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Research article

δ-connectedness modulo an ideal

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Abstract: The aim of this paper is to introduce the notion of δ-connectedness modulo an ideal in proximity spaces. A sufficient condition for a δ-connected modulo an ideal proximity space to be connected modulo an ideal is defined. The notion of δ-connectedness modulo a proximal property is defined, and several results for δ-connectedness modulo compactness and modulo pseudocompactness are obtained. δ-perfect map for proximity spaces is defined and it is shown that the class of δ-perfect maps is properly contained in the class of perfect maps, and some results about δ-perfect maps are substantiated.

Keywords: δ-connected; δ-continuous map; ideal; Smirnov compactification; proximal property
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1. Introduction

In 1951, Efremovic [2] initiated the theory of a proximity space as a natural generalization of metric space and a topological group. Smirnov [14, 15], Naimpally [9] and Warrack [10] did the most substantial and comprehensive work in proximity spaces.

An ideal $\mathcal{I}$ in a set X is a non-empty collection of subsets of X which is hereditary and closed under finite unions. The notion of an ideal in topological spaces was introduced by Kuratowski [6] and Vaidyanathswamy [16]. In the last few years, many authors studied the various notions of topological space using ideals.

Connectedness plays a crucial role in topological spaces. The notion of connectedness in proximity spaces, namely, δ-connectedness was introduced by Mrówka et al. [7]. Dimitrijević et al. [1] defined local δ-connectedness, treelike proximity spaces to study δ-connectedness in detail. Singh et al. defined sum δ-connectedness in proximity spaces [11] and S-δ-connectedness in S-proximity spaces [12]. In 2016, Koushesh [5] defined the generalized notion of connectedness using an ideal, namely, connectedness modulo an ideal in topological spaces. Recently, Singh et al. [13] have introduced the notion of connectedness in ideal proximity spaces.

In this paper, we introduce the notion of δ-connectedness modulo an ideal in proximity spaces by utilizing an ideal which naturally extends δ-connectedness in its usual sense. In Section 2, we recall some basic definitions and results which are used in the subsequent sections. In Section 3, we define δ-connectedness modulo an ideal with some characterizations, and provide an example to show that a proximity space which is δ-connected modulo an ideal need not be connected modulo an ideal. For a given proximity space X and an ideal $\mathcal{I}$, a subspace $\gamma_{\mathcal{I}}X$ of X^* is defined, where X^* is the Smirnov compactification of X, and use this subspace to find a necessary and sufficient condition for X to be δ-connected modulo an ideal in terms of it's Smirnov compactification. Further, we show that if a completely regular space X is δ-connected modulo an ideal with respect to all possible compatible proximities δ, then it is also connected modulo an ideal. Finally, the notion of $\mathcal{U}$-connectedness modulo an ideal in uniform spaces is defined and it is shown that it is equivalent to the δ-connectedness modulo an ideal. In the last section, we define δ-connectedness modulo a proximal property, a particular case of δ-connectedness modulo an ideal and use compactness and pseudocompactness proximal properties to illustrate the general results of Section 3. We observe that connectedness modulo a topological property [4] implies δ-connectedness modulo a proximal property. δ-perfect map in proximity spaces is defined and it is shown by an example the class of δ-perfect maps is properly contained in the class of perfect maps and further substantiate some of it's characterizations. At last, we give an example to show that the product of two δ-connected modulo an ideal proximity spaces need not be δ-connected modulo an ideal.

2. Preliminaries

Now let us recall some basic definitions and results which are going to be used in the subsequent sections.

Definition 2.1. *[10] A binary relation δ on the power set $\mathcal{P}(X)$ of X is said to be a proximity on X, if the following axioms are satisfied for all P, Q, R and the empty set ϕ in $\mathcal{P}(X)$:*

(1) $(\phi, P) \notin \delta$;
(2) If $P \cap Q \neq \phi$, *then* $(P, Q) \in \delta$;
(3) If $(P, Q) \in \delta$, *then* $(Q, P) \in \delta$;
(4) $(P, Q \cup R) \in \delta$ *if and only if* $(P, Q) \in \delta$ *or* $(P, R) \in \delta$;
(5) If $(P, Q) \notin \delta$, *then there exists a subset* R *of* X *such that* $(P, R) \notin \delta$ *and* $(X \backslash R, Q) \notin \delta$.

The pair (X, δ) is called a proximity space. Throughout the paper, a proximity space (X, δ) is denoted by X.

Definition 2.2. *[9, 10] A proximity space X is said to be separated if for any $x, y \in X$, $(\{x\}, \{y\}) \in \delta \Rightarrow x = y$.*

All proximity spaces considered here are assumed to be separated.

Definition 2.3. *[10] Let X be a proximity space and P be a subset of X. Then P is said to be δ-closed if $(\{x\}, P) \in \delta \Rightarrow x \in P$.*

Proposition 2.1. *[10] The collection of the complements of all δ-closed sets forms a topology $\mathcal{T}_\delta$ on a proximity space X.*

Proposition 2.2. *[10] Let X be a proximity space. Then the closure $C(P)$ of P with respect to $\mathcal{T}_\delta$ is given by* $C(P) = \{x \in X : (\{x\}, P) \in \delta\}$.

Corollary 2.1. *[10] Let X be a proximity space. Then $M \in \mathcal{T}_\delta$ if and only if $(\{x\}, X\backslash M) \notin \delta$ for every $x \in M$.*

By Proposition 2.2, a set F is δ-closed if $C(F) = F$. From Corollary 2.1, a set U is δ-open if $(\{x\}, X\backslash U) \notin \delta$, for every $x \in U$. In a proximity space, a δ-open (or a δ-closed) set stands for an open (or a closed) set with respect to the generated topology $\mathcal{T}_\delta$.

Definition 2.4. *[10] Let X be a proximity space and $\mathcal{T}$ be a topology on X. Then δ is said to be compatible with $\mathcal{T}$ if the generated topology $\mathcal{T}_\delta$ and $\mathcal{T}$ are equal.*

Definition 2.5. *[10] Let X be a proximity space. Then a subset N of X is said to be a p-neighbourhood (or δ-neighbourhood) of $M \subset X$ if $(M, X\backslash N) \notin \delta$.*

Definition 2.6. *[10] Let (X, δ) and (Y, δ') be two proximity spaces. Then a map $f : (X, \delta) \longrightarrow (Y, \delta')$ is said to be δ-continuous (or p-continuous) if $(f(P), f(Q)) \in \delta'$ whenever $(P, Q) \in \delta$, for all $P, Q \subset X$.*

Definition 2.7. *[9] Let X be a non-empty set and $P, Q \subset X$. Then the discrete proximity on X is a binary relation δ on $\mathcal{P}(X)$ defined by $(P, Q) \in \delta$ if and only if $P \cap Q \neq \phi$.*

Definition 2.8. *[7] Let X be a proximity space. Then X is said to be δ-connected if every δ-continuous map from X to a discrete proximity space is constant.*

Theorem 2.1. *[7] Let X be a proximity space. Then the following statements are equivalent:*

(1) X is δ-connected.
(2) $(P, X\backslash P) \in \delta$ for each non-empty subset P with $P \neq X$.
(3) For every δ-continuous real-valued function f, the image $f(X)$ is dense in some interval of R.
(4) If $X = P \cup Q$ and $(P, Q) \notin \delta$, then either $P = \phi$ or $Q = \phi$.

The Smirnov compactification [10]: Every separated proximity space with its induced topology is a dense subspace of a unique (up to proximity isomorphism) compact and Hausdorff space, called the Smirnov compactification. In fact, given a completely regular Hausdorff space X, compatible proximities are in one to one correspondence with Hausdorff compactifications of X.

Each compactification Y of X, produces a proximity δ on X which is compatible with the topology of the space X and each proximity mapping from (X, δ) to an arbitrary compact space T can be extended (may not be continuously) to Y. Conversely, each proximity space (X, δ) admits a Smirnov compacification (X^*, δ^*) with the aforementioned property.

Definition 2.9. *[3] A completely regular space X is called pseudocompact if every real-valued continuous map on X is bounded.*

Definition 2.10. *[8] Let X and Y be spaces. Then a map $f : X \longrightarrow Y$ is called perfect if it is a closed, continuous and surjective such that $f^{-1}(u)$ is compact for each $u \in Y$.*

Definition 2.11. *[5] Let $\mathcal{I}$ be an ideal in a space X. A map $f : X \longrightarrow [0, 1]$ is called 2-valued modulo $\mathcal{I}$ if*

(1) $f^{-1}(0)$ and $f^{-1}(1)$ are not in $\mathcal{I}$.

(2) $X \backslash (f^{-1}(0) \cup f^{-1}(1))$ *is in* $\mathcal{I}$.

A space X *is said to be connected modulo* $\mathcal{I}$ *if there is no continuous map which is* 2*-valued modulo* $\mathcal{I}$.

Theorem 2.2. *[5] Let* X^* *be the Stone-Čech compactification of a Tychonoff space* X *and* $\mathcal{I}$ *be an ideal in* X. *Let* $\gamma_{\mathcal{I}} X = \bigcup\{int_{X^*} Cl_{X^*}(W) : Cl_X(W) \in \mathcal{I}\}$. *Then* X *is connected modulo* $\mathcal{I}$ *if and only if* $X^* \backslash \gamma_{\mathcal{I}} X$ *is connected (i.e. every continuous map from* $X^* \backslash \gamma_{\mathcal{I}} X$ *to a discrete space is constant).*

Definition 2.12. *[17] Let* (X, δ) *be a proximity space and* $f : X \longrightarrow Y$ *be a surjective map, where* Y *is any set. Then the quotient proximity on* Y *is the finest proximity such that the map* f *is* δ*-continuous. When* Y *has the quotient proximity,* f *is called a* δ*-quotient map.*

3. δ-connectedness modulo an ideal

Definition 3.1. *Let* $\mathcal{I}$ *be an ideal in a proximity space* X. *A map* $f : X \longrightarrow [0, 1]$ *is called* 2*-valued modulo* $\mathcal{I}$ *if*

(1) $f^{-1}(0)$ *and* $f^{-1}(1)$ *are not in* $\mathcal{I}$.
(2) $X \backslash (f^{-1}(0) \cup f^{-1}(1))$ *is in* $\mathcal{I}$.

A proximity space X *is said to be* δ*-connected modulo* $\mathcal{I}$ *(or* δ*-*$\mathcal{I}$*-connected) if there is no* δ*-continuous map which is* 2*-valued modulo* $\mathcal{I}$.

Definition 3.2. *In a proximity space* X, *a pair* P, Q *of subsets of* X *is said to be completely* δ*-separated if* P *and* Q *can be separated by a* δ*-continuous map.*

Definition 3.3. *Let* $\mathcal{I}$ *be an ideal in a proximity space* X. *A pair* P, Q *of completely* δ*-separated subsets of* X *is said to be a* δ*-separation modulo* $\mathcal{I}$ *for* X *if*

(1) P *and* Q *are not in* $\mathcal{I}$.
(2) $X \backslash (P \cup Q)$ *is in* $\mathcal{I}$.

In the next theorem, we give a characterization of δ-connectedness modulo $\mathcal{I}$.

Theorem 3.1. *Let* $\mathcal{I}$ *be an ideal in a proximity space* X. *Then the following statements are equivalent:*

(1) X *is* δ*-connected modulo* $\mathcal{I}$.
(2) X *does not have any* δ*-separation modulo* $\mathcal{I}$.

Proof. (1) ⇒ (2). Let P, Q be a δ-separation modulo $\mathcal{I}$ for X and $f : X \longrightarrow [0, 1]$ be a δ-continuous map such that $f(P) = \{0\}$ and $f(Q) = \{1\}$. Then $f^{-1}(0)$ and $f^{-1}(1)$ are not in $\mathcal{I}$ and $X \backslash (f^{-1}(0) \cup f^{-1}(1)) \in \mathcal{I}$. Thus, X is not δ-connected modulo $\mathcal{I}$.

(2) ⇒ (1). Suppose X is not δ-connected modulo $\mathcal{I}$. Then there exists a δ-continuous, 2-valued modulo $\mathcal{I}$ map $f : X \longrightarrow [0, 1]$. If $P = f^{-1}(0)$ and $Q = f^{-1}(1)$, then the pair P, Q forms a δ-separation modulo $\mathcal{I}$.

We conclude that the notion of δ-connectedness modulo an ideal naturally extends δ-connectedness of a proximity space X.

Example 3.1. *Let $\mathcal{I}$ be an ideal in a proximity space X which contains only the empty set ϕ. Then the notion of* 2*-valued modulo $\mathcal{I}$ mapping coincides with the notion of* 2*-valued mapping and the notion of δ-separation modulo $\mathcal{I}$ coincides with the notion of δ-separation. Hence X is δ-connected modulo $\mathcal{I}$ if and only if X is δ-connected.*

The following examples show that there may exist ideals other than ϕ for which δ-connectedness modulo an ideal and δ-connectedness agree.

Example 3.2. *Let X be a proximity space with no isolated points and $\mathcal{F}$ be an ideal which consists of all finite subsets of X. Then a δ-continuous mapping $f : X \longrightarrow [0, 1]$ is* 2*-valued modulo $\mathcal{F}$ if and only if $f(X) = \{0, 1\}$. Assume that f is* 2*-valued modulo $\mathcal{F}$. Then $f^{-1}(0)$ and $f^{-1}(1)$ are not finite, $W = X\backslash(f^{-1}(0) \cup f^{-1}(1))$ is finite. As W is δ-open and X is separated with no isolated points, so it has to be infinite if it is non-empty. Therefore $W = \phi$. Thus, $f(X) = \{0, 1\}$. Now suppose that $f(X) = \{0, 1\}$. Then it is* 2*-valued modulo $\mathcal{F}$.*

Example 3.3. *Let $\mathcal{I}$ be the collection of all the subsets U of a proximity space X such that $int_\delta cl_\delta(U) = \phi$. Then $\mathcal{I}$ is an ideal in X. To show that δ-connectedness modulo $\mathcal{I}$ agrees with δ-connectedness, let $f : X \longrightarrow [0, 1]$ be a δ-continuous,* 2*-valued modulo $\mathcal{I}$ map. Since $f^{-1}(0)$ and $f^{-1}(1)$ are not in $\mathcal{I}$, $W = X\backslash(f^{-1}(0) \cup f^{-1}(1))$ is in $\mathcal{I}$. As W is δ-open in X, so $W = \phi$. Thus, f is a* 2*-valued map. Every* 2*-valued δ-continuous map is a* 2*-valued modulo $\mathcal{I}$ map.*

Above examples motivate us for the following theorem which indeed give an answer for how bigger this augmentation of an ideal can be.

Theorem 3.2. *Let X be a δ-connected modulo $\mathcal{I}$ proximity space where $\mathcal{I}$ is an ideal in X. Then there exists a maximal ideal $\mathcal{M}$ (with respect to set inclusion $\subseteq$) containing $\mathcal{I}$ such that X is δ-connected modulo $\mathcal{M}$.*

Proof. Let $\mathcal{F}$ be the family of all those ideals $\mathcal{J}$ in X such that $\mathcal{I} \subseteq \mathcal{J}$ with X is δ-connected modulo $\mathcal{J}$. Then it is a partially ordered set with respect to set inclusion $\subseteq$. It is to show that every non-empty chain in $\mathcal{F}$ admits an upper bound in $\mathcal{F}$. Let C be a non-empty chain in $\mathcal{F}$ and $\mathcal{I}^* = \bigcup_{\mathcal{J}\in C} \mathcal{J}$. Then $\mathcal{I}^*$ is an ideal in X and $\mathcal{I} \subset \mathcal{I}^*$. Suppose X is not δ-connected modulo $\mathcal{I}^*$. Then there exists a δ-continuous map $f : X \longrightarrow [0, 1]$ such that $f^{-1}(0)$ and $f^{-1}(1)$ are not in $\mathcal{I}^*$, $X\backslash(f^{-1}(0) \cup f^{-1}(1))$ is in $\mathcal{I}^*$. Now there is some $\mathcal{J} \in C$ such that $X\backslash(f^{-1}(0) \cup f^{-1}(1))$ is in $\mathcal{J}$, and $f^{-1}(0)$ and $f^{-1}(1)$ are not in $\mathcal{J}$. Hence X is not δ-connected modulo $\mathcal{J}$, a contradiction.

Since every δ-continuous map is continuous, so every connected modulo $\mathcal{I}$ proximity space is δ-connected modulo $\mathcal{I}$. However, a δ-connected modulo $\mathcal{I}$ proximity space need not be connected modulo $\mathcal{I}$. Hence δ-connected modulo $\mathcal{I}$ extends the notion of connected modulo $\mathcal{I}$.

Example 3.4. *Let $\mathbb{Q}$ be a usual proximity subspace of $\mathbb{R}$ and $\mathcal{F}$ be an ideal in $\mathbb{Q}$ which consists of all the finite subsets of $\mathbb{Q}$. $\mathbb{Q}$ is a separated proximity subspace with no isolated point. By Example* 3.2*, a δ-continuous mapping $f : \mathbb{Q} \longrightarrow [0, 1]$ is* 2*-valued modulo $\mathcal{F}$ if and only if $f(\mathbb{Q}) = \{0, 1\}$. Thus, $\mathbb{Q}$ is δ-connected modulo $\mathcal{F}$ as $\mathbb{Q}$ is δ-connected. As $\mathbb{Q}$ is not connected, it is not connected modulo $\mathcal{F}$.*

Recall that for a mapping f from a set X to set Y, and an ideal $\mathcal{I}$ in Y, the collection $f^{-1}(\mathcal{I}) = \{U \subseteq X : f(U) \in \mathcal{I}\}$ is an ideal in X.

Theorem 3.3. *Let f be a δ-continuous, surjective map from a proximity space (X, δ_X) to (Y, δ_Y) and $\mathcal{I}$ be an ideal in Y. If X is δ-connected modulo $f^{-1}(\mathcal{I})$, then Y is also δ-connected modulo $\mathcal{I}$.*

Proof. Suppose Y is not δ-connected modulo $\mathcal{I}$, then there exists a δ-continuous map $g : Y \longrightarrow [0, 1]$ which is 2-valued modulo $\mathcal{I}$. Therefore $h = g \circ f$ being a composition of two δ-continuous maps is δ-continuous. Also, $h^{-1}(0)$ and $h^{-1}(1)$ are not in $f^{-1}(\mathcal{I})$. Since f is a surjective map, $f(X\backslash(h^{-1}(0) \cup h^{-1}(1))) = f(f^{-1}(Y\backslash(g^{-1}(0)\cup g^{-1}(1)))) = Y\backslash(g^{-1}(0)\cup g^{-1}(1))$. As $Y\backslash(g^{-1}(0)\cup g^{-1}(1))$ is in $\mathcal{I}$, $X\backslash(h^{-1}(0)\cup h^{-1}(1))$ is in $f^{-1}(\mathcal{I})$. Thus, $h : X \longrightarrow [0, 1]$ is a 2-valued modulo $f^{-1}(\mathcal{I})$ map. Hence X is not δ-connected modulo $f^{-1}(\mathcal{I})$, a contradiction.

Lemma 3.1. *Let W be a subspace of a proximity space X and $\mathcal{I}$ be an ideal in X. If W is δ-connected modulo $\mathcal{I}|_W$ and $f : X \longrightarrow [0, 1]$ a δ-continuous, 2-valued modulo $\mathcal{I}$ map. Then either $W \cap f^{-1}(0)$ or $W \cap f^{-1}(1)$ in $\mathcal{I}$.*

Proof. $f|_W : W \longrightarrow [0, 1]$ is a δ-continuous map which is not a 2-valued modulo $\mathcal{I}|_W$ as W is δ-connected modulo $\mathcal{I}|_W$. Since $X\backslash(f^{-1}(0) \cup f^{-1}(1))$ is in $\mathcal{I}$, so $W \cap (X\backslash(f^{-1}(0) \cup f^{-1}(1)))$ is in $\mathcal{I}|_W$. Since $(f|_W)^{-1}(0) = W \cap f^{-1}(0)$ and $(f|_W)^{-1}(1) = W \cap f^{-1}(1)$, therefore $W\backslash((f|_W)^{-1}(0) \cup (f|_W)^{-1}(1)) = W \cap (X\backslash(f^{-1}(0) \cup f^{-1}(1)))$. Thus, $W\backslash((f|_W)^{-1}(0) \cup (f|_W)^{-1}(1))$ is in $\mathcal{I}|_W$. Hence the only possiblity is that either $W \cap f^{-1}(0)$ or $W \cap f^{-1}(1)$ is in $\mathcal{I}$.

Theorem 3.4. *Let $\{X_i : 1 \leq i \leq n\}$ be a family of subspaces of a proximity space X such that $X = \bigcup_{i=1}^{n} X_i$ and $\mathcal{I}$ be an ideal in X. If X_i's are δ-connected modulo $\mathcal{I}|_{X_i}$ for each $1 \leq i \leq n$ and $W = \bigcap_{i=1}^{n} X_i$ is δ-connected modulo $\mathcal{I}|_W$ such that it is not in $\mathcal{I}$, then X is δ-connected modulo $\mathcal{I}$.*

Proof. Let $f : X \longrightarrow [0, 1]$ be a δ-continuous, 2-valued modulo $\mathcal{I}$ map. So, $f|_W$ is also δ-continuous. But it is not 2-valued modulo $\mathcal{I}|_W$ as W is δ-connected modulo $\mathcal{I}|_W$. Therefore by Lemma 3.1, either $W \cap f^{-1}(0)$ or $W \cap f^{-1}(1)$ is in $\mathcal{I}$. Without loss of generality, suppose $W \cap f^{-1}(0)$ is in $\mathcal{I}$. Likewise, either $X_i \cap f^{-1}(0)$ or $X_i \cap f^{-1}(1)$ is in $\mathcal{I}$, for each $1 \leq i \leq n$. If, for a fixed natural number i with $1 \leq i \leq n$, $X_i \cap f^{-1}(1)$ is in $\mathcal{I}$, then $W \cap f^{-1}(1)$ and $W\backslash(f^{-1}(0) \cup f^{-1}(1))$ are in $\mathcal{I}$. Thus, W being union of $W \cap f^{-1}(0)$, $W \cap f^{-1}(1)$ and $W\backslash(f^{-1}(0) \cup f^{-1}(1))$ is in $\mathcal{I}$, which is absurd. Hence $X_i \cap f^{-1}(0)$ is in $\mathcal{I}$ for each i, which implies $f^{-1}(0)$ is in $\mathcal{I}$, a contradiction.

However, the above theorem does not hold if the union of X_i's is not finite.

Let us recall that for an infinite cardinal κ, an ideal $\mathcal{I}$ in a set X is said to be κ-complete if for every subfamily $\mathcal{J} \subset \mathcal{I}$ of cardinality $\leq \kappa$, $\bigcup \mathcal{J}$ is in $\mathcal{I}$.

Theorem 3.5. *Let $\mathcal{I}$ be a κ-complete ideal in a proximity space X and $X = \bigcup_{i<\kappa} X_i$, where $\{X_i : i < \kappa\}$ is a collection of δ-connected modulo $\mathcal{I}|_{X_i}$ subspaces of X. If $W = \bigcap_{i<\kappa} X_i$ is δ-connected modulo $\mathcal{I}|_W$ subspace which is not in $\mathcal{I}$, then X is δ-connected modulo $\mathcal{I}$.*

The closure of an ideal $\mathcal{I}$ in a proximity space X is defined as $Cl_\delta(\mathcal{I}) = \{Cl_\delta(A) : A \in \mathcal{I}\}$.

Theorem 3.6. *Let $\mathcal{I}$ be an ideal in a proximity space X and W be a subspace of X such that $Cl_\delta(W) = X$ and $Cl_\delta(\mathcal{I}|_W) \subseteq \mathcal{I}$. If W is δ-connected modulo $\mathcal{I}|_W$, then X is δ-connected modulo $\mathcal{I}$.*

Proof. Suppose X is not δ-connected modulo $\mathcal{I}$. Then there is a δ-continuous 2-valued modulo $\mathcal{I}$ map $f : X \longrightarrow [0, 1]$. Therefore $f|_W : W \longrightarrow [0, 1]$ is also δ-continuous, however by hypothesis, it is not 2-valued modulo $\mathcal{I}|_W$. Now $W\backslash((f|_W)^{-1}(0)\cup(f|_W)^{-1}(1)) = W\cap(X\backslash(f^{-1}(0)\cup f^{-1}(1)))$ is in $\mathcal{I}|_W$. Thus, either

$(f|_W)^{-1}(0) = W \cap f^{-1}(0)$ or $(f|_W)^{-1}(1) = W \cap f^{-1}(1)$ has to be in $\mathcal{I}|_W$. If $(f|_W)^{-1}(0) = W \cap f^{-1}(0)$ is in $\mathcal{I}|_W$, then $W \cap f^{-1}([0,1)) = [W \cap f^{-1}(0)] \cup [W \backslash (f^{-1}(0) \cup f^{-1}(1)]$ is in $\mathcal{I}|_W$ which implies $Cl_\delta(W \cap f^{-1}([0,1))$ is in $Cl_\delta(\mathcal{I}|_W)$. Therefore $Cl_\delta f^{-1}([0,1))$ is in $\mathcal{I}$. Hence $f^{-1}(0)$ is in $\mathcal{I}$, a contradiction. Similarly, if $(f|_W)^{-1}(1) = W \cap f^{-1}(1)$ is in $\mathcal{I}|_W$, then $f^{-1}(1)$ is in $\mathcal{I}$, a contradiction.

Definition 3.4. *Let $\mathcal{I}$ be an ideal in a proximity space X and X^* be the Smirnov compactification of X. Then the subspace $\gamma_{\mathcal{I}} X$ is defined as $\gamma_{\mathcal{I}} X = \bigcup \{int_{\delta^*} Cl_{\delta^*}(W) : Cl_\delta(W) \in \mathcal{I}\}$.*

Theorem 3.7. *Let (X^*, δ^*) be the Smirnov compactification of a proximity space X and $\mathcal{I}$ be an ideal in X. Then X is δ-connected modulo $\mathcal{I}$ if and only if $X^* \backslash \gamma_{\mathcal{I}} X$ is δ-connected.*

Proof. Suppose $X^* \backslash \gamma_{\mathcal{I}} X$ is not δ-connected. Then there is a pair of non-empty sets P, Q such that $X^* \backslash \gamma_{\mathcal{I}} X = P \cup Q$ with $(P, Q) \notin \delta^*$ and a δ-continuous map $h : X^* \longrightarrow [0,1]$ such that $h(P) = \{0\}$ and $h(Q) = \{1\}$. Let $M = X \cap h^{-1}([0, 1/4])$ and $N = X \cap h^{-1}([3/4, 1])$. Now it is to show that the pair M, N forms a δ-separation modulo $\mathcal{I}$ for X. Define $i : [0,1] \longrightarrow [0,1]$ such as:

$$i(x) = \begin{cases} 0 & : \quad 0 \leq x \leq 1/4; \\ 2x - 1/2 & : \quad 1/4 < x \leq 3/4; \\ 1 & : \quad 3/4 < x \leq 1. \end{cases}$$

Obviously, i is a δ-continuous map. Since $g = h|_X : X \longrightarrow [0,1]$ being restriction of a δ-continuous map is δ-continuous, therefore $i \circ g : X \longrightarrow [0,1]$ is also a δ-continuous map with $i \circ g(M) = \{0\}$ and $i \circ g(N) = \{1\}$. Thus, M and N are completely δ-separated. Next claim is that M and N are not in $\mathcal{I}$. Suppose N is in $\mathcal{I}$. Put $W = X \cap h^{-1}((3/4, 1])$, $Cl_\delta(W) = N$ is in $\mathcal{I}$. Thus, $int_{\delta^*} Cl_{\delta^*}(W) \subseteq \gamma_{\mathcal{I}} X$. $Q \subseteq h^{-1}((3/4, 1])$ and $h^{-1}((3/4, 1]) \subseteq int_{\delta^*} Cl_{\delta^*}(X \cap h^{-1}((3/4, 1]))$ as $Cl_{\delta^*}(X \cap h^{-1}((3/4, 1])) = Cl_{\delta^*} h^{-1}((3/4, 1])$. Therefore, $Q \subseteq \gamma_{\mathcal{I}} X$, a contradiction. Similarly, it can be shown that M is not in $\mathcal{I}$. Further, it is to show that $X \backslash (M \cup N)$ is in $\mathcal{I}$. Since h takes only 0 or 1 on $X^* \backslash \gamma_{\mathcal{I}} X$, therefore $h^{-1}([1/4, 3/4]) \subseteq \gamma_{\mathcal{I}} X$. Thus, $h^{-1}([1/4, 3/4]) \subseteq int_{\delta^*} Cl_{\delta^*}(W_1) \cup int_{\delta^*} Cl_{\delta^*}(W_2) \cup \cdots \cup int_{\delta^*} Cl_{\delta^*}(W_k)$, where each W_i for $1 \leq i \leq k$ is such that $Cl_\delta(W_i)$ is in $\mathcal{I}$. So, $X \backslash (M \cup N) \subseteq X \cap h^{-1}([1/4, 3/4]) \subseteq Cl_\delta(W_1) \cup Cl_\delta(W_2) \cup \cdots \cup Cl_\delta(W_k)$. Since $Cl_\delta(W_i)$ is in $\mathcal{I}$, for all $1 \leq i \leq k$, so $X \backslash (M \cup N)$ is in $\mathcal{I}$.

Conversely, Suppose X is not δ-connected modulo $\mathcal{I}$. Then there is a pair M, N which constitutes a δ-separation modulo $\mathcal{I}$. Let $g : X \longrightarrow [0,1]$ be a δ-continuous map such that $g(M) = \{0\}$ and $g(N) = \{1\}$. Since $[0,1]$ is compact, so there is a δ-continuous map $h : X^* \longrightarrow [0,1]$ such that $h|_X = g$. Let $W = g^{-1}((p,q))$ for $0 < p < q < 1$, $Cl_\delta(W) \subseteq X \backslash (M \cup N)$ is in $\mathcal{I}$. So, $int_{\delta^*} Cl_{\delta^*}(W) \subseteq \gamma_{\mathcal{I}} X$. $h^{-1}((p,q)) \subseteq int_{\delta^*} Cl_{\delta^*}(W)$. Therefore $h^{-1}((p,q)) \subseteq \gamma_{\mathcal{I}} X$ for every $0 < p < q < 1$ which implies $h^{-1}((0,1)) \subseteq \gamma_{\mathcal{I}} X$. Thus, conclude that $X^* \backslash \gamma_{\mathcal{I}} X = (h^{-1}(0) \backslash \gamma_{\mathcal{I}} X) \cup (h^{-1}(1) \backslash \gamma_{\mathcal{I}} X)$. Put $P = h^{-1}(0) \backslash \gamma_{\mathcal{I}} X$ and $Q = h^{-1}(1) \backslash \gamma_{\mathcal{I}} X$. Now the claim is that the pair P and Q forms a δ-separation for $X^* \backslash \gamma_{\mathcal{I}} X$. Let $Q = \phi$. Then $h^{-1}(1) \subseteq \gamma_{\mathcal{I}} X$. Thus, $h^{-1}(1) \subseteq int_{\delta^*} Cl_{\delta^*}(W_1) \cup int_{\delta^*} Cl_{\delta^*}(W_2) \cup \cdots \cup int_{\delta^*} Cl_{\delta^*}(W_k)$ where each W_i for $1 \leq i \leq k$ is such that $Cl_\delta(W_i)$ is in $\mathcal{I}$. Hence $g^{-1}(1) = X \cap h^{-1}(1) \subseteq Cl_\delta(W_1) \cup Cl_\delta(W_2) \cup \cdots \cup Cl_\delta(W_k)$. Since $Cl_\delta(W_i)$ is in $\mathcal{I}$, for all $1 \leq i \leq k$, therefore $g^{-1}(1)$ is in $\mathcal{I}$, a contradiction. A similar proof gives $P \neq \phi$. Finally, it is to show that $(P, Q) \notin \delta^*$. Since P is compact and Q is δ-closed in $X^* \backslash \gamma_{\mathcal{I}} X$. Therefore $(P, Q) \notin \delta^*|_{X^* \backslash \gamma_{\mathcal{I}} X}$ implies $(P, Q) \notin \delta^*$.

Corollary 3.1. *Let (X^*, δ^*) be the Stone-Čech compactification of a proximity space (X, δ) and $\mathcal{I}$ be an ideal in X, then X is δ-connected modulo $\mathcal{I}$ if and only if $X^* \backslash \gamma_{\mathcal{I}} X$ is δ-connected.*

From the above corollary, we conclude that if $\mathcal{I} = \{\phi\}$, then $\gamma_{\mathcal{I}} X = \phi$. Thus, X is δ-connected if and only if X^* is δ-connected.

Theorem 3.8. *Let $(X, \mathcal{T})$ be a completely regular space and $\mathcal{I}$ be an ideal in X. If X is δ-connected modulo $\mathcal{I}$ with respect to all the compatible proximities δ on X, then X is connected modulo $\mathcal{I}$.*

Proof. Let C be a collection of all proximities on X which are compatible with $\mathcal{T}$. Then there exists the finest proximity (Stone-Čech proximity) δ_0 in C which is compatible with $\mathcal{T}$. Thus, by hypothesis, (X, δ_0) is δ-connected modulo $\mathcal{I}$. According to Smirnov's theorem, the ordinary compactification corresponding to δ_0 is the Stone-Čech compactification (X^*, δ_0^*). Therefore by Corollary 3.1, $X^* \backslash \gamma_{\mathcal{I}} X$ is δ-connected. Since $X^* \backslash \gamma_{\mathcal{I}} X$ is compact, so it is also connected. Hence by Theorem 2.2, X is connected modulo $\mathcal{I}$.

Definition 3.5. *Let $\mathcal{I}$ be an ideal in a proximity space X. Then X is said to be δ-local modulo $\mathcal{I}$ if for each $x \in X$, there exists a p-neighbourhood W of x such that $Cl_\delta(W)$ is in $\mathcal{I}$.*

Theorem 3.9. *Let $\mathcal{I}$ be an ideal in a proximity space X and (X^*, δ^*) be the Smirnov compactification of (X, δ). Then X is δ-local modulo $\mathcal{I}$ if and only if $X^* \backslash \gamma_{\mathcal{I}} X$ is contained in $X^* \backslash X$.*

Proof. Let X be δ-local modulo $\mathcal{I}$ and $x \in X$. Then there exists a p-neighbourhood W of x such that $Cl_\delta(W) \in \mathcal{I}$. Then $(\{x\}, X \backslash W) \notin \delta$ as W is p-nbhd of x. Therefore there exists a continuous map $g : X \longrightarrow [0, 1]$ such that $g(x) = 0$ and $g(X \backslash W) = \{1\}$. Suppose $h : X^* \longrightarrow [0, 1]$ is a continuous map such that $h|_X = g$. Since for every $0 < p < 1$, $h^{-1}([0, p))$ is δ-open in X^* and $Cl_{\delta^*}(X) = X^*$, therefore $Cl_{\delta^*} h^{-1}([0, p)) = Cl_{\delta^*}(X \cap h^{-1}([0, p)))$. Also $X \cap h^{-1}([0, p)) = g^{-1}([0, p))$ is contained in W. Thus, $x \in int_{\delta^*} Cl_{\delta^*}(W)$, because $x \in h^{-1}([0, p))$ and $h^{-1}([0, p)) \subseteq int_{\delta^*} Cl_{\delta^*} h^{-1}([0, p))$. Hence $x \in \gamma_{\mathcal{I}} X$.

Conversely, assume that $X^* \backslash \gamma_{\mathcal{I}} X \subseteq X^* \backslash X$. Let $x \in X$. Then $x \in \gamma_{\mathcal{I}} X$, so there exists $U \subseteq X$ such that $Cl_\delta(U) \in \mathcal{I}$ and $x \in int_{\delta^*} Cl_{\delta^*}(U)$. Define $W = X \cap int_{\delta^*} Cl_{\delta^*}(U)$. Then $(\{x\}, X \backslash W) \notin \delta$, so W is a p-neighbourhood of x. Since $Cl_\delta(W) \subseteq Cl_\delta(U) \in \mathcal{I}$, $Cl_\delta(W) \in \mathcal{I}$. Hence X is δ-local modulo $\mathcal{I}$.

Definition 3.6. *Let $(X, \mathcal{U})$ be a uniform space and $\mathcal{I}$ be an ideal in X. Then X is called $\mathcal{U}$-$\mathcal{I}$-connected (or $\mathcal{U}$-connected modulo $\mathcal{I}$) if there is no uniformily continuous map which is 2-valued modulo $\mathcal{I}$.*

Proposition 3.1. *Let $\mathcal{I}$ be an ideal in a uniform space $(X, \mathcal{U})$ and $\delta = \delta_{\mathcal{U}}$ be the proximity generated by $\mathcal{U}$. Then X is δ-connected modulo $\mathcal{I}$ if and only if it is $\mathcal{U}$-connected modulo $\mathcal{I}$.*

Proof. Since $[0, 1]$ is totally bounded, therefore a map $f : X \longrightarrow [0, 1]$ is uniformly continuous if and only if it is δ-continuous. Thus, X is δ-connected modulo $\mathcal{I}$ if and only if it is $\mathcal{U}$-connected modulo $\mathcal{I}$.

4. δ-connectedness modulo a proximal property

In this section, we prove all the results analogous to [4] connectedness modulo a topological property in proximity sense.

A property is called proximal property [10] if it is preserved under δ-homeomorphism (p-isomorphism).

Definition 4.1. *A proximal property $\mathcal{P}$ is called*

(1) *closed hereditary if U is a δ-closed subspace of a proximity space X and X has $\mathcal{P}$, then U also has $\mathcal{P}$.*
(2) *preserved under finite unions of closed subspaces if X is a proximity space such that $X = \bigcup_{i=1}^{n} U_i$ where each U_i is δ-closed subspace with $\mathcal{P}$, then X has also $\mathcal{P}$.*
(3) *invariant (and inversely invariant, respectively) under a family $\mathcal{F}$ of surjective maps from a proximity space X to a proximity space Y if $f : X \longrightarrow Y$ is any surjective map in $\mathcal{F}$ and X (Y, respectively) has $\mathcal{P}$, then Y (X, respectively) has also $\mathcal{P}$.*

The empty proximity space ϕ always has a proximal property $\mathcal{P}$. We always assume that for every proximal property $\mathcal{P}$, there is a proximity space having $\mathcal{P}$ *i.e* every proximal property is non-empty.

Definition 4.2. *Let $\mathcal{P}$ be a proximal property and X be a proximity space. Then $\mathcal{I}_{\mathcal{P}}(X)$ is defined as $\mathcal{I}_{\mathcal{P}}(X) = \{W \subseteq X : Cl_{\delta}(W) \text{ has } \mathcal{P}\}$.*

Theorem 4.1. *Let $\mathcal{P}$ be a closed hereditary proximal property preserved under finite unions of closed subspaces. Then $\mathcal{I}_{\mathcal{P}}(X)$ is an ideal in X.*

Proof. $\mathcal{I}_{\mathcal{P}}(X)$ is non-empty as the empty proximity space ϕ always has a proximal property $\mathcal{P}$. Let $W \in \mathcal{I}_{\mathcal{P}}(X)$ and $U \subset W$. Then $Cl_{\delta}(U) \subset Cl_{\delta}(W)$ and $Cl_{\delta}(W)$ has $\mathcal{P}$. Therefore $Cl_{\delta}(U)$ has $\mathcal{P}$. So, $U \in \mathcal{I}_{\mathcal{P}}(X)$. Let $U, V \in \mathcal{I}_{\mathcal{P}}(X)$, then both $Cl_{\delta}(U)$ and $Cl_{\delta}(V)$ have $\mathcal{P}$. Since $Cl_{\delta}(U \cup V) = Cl_{\delta}(U) \cup Cl_{\delta}(V)$ and $\mathcal{P}$ is preserved under finite unions of closed subspaces, $U \cup V \in \mathcal{I}_{\mathcal{P}}(X)$.

Definition 4.3. *Let $\mathcal{P}$ be a closed hereditary proximal property and it is preserved under finite unions of closed subspaces. Then a proximity space X is called δ-connected modulo $\mathcal{P}$ (or δ-$\mathcal{P}$-connected) if X is δ-connected modulo $\mathcal{I}_{\mathcal{P}}(X)$.*

From Section 3, for an ideal $\mathcal{I}_{\mathcal{P}}(X)$, a δ-$\mathcal{P}$-separation for X is a pair P, Q of completely δ-separated sets such that $P, Q \notin \mathcal{I}_{\mathcal{P}}(X)$ and $X \backslash (P \cup Q) \in \mathcal{I}_{\mathcal{P}}(X)$.

Theorem 4.2. *Let $\mathcal{P}$ be a closed hereditary proximal property and it is preserved under finite unions of closed subspaces. Then for a proximity space X, the following are equivalent:*

(1) *X is δ-$\mathcal{P}$-connected.*
(2) *X does not have any δ-$\mathcal{P}$-separation.*

Let $\mathcal{P}$ be a proximal property and a proximity space X has $\mathcal{P}$ if and only if $X = \phi$, then the concept of δ-$\mathcal{P}$-connectedness and δ-connectedness is same. Hence the notion of δ-$\mathcal{P}$-connectedness extends the notion of δ-connectedness.

Note that if $\mathcal{P}$ is a closed hereditary proximal property and it is preserved under finite unions of closed subspaces, then a proximity space X is δ-$\mathcal{P}$-connected if it has $\mathcal{P}$.

Every topological property is a proximal property. Therefore if $\mathcal{P}$ is any topological property, then every δ-$\mathcal{P}$-separation of a proximity space X is a $\mathcal{P}$-separation [4]. Thus, every $\mathcal{P}$-connected (or connected modulo $\mathcal{P}$ [4]) proximity space is δ-$\mathcal{P}$-connected.

Theorem 4.3. *Let X^* be the Smirnov compactification of a locally compact proximity space X and $\mathcal{P}$ be the compactness property. Then X is δ-$\mathcal{P}$-connected if and only if $X^* \backslash X$ is δ-connected.*

Proof. By Theorem 3.7, it is only to show that $\gamma_{\mathcal{I}_{\mathcal{P}}(X)} X = X$, where $\mathcal{I}_{\mathcal{P}}(X)$ is the ideal generated by $\mathcal{P}$. Since X is locally compact, every δ-open set W in X is δ-open in X^*. Let $x \in X$, then there exists a δ-open set W in X, hence in X^*, such that $Cl_\delta(W)$ is compact. Therefore, $W \in \gamma_{\mathcal{I}_\mathcal{P}(X)}X$ and $x \in \gamma_{\mathcal{I}_\mathcal{P}(X)}X$. Now for $y \in \gamma_{\mathcal{I}_\mathcal{P}(X)}X$, there exists $U \subset X$ such that $Cl_\delta(U) \in \mathcal{I}_\mathcal{P}(X)$ and $y \in int_{\delta^*}Cl_{\delta^*}(U) \subset Cl_{\delta^*}(Cl_\delta(U)) = Cl_\delta(U) \subset X$. Thus, $y \in X$.

Corollary 4.1. *Let $X = \mathbb{R}^n$ and X^* be the Smirnov compactification of X. Then $X^*\backslash X$ is δ-connected if and only if $n \geq 2$.*

Example 4.1. *Let $\mathcal{P}$ be the compactness property. Then by Theorem* 4.3 *and Corollary* 4.1*, $\mathbb{R}^n$ is δ-$\mathcal{P}$-connected if and only if $n \geq 2$.*

If Y is a δ-connected, dense subspace of a proximity space X, then X is always δ-connected. However, it may not be true in δ-connectedness modulo an ideal as a general notion of δ-connectedness.

Example 4.2. *Let $X = [-1, 1] \times \mathbb{R}$ be a usual proximity subspace of $\mathbb{R}^2$ and $\mathcal{P}$ be the compactness property. Let $Y = (-1, 1) \times \mathbb{R}$. Then Y is δ-$\mathcal{P}$-connected, dense subspace of X. The pair $P = [-1, 1] \times (-\infty, -2]$, $Q = [-1, 1] \times [2, \infty)$ forms a δ-$\mathcal{P}$-separation for X.*

Definition 4.4. *For given proximity spaces X and Y, a map $f : X \longrightarrow Y$ is said to be δ-perfect if it is a δ-closed (i.e, a closed map with respect to proximities), δ-continuous and surjective map such that $f^{-1}(u)$ is compact for each $u \in Y$.*

Example 4.3. *Let δ_d and δ_0 be two different proximities on $\mathbb{R}$ such that*
(i) $(A, B) \in \delta_d$ if and only if $d(A, B) = 0$; for all subsets $A, B \subset \mathbb{R}$.
(ii) $(A, B) \in \delta_0$ if and only if $Cl(A) \cap Cl(B) \neq \phi$; for all subsets $A, B \subset \mathbb{R}$.
Then $\mathcal{T}(\delta_d) = \mathcal{T}(\delta_0)$.

Let $I : (\mathbb{R}, \delta_d) \longrightarrow (\mathbb{R}, \delta_0)$ be the identity map. Then I is not δ-continuous. Suppose $A = \{n : n \in \mathbb{N}\}$ and $B = \{n - \frac{1}{n} : n \in \mathbb{N}\}$. Then $(A, B) \in \delta_d$ but $(A, B) \notin \delta_0$. Therefore I is not δ-continuous. Thus, I is not a δ-perfect map. However, I is a homeomorphism. Therefore it is a perfect map.

Proposition 4.1. *For a given map $f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$, the following statements hold:*

(1) If f is a δ-homeomorphism, then it is a δ-perfect map.
(2) If f is an injective δ-perfect map, then it is a δ-homeomorphism.
(3) If f is a perfect map and both X, Y are compact Hausdorff spaces, then it is a δ-perfect map.

The inverse image of a δ-connected proximity space under a δ-perfect map need not be δ-connected.

Example 4.4. *Let X be any finite discrete space and Y be a proximity space consisting of only one element. Then the constant map from X to Y is a δ-perfect map. But X is not δ-connected.*

A δ-perfect map need not be δ-open.

Example 4.5. *Let $f : [0, 1] \cup [2, 3] \longrightarrow [0, 3]$ be a map defined by $f(x) = x + 1$ if $x \in [0, 1]$ and $f(x) = x$ if $x \in [2, 3]$. Since both $[0, 1] \cup [2, 3]$ and $[0, 3]$ are compact Hausdorff spaces, therefore every perfect map is δ-perfect. Since f is a closed, continuous and surjective map. So, it is perfect. However, f is not δ-open.*

Proposition 4.2. *Every δ-perfect map is a δ-quotient map.*

Proof. Let $f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$ be a δ-perfect map. It is to prove that δ_Y is the finest proximity on Y such that f is δ-continuous. Let δ'_Y be any other proximity on Y such that f is δ-continuous. If $(A, B) \notin \delta'_Y$, then $(f^{-1}(A), f^{-1}(B)) \notin \delta_X$. Therefore, $x \notin Cl_{\delta_X}(f^{-1}(B))$ for all $x \in f^{-1}(A)$ which implies $f(x) \notin f(Cl_{\delta_X}(f^{-1}(B)))$ for all $f(x) \in A$. Since f is δ-closed and surjective, therefore $Cl_{\delta_Y} f(f^{-1}(B)) = Cl_{\delta_Y}(B) \subset f(Cl_{\delta_X}(f^{-1}(B)))$. Thus, $f(x) \notin Cl_{\delta_Y}(B)$ for all $f(x) \in A$. Hence $(A, B) \notin \delta_Y$.

Theorem 4.4. *Let $\mathcal{P}$ be a proximal property which is invariant and inversely invariant under δ-closed, δ-continuous surjective maps, and $\mathcal{P}$ is closed hereditary and preserved under finite unions of closed subspaces. Then δ-$\mathcal{P}$-connectedness is invariant under δ-closed, δ-continuous surjective maps.*

Proof. For proximity spaces X and Y, let $f : X \longrightarrow Y$ be any δ-closed, δ-continuous, surjective map and X is δ-$\mathcal{P}$-connected *i.e* X is δ-connected modulo $\mathcal{I}_{\mathcal{P}}(X)$. It is to prove that Y is δ-connected modulo $\mathcal{I}_{\mathcal{P}}(Y)$. Use Theorem 3.3 and hypothesis to show that $\mathcal{I}_{\mathcal{P}}(X) = f^{-1}(\mathcal{I}_{\mathcal{P}}(Y))$. If $U \in \mathcal{I}_{\mathcal{P}}(X)$, then $Cl_\delta(U)$ has $\mathcal{P}$. So, $f(Cl_\delta(U))$ has $\mathcal{P}$. Now observe that $Cl_{\delta'} f(U) \subseteq f(Cl_\delta(U))$ where δ' is a proximity on Y. Therefore $Cl_{\delta'} f(U)$ has $\mathcal{P}$, *i.e* $f(U)$ is in $\mathcal{I}_{\mathcal{P}}(Y)$. Thus, $U \in f^{-1}(\mathcal{I}_{\mathcal{P}}(Y))$. On the other hand, if $V \in f^{-1}(\mathcal{I}_{\mathcal{P}}(Y))$, then $f(V) \in \mathcal{I}_{\mathcal{P}}(Y)$. Therefore $Cl_{\delta'} f(V)$ has $\mathcal{P}$. Thus, $f^{-1}(Cl_{\delta'} f(V))$ has $\mathcal{P}$. Note that $Cl_\delta(V) \subseteq f^{-1}(Cl_{\delta'} f(V))$. Therefore $Cl_\delta(V)$ has $\mathcal{P}$. Hence $V \in \mathcal{I}_{\mathcal{P}}(X)$.

Lemma 4.1. *Let X be a proximity space and $\mathcal{P}$ be a proximal property which is closed hereditary and preserved under finite unions of closed subspaces. If U is any δ-closed subset of X, then $\mathcal{I}_{\mathcal{P}}(X)|_U = \mathcal{I}_{\mathcal{P}}(U)$.*

Proof. If $W \in \mathcal{I}_{\mathcal{P}}(X)|_U$, then $W = V \cap U$ where $V \in \mathcal{I}_{\mathcal{P}}(X)$. Therefore $Cl_{\delta_X}(V)$ has $\mathcal{P}$ and $Cl_{\delta_U}(W) = Cl_{\delta_X}(W)$, U is δ-closed in X. Since $Cl_{\delta_X}(W) \subset Cl_{\delta_X}(V)$ and $\mathcal{P}$ is closed hereditary, therefore $Cl_{\delta_X}(W)$ has $\mathcal{P}$. Thus, $Cl_{\delta_U}(W)$ has $\mathcal{P}$. Hence, $W \in \mathcal{I}_{\mathcal{P}}(U)$. On the other hand, if $W \in \mathcal{I}_{\mathcal{P}}(U)$, then $Cl_{\delta_U}(W)$ has $\mathcal{P}$. Since $Cl_{\delta_U}(W) = Cl_{\delta_X}(W)$, therefore $Cl_{\delta_X}(W)$ has $\mathcal{P}$. So, $W \in \mathcal{I}_{\mathcal{P}}(X)$. Thus, $W \in \mathcal{I}_{\mathcal{P}}(X)|_U$ as $W \subset U$.

Theorem 4.5. *Let X be a proximity space and $\mathcal{P}$ be a proximal property which is closed hereditary and preserved under finite unions of closed subspaces. Let each X_i for $1 \leq i \leq n$ be a δ-closed and δ-$\mathcal{P}$-connected subspace of X with $X = \bigcup_{i=1}^n X_i$. If $\bigcap_{i=1}^n X_i$ is δ-$\mathcal{P}$-connected and does not has $\mathcal{P}$, then X is δ-$\mathcal{P}$-connected.*

Proof. Use Lemma 4.1 and Theorem 3.4 to conclude the proof.

Definition 4.5. *Let X be a proximity space and $\mathcal{P}$ be pseudocompactness. Then $\mathcal{U}_{\mathcal{P}}(X)$ can be defined as:*

$\mathcal{U}_{\mathcal{P}}(X) = \{U :$ *there exists a δ-open set $W \subseteq X$ such that $Cl_\delta(W)$ has $\mathcal{P}$, and $U \subseteq W$*$\}$

Definition 4.6. *Let $\mathcal{P}$ be pseudocompactness. A proximity space X is called δ-$\mathcal{P}$-connected if X is δ-connected modulo $\mathcal{U}_{\mathcal{P}}(X)$.*

If $\mathcal{P}$ is the pseudocompactness property, then every pseudocompact proximity space is always δ-$\mathcal{P}$-connected.

Theorem 4.6. *Let $\mathcal{P}$ be pseudocompactness. Then δ-$\mathcal{P}$-connectedness is invariant under δ-perfect, δ-open maps.*

Proof. For proximity spaces X and Y, let $f : X \longrightarrow Y$ be any δ-perfect, δ-open map and X is δ-$\mathcal{P}$-connected, *i.e* X is δ-connected modulo $\mathcal{U}_{\mathcal{P}}(X)$. It is to show that Y is δ-connected modulo $\mathcal{U}_{\mathcal{P}}(Y)$. Using Theorem 3.3 and hypothesis, it suffices to show that $\mathcal{U}_{\mathcal{P}}(X) = f^{-1}(\mathcal{U}_{\mathcal{P}}(Y))$. If $U \in \mathcal{U}_{\mathcal{P}}(X)$, then $U \subseteq W$ for some δ-open subset W of X such that $Cl_{\delta}(W)$ is pseudocompact. So, $f(U) \subseteq f(W)$ and $f(W)$ is δ-open subset of Y as f is δ-open. Since f is δ-continuous, therefore $f(Cl_{\delta}(W))$ is pseudocompact. Also, $Cl_{\delta'} f(W) \subseteq f(Cl_{\delta}(W))$ as f is δ-perfect. Thus, $Cl_{\delta'} f(W)$ being regular closed subset of $f(Cl_{\delta}(W))$ is pseudocompact. Therefore, $f(U) \in \mathcal{U}_{\mathcal{P}}(Y)$. On the other hand, if $V \in f^{-1}(\mathcal{U}_{\mathcal{P}}(Y))$, then $f(V) \in \mathcal{U}_{\mathcal{P}}(Y)$. Therefore, $f(V) \subseteq Q$ for some δ-open subset Q of Y such that $Cl_{\delta'}(Q)$ is pseudocompact. Thus, $V \subseteq f^{-1}(Q)$ and $f^{-1}(Q)$ is δ-open. It suffices to show that $Cl_{\delta} f^{-1}(Q)$ is pseudocompact. Since every regular closed subspace of a pseudocompact space is pseudocompact, therefore, $Cl_{\delta} f^{-1}(Q)$ is pseudocompact.

In the following example, we show that the product of two δ-connected modulo an ideal proximity spaces need not be δ-connected modulo an ideal.

Example 4.6. *Let $X_1 = \{0, 1\}$ and $X_2 = [0, \infty)$ be subspaces of $\mathbb{R}$ with the usual subspace proximity, and $\mathcal{P}$ be the pseudocompactness property. Then X_1 and X_2 are δ-$\mathcal{P}$-connected. The product $X = X_1 \times X_2$ is not δ-$\mathcal{P}$-connected as $P = \{0\} \times [0, \infty)$ and $Q = \{1\} \times [0, \infty)$ forms a δ-separation modulo $\mathcal{U}_{\mathcal{P}}(X)$.*

5. Conclusions

In this paper, we have defined the notion of δ-connectedness modulo an ideal in proximity spaces by using an ideal to generalize the notions of δ-connectedness in proximity spaces and connectedness modulo an ideal in topological spaces.

Conflict of interest

The authors declare no conflict of interest.

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