c,Sb)\} \end{array}\right)$$
$$= F\left(\max\left\{6, 3, 6, 9, 0, \left[\frac{33}{24}\right], \left[\frac{33}{120}\right]6\right\} + L(0)\right)$$
$$= F(\max\{6, 3, 6, 9, 0, 1.38, 1.65\})$$
$$= F(9) = \ln(9) \approx 2.197.$$

If $r = b$, $w = \rho$, then

$$F(H(Sb, S\rho)) = F(d(a,b)) = \ln(3) \approx 1.0986 \text{ and}$$
$$F(M_3(b,\rho) + LN_2(b,\rho))$$
$$= F\left(\begin{array}{l} \max\left\{\begin{array}{l} d(b,\rho), d(b,Sb), d(\rho,S\rho), d(b,Sb)\,d(\rho,Sb), d(b,S\rho)\,d(\rho,Sb), \\ \dfrac{d(b,S\rho)+d(\rho,Sb)}{2s}, \left[\dfrac{d(b,S\rho)+d(\rho,Sb)}{s(1+d(b,Sb)+d(\rho,S\rho))}\right]d(b,\rho) \end{array}\right\} \\ +L\min\{d(b,Sb), d(\rho,S\rho), d(b,S\rho), d(\rho,Sb)\} \end{array}\right)$$
$$= F\left(\max\left\{8, 3, 8, 36, 0, \frac{132}{24}, \left[\frac{132}{144}\right]8\right\} + L\{3, 8, 0, 12\}\right)$$
$$= F(36) = \ln(36) \approx 3.584.$$

If $r = b$, $w = e$, then

$$\begin{aligned}
&F(H(Sb,Se)) = F(H(a,\{c,\rho\})) = \ln(12) \approx 2.4849 \text{ and}\\
&F(M_3(b,e) + LN_2(b,e))\\
&= F\left(\begin{array}{l} \max\left\{\begin{array}{l} d(b,e), d(b,Sb), d(e,Se), d(b,Sb)\, d(e,Sb), d(b,Se)\, d(e,Sb),\\ \dfrac{d(b,Se)+d(e,Sb)}{2s}, \left[\dfrac{d(b,Se)+d(e,Sb)}{s(1+d(b,Sb)+d(e,Se))}\right] d(b,e) \end{array}\right\}\\ +L\min\{d(b,Sb), d(e,Se), d(b,Se), d(e,Sb)\} \end{array}\right)\\
&= F\left(\max\left\{9,3,9,3(12),96,\frac{220}{24},\left[\frac{220}{156}\right]9\right\} + L\min\{3,9,8,12\}\right)\\
&= F(96 + 2(3)) = \ln(102) \approx 4.625.
\end{aligned}$$

Hence, for any $\tau \in (0, 0.5492)$,

$$2\tau + F(H(Sr,Sw)) \leq F(M_3(r,w) + LN_2(r,w))$$

holds for all $r, w \in X$ and for any $L = 2$. That is, all the assumptions of Theorem 2.3 are met, so there exists a fixed point $a \in S(a)$.

Following result is the metric version of Theorem 2.3.

Corollary 2.6. *Let (X,d) be a complete metric space and $S : X \to C_B(X)$ a set-valued mapping. Suppose there is a function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$, $c \in (0,1)$, $\tau \in \mathbb{R}_+$ satisfying (A_1)–(A_3), (A_6) and*

$$2\tau + F(H(Sr,Sw)) \leq F(M_4(r,w) + LN_2(r,w)),$$

for all $r, w \in X$ with $Sr \neq Sw$ and for some $L \geq 0$. Then $F^{ix}(S)$ is non-empty.

Proof. Take $s = 1$ in Theorem 2.3. □

Abbas et al. [3] proved a coincidence point theorem for generalized set-valued (f, L)-almost F-contraction and the following is one of the corollary of their main result.

Corollary 2.7. *[3] Let (X,d) be a complete metric space and $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a function with $\tau \in (0,\infty)$ satisfying (A_1)–(A_3) and (A_6). There is a mapping $S : X \to C_B(X)$ with $L \geq 0$, satisfying*

$$2\tau + F(H(Sr,Sw)) \leq F(M_5(r,w) + LN_2(r,w)),$$

for every $r, w \in X$ with $Sr \neq Sw$. Then $F^{ix}(S)$ is non-empty.

Remark 2.2. *Note that the Corollary 2.7 is the corollary of Corollary 2.6.*

In the next example, we show that the Corollary 2.6 properly generalize the Corollary 2.7.

Example 2.3. Let $X = \{1,2,3,4,5\}$ and set a mapping $d : X \times X \to \mathbb{R}^+$ as

$$\begin{aligned}
d(r,r) &= 0, \text{ for all } r \in X,\\
d(r,w) &= d(w,r), \text{ for all } r, w \in X,
\end{aligned}$$

$$\begin{aligned} d(1,2) &= d(1,3) = d(1,4) = d(2,3) = d(2,4) = 2, \\ d(1,5) &= d(2,4) = d(2,5) = d(3,4) = 3, \\ d(3,5) &= d(4,5) = 1.5, \end{aligned}$$

so d is a metric on X and it is a complete metric. Set $L = 2$ and define a mapping $S : X \longrightarrow CL(X)$ as

$$Sr = \begin{cases} \{1\}, & \text{if } r = 1, 2, \\ \{5\}, & \text{if } r = 3, 4, \\ \{3, 4\}, & \text{if } r = 5. \end{cases}$$

For $r \in \{1, 2\}$ and $w \in \{3, 4, 5\}$, there is following cases:

If $r = 1$, $w = 3$, then

$$\begin{aligned} & F(H(S1, S3)) = F(d(1,5)) = \ln(3) \approx 1.0986, \text{ and} \\ & F(M_4(1,3) + LN_2(1,3)) \\ = F & \left(\begin{array}{l} \max \left\{ \begin{array}{l} d(1,3), d(1,S1), d(3,S3), d(1,S1)\, d(3,S1), d(1,S3)\, d(3,S1), \\ \dfrac{d(1,S3) + d(3,S1)}{2}, \left[\dfrac{d(1,S3) + d(3,S1)}{1 + d(1,S1) + d(3,S3)} \right] d(1,3) \end{array} \right\} \\ + L \min \{ d(1,S1), d(3,S3), d(1,S3), d(3,S1) \} \end{array} \right) \\ = {} & F(\max\{2, 0, 1.5, 0, 6, 2.5, 2(2)\} + L \min\{0, 1.5, 3, 2\}) \\ = {} & F(6 + L(0)) = F(6) = \ln(6) \approx 1.792. \end{aligned}$$

If $r = 1$, $w = 4$, then

$$\begin{aligned} & F(H(S1, S4)) = F(d(1,5)) = \ln(3) \approx 1.0986, \text{ and} \\ & F(M_4(1,4) + LN_2(1,4)) \\ = F & \left(\begin{array}{l} \max \left\{ \begin{array}{l} d(1,4), d(1,S1), d(4,S4), d(1,S1)\, d(4,S1), d(1,S4)\, d(4,S1), \\ \dfrac{d(1,S4) + d(4,S1)}{2}, \left[\dfrac{d(1,S4) + d(4,S1)}{1 + d(1,S1) + d(4,S4)} \right] d(1,4) \end{array} \right\} \\ + L \min \{ d(1,S1), d(4,S4), d(1,S4), d(4,S1) \} \end{array} \right) \\ = {} & F(\max\{2, 0, 1.5, 0, 3(2), 2.5, 2(2)\} + L \min\{0, 1.5, 3, 2\}) \\ = {} & F(6 + L(0)) = F(6) = \ln(6) \approx 1.792. \end{aligned}$$

If $r = 1$, $w = 5$, then

$$\begin{aligned} & F(H(S1, S5)) = F(d(1, \{3,4\})) = \ln(2) \approx 0.693, \text{ and} \\ & F(M_4(1,5) + LN_2(1,5)) \\ = F & \left(\begin{array}{l} \max \left\{ \begin{array}{l} d(1,5), d(1,S1), d(5,S5), d(1,S1)\, d(5,S1), d(1,S5)\, d(5,S1), \\ \dfrac{d(1,S5) + d(5,S1)}{2}, \left[\dfrac{d(1,S5) + d(5,S1)}{1 + d(1,S1) + d(5,S5)} \right] d(1,5) \end{array} \right\} \\ + L \min \{ d(1,S1), d(5,S5), d(1,S5), d(5,S1) \} \end{array} \right) \\ = {} & F(\max\{3, 0, 1.5, 0, 2(3), 2.5, 2(3)\} + L \min\{0, 1.5, 2, 3\}) \\ = {} & F(6 + L(0)) = F(6) = \ln(6) \approx 1.792. \end{aligned}$$

If $r = 2$, $w = 3$, then

$$\begin{aligned}
&F\left(H\left(S2, S3\right)\right) = F\left(d\left(1,5\right)\right) = \ln\left(3\right) \approx 1.0986, \text{ and}\\
&F\left(M_4\left(2,3\right) + LN_2\left(2,3\right)\right)\\
= F&\left(\begin{array}{l}\max\left\{\begin{array}{l} d\left(2,3\right), d\left(2,S2\right), d\left(3,S3\right), d\left(2,S2\right)d\left(3,S2\right), d\left(2,S3\right)d\left(3,S2\right),\\ \dfrac{d\left(2,S3\right)+d\left(3,S2\right)}{2}, \left[\dfrac{d\left(2,S3\right)+d\left(3,S2\right)}{1+d\left(2,S2\right)+d\left(3,S3\right)}\right]d\left(2,3\right)\end{array}\right\}\\ +L\min\left\{d\left(2,S2\right), d\left(3,S3\right), d\left(2,S3\right), d\left(3,S2\right)\right\}\end{array}\right)\\
= F&\left(\max\left\{2,2,1.5,4,6,2.5,(1.11)2\right\} + L\min\left\{2,1.5,3,2\right\}\right)\\
= F&\left(6 + L\left(1.5\right)\right) = F\left(9\right) = \ln\left(9\right) \approx 2.197.
\end{aligned}$$

If $r = 2$, $w = 4$, then

$$\begin{aligned}
&F\left(H\left(S2, S4\right)\right) = F\left(d\left(1,5\right)\right) = \ln\left(3\right) \approx 1.0986, \text{ and}\\
&F\left(M_4\left(2,4\right) + LN_2\left(2,4\right)\right)\\
= F&\left(\begin{array}{l}\max\left\{\begin{array}{l} d\left(2,4\right), d\left(2,S2\right), d\left(4,S4\right), d\left(2,S2\right)d\left(4,S2\right), d\left(2,S4\right)d\left(4,S2\right),\\ \dfrac{d\left(2,S4\right)+d\left(4,S2\right)}{2}, \left[\dfrac{d\left(2,S4\right)+d\left(4,S2\right)}{1+d\left(2,S2\right)+d\left(4,S4\right)}\right]d\left(2,4\right)\end{array}\right\}\\ +L\min\left\{d\left(2,S2\right), d\left(4,S4\right), d\left(2,S4\right), d\left(4,S2\right)\right\}\end{array}\right)\\
= F&\left(\max\left\{3,2,1.5,4,6,2.5,(1.11)3\right\} + L\min\left\{2,1.5,3,2\right\}\right)\\
= F&\left(6 + L\left(1.5\right)\right) = F\left(9\right) = \ln\left(9\right) \approx 2.197.
\end{aligned}$$

If $r = 2$, $w = 5$, then

$$\begin{aligned}
&F\left(H\left(S2, S5\right)\right) = F\left(d\left(1,\{3,4\}\right)\right) = \ln\left(2\right) \approx 0.693, \text{ and}\\
&F\left(M_4\left(2,5\right) + LN_2\left(2,5\right)\right)\\
= F&\left(\begin{array}{l}\max\left\{\begin{array}{l} d\left(2,5\right), d\left(2,S2\right), d\left(5,S5\right), d\left(2,S2\right)d\left(5,S2\right), d\left(2,S5\right)d\left(5,S2\right),\\ \dfrac{d\left(2,S5\right)+d\left(5,S2\right)}{2}, \left[\dfrac{d\left(2,S5\right)+d\left(5,S2\right)}{1+d\left(2,S2\right)+d\left(5,S5\right)}\right]d\left(2,5\right)\end{array}\right\}\\ +L\min\left\{d\left(2,S2\right), d\left(5,S5\right), d\left(2,S5\right), d\left(5,S2\right)\right\}\end{array}\right)\\
= F&\left(\max\left\{3,2,1.5,6,6,2.5,(1.11)3\right\} + L\min\left\{2,1.5,2,3\right\}\right)\\
= F&\left(6 + L\left(1.5\right)\right) = F\left(9\right) = \ln\left(9\right) \approx 2.197.
\end{aligned}$$

Hence, for all $\tau \in (0, 0.3467]$,

$$2\tau + F\left(H\left(Sr, Sw\right)\right) \leq F\left(M_4\left(r,w\right) + LN_2\left(r,w\right)\right)$$

holds for all $r, w \in X$ and for any $L = 2$. Hence, all the assumptions of Corollary 2.6 are satisfied, so there exist a fixed point $1 \in S(1)$. On the other hand, if $r = 1$ and $w = 3$, then

$$\begin{aligned} &F(H(S1, S3)) = F(d(1,5)) = \ln(3) \approx 1.0986, \text{ and} \\ &F(M_5(1,3) + LN_2(1,3)) \\ &= F\left(\begin{array}{l} \max\left\{d(1,3), d(1,S1), d(3,S3), \dfrac{d(1,S3)+d(3,S1)}{2}\right\} \\ +L\min\{d(1,S1), d(3,S3), d(1,S3), d(3,S1)\} \end{array}\right) \\ &= F(\max\{2, 0, 1.5, 2.5\} + L\min\{0, 1.5, 3, 2\}) \\ &= F(2.5) = \ln(2.5) \approx 0.916. \end{aligned}$$

Hence, there is no $\tau > 0$ and $L \geq 0$ such that

$$2\tau + F(H(S1, S3)) \leq F(M_5(1,3) + LN_2(1,3)).$$

That is

$$2\tau + F(H(S1, S3)) \nleq F(M_5(1,3) + LN_2(1,3)),$$

for all $\tau > 0$ and $L \geq 0$. Hence, Corollary 2.7 is not applicable in this example.

Remark 2.3. *Above example provides a situation where Corollary 2.7 is not applicable while Corollary 2.6 is applicable. Note that Corollary 2.6 is a consequence of our main result (Theorem 2.3).*

3. Conclusions

In this paper, we have introduced generalized set-valued F-contractions of b-metric spaces and presented the results about the existence of non-empty fixed point sets of newly introduced mappings. Our results improve some already existing very important results in the literature. Examples show that the new results offer proper generalizations. It is worth noting that by setting b-metric constant equal to one, we obtain some specific cases showing notable enhancement of existing results yet in metric spaces (see Corollary 2.6 and Example 2.3 above). It would be interesting to investigate these results in the framework of asymmetric distance spaces.

Acknowledgments

The authors are grateful for the funding provided by Basque Government for the grant number IT1207-19.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. M. Abbas, B. Ali, S. Romaguera, Fixed and periodic points of generalized contractions in metric spaces, *Fixed Point Theory Appl.*, **2013** (2013), 1–11. https://doi.org/10.1186/1687-1812-2013-243

2. M. Abbas, B. Ali, S. Romaguera, Generalized contraction and invariant approximation results on nonconvex subsets of normed spaces, *Abstr. Appl. Anal.*, **2014** (2014), 1–5. https://doi.org/10.1155/2014/391952
3. M. Abbas, B. Ali, S. Romaguera, Coincidence points of generalized multivalued (f, L)-almost F-contraction with applications, *J. Nonlinear Sci. Appl.*, **8** (2015), 919–934.
4. M. Abbas, B. Ali, O. Rizzo, C. Vetro, Fuzzy fixed points of generalized F_2-Geraghty type fuzzy mappings and complementary results, *Nonlinear Anal. Model. Control*, **21** (2016), 274–292. https://doi.org/10.15388/NA.2016.2.9
5. O. Acar, G. Durmaz, G. Minak, Generalized multivalued F-contraction on complete metric spaces, *Bull. Iranian Math. Soc.*, **40** (2014), 1469–1478.
6. I. Altun, G. Durmaz, G. Mınak, S. Romaguera, Multivalued almost F-contractions on complete metric spaces, *Filomat*, **30** (2016), 441–448. https://doi.org/10.2298/FIL1602441A
7. E. Ameer, M. Arshad, Two new fixed point results for generalized Wardowski type contractions, *J. Ana. Num. Theor.*, **5** (2017), 63–71.
8. T. V. An, N. V. Dung, Z. Kadelburg, S. Radenović, Various generalizations of metric spaces and fixed point theorems, *RACSAM*, **109** (2015), 175–198. https://doi.org/10.1007/s13398-014-0173-7
9. T. V. An, L. Q. Tuyen, N. V. Dung, Stone-type theorem on b-metric spaces and applications, *Topology Appl.*, **185–186** (2015), 50–64. https://doi.org/10.1016/j.topol.2015.02.005
10. H. Aydi, E. Karapinar, H. Yazidi, Modified F-contractions via α-admissible mappings and application to integral equations, *Filomat*, **31** (2017), 1141–1148. https://doi.org/10.2298/FIL1705141A
11. I. A. Bakhtin, The contraction mapping principle in quasi-metric spaces, *Funct. Anal.*, **30** (1989), 26–37.
12. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.*, **3** (1922), 133–181.
13. V. Berinde, Generalized contractions in quasimetric spaces, In: *Seminar on fixed point theory*, **3** (1993), 3–9.
14. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostrav.*, **1** (1993), 5–11.
15. S. Czerwik, K. Dłutek, S. L. Singh, Round-off stability of iteration procedures for operators in b-metric spaces, *J. Natur. Phys. Sci.*, **11** (1997), 87–94.
16. S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, *Atti Sem. Mat. Fis. Univ. Modena*, **46** (1998), 263–276.
17. E. Karapınar, A. Fulga, R. P. Agarwal, A survey: F-contractions with related fixed point results, *J. Fixed Point Theory Appl.*, **22** (2020), 1–58. https://doi.org/10.1007/s11784-020-00803-7
18. R. Miculescu, A. Mihail, New fixed point theorems for set-valued contractions in b-metric spaces, *J. Fixed Point Theory Appl.*, **19** (2017), 2153–2163. https://doi.org/10.1007/s11784-016-0400-2
19. L. V. Nguyen, L. T. Phuong, N. T. Hong, X. L. Qin, Some fixed point theorems for multivalued mappings concerning F-contractions, *J. Fixed Point Theory Appl.*, **20** (2018), 1–13. https://doi.org/10.1007/s11784-018-0621-7

20. H. Piri, P. Kumam, Some fixed point theorems concerning F-contraction in complete metric spaces, *Fixed Point Theory Appl.*, **2014** (2014), 1–11. https://doi.org/10.1186/1687-1812-2014-210

21. P. D. Proinov, Fixed point theorems for generalized contractive mappings in metric spaces, *J. Fixed Point Theory Appl.*, **22** (2020), 1–27. https://doi.org/10.1007/s11784-020-0756-1

22. S. L. Singh, B. Prasad, Some coincidence theorems and stability of iterative procedures, *Comput. Math. Appl.*, **55** (2008), 2512–2520. https://doi.org/10.1016/j.camwa.2007.10.026

23. F. Skof, Theoremi di punto fisso per applicazioni negli spazi metrici, *Atti. Acad. Sci. Torino Cl. Sci. Fis. Mat. Natur.*, **111** (1977), 323–329.

24. T. Suzuki, Basic inequality on a b-metric space and its applications, *J. Inequal. Appl.*, **2017** (2017), 1–11. https://doi.org/10.1186/s13660-017-1528-3

25. T. Suzuki, Fixed point theorems for single- and set-valued F-contractions in b-metric spaces, *J. Fixed Point Theory Appl.*, **20** (2018), 1–12. https://doi.org/10.1007/s11784-018-0519-4

26. D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 1–6. https://doi.org/10.1186/1687-1812-2012-94

AIMS Press