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Research article

On constructing almost complex Norden metric structures

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Abstract: For a given almost contact Norden metric structure on a smooth manifold M, one can obtain an almost complex Norden metric structure on $M \times \mathbb{R}$. In this work, we study this construction in details and give the relations between the classes of these structures. Furthermore, we give examples of almost complex Norden metric structures of which the existence are guaranteed by the results of the paper.

Keywords: almost complex B-metric structure; almost contact B-metric structure; Norden metric
Mathematics Subject Classification: 53B30, 53B35, 53D15

1. Introduction

Contact and complex structures on smooth manifolds are studied with details in [3]. In the literature, there are many studies on these structures and their classifications with compatible (semi-)Riemannian metrics. For instance, almost contact metric structures and their classification is studied in [2, 4, 8]; almost Hermitian structures were examined in [1]; in [5], almost contact structures with Norden metric are classified. Likewise, almost complex Norden metric structures and their classification are given in [6]. In this paper, we study how to obtain an almost complex Norden metric structures by given almost contact Norden metric structures and give the correspondence between the classes of these structures. In the final section, we give some examples about the existence of the induced almost complex Norden metric structures by the results of the paper.

2. Preliminaries

Definition 2.1. *Let M be a $2n+1$ dimensional C^∞ manifold and ξ, η and ϕ be a vector field, a 1-form and a $(1,1)$ tensor field respectively on M with*

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{2.1}$$

then (ϕ, ξ, η) *is called an almost contact structure on M and the manifold M is said to be an almost contact manifold.*

Furthermore, if M is also equipped with a metric ρ of signature $(n+1, n)$, satisfying

$$\rho(\phi(U), \phi(V)) = -\rho(U, V) + \eta(U)\eta(V), \tag{2.2}$$

where $U, V \in \mathfrak{X}(M)$, then ρ is said to be a compatible metric with the structure (ϕ, ξ, η) and M is called an almost contact Norden metric manifold (or almost contact B-metric manifold) with the structure (ϕ, ξ, η, ρ).

For a given almost contact Norden metric manifold $(M, \phi, \xi, \eta, \rho)$, the followings hold:

$$\phi^2 = -id + \eta \otimes \xi \quad ; \quad \eta(\xi) = 1. \tag{2.3}$$

It follows,

$$\eta \circ \phi = 0, \quad \phi\xi = 0, \quad \rho(\phi U, V) = \rho(U, \phi V), \quad \eta(U) = \rho(U, \xi). \tag{2.4}$$

Moreover, ϕ has rank $2n$ [5].

Definition 2.2. *The* $(0,3)$*-tensor field* α *on M given with*

$$\alpha(U, V, W) = \rho((\nabla_U \phi)V, W), \tag{2.5}$$

is called the fundamental tensor of the structure,where $U, V, W \in \mathfrak{X}(M)$ *and* ∇ *is the Levi-Civita connection of* ρ *.*

Proposition 2.1. *By the definition of* ∇*, we have*

$$(\nabla_U \eta)(V) = \rho(\nabla_U \xi, V) = \alpha(U, \phi V, \xi). \tag{2.6}$$

Also from the Eqs (2.1) and (2.4), it follows immediately

$$\begin{aligned} &\alpha(U, V, W) = \alpha(U, V, W), \\ &\alpha(U, \phi V, \phi W) = \alpha(U, V, W) - \eta(V)\alpha(U, \xi, W) - \eta(W)\alpha(U, V, \xi), \\ &\alpha(U, \xi, \xi) = 0, \end{aligned} \tag{2.7}$$

where $U, V, W \in \mathfrak{X}(M)$. The 1-forms below, called "Lee forms", are associated with α:

$$\theta(u) = \rho^{ij}\alpha(e_i, e_j, u); \quad \theta^*(u) = \rho^{ij}\alpha(e_i, \phi e_j, u); \quad \omega(u) = \alpha(\xi, \xi, u), \tag{2.8}$$

where $u \in T_pM$, $\{e_i, \xi\}$ $(i = 1, 2, \cdots, 2n)$ is a basis of T_pM, and (ρ^{ij}) is the inverse matrix of (ρ_{ij}).

In [5], a classification of almost contact Norden metric manifolds with respect to the fundamental tensor α is studied , where the defining relations of the eleven basic classes $\mathcal{F}_i, (i = 1, 2, \cdots, 11)$ are given as follow Table 1:

Table 1. Basic classes of almost contact Norden metric structures.

$\mathcal{F}_1$	$\alpha(U,V,W)=\frac{1}{2n}[\rho(U,\phi V)\theta(\phi W)+\rho(U,\phi W)\theta(\phi V)+\rho(\phi U,\phi W)\theta(\phi^2 V)+\rho(\phi U,\phi V)\theta(\phi^2 W)]$
$\mathcal{F}_2$	$\alpha(\xi,V,W)=\alpha(U,\xi,W)=0,\quad \alpha(U,V,\phi W)+\alpha(V,W,\phi U)+\alpha(W,U,\phi V)=0,\quad \theta=0$
$\mathcal{F}_3$	$\alpha(\xi,V,W)=\alpha(U,\xi,W)=0,\quad \alpha(U,V,W)+\alpha(V,W,U)+\alpha(W,U,V)=0$
$\mathcal{F}_4$	$\alpha(U,V,W)=-\frac{\theta(\xi)}{2n}[\rho(\phi U,\phi W)\eta(V)+\rho(\phi U,\phi V)\eta(W)]$
$\mathcal{F}_5$	$\alpha(U,V,W)=-\frac{\theta^*(\xi)}{2n}[\rho(U,\phi W)\eta(V)+\rho(U,\phi V)\eta(W)]$
$\mathcal{F}_6$	$\alpha(U,V,W)=-\alpha(\phi U,\phi V,W)-\alpha(\phi U,V,\phi W)=-\alpha(V,W,U)+\alpha(W,U,V)-2\alpha((\phi U,\phi V,W),\quad \theta(\xi)=\theta^*(\xi)=0$
$\mathcal{F}_7$	$\alpha(U,V,W)=-\alpha(\phi U,\phi V,W)-\alpha(\phi U,V,\phi W)=-\alpha(V,W,U)-\alpha(W,U,V)$
$\mathcal{F}_8$	$\alpha(U,V,W)=\alpha(\phi U,\phi V,W)+\alpha(\phi U,V,\phi W)=-\alpha(V,W,U)+\alpha(W,U,V)+2\alpha(\phi U,\phi V,W)$
$\mathcal{F}_9$	$\alpha(U,V,W)=\alpha(\phi U,\phi V,W)+\alpha(\phi U,V,\phi W)=-\alpha(V,W,U)-\alpha(W,U,V)$
$\mathcal{F}_{10}$	$\alpha(U,V,W)=\eta(U)\alpha(\xi,\phi V,\phi W)$
$\mathcal{F}_{11}$	$\alpha(U,V,W)=\eta(U)(\eta(W)\omega(V)+\eta(V)\omega(W))$

The class $\mathcal{F}_0$ is determined by the condition $\alpha=0$.

By setting specific choices for U,V,W in the defining relations $\mathcal{F}_i$'s, one can obtain the following results.

Proposition 2.2. *Let (M,ϕ,ξ,η,ρ) be an almost contact Norden metric manifold. Then we have,*

(1) The Reeb vector field ξ is parallel only in the classes $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_{10}$ and in their direct sums.
(2) The Reeb vector field ξ is Killing only in the classes $\mathcal{F}_7$, $\mathcal{F}_8$ and in their direct sum.
(3) The Reeb vector field ξ satisfies the relation $\rho(\nabla_U\xi,V)=\rho(\nabla_V\xi,U)$ for any $U,V\in\mathfrak{X}(M)$, in the classes $\mathcal{F}_4$, $\mathcal{F}_5$, $\mathcal{F}_6$, $\mathcal{F}_9$ and in their direct sums.

Definition 2.3. *Let N^{2n} be a C^∞ manifold with an almost complex structure J and a semi-Riemannian metric h of signature (n,n) that holds*

$$J^2=-id,\quad h(JU,JV)=-h(U,V), \tag{2.9}$$

for arbitrary $U,V\in\mathfrak{X}(N)$. Then (N,J,h) is called an almost complex Norden metric manifold (or almost complex B-metric manifold) [5].

The metric h satisfies

$$h(JU,V)=h(U,JV), \tag{2.10}$$

for all vector fields U,V.

Definition 2.4. *The fundamental tensor F of the manifold (N,J,h) is given with :*

$$F(U,V,W)=h((\tilde{\nabla}_U J)V,W), \tag{2.11}$$

where U,V,W are smooth vector fields and $\tilde{\nabla}$ is the Levi-Civita connection of h.

The tensor F satisfies

$$\begin{aligned} F(U,V,W) &= F(U,W,V) \\ F(U,V,W) &= F(U,JV,JW). \end{aligned} \tag{2.12}$$

Definition 2.5. *For a given $p \in (N,J,h)$ and a basis $\{e_1,\cdots,e_{2n}\}$ of T_pN, the Lee form φ associated with the tensor F is defined as*

$$\varphi(u) = h^{ij}F(e_i,e_j,u), \tag{2.13}$$

where (h^{ij}) is the the inverse of (h) and $u \in T_pN$.

In [6], almost complex Norden metric structures are classified with respect to $\tilde{\nabla}J$. Due to this classification, three basic classes $\mathcal{W}_i, (i = 1,2,3)$ and so 2^3 invariant subspaces are obtained. These subspaces are given with the relations below. For $U,V,W \in \mathfrak{X}(N)$,

$$\begin{aligned} \mathcal{W}_0 &: F(U,V,W) = 0, \\ \mathcal{W}_1 &: F(U,V,W) = \frac{1}{2n}[h(U,V)\varphi(W) + h(U,W)\varphi(V) \\ &\qquad + h(U,JV)\varphi(JW) + h(U,JW)\varphi(JV)], \\ \mathcal{W}_2 &: F(U,V,JW) + F(V,W,JU) + F(W,U,JV) = 0,\ \varphi = 0, \\ \mathcal{W}_3 &: F(U,V,W) + F(V,W,U) + F(W,U,V) = 0, \end{aligned}$$

$$\begin{aligned} &\mathcal{W}_1 \oplus \mathcal{W}_2 : F(U,V,JW) + F(V,W,JU) + F(W,U,JV) = 0, \\ &\mathcal{W}_2 \oplus \mathcal{W}_3 : \varphi = 0, \\ &\mathcal{W}_1 \oplus \mathcal{W}_3 : F(U,V,W) + F(V,W,U) + F(W,U,V) = \frac{1}{n}[h(U,V)\varphi(W) \\ &\qquad + h(W,U)\varphi(V) + h(V,W)\varphi(U) + h(U,JV)\varphi(JW) \\ &\qquad + h(V,JW)\varphi(JU) + h(W,JU)\varphi(JV)], \\ &\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 : \text{(The whole class)}. \end{aligned}$$

3. Induced almost complex Norden metric structure

Let $(M^{2n+1},\phi,\xi,\eta,\rho)$ be an almost contact Norden metric manifold. Define J as

$$J(\tilde{U}) := (\phi U - a\xi, \eta(U)\frac{d}{dt}), \tag{3.1}$$

where $U \in \mathfrak{X}(M)$, $\tilde{U} = (U, a\frac{d}{dt}) \in \mathfrak{X}(M\times\mathbb{R})$. Here, a is a real valued smooth function on $M\times\mathbb{R}$, and t is the coordinate of $\mathbb{R}$. It is known that $(M\times\mathbb{R},J,h)$ is an almost complex Norden metric manifold with the metric h

$$h(\tilde{U},\tilde{V}) := \rho(U,V) - ab, \tag{3.2}$$

where $\tilde{U} = (U, a\frac{d}{dt}), \tilde{V} = (V, b\frac{d}{dt})$ [9].

In this work, this structure is said to be the induced almost complex Norden metric structure.

Unless otherwise stated, throughout the paper, we will use the notation $\tilde{M}$ for $M\times\mathbb{R}$, and $\tilde{U}, \tilde{V}, \tilde{W}, ...$ for the vector fields $(U, a\frac{d}{dt}), (V, b\frac{d}{dt}), (W, c\frac{d}{dt}), ...$ on $\tilde{M}$.

Let $(M, \phi, \xi, \eta, \rho)$ be an almost contact Norden metric manifold and $(\tilde{M}, J, h)$ be the induced almost complex Norden metric manifold. Then, we have the following relations [9]

$$\tilde{\nabla}_{\tilde{U}}\tilde{V} = (\nabla_U V, (U[b] + a\frac{db}{dt})\frac{d}{dt}), \tag{3.3}$$

$$(\tilde{\nabla}_{\tilde{U}}J)(\tilde{V}) = ((\nabla_U\phi)(V) - b\nabla_U\xi, (\nabla_U\eta)(V)\frac{d}{dt}), \tag{3.4}$$

$$F(\tilde{U}, \tilde{V}, \tilde{W}) = \alpha(U, V, W) - c\alpha(U, \xi, \phi V) - b\alpha(U, \xi, \phi W), \tag{3.5}$$

where ∇ and $\tilde{\nabla}$ denote the Levi-Civita connections of ρ and h respectively. From the Eq (3.5) we can obtain the following equalities, that will be used later.

$$F(\tilde{U}, \tilde{V}, J\tilde{W}) = \alpha(U, V, \phi W) - c\alpha(U, V, \xi) + b\alpha(U, W, \xi) - \eta(W)\alpha(U, \xi, \phi V). \tag{3.6}$$

Let $\{e_i, \xi\}, (i = 1, \cdots, 2n)$ be a basis for T_pM, $p \in (M, \phi, \xi, \eta, \rho)$. Then, one can construct the basis $\{\tilde{e}_i = (e_i, 0), \tilde{e}_{2n+1} = (\xi, 0), \tilde{e}_{2n+2} = (0, \frac{d}{dt})\}, (i = 1, \cdots, 2n)$ for the manifold $(M \times \mathbb{R}, J, h)$. Thus, the inverse matrix of the metric h becomes:

$$(h^{ij}) = \left(\begin{array}{c|c} (\rho^{ij}) & 0 \\ \hline 0 & -1 \end{array}\right),$$

where (ρ^{ij}) is the inverse matrix of ρ.

With the basis above, after long but direct calculation, we can state the following lemma.

Lemma 3.1. *The Lee form φ of the manifold $(M \times \mathbb{R}, J, h)$ can be stated as:*

$$\varphi(\tilde{U}) = \theta(U) + \rho^{\xi\xi}\omega(U) - a\theta^*(\xi) - a\rho^{i\xi}\omega(\phi(e_i)) + \rho^{i\xi}\alpha(e_i, \xi, U) + \rho^{i\xi}\alpha(\xi, e_i, U). \tag{3.7}$$

4. Results

In the paper [9], it is shown that if the $(M, \phi, \xi, \eta, \rho)$ is of class $\mathcal{F}_0$, $\mathcal{F}_2$, $\mathcal{F}_3$, then $(\tilde{M}, J, h)$ is in the class $\mathcal{W}_0$, $\mathcal{W}_1 \oplus \mathcal{W}_2$, $\mathcal{W}_3$ respectively. In this study, we focus on the remaining classes.

Shortly, $(M, \phi, \xi, \eta, \rho)$ and the induced manifold $(\tilde{M}, J, h)$ will be denoted by M and $\tilde{M}$ respectively.

Theorem 4.1. *Let M and $\tilde{M}$ be given as above. We have the followings:*

(1) If M is in $\mathcal{F}_1$, then $\tilde{M}$ is in $\mathcal{W}_1 \oplus \mathcal{W}_2$.
(2) If M is in $\mathcal{F}_2$, then $\tilde{M}$ is in $\mathcal{W}_2$.
(3) If M is in $\mathcal{F}_3$, then $\tilde{M}$ is in $\mathcal{W}_3$.
(4) If M is in $\mathcal{F}_4$, then $\tilde{M}$ is in $\mathcal{W}_1 \oplus \mathcal{W}_2$.

(5) If M is in $\mathcal{F}_5$, then $\tilde{M}$ is in $\mathcal{W}_1 \oplus \mathcal{W}_2$.
(6) If M is in $\mathcal{F}_6$, then $\tilde{M}$ is in $\mathcal{W}_1 \oplus \mathcal{W}_2$.
(7) If M is in $\mathcal{F}_7$, then $\tilde{M}$ is in $\mathcal{W}_3$.
(8) If M is in $\mathcal{F}_8$, then $\tilde{M}$ is in $\mathcal{W}_1 \oplus \mathcal{W}_2$.
(9) If M is in $\mathcal{F}_9$, then $\tilde{M}$ is in $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$.
(not in one of the classes $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_1 \oplus \mathcal{W}_2$, $\mathcal{W}_2 \oplus \mathcal{W}_3$, $\mathcal{W}_1 \oplus \mathcal{W}_3$).
(10) If M is in $\mathcal{F}_{10}$, then $\tilde{M}$ is in $\mathcal{W}_2 \oplus \mathcal{W}_3$ or $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$.
(11) If M is in $\mathcal{F}_{11}$, then $\tilde{M}$ is in $\mathcal{W}_1 \oplus \mathcal{W}_2$.

Proof. (1) Let M be in $\mathcal{F}_1$. From the defining relation of the class $\mathcal{F}_1$, by direct calculation we get $\alpha(\xi, U, V) = 0$ and $\alpha(U, \xi, W) = 0$ for any $U, V, W \in \mathfrak{X}(M)$. Moreover, since ξ is parallel in $\mathcal{F}_1$, we obtain

$$F(\tilde{U}, \tilde{V}, \tilde{W}) = \alpha(U, V, W) \tag{4.1}$$

from the Eq (3.5). By this equation, one can see that the equation $F(\tilde{U}, \tilde{V}, J\tilde{W}) + F(\tilde{V}, \tilde{W}, J\tilde{W}) + F(\tilde{W}, \tilde{U}, J\tilde{V}) = 0$ holds. So $\tilde{M}$ is in $\mathcal{W}_1 \oplus \mathcal{W}_2$.

Remark that the structure $\tilde{M}$ is neither in $\mathcal{W}_1$, nor $\mathcal{W}_2$. Suppose that $\tilde{M} \in \mathcal{W}_1$. By setting $\tilde{U} = (\xi, 0\frac{d}{dt})$, $\tilde{V} = (\xi, 0\frac{d}{dt})$, $\tilde{W} = (W, 0\frac{d}{dt})$ in the defining relation of $\mathcal{W}_1$, we get $\theta(W) = 0$ that implies $\alpha(U, V, W) = 0$ in the defining relation of $\mathcal{F}_1$. This contradicts with the non-triviality of α in $\mathcal{F}_1$. So, $\tilde{M}$ is not in $\mathcal{W}_1$. Similarly, by assuming $\tilde{M} \in \mathcal{W}_2$, we get $\varphi(U, a\frac{d}{dt}) = \theta(U) = 0$ which also implies $\alpha(U, V, W) = 0$. Hence, $\tilde{M}$ is not of the class $\mathcal{W}_2$.

(2) Let M be in $\mathcal{F}_2$. Since $\alpha(\xi, V, W) = \alpha(U, \xi, W) = 0$, the terms $\omega(U)$, $\theta^*(\xi)$, $\omega(\phi(e_i))$ vanish. So, we get $\varphi = \theta = 0$. On the other hand, as ξ is Killing in $\mathcal{F}_2$, $F(\tilde{U}, \tilde{V}, J\tilde{W}) = \alpha(U, V, \phi W)$. Thus, $F(\tilde{U}, \tilde{V}, J\tilde{W}) + F(\tilde{V}, \tilde{W}, J\tilde{U}) + F(\tilde{W}, \tilde{U}, J\tilde{V}) = 0$, that is $\tilde{M} \in \mathcal{W}_2$.

Remark that, in [9], it was shown that if M is in $\mathcal{F}_2$, the induced structure $\tilde{M}$ is of the class $\mathcal{W}_1 \oplus \mathcal{W}_2$. In this theorem, we improved this statement as $\mathcal{W}_2$.

(3) The proof follows by routine calculations by using the defining relations and the Eq (3.5).

(4) Let M be in $\mathcal{F}_4$. After a usual but long calculation with considering (3.6), one can see that $F(\tilde{U}, \tilde{V}, J\tilde{W}) + F(\tilde{V}, \tilde{W}, J\tilde{U}) + F(\tilde{W}, \tilde{U}, J\tilde{V}) = 0$, i.e., $\tilde{M}$ is of the class $\mathcal{W}_1 \oplus \mathcal{W}_2$.

Note that, the structure $\tilde{M}$ is neither in $\mathcal{W}_1$, nor $\mathcal{W}_2$. Assume that $\tilde{M}$ is in $\mathcal{W}_1$. Then, we get

$$\begin{aligned}
&-\frac{\theta(\xi)}{2n}\{\eta(V)\rho(\phi U, \phi V) + \eta(W)\rho(\phi U, \phi V) - c\alpha(U, \xi, \phi V) - b\alpha(U, \xi, \phi W)\} \\
&= \frac{1}{2(n+1)}\{h(\tilde{U}, \tilde{V})\varphi(\tilde{W}) + h(\tilde{U}, \tilde{W})\varphi(\tilde{V}) + h(\tilde{U}, J(\tilde{V})\varphi(J\tilde{W}) \\
&+ h(\tilde{U}, J\tilde{W})\varphi(J\tilde{V})\}.
\end{aligned} \tag{4.2}$$

If we choose $\tilde{U} = (\xi, 0\frac{d}{dt})$, $\tilde{V} = (\xi, b\frac{d}{dt})$, $\tilde{W} = (\xi, c\frac{d}{dt})$, where $bc \neq 1$, then the left hand side of (4.2) vanishes. However, the right hand side becomes $2\theta(\xi)(1 - bc)$ which must be non-zero. In conclusion, $\tilde{M}$ can not be in $\mathcal{W}_1$. Similarly, if we set $\tilde{U} = (\xi, 0)$, by the Lemma 3.1, we get $\varphi(\tilde{U}) = \theta(\xi)$ which is non-zero since $M \in \mathcal{F}_4$. Thus, $\tilde{M}$ can not be of the class $\mathcal{W}_2$.

(5) The proof follows very similar with the Proof (4).

(6) Let M be in $\mathcal{F}_6$. From the defining relation of $\mathcal{F}_6$, we have

$$\alpha(\xi, V, W) = 0, \ \alpha(U, V, \xi) = \alpha(V, U, \xi) = -\alpha(\phi U, \phi V, \xi), \tag{4.3}$$

for any vector fields U, V, W. So, by the Eq (3.6),

$$\begin{aligned}
&F(\tilde{U},\tilde{V},J\tilde{W})+F(\tilde{V},\tilde{W},J\tilde{U})+F(\tilde{W},\tilde{U},J\tilde{V})\\
&=\alpha(U,V,\phi W)+\alpha(V,W,\phi U)+\alpha(W,U,\phi V)-\eta(W)\alpha(U,\xi,\phi V)\\
&-\eta(U)\alpha(V,\xi,\phi W)-\eta(V)\alpha(W,\xi,\phi U).
\end{aligned}\tag{4.4}$$

On the other hand, from the relations in (4.3), after direct calculation, we get

$$\alpha(U,V\phi W)=\eta(V)\alpha(W,\xi,\phi U).$$

Thus, the induced structure in $\tilde{M}$ is of the class $\mathcal{W}_1\oplus\mathcal{W}_2$.

Also, by setting $\tilde{U}=(\xi,0)$, $\tilde{V}=(V,0)$, $\tilde{W}=(W,0)$ in the defining relation of the class $\mathcal{W}_1$ provides $\alpha=0$. Thus, $\tilde{M}$ is not of the class $\mathcal{W}_1$.

(7) Let M be in $\mathcal{F}_7$. After routine calculation, by using (3.5) we obtain,

$$\begin{aligned}
&F(\tilde{U},\tilde{V},\tilde{W})+F(\tilde{V},\tilde{W},\tilde{U})+F(\tilde{W},\tilde{U},\tilde{V})\\
&=\{\alpha(U,V,W)+\alpha(V,W,U)+\alpha(W,U,V)\}\\
&-a(\alpha(V,\xi,\phi W)+\alpha(W,\xi,\phi V)-b(\alpha(U,\xi,\phi W)+\alpha(W,\xi,\phi U))\\
&-c(\alpha(U,\xi,\phi V)+\alpha(V,\xi,\phi U)).
\end{aligned}\tag{4.5}$$

Since ξ is Killing in $\mathcal{F}_7$, $\alpha(U,\xi,\phi V)+\alpha(V,\xi,\phi U))=0$, for any U, V. On the other hand, the summation of the first three terms of the right hand side of (4.5) vanishes by the defining relation of $\mathcal{F}_7$. Thus, $\tilde{M}$ is of the class $\mathcal{W}_3$.

(8) The proof is very similar with the Proof (6), by means of the following relations:

$$\alpha(\xi,V,W)=0,\ \alpha(U,V,\xi)=\alpha(V,U,\xi)=\alpha(\phi U,\phi V,\xi),\tag{4.6}$$

that holds for the structures in $\mathcal{F}_8$. If $M\in\mathcal{F}_8$, we get $\varphi(\tilde{U})=\theta(U)+\rho^{i\xi}\alpha(e_i,\xi,U)$, since $\theta^*(\xi)=0$. Assume that $\tilde{M}\in\mathcal{W}_1$. Then for $\tilde{U}=(\xi,0)$, $\tilde{V}=(V,0)$, $\tilde{W}=(W,0)$ in the defining relation of $\mathcal{W}_1$, we get

$$\frac{1}{2n}\{\eta(V)\varphi(W,0)+\eta(W)\varphi(V,0)\}=0.$$

By setting $V=W$ in this equation, we obtain $\alpha=0$, that is not true since α is non-trivial in $\mathcal{F}_8$. So, the induced manifold $\tilde{M}$ is not of the class $\mathcal{W}_1$.

(9) Let M be in $\mathcal{F}_9$. Then, from the defining relation we have

$$\alpha(\xi,V,W)=0,\ \alpha(U,V,\xi)=\alpha(\phi U,\phi V,\xi)=-\alpha(V,U,\xi),\tag{4.7}$$

and then,

$$\alpha(U,\xi,\phi V)=\alpha(V,\xi,\phi U).\tag{4.8}$$

Assume that $\tilde{M}\in\mathcal{W}_1$. Then for $\tilde{U}=\tilde{V}=\tilde{W}=(\xi,0\frac{d}{dt})$, from the defining relation of $\mathcal{W}_1$, we get $0=\frac{\theta(\xi)}{n+1}$, which is not true since $\theta(\xi)$ is non-zero in the class $\mathcal{F}_9$. Hence, $\tilde{M}$ is not in $\mathcal{W}_1$.

Let $\tilde{U}=(\xi,0\frac{d}{dt})$. It is easy to see that $\varphi(\tilde{U})=\theta(\xi)$. Since $\theta(\xi)$ is non-zero in $\mathcal{F}_9$, so is not φ. So,

$\tilde{M}$ is neither in $\mathcal{W}_2$, nor $\mathcal{W}_2 \oplus \mathcal{W}_3$.

Assume that $\tilde{M} \in \mathcal{W}_3$. By using the defining relation of $\mathcal{W}_3$, the following equation holds:

$$\begin{aligned}&F(\tilde{U}, \tilde{V}, \tilde{W}) + F(\tilde{V}, \tilde{W}, \tilde{U}) + F(\tilde{W}, \tilde{U}, \tilde{V})\\&= -2c\alpha(U, \xi, \phi V) - 2b\alpha(U, \xi, \phi W) - 2a\alpha(V, \xi, \phi W)\\&= 0.\end{aligned} \tag{4.9}$$

Set $\tilde{U} = (U, 0\frac{d}{dt})$, $\tilde{V} = (V, 0\frac{d}{dt})$, $\tilde{W} = (0, \frac{d}{dt})$ in the Eq (4.9). Then we get $\alpha(U, \xi, \phi V) = 0$, that implies ξ to be parallel. However, ξ is not parallel in the class $\mathcal{F}_9$. So, $\tilde{M}$ can not be of the class $\mathcal{W}_3$. Suppose that $\tilde{M}$ is of the class $\mathcal{W}_1 \oplus \mathcal{W}_2$. After direct calculation, we get $\alpha(U, V, \xi) = 0$, for $\tilde{U} = (U, 0\frac{d}{dt})$, $\tilde{V} = (V, 0\frac{d}{dt})$, $\tilde{W} = (\xi, \frac{d}{dt})$, which is not true as ξ is not parallel in $\mathcal{F}_9$. So, $\tilde{M}$ is not in $\mathcal{W}_1 \oplus \mathcal{W}_2$.

To see that $\tilde{M}$ is not in $\mathcal{W}_1 \oplus \mathcal{W}_3$, by choosing $\tilde{U} = \tilde{V} = \tilde{W} = (\xi, 0\frac{d}{dt})$ in the defining relation of $\mathcal{W}_1 \oplus \mathcal{W}_3$, we get $0 = \frac{3\theta(\xi)}{n+1}$, which does not hold as $\theta(\xi)$ is non-zero in $\mathcal{F}_9$. Thus, $\tilde{M}$ is not in $\mathcal{W}_1 \oplus \mathcal{W}_3$.

(10) Let M be in $\mathcal{F}_{10}$. From the defining relation, the followings hold:

$$\alpha(\xi, V, W) = \alpha(\xi, \phi V, \phi W), \ \alpha(U, V, \xi) = 0. \tag{4.10}$$

By using (4.10) in the Eqs (3.5) and (3.6), we obtain:

$$F(\tilde{U}, \tilde{V}, \tilde{W}) = \alpha(U, V, W), \tag{4.11}$$

and

$$F(\tilde{U}, \tilde{V}, J\tilde{W}) = \alpha(U, V, \phi W), \tag{4.12}$$

respectively. Assume that $\tilde{M}$ is in $\mathcal{W}_1 \oplus \mathcal{W}_2$. If we choose $U = \xi$ and substitute W with ϕW in the defining relation of $\mathcal{W}_1 \oplus \mathcal{W}_2$, by the Eq (4.12), we get $\alpha(\xi, V, W) = 0$, which is not true since α is non-trivial in $\mathcal{F}_{10}$. So, $\tilde{M}$ is not in the classes $\mathcal{W}_1 \oplus \mathcal{W}_2$, $\mathcal{W}_1$, $\mathcal{W}_2$. Let $\tilde{M}$ be in $\mathcal{W}_1 \oplus \mathcal{W}_3$. By the Eq (4.11), The defining relation becomes:

$$\alpha(\xi, U, V) = \frac{1}{n+1}\{\eta(U)\varphi((V, 0\frac{d}{dt}) + \eta(V)\varphi(U, 0\frac{d}{dt})\}. \tag{4.13}$$

By substituting U and V with ϕU and ϕV respectively in (4.13), we get $\alpha(\xi, \phi U, \phi V) = 0$, that is not true since α is non-trivial in $\mathcal{F}_{10}$. Thus, $\tilde{M}$ can not be in the classes $\mathcal{W}_1 \oplus \mathcal{W}_3$ or $\mathcal{W}_3$.

As a result, $\tilde{M}$ can only be in $\mathcal{W}_2 \oplus \mathcal{W}_3$ or $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$.

(11) Let M be in $\mathcal{F}_{11}$. By the aid of the Eq (3.6), it can be seen that the defining relation of the class $\mathcal{W}_1 \oplus \mathcal{W}_2$ holds.

□

5. Some examples

Example 1. Consider the real connected Lie group G of dimension five and its Lie algebra $\mathfrak{g}$ with the following non-zero brackets:

$$\begin{aligned}&[x_1, x_2] = -[x_3, x_4] = -\lambda_1 x_1 - \lambda_2 x_2 + \lambda_3 x_3 \lambda_4 x_4 + 2\mu_1 x_5,\\&[x_1, x_4] = -[x_2, x_3] = -\lambda_3 x_1 - \lambda_4 x_2 - \lambda_1 x_3 - \lambda_2 x_4 + 2\mu_2 x_5,\end{aligned}$$

where $\{x_i\}$ is a basis of left-invariant vector fields on G and $\lambda_i, \mu_j \in \mathbb{R}, (i = 1, 2, 3, 4; j = 1, 2)$. It is shown in [10] that the almost contact Norden metric structure given with:

$$\begin{aligned}
&\phi(x_1) = x_3, \phi(x_2) = x_4, \phi(x_3) = -x_1, \phi(x_4) = -x_2, \phi(x_5) = 0;\\
&\xi = x_5; \eta(x_i) = 0\,(i = 1, 2, 3, 4),\ \eta(x_5) = 1;\\
&g(x_1, x_1) = g(x_2, x_2) = g(x_5, x_5) = 1 = -g(x_3, x_3) = -g(x_4, x_4);\\
&g(x_i, x_j) = 0,\ i, j \in \{1, 2, 3, 4, 5\};\ i \neq j,
\end{aligned}$$

belongs to the class $\mathcal{F}_7$ (see [10] for the details and the proof). Thus, by the Theorem 4.1 (7), the derived almost complex Norden metric structure on $G \times \mathbb{R}$ that is constructed by (3.1) and (3.2) is of the class $\mathcal{W}_3$.
Indeed, under the consideration of the global basis
$\{\tilde{x}_i = (x_i, 0), \tilde{x}_6 = (0, \frac{d}{dt}\}(i = 1, ..., 5)$, one can obtain the relation between the components of the Levi-Civita connections as:

$$\begin{aligned}
\tilde{\nabla}_{\tilde{x}_i}\tilde{x}_j =&(\nabla_{x_i}x_j, 0),\ (i, j = 1, 2, ..., 5);\\
\tilde{\nabla}_{\tilde{x}_i}\tilde{x}_6 =&\tilde{\nabla}_{\tilde{x}_6}\tilde{x}_i = 0, (i = 1, 2, ..., 6),
\end{aligned}$$

where ∇ and $\tilde{\nabla}$ address G and $G \times \mathbb{R}$ respectively. Thus, by using (3.5), after a routine calculation, we get

$$\mathfrak{S}_{\tilde{U}\tilde{V}\tilde{W}} F(\tilde{U}, \tilde{V}, \tilde{W}) = 0,$$

for any $\tilde{U} = \sum u_i\tilde{x}_i$, $\tilde{V} = \sum v_i\tilde{x}_i$, $\tilde{W} = \sum w_i\tilde{x}_i$ in $\mathfrak{X}(G \times \mathbb{R})$. So, the defining relation of $\mathcal{W}_3$ holds.
Note that the vector fields $\tilde{\xi} = (\xi, 0\frac{d}{dt})$, $\tilde{U} = (0, \frac{d}{dt})$ on $G \times \mathbb{R}$ are Killing and parallel, respectively. So, we can state

Corollary 5.1. *There exists an almost complex Norden metric manifold of the class $\mathcal{W}_3$, possessing a Killing vector field and a parallel vector field.*

Example 2. Consider the complex Riemannian manifold $\mathbb{R}^{2n+2}$ with canonical complex structure J and the metric ρ:

$$\rho(U, U) = -\delta_{ij}\lambda^i\lambda^j + \delta_{ij}\mu^i\mu^j,$$

where $U = \lambda^i\frac{\partial}{\partial x^i} + \mu^i\frac{\partial}{\partial y^i}$, and the time-like hyper-surface S^{2n+1} by identifying the point $p \in \mathbb{R}^{2n+2}$ with its position vector W satisfying $\rho(W, W) = -1$. Under the conditions

$$\xi = \lambda W + \mu JW,\ \rho(W, \xi)\ \rho(\xi, \xi) = 1,$$

we have the unique decomposition

$$JU = \phi U + \eta(U)J\xi,$$

where $U \in T_pS^{2n+1}$, ϕU is the projection of JU into T_pS^{2n+1} with respect to $J\xi$ and η is a one-form in T_pS^{2n+1}.

It is shown in [5] that, (ϕ, ξ, η, ρ) is an almost contact B- metric structure on S^{2n+1} with the construction above. Moreover, this structure is in the class $\mathcal{F}_4 \oplus \mathcal{F}_5$. In other words, satisfies the relation:

$$\begin{aligned}
\alpha(U, V, W) =& \frac{\theta(\xi)}{2n}\{\eta(V)\rho(\phi U, \phi W) + \eta(W)\rho(\phi U, \phi V)\}\\
&- \frac{\theta^*(\xi)}{2n}\{\eta(V)\rho(\phi U, W) + \eta(W)\rho(\phi U, V)\}. \qquad (5.1)
\end{aligned}$$

Without any need to (5.1) and long calculations, we can state that the almost complex Norden metric structure on $S^{2n+1} \times \mathbb{R}$ constructed with (3.1) and (3.2) is of the class $\mathcal{W}_1 \oplus \mathcal{W}_2$ by the Theorem 4.1 (4) and (5).

Example 3. Let M be the hyper-surface of the almost complex Norden metric manifold $(\mathbb{R}^{2n+2}, J, \rho)$ given in the above example with

$$M: \ \rho(W, JW) = 0; \ \rho(W, W) = ch^2 t, \ t > 0.$$

Take the vector field $N = \frac{1}{cht} JW$, that obviously holds $\rho(N, N) - 1$. Choose $\xi = -JN = \frac{1}{cht} W$, then for any $U \in T_p M$, we haver the unique decomposition

$$JU = \phi U + \eta(U)\xi,$$

where ϕU is the orthogonal projection of JU and η is a 1-form. In [5], it is shown with details that, (ϕ, ξ, η, ρ) is an almost contact Norden metric structure on M. Moreover, this structure is of the class $\mathcal{F}_5$. So, by the Theorem 4.1 (5), the induced almost complex B- metric structure on the manifold $M \times \mathbb{R}$ by means of the Eqs (3.1) and (3.2) is in the class $\mathcal{W}_1 \oplus \mathcal{W}_2$.

Example 4. In [11], S. Ivanov et al. give the defining relation of an almost contact Norden metric manifold $(M, \phi, \xi, \eta, \rho)$ to be Sasaki-like as:

$$(\nabla_U \phi)(V) = -\rho(U, V)\xi - \eta(V)U + 2\eta(U)\eta(V)\xi, \tag{5.2}$$

or equivalently,

$$(\nabla_U \phi)(V) = \rho(\phi U, \phi V)\xi + \eta(V)\phi^2 U. \tag{5.3}$$

Also, they considered the Lie group G of dimension five with the basis of left-invariant vector fields $\{x_0, x_1, x_2, x_3, x_4\}$ with the commutators

$$\begin{aligned}
[x_0, x_1] =& kx_2 + x_3 + lx_4, \quad [x_0, x_2] = -kx_1 - lx_3 + x_4, \\
[x_0, x_3] =& -x_1 - lx_2 + kx_4, \quad [x_0, x_4] = lx_1 - x_2 - kx_3, \ \ k, l \in \mathbb{R},
\end{aligned}$$

and show that the almost contact Norden metric structure (ϕ, ξ, η, ρ) given by

$$\begin{aligned}
&\xi = x_0, \ \phi x_1 = x_3, \ \phi x_2 = x_4, \ \phi x_3 = -x_1, \ \phi x_4 = -x_2 \\
&\rho(x_0, x_0) = \rho(x_1, x_1) = \rho(x_2, x_2) = -\rho(x_3, x_3) = -\rho(x_4, x_4) = 1 \\
&\rho(x_i, x_j) = 0, (i \neq j)
\end{aligned}$$

is Sasaki-like. It is known that the class of Sasaki-like structures is a subclass of the basic class $\mathcal{F}_4$ (with $\theta = 2n\eta$ and $\theta^* = \omega = 0$) [12]. Hence the Theorem 4.1 (4) enables us to state that the almost complex Norden metric manifold $(G \times \mathbb{R}, J, h)$, obtained by (3.1) and (3.2) is of the class $\mathcal{W}_1 \oplus \mathcal{W}_2$. Since structure on G is of the subclass of $\mathcal{F}_4$, one may question if the induced structure on $G \times \mathbb{R}$ is either in the class $\mathcal{W}_1$, or $\mathcal{W}_2$. The answer is "no". Indeed, from the Eq (5.3), we get

$$\alpha(U, V, W) = \eta(W)\rho(\phi U, \phi V) + \eta(V)\rho(\phi U, \phi W), \tag{5.4}$$

and by the Eq (3.5), we obtain

$$F(\tilde{U}, \tilde{V}, \tilde{W}) = \alpha(U, V, W), \tag{5.5}$$

since $\alpha(U, \xi, \phi V) = \alpha(U, \xi, \phi W) = 0$.

Assume that $G \times \mathbb{R}$ is in $\mathcal{W}_1$. So, the following equation holds:

$$F(\tilde{U}, \tilde{V}, \tilde{W}) = \frac{1}{n+1}\{h(\tilde{U}, \tilde{V})\varphi(\tilde{W}) + h(\tilde{U}, \tilde{W})\varphi(\tilde{V}) + h(\tilde{U}, J\tilde{V})\varphi(J\tilde{W}) + h(\tilde{U}, J\tilde{W})\varphi(J\tilde{V})\}.$$

By setting $\tilde{U} = (\xi, 0\frac{d}{dt})$, $\tilde{V} = (\xi, b\frac{d}{dt})$, $\tilde{W} = (\xi, c\frac{d}{dt})$ with $bc \neq 1$ in this equation, as the left-hand side vanishes, the right-hand side becomes $n(bc-1)$. So, $G \times \mathbb{R}$ can not be in $\mathcal{W}_1$.

If we choose $\tilde{U} = (\xi, 0\frac{d}{dt})$ in the Eq (3.7), we get $\varphi(\tilde{U}) = \theta(\xi)$ that is non-zero since the structure is Sasaki-like. Thus, $G \times \mathbb{R}$ is not in $\mathcal{W}_2$.

Example 5. Let L be a real connected Lie group of dimension five and $\mathfrak{g}$ be the associated Lie algebra equipped with a global basis $\{x_1, ..., x_5\}$ of left-invariant vector fields, satisfying

$$[x_5, x_i] = -\lambda_i x_i - \lambda_{i+2} x_{i+2},\ [x_5, x_{i+2}] = -\lambda_{i+2} x_i + \lambda_i x_{i+2},$$

where λ_i's are real constants, $i = 1, 2$ and $[x_j, x_k] = 0$ in other cases.
It is shown that the quadruple (ϕ, ξ, η, ρ) given with

$$\phi x_1 = x_3,\ \phi x_2 = x_4,\ \phi x_3 = -x_1,\ \phi x_4 = -x_2,\ \phi x_5 = 0;\ \xi = x_5,$$

$$\eta(x_i) = 0,\ i = 1, 2, 3, 4;\ \eta(x_5) = 1,$$

$$\rho(x_1, x_1) = \rho(x_2, x_2) = \rho(x_5, x_5) = 1 = -\rho(x_3, x_3) = -\rho(x_4, x_4),\ \rho(x_i, x_j) = 0,$$

for $i \neq j$, is an almost contact Norden metric structure on L and moreover this structure in the class $\mathcal{F}_9 \oplus \mathcal{F}_{10}$ [13]. So, by the Theorem 4.1 (9) and (10), we are able to state that the induced almost complex Norden metric structure on the Lie group $\tilde{L} = L \times \mathbb{R}$ can only be in the class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ or $\mathcal{W}_2 \oplus \mathcal{W}_3$.

6. Conclusions

In this paper, we discuss the construction of almost complex Norden metric structure on $M \times \mathbb{R}$, where M is an almost contact Norden metric manifold. After analysing the relations of the basic classes, we state several examples by means of the results of the paper.

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Conflict of interest

Author states no conflict of interest.

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