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Research article

Fixed point results of a new family of hybrid contractions in generalised metric space with applications

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Abstract: In this manuscript, a novel general family of contraction, called hybrid-interpolative Reich-Istrăţescu-type (G-α-μ)-contraction is introduced and some fixed point results in generalised metric space that are not deducible from their akin in metric spaces are obtained. The preeminence of this class of contraction is that its contractive inequality can be extended in a variety of manners, depending on the given parameters. Consequently, a number of corollaries that reduce our result to other well-known results in the literature are highlighted and analysed. Substantial examples are constructed to validate the assumptions of our obtained theorems and to show their distinction from corresponding results. Additionally, one of our obtained corollaries is applied to set up unprecedented existence conditions for solution of a family of integral equations.

Keywords: G-metric; fixed point; hybrid contraction; integral equation
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1. Introduction

The prominent Banach contraction in metric spaces has laid a solid foundation for fixed point theory in metric space. The basic idea of the contraction mapping principle has been fine-tuned in several domains (see, e.g. [12,13,15]). The applications of fixed point range across inequalities, approximation theory, optimization and so on. Researchers in this area have introduced several new concepts in metric space and obtained a great deal of fixed point results for linear and nonlinear contractions. Recently, Karapınar et al. [7] introduced a new notion of hybrid contraction which is a unification of some existing linear and nonlinear contractions in metric space.

On the other hand, Mustafa [8] pioneered an extension of metric space by the name, generalized metric space (or more precisely, G-metric space) and proved some fixed point results for Banach-type contraction mappings. This new generalization was brought to spotlight by Mustafa and Sims [9]. Subsequently, Mustafa et al. [11] obtained some engrossing fixed point results for Lipschitzian-type mappings on G-metric space. However, Jleli and Samet [5], as well as Samet et al. [16] noted that most of the fixed point results in G-metric space are direct consequences of existence results in corresponding metric space. Jleli and Samet [5] further observed that if a G-metric is consolidated into a quasi-metric, then the resultant fixed point results become the known fixed point results in the setting of quasi-metric space. Motivated by the latter observation, many investigators (see for instance, [3, 6]) have established techniques of obtaining fixed point results in symmetric G-metric space that are not deducible from their ditto ones in metric space or quasi-metric space.

Following the existing literature, we realize that hybrid fixed point results in G-metric space are not adequately investigated. Hence, motivated by the ideas in [3, 6, 7], we introduce a new concept of hybrid-interpolative Reich-Istrăţescu-type (G-α-μ)-contraction in G-metric space and prove some related fixed point theorems. An example is constructed to demonstrate that our result is valid, an improvement of existing result and the main ideas obtained herein do not reduce to any existence result in metric space. A corollary is presented to show that the concept proposed herein is a generalization and improvement of well-known fixed point result in metric space. Finally, one of our obtained corollaries is applied to establish novel existence conditions for solution of a class of integral equations.

2. Preliminaries

In this section, we will present some fundamental notations and results that will be deployed subsequently.

All through, every set Φ is considered non-empty, $\mathbb{N}$ is the set of natural numbers, $\mathbb{R}$ represents the set of real numbers and $\mathbb{R}_+$, the set of non-negative real numbers.

Definition 2.1. [9] Let Φ be a non-empty set and let $G : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be a function satisfying:

(G_1) $G(r, s, t) = 0$ if $r = s = t$;
(G_2) $0 < G(x, r, s)$ for all $r, s \in \Phi$ with $r \neq s$;
(G_3) $G(r, r, s) \leq G(r, s, t)$, for all $r, s, t \in \Phi$ with $t \neq s$;
(G_4) $G(r, s, t) = G(r, t, s) = G(s, r, t) = ...$ (symmetry in all variables);
(G_5) $G(r, s, t) \leq G(r, u, u) + G(u, s, t)$, for all $r, s, t, u \in \Phi$ (rectangular inequality).

Then the function G is called a generalised metric, or more precisely, a G-metric on Φ, and the pair (Φ, G) is called a G-metric space.

Example 2.2. [11] Let (Φ, d) be a usual metric space, then (Φ, G_p) and (Φ, G_m) are G-metric space, where

$$G_p(r, s, t) = d(r, s) + d(s, t) + d(r, t) \quad \forall r, s, t \in \Phi. \tag{2.1}$$

$$G_m(r,s,t) = max\{d(r,s), d(s,t), d(r,t)\} \quad \forall r,s,t \in \Phi. \tag{2.2}$$

Definition 2.3. [11] Let (Φ, G) be a G-metric space and let $\{r_n\}$ be a sequence of points of Φ. Then $\{r_n\}$ is said to be G-convergent to r if $\lim_{n,m\to\infty} G(r, r_n, r_m) = 0$; that is, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(r, r_n, r_m) < \epsilon$, $\forall n, m \geq n_0$. We refer to r as the limit of the sequence $\{r_n\}$.

Proposition 1. *[11] Let (Φ, G) be a G-metric space. Then the following are equivalent:*

(i) $\{r_n\}$ *is G-convergent to r.*
(ii) $G(r, r_n, r_m) \longrightarrow 0$*, as* $n, m \to \infty$.
(iii) $G(r_n, r, r) \longrightarrow 0$*, as* $n \to \infty$.
(iv) $G(r_n, r_n, r) \longrightarrow 0$*, as* $n \to \infty$.

Definition 2.4. [11] Let (Φ, G) be a G-metric space. A sequence $\{r_n\}$ is called G-Cauchy if for any $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that $G(r_n, r_m, r_l) < \epsilon$, $\forall n, m, l \geq n_0$, that is, $G(r_n, r_m, r_l) \longrightarrow 0$, as $n, m, l \to \infty$.

Proposition 2. *[11] If (Φ, G) is a G-metric space, the following statements are equivalent:*

(i) The sequence $\{r_n\}$ *is G-Cauchy.*
(ii) For every $\epsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that* $G(r_n, r_m, r_m) < \epsilon$*,* $\forall n, m \geq n_0$.

Definition 2.5. [11] Let (Φ, G) and (Φ', G') be two G-metric spaces and $f : (\Phi, G) \longrightarrow (\Phi', G')$ be a function. Then f is G-continuous at a point $u \in \Phi$ if and only if for any $\epsilon > 0$, there exists $\delta > 0$ such that $r, s \in \Phi$; and $G(u, r, s) < \delta$ implies $G'(f(u), f(r), f(s)) < \epsilon$. A function f is G-continuous on Φ if and only if it is G-continuous at all $u \in \Phi$.

Proposition 3. *[11] Let (Φ, G) and (Φ', G') be two G-metric spaces. Then a function $f : (\Phi, G) \longrightarrow (\Phi', G')$ is said to be G-continuous at a point $r \in \Phi$ if and only if it is G-sequentially continuous at r, that is, whenever $\{r_n\}$ is G-convergent to r, $\{fr_n\}$ is G-convergent to fr.*

Definition 2.6. [11] A G-metric space (Φ, G) is called symmetric G-metric space if

$$G(r, r, s) = G(s, r, r) \quad \forall r, s \in \Phi.$$

Proposition 4. *[11] Let (Φ, G) be a G-metric space. Then the function $G(r, s, t)$ is jointly continuous in all variables.*

Proposition 5. *[11] Every G-metric space (Φ, G) defines a metric space (Φ, d_G) by*

$$d_G(r, s) = G(r, s, s) + G(s, r, r) \quad \forall r, s \in \Phi. \tag{2.3}$$

Note that for a symmetric G-metric space (Φ, G),

$$(\Phi, d_G) = 2G(r, s, s) \quad \forall r, s \in \Phi. \tag{2.4}$$

On the other hand, if (Φ, G) is not symmetric, then by the G-metric properties,

$$\frac{3}{2}G(r, s, s) \leq d_G(r, s) \leq 3G(r, s, s) \quad \forall r, s \in \Phi, \tag{2.5}$$

and that in general, these inequalities are sharp.

Definition 2.7. [11] A G-metric space (Φ, G) is referred to as G-complete (or complete G-metric) if every G-Cauchy sequence in (Φ, G) is G-convergent in (Φ, G).

Proposition 6. *[11] A G-metric space (Φ, G) is G-complete if and only if (Φ, d_G) is a complete metric space.*

Popescu [14] gave the following definition in the setting of metric space.

Definition 2.8. [14] Let $\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be a function. A self-mapping $\Gamma : \Phi \longrightarrow \Phi$ is referred to as α-orbital admissible if for all $r \in \Phi$,

$$\alpha(r, \Gamma r) \geq 1 \Rightarrow \alpha(\Gamma r, \Gamma^2 r) \geq 1.$$

We modify the above definitions in the framework of G-metric space as follows:

Definition 2.9. Let $\alpha : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be a function. A self-mapping $\Gamma : \Phi \longrightarrow \Phi$ is called $(G\text{-}\alpha)$-orbital admissible if for all $r \in \Phi$,

$$\alpha(r, \Gamma r, \Gamma^2 r) \geq 1 \text{ implies } \alpha(\Gamma r, \Gamma^2 r, \Gamma^3 r) \geq 1.$$

Definition 2.10. [2] Let $\alpha : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be a mapping. The set Φ is called regular with respect to α if and only if for a sequence $\{r_n\}$ in Φ such that $\alpha(r_n, r_{n+1}, r_{n+2}) \geq 1$, for all n and $r_n \to r \in \Phi$ as $n \to \infty$, we have $\alpha(r_n, r, r) \geq 1$ for all n.

The following Propositions 7 and 8 were studied in [7], where the constant $C \in \mathbb{R}_+$. However, we noticed that the arguments in their proofs are only valid if we restrict $\mathbb{R}_+$ to $[0, 1)$. Hence, we reexamine them in the latter interval.

Proposition 7. *Given $C \in [0, 1)$, let $\{\rho_n\} \subset \mathbb{R}_+$ be a sequence such that*

$$\rho_{n+2} \leq C \max\{\rho_n, \rho_{n+1}\} \quad \forall n \in \mathbb{N}. \tag{2.6}$$

Let $K = \max\{\rho_0, \rho_1\}$. Then

$$\rho_{2n} \leq C^n K, \quad \rho_{2n+1} \leq C^n K \quad \forall n \geq 1. \tag{2.7}$$

Proof. The proof is by induction.
For $n = 0$, we have from (2.6) that

$$\rho_2 \leq C \max\{\rho_0, \rho_1\} = CK.$$

For $n = 1$, (2.6) becomes

$$\begin{aligned} \rho_3 \leq C \max\{\rho_1, \rho_2\} &\leq C \max\{\rho_1, C \max\{\rho_0, \rho_1\}\} \\ &\leq C \max\{\rho_1, CK\} \leq CK. \end{aligned}$$

Suppose that (2.7) holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned} \rho_{2n+2} &\leq C \max\{\rho_{2n}, \rho_{2n+1}\} \leq C \max\{C^n K, C^n K\} = C^{n+1} K; \\ \rho_{2n+3} &\leq C \max\{\rho_{2n+1}, \rho_{2n+2}\} \leq C \max\{C^n K, C^{n+1} K\} = C^{n+1} K. \end{aligned}$$

This completes the induction. Hence, the proof. □

Lemma 2.11. *Let $\{r_n\}$ be a sequence in a G-metric space (Φ, G). Suppose that there exists $C \in [0, 1)$ such that*

$$G(r_{n+2}, r_{n+3}, r_{n+4}) \leq C \max\{G(r_n, r_{n+1}, r_{n+2}), G(r_{n+1}, r_{n+2}, r_{n+3})\} \quad \forall n \in \mathbb{N}.$$

Then $\{r_n\}$ is a G-Cauchy sequence in (Φ, G).

Proof. Consider the sequence $\{\rho_n\}$ in Φ defined by

$$\rho_n = G(r_n, r_{n+1}, r_{n+2}) \quad \forall n \in \mathbb{N}.$$

Then by the hypothesis, $\{\rho_n\}$ satisfies (2.6). Hence, by Proposition 7, we obtain

$$G(r_{2n}, r_{2n+1}, r_{2n+2}) = \rho_{2n} \leq C^n K;$$
$$G(r_{2n+1}, r_{2n+2}, r_{2n+3}) = \rho_{2n+1} \leq C^n K,$$

for all $n \in \mathbb{N}$, where $K = \max\{\rho_0, \rho_1\}$. In particular,

$$G(r_{2n}, r_{2n+1}, r_{2n+2}) + G(r_{2n+1}, r_{2n+2}, r_{2n+3}) \leq 2C^n K \quad \forall n \in \mathbb{N}. \tag{2.8}$$

If $C = 0$ or $K = 0$, then $\{r_n\}_{n\geq 2}$ is constant, hence G-Cauchy. Assume that $C > 0$ and $K > 0$. To see that $\{r_n\}_{n\in\mathbb{N}}$ is G-Cauchy in (Φ, G), take arbitrary $\epsilon > 0$. Since $\frac{\epsilon}{2K} > 0$ and $0 < C < 1$, then we can find $n_0 \in \mathbb{N}$ such that

$$\sum_{i=n_0}^{\infty} C^i < \frac{\epsilon}{2K}.$$

In particular,

$$2K \sum_{i=n_0}^{p} C^i < 2K \sum_{i=n_0}^{\infty} C^i < \epsilon \quad \forall p \in \mathbb{N}; \quad p \geq n_0.$$

Let $l, m, n \in \mathbb{N}$, where $2n_0 \leq n < m < l$. Let $p \in \mathbb{N}$ be such that $p \geq n_0 + 2$ and $2p \geq l$. Then by rectangular inequality and (2.8), we have

$$\begin{aligned} G(r_n, r_m, r_l) &\leq \sum_{j=n}^{l-2} G(r_j, r_{j+1}, r_{j+2}) \leq \sum_{j=2n_0}^{2p-2} G(r_j, r_{j+1}, r_{j+2}) \\ &= \sum_{q=n_0}^{p-2} \left[G(r_{2q}, r_{2q+1}, r_{2q+2}) + G(r_{2q+1}, r_{2q+2}, r_{2q+3})\right] \\ &\leq \sum_{q=n_0}^{p-2} 2C^q K = 2K \sum_{q=n_0}^{p} C^q < 2K \sum_{q=n_0}^{\infty} C^q < \epsilon. \end{aligned}$$

This completes the proof. □

Proposition 8. *Given* $C \in [0,1)$ *and* $\sigma \in (0,1)$, *let* $\{\rho_n\} \subset (0,1)$ *be a sequence such that*

$$\rho_{n+2} \leq C \max\{\rho_n, \rho_{n+1}\}^{\sigma} \quad \forall n \in \mathbb{N}. \tag{2.9}$$

Let $K = \max\{\rho_0, \rho_1\}$. *Then*

$$\rho_{2n} \leq C^{1+\sigma+\sigma^2+...+\sigma^{n-1}} K^{\sigma^n}, \quad \rho_{2n+1} \leq C^{1+\sigma+\sigma^2+...+\sigma^{n-1}} K^{\sigma^n} \quad \forall n \geq 1. \tag{2.10}$$

Therefore,

$$\limsup_{n\to\infty} \rho_n \leq C^{\frac{1}{1-\sigma}}.$$

Proof. The result holds if $C = 0$. Assume $C > 0$. Since $\sigma \in (0,1)$, then obviously,

$$\sigma^{m+1} < \sigma^m < \sigma^{m-1} < ... < \sigma^2 < \sigma \quad \forall m \in \mathbb{N}; \quad m \geq 1.$$

Therefore, since $K \in (0,1)$, we have

$$K^{\sigma^{m+1}} < K^{\sigma^m} < K^{\sigma^{m-1}} < ... < K^{\sigma^2} < K^{\sigma} \quad \forall m \in \mathbb{N}; \quad m \geq 1.$$

Consider (2.9) and let $n = 0$. Then

$$\rho_2 \leq C \max\{\rho_0, \rho_1\}^{\sigma} \leq CK^{\sigma}.$$

For $n = 1$, we obtain

$$\begin{aligned}
\rho_3 &\leq C \max\{\rho_1, \rho_2\}^{\sigma} \leq C \max\{\rho_1, CK^{\sigma}\}^{\sigma} \leq C \max\{K, CK^{\sigma}\}^{\sigma} \\
&= C \max\{K^{\sigma}, C^{\sigma} K^{\sigma^2}\} \leq C \max\{K^{\sigma}, C^{\sigma} K^{\sigma}\} \\
&= CK^{\sigma} \max\{1, C^{\sigma}\} = CK^{\sigma}.
\end{aligned}$$

Now, assume that (2.10) holds for some $n \in \mathbb{N}$. We then prove that it holds for $n+1$. Therefore,

$$\begin{aligned}
\rho_{2n+2} &\leq C \max\{\rho_{2n}, \rho_{2n+1}\}^{\sigma} \\
&\leq C \max\{C^{1+\sigma+\sigma^2+...+\sigma^{n-1}} K^{\sigma^n}, C^{1+\sigma+\sigma^2+...+\sigma^{n-1}} K^{\sigma^n}\}^{\sigma} \\
&= C(C^{1+\sigma+\sigma^2+...+\sigma^{n-1}} K^{\sigma^n})^{\sigma} = C(C^{\sigma+\sigma^2+...+\sigma^n} K^{\sigma^{n+1}}) \\
&= C^{1+\sigma+\sigma^2+...+\sigma^n} K^{\sigma^{n+1}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\rho_{2n+3} &\leq C \max\{\rho_{2n+1}, \rho_{2n+2}\}^{\sigma} \\
&\leq C \max\{C^{1+\sigma+\sigma^2+...+\sigma^{n-1}} K^{\sigma^n}, C^{1+\sigma+\sigma^2+...+\sigma^n} K^{\sigma^{n+1}}\}^{\sigma} \\
&= C \max\{C^{\sigma+\sigma^2+...+\sigma^n} K^{\sigma^{n+1}}, C^{\sigma+\sigma^2+...+\sigma^{n+1}} K^{\sigma^{n+2}}\} \\
&\leq C \max\{C^{\sigma+\sigma^2+...+\sigma^n} K^{\sigma^{n+1}}, C^{\sigma+\sigma^2+...+\sigma^{n+1}} K^{\sigma^{n+1}}\} \\
&= CK^{\sigma^{n+1}} \max\{C^{\sigma+\sigma^2+...+\sigma^n}, C^{\sigma+\sigma^2+...+\sigma^{n+1}}\}
\end{aligned}$$

$$= CK^{\sigma^{n+1}}(C^{\sigma+\sigma^2+...+\sigma^n}) = C^{1+\sigma+\sigma^2+...+\sigma^n}K^{\sigma^{n+1}}.$$

This completes the induction, therefore, verifying (2.10). Noting that $\{\sigma^n\}_{n\in\mathbb{N}} \to 0$ as $n \to \infty$, it is obvious that $\{K^{\sigma^n}\}_{n\in\mathbb{N}} \to K^0 = 1$. Also,

$$\lim_{n\to\infty}(1 + \sigma + \sigma^2 + ... + \sigma^n) \leq \sum_{i=0}^{\infty}\sigma^i = \frac{1}{1-\sigma}.$$

Therefore,

$$\lim_{n\to\infty} C^{1+\sigma+\sigma^2+...+\sigma^n} = C^{(\frac{1}{1-\sigma})}.$$

□

Corollary 1. *Let $\{r_n\}$ be a sequence in a G-metric space (Φ, G). Assume that there exist $C \in [0, 1)$ and $\theta, \eta \in [0, 1]$ with $\theta + \eta = 1$ such that*

$$G(r_{n+2}, r_{n+3}, r_{n+4}) \leq C\left[G(r_n, r_{n+1}, r_{n+2})^{\theta} \cdot G(r_{n+1}, r_{n+2}, r_{n+3})^{\eta}\right] \quad \forall n \in \mathbb{N}.$$

Then $\{r_n\}$ is a G-Cauchy sequence in (Φ, G).

Proof. Notice that

$$\begin{aligned}
G(r_{n+2}, r_{n+3}, r_{n+4}) &\leq C\left[G(r_n, r_{n+1}, r_{n+2})^{\theta} \cdot G(r_{n+1}, r_{n+2}, r_{n+3})^{\eta}\right] \\
&\leq C[\max\{G(r_n, r_{n+1}, r_{n+2}), G(r_{n+1}, r_{n+2}, r_{n+3})\}^{\theta} \\
&\quad \cdot \max\{G(r_n, r_{n+1}, r_{n+2}), G(r_{n+1}, r_{n+2}, r_{n+3})\}^{\eta}] \\
&= C\max\{G(r_n, r_{n+1}, r_{n+2}), G(r_{n+1}, r_{n+2}, r_{n+3})\}^{\theta+\eta} \\
&= C\max\{G(r_n, r_{n+1}, r_{n+2}), G(r_{n+1}, r_{n+2}, r_{n+3})\}.
\end{aligned}$$

Then the result follows from Lemma 2.11. □

Karapınar et al. [7] gave the following definition of hybrid-interpolative Reich-Istrăţescu-type contraction in metric space.

Definition 2.12. [7] Let (Φ, d) be a metric space and let $\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be a function. A self-mapping $\Gamma : \Phi \longrightarrow \Phi$ is called hybrid-interpolative Reich-Istrăţescu-type contraction if for some $q \in \mathbb{R}_+$, there exist constants $\mu \in (0, 1)$, $\delta \geq 0$ and $\lambda_i \geq 0$ with $i = 1, 2, ..., 5$ such that for all distinct $r, s \in \Phi \backslash Fix(\Gamma)$,

$$\alpha(r, s)d(\Gamma^2 r, \Gamma^2 s) \leq \mu M(r, s), \tag{2.11}$$

where

$$M(r,s) = \begin{cases} \Big[\lambda_1 d(r,s)^q + \lambda_2 d(r,\Gamma r)^q + \lambda_3 d(s,\Gamma s)^q + \lambda_4 d(\Gamma r,\Gamma s)^q + \lambda_5 d(\Gamma r,\Gamma^2 r)^q \\ +\delta d(\Gamma s,\Gamma^2 s)^q\Big]^{\frac{1}{q}}, \qquad for \quad q>0, \quad with \quad \sum_{i=1}^5 \lambda_i + \delta \leq 1; \\ \\ [d(r,s)]^{\lambda_1}\cdot[d(r,\Gamma r)]^{\lambda_2}\cdot[d(s,\Gamma s)]^{\lambda_3}\cdot[d(\Gamma r,\Gamma s)]^{\lambda_4}\cdot[d(\Gamma r,\Gamma^2 r)]^{\lambda_5} \\ \cdot[d(\Gamma s,\Gamma^2 s)]^{\delta}, \qquad for \quad q=0, \quad with \quad \sum_{i=1}^5 \lambda_i + \delta = 1, \end{cases} \tag{2.12}$$

and $Fix(\Gamma) = \{r \in \Phi : \Gamma r = r\}$.

3. Main results

In this section, we introduce a new concept of hybrid-interpolative Reich-Istrăţescu-type (G-α-μ)-contraction in G-metric space.

Definition 3.1. Let (Φ, G) be a G-metric space and let $\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be a function. A self-mapping $\Gamma : \Phi \longrightarrow \Phi$ is called hybrid-interpolative Reich-Istrăţescu-type (G-α-μ)-contraction if for some $q \in \mathbb{R}_+$, there exist constants $\mu \in (0,1)$, $\delta \geq 0$ and $\lambda_i \geq 0$ with $i = 1, 2, ..., 5$ such that for all $r, s \in \Phi \backslash Fix(\Gamma)$,

$$\alpha(r, s, \Gamma s) G(\Gamma^2 r, \Gamma^2 s, \Gamma^3 s) \leq \mu M(r, s, \Gamma s), \tag{3.1}$$

where

$$M(r,s,\Gamma s) = \begin{cases} \Big[\lambda_1 G(r,s,\Gamma s)^q + \lambda_2 G(r,\Gamma r,\Gamma^2 r)^q + \lambda_3 G(s,\Gamma s,\Gamma^2 s)^q \\ +\lambda_4 G(\Gamma r,\Gamma s,\Gamma^2 s)^q + \lambda_5 G(\Gamma r,\Gamma^2 r,\Gamma^3 r)^q + \delta G(\Gamma s,\Gamma^2 s,\Gamma^3 s)^q\Big]^{\frac{1}{q}}, \\ \qquad\qquad for \quad q > 0, \quad with \quad \sum_{i=1}^5 \lambda_i + \delta \leq 1; \\ \\ [G(r,s,\Gamma s)]^{\lambda_1} \cdot [G(r,\Gamma r,\Gamma^2 r)]^{\lambda_2} \cdot [G(s,\Gamma s,\Gamma^2 s)]^{\lambda_3} \cdot [G(\Gamma r,\Gamma s,\Gamma^2 s)]^{\lambda_4} \\ \cdot [G(\Gamma r,\Gamma^2 r,\Gamma^3 r)]^{\lambda_5} \cdot [G(\Gamma s,\Gamma^2 s,\Gamma^3 s)]^{\delta}, \\ \qquad\qquad for \quad q = 0, \quad with \quad \sum_{i=1}^5 \lambda_i + \delta = 1, \end{cases} \tag{3.2}$$

and $Fix(\Gamma) = \{r \in \Phi : \Gamma r = r\}$.

Our main result is the following.

Theorem 3.2. *Let (Φ, G) be a complete G-metric space and let $\Gamma : \Phi \longrightarrow \Phi$ be a hybrid-interpolative Reich-Istrăţescu-type (G-α-μ)-contraction satisfying the following conditions:*

(i) Γ *is* (G-α)*-orbital admissible;*
(ii) Γ *is continuous;*
(iii) there exists $r_0 \in \Phi$ *such that* $\alpha(r_0, \Gamma r_0, \Gamma^2 r_0) \geq 1$.

Then Γ *has at least a fixed point in* Φ.

Proof. Let $r_0 \in \Phi$ be such that $\alpha(r_0, \Gamma r_0, \Gamma^2 r_0) \geq 1$. Since Γ is (G-α)-orbital admissible, then $\alpha(\Gamma r_0, \Gamma^2 r_0, \Gamma^3 r_0) \geq 1$ and by induction, we have $\alpha(\Gamma^n r_0, \Gamma^{n+1} r_0, \Gamma^{n+2} r_0) \geq 1$ for any $n \in \mathbb{N}$. Let $\{r_n\}$ be a sequence in Φ defined by $r_n = \Gamma^n r_0$ for all $n \in \mathbb{N}$. If there exists some $m \in \mathbb{N}$ such that $\Gamma r_m = r_{m+1} = r_m$, then clearly, r_m is a fixed point of Γ. Assume now that $r_n \neq r_{n+1}$ for any $n \in \mathbb{N}$. Since Γ is hybrid-interpolative Reich-Istrăţescu-type (G-α-μ)-contraction, then we have from (3.1) that

$$\begin{aligned} G(r_{n+2}, r_{n+3}, r_{n+4}) &\leq \alpha(r_n, r_{n+1}, \Gamma r_{n+1}) G(\Gamma^2 r_n, \Gamma^2 r_{n+1}, \Gamma^3 r_{n+1}) \\ &\leq \mu M(r_n, r_{n+1}, \Gamma r_{n+1}). \end{aligned} \tag{3.3}$$

We now consider the following cases:

Case 1: For $q > 0$, we have

$$M(r_n, r_{n+1}, \Gamma r_{n+1}) = [\lambda_1 G(r_n, r_{n+1}, \Gamma r_{n+1})^q + \lambda_2 G(r_n, \Gamma r_n, \Gamma^2 r_n)^q$$

$$
\begin{aligned}
&+ \lambda_3 G(r_{n+1}, \Gamma r_{n+1}, \Gamma^2 r_{n+1})^q + \lambda_4 G(\Gamma r_n, \Gamma r_{n+1}, \Gamma^2 r_{n+1})^q \\
&+ \lambda_5 G(\Gamma r_n, \Gamma^2 r_n, \Gamma^3 r_n)^q + \delta G(\Gamma r_{n+1}, \Gamma^2 r_{n+1}, \Gamma^3 r_{n+1})^q]^{\frac{1}{q}} \\
&= [\lambda_1 G(r_n, r_{n+1}, r_{n+2})^q + \lambda_2 G(r_n, r_{n+1}, r_{n+2})^q + \lambda_3 G(r_{n+1}, r_{n+2}, r_{n+3})^q \\
&+ \lambda_4 G(r_{n+1}, r_{n+2}, r_{n+3})^q + \lambda_5 G(r_{n+1}, r_{n+2}, r_{n+3})^q \\
&+ \delta G(r_{n+2}, r_{n+3}, r_{n+4})^q]^{\frac{1}{q}} \\
&= [(\lambda_1 + \lambda_2) G(r_n, r_{n+1}, r_{n+2})^q + (\lambda_3 + \lambda_4 + \lambda_5) G(r_{n+1}, r_{n+2}, r_{n+3})^q \\
&+ \delta G(r_{n+2}, r_{n+3}, r_{n+4})^q]^{\frac{1}{q}} \\
&\leq [(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) \max\{G(r_n, r_{n+1}, r_{n+2})^q, G(r_{n+1}, r_{n+2}, r_{n+3})^q\} \\
&+ \delta G(r_{n+2}, r_{n+3}, r_{n+4})^q]^{\frac{1}{q}}.
\end{aligned} \tag{3.4}
$$

Therefore, (3.3) becomes

$$
\begin{aligned}
G(r_{n+2}, r_{n+3}, r_{n+4})^q \leq \mu^q [&(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) \\
&\max\{G(r_n, r_{n+1}, r_{n+2})^q, G(r_{n+1}, r_{n+2}, r_{n+3})^q\} \\
&+ \delta G(r_{n+2}, r_{n+3}, r_{n+4})^q]
\end{aligned}
$$

so that

$$
\begin{aligned}
(1 - \mu^q \delta) G(r_{n+2}, r_{n+3}, r_{n+4})^q &\leq \mu^q (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) \\
&\quad \max\{G(r_n, r_{n+1}, r_{n+2})^q, G(r_{n+1}, r_{n+2}, r_{n+3})^q\} \\
&\leq \mu^q (1 - \delta) \max\{G(r_n, r_{n+1}, r_{n+2}), G(r_{n+1}, r_{n+2}, r_{n+3})\}^q,
\end{aligned}
$$

implying that for all $n \in \mathbb{N}$,

$$
G(r_{n+2}, r_{n+3}, r_{n+4})^q \leq \left(\frac{\mu^q (1 - \delta)}{1 - \mu^q \delta} \right) \max\{G(r_n, r_{n+1}, r_{n+2}), G(r_{n+1}, r_{n+2}, r_{n+3})\}^q,
$$

which is equivalent to

$$
G(r_{n+2}, r_{n+3}, r_{n+4}) \leq C \max\{G(r_n, r_{n+1}, r_{n+2}), G(r_{n+1}, r_{n+2}, r_{n+3})\},
$$

where

$$
C = \left(\frac{\mu^q (1 - \delta)}{1 - \mu^q \delta} \right) \in (0, 1).
$$

Hence, by Lemma 2.11, $\{r_n\}_{n \in \mathbb{N}}$ is a G-Cauchy sequence in (Φ, G).
Case 2: For $q = 0$, we have

$$
\begin{aligned}
M(r_n, r_{n+1}, \Gamma r_{n+1}) &= G(r_n, r_{n+1}, \Gamma r_{n+1})^{\lambda_1} \cdot G(r_n, \Gamma r_n, \Gamma^2 r_n)^{\lambda_2} \\
&\quad \cdot G(r_{n+1}, \Gamma r_{n+1}, \Gamma^2 r_{n+1})^{\lambda_3} \cdot G(\Gamma r_n, \Gamma r_{n+1}, \Gamma^2 r_{n+1})^{\lambda_4} \\
&\quad \cdot G(\Gamma r_n, \Gamma^2 r_n, \Gamma^3 r_n)^{\lambda_5} \cdot G(\Gamma r_{n+1}, \Gamma^2 r_{n+1}, \Gamma^3 r_{n+1})^{\delta} \\
&= G(r_n, r_{n+1}, r_{n+2})^{\lambda_1} \cdot G(r_n, r_{n+1}, r_{n+2})^{\lambda_2} \\
&\quad \cdot G(r_{n+1}, r_{n+2}, r_{n+3})^{\lambda_3} \cdot G(r_{n+1}, r_{n+2}, r_{n+3})^{\lambda_4} \\
&\quad \cdot G(r_{n+1}, r_{n+2}, r_{n+3})^{\lambda_5} \cdot G(r_{n+2}, r_{n+3}, r_{n+4})^{\delta}
\end{aligned}
$$

$$= G(r_n, r_{n+1}, r_{n+2})^{(\lambda_1+\lambda_2)} \cdot G(r_{n+1}, r_{n+2}, r_{n+3})^{(\lambda_3+\lambda_4+\lambda_5)} \cdot G(r_{n+2}, r_{n+3}, r_{n+4})^{\delta}.$$

Therefore, (3.3) becomes

$$G(r_{n+2}, r_{n+3}, r_{n+4}) \leq \mu G(r_n, r_{n+1}, r_{n+2})^{(\lambda_1+\lambda_2)} \cdot G(r_{n+1}, r_{n+2}, r_{n+3})^{(\lambda_3+\lambda_4+\lambda_5)} \cdot G(r_{n+2}, r_{n+3}, r_{n+4})^{\delta}. \tag{3.5}$$

If $\delta = 1$, then $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$, and so we obtain

$$0 < G(r_{n+2}, r_{n+3}, r_{n+4}) \leq \mu G(r_{n+2}, r_{n+3}, r_{n+4}),$$

which is a contradiction. Therefore, $\delta < 1$, so that $\sum_{i=1}^{5} \lambda_i = 1 - \delta > 0$, implying that

$$\theta = \frac{\lambda_1 + \lambda_2}{1 - \delta}, \quad \eta = \frac{\lambda_3 + \lambda_4 + \lambda_5}{1 - \delta}$$

satisfying $\theta + \eta = 1$. Hence, (3.5) becomes

$$G(r_{n+2}, r_{n+3}, r_{n+4})^{(1-\delta)} \leq \mu G(r_n, r_{n+1}, r_{n+2})^{(\lambda_1+\lambda_2)} \cdot G(r_{n+1}, r_{n+2}, r_{n+3})^{(\lambda_3+\lambda_4+\lambda_5)}$$

so that

$$\begin{aligned} G(r_{n+2}, r_{n+3}, r_{n+4}) &\leq \mu^{\left(\frac{1}{1-\delta}\right)} G(r_n, r_{n+1}, r_{n+2})^{\left(\frac{\lambda_1+\lambda_2}{1-\delta}\right)} \cdot G(r_{n+1}, r_{n+2}, r_{n+3})^{\left(\frac{\lambda_3+\lambda_4+\lambda_5}{1-\delta}\right)} \\ &= \mu^{\left(\frac{1}{1-\delta}\right)} G(r_n, r_{n+1}, r_{n+2})^{\theta} \cdot G(r_{n+1}, r_{n+2}, r_{n+3})^{\eta} \end{aligned}$$

for all n in $\mathbb{N}$. Noting that $\mu \in (0, 1)$, we have

$$0 < 1 - \delta \leq 1 \Rightarrow 1 \leq \frac{1}{1 - \delta} \Rightarrow \mu^{\left(\frac{1}{1-\delta}\right)} \leq \mu < 1.$$

Hence, since $0 < \mu^{\left(\frac{1}{1-\delta}\right)} < 1$ and $\theta + \eta = 1$, then by Corollary 1, we can conclude that $\{r_n\}_{n\in\mathbb{N}}$ is G-Cauchy in (Φ, G).

Therefore, for all $q \geq 0$, we have established that $\{r_n\}_{n\in\mathbb{N}}$ is a G-Cauchy sequence in (Φ, G) and so by the completeness of (Φ, G), there exists a point c in Φ such that $\lim_{n\to\infty} r_n = c$. Moreover, since Γ is continuous, then we can conclude that $\Gamma c = c$, that is, c is a fixed point of Γ. □

Theorem 3.3. *Let (Φ, G) be a complete G-metric space and let $\Gamma : \Phi \longrightarrow \Phi$ be a hybrid-interpolative Reich-Istrăţescu-type $(G$-α-$\mu)$-contraction satisfying the following conditions:*

(i) Γ is $(G$-$\alpha)$-orbital admissible;
(ii) there exists $r_0 \in \Phi$ such that $\alpha(r_0, \Gamma r_0, \Gamma^2 r_0) \geq 1$;
(iii) Γ^3 is continuous and $\alpha(r, \Gamma r, \Gamma^2 r) \geq 1$ for any $r \in Fix(\Gamma^3)$.

Then Γ has at least a fixed point in Φ.

Proof. Let $r_0 \in \Phi$ be arbitrary and define a sequence $\{r_n\}_{n\in\mathbb{N}}$ in Φ by $r_n = \Gamma^n r_0$. We have shown in Theorem 3.2 that there exists $c \in \Phi$ such that $r_n \to c$. Since Γ^3 is continuous, then

$$\Gamma^3 c = \lim_{n\to\infty} \Gamma^3 r_n = c, \tag{3.6}$$

that is, c is a fixed point of Γ^3. This implies that Γ^3 has at least one fixed point in Φ, that is, $Fix(\Gamma^3)$ is nonempty. Moreover,

$$\Gamma^4 c = \Gamma c. \tag{3.7}$$

To see that c is a fixed point of Γ, assume contrary that $\Gamma c \neq c$. In this case, Γc is not a fixed point of Γ^3 either, since $\Gamma c = \Gamma^3 c = c$, which is a contradiction. Also, by (3.1), we have

$$\begin{aligned} G(c,\Gamma c,\Gamma^2 c) &\leq \alpha(c,\Gamma c,\Gamma^2 c)G(c,\Gamma c,\Gamma^2 c) \\ &= \alpha(c,\Gamma c,\Gamma^2 c)G(\Gamma^3 c,\Gamma^4 c,\Gamma^2 c) \\ &= \alpha(c,\Gamma c,\Gamma^2 c)G(\Gamma^3 c,\Gamma c,\Gamma^2 c) \\ &= \alpha(c,\Gamma c,\Gamma^2 c)G(\Gamma c,\Gamma^2 c,\Gamma^3 c) \\ &\leq \mu M(c,\Gamma c,\Gamma^2 c). \end{aligned} \tag{3.8}$$

Considering Case 1, we obtain

$$\begin{aligned} M(c,\Gamma c,\Gamma^2 c) = &\ \big[\lambda_1 G(c,\Gamma c,\Gamma^2 c)^q + \lambda_2 G(c,\Gamma c,\Gamma^2 c)^q + \lambda_3 G(\Gamma c,\Gamma^2 c,\Gamma^3 c)^q \\ &+ \lambda_4 G(\Gamma c,\Gamma^2 c,\Gamma^3 c)^q + \lambda_5 G(\Gamma c,\Gamma^2 c,\Gamma^3 c)^q + \delta G(\Gamma^2 c,\Gamma^3 c,\Gamma^4 c)^q\big]^{\frac{1}{q}} \\ = &\ \big[\lambda_1 G(c,\Gamma c,\Gamma^2 c)^q + \lambda_2 G(c,\Gamma c,\Gamma^2 c)^q + \lambda_3 G(c,\Gamma c,\Gamma^2 c)^q \\ &+ \lambda_4 G(c,\Gamma c,\Gamma^2 c)^q + \lambda_5 G(c,\Gamma c,\Gamma^2 c)^q + \delta G(c,\Gamma c,\Gamma^2 c)^q\big]^{\frac{1}{q}} \\ = &\ \big[(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\delta)G(c,\Gamma c,\Gamma^2 c)^q\big]^{\frac{1}{q}} \\ = &\ (\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\delta)^{\frac{1}{q}} G(c,\Gamma c,\Gamma^2 c) \\ \leq &\ G(c,\Gamma c,\Gamma^2 c) \end{aligned}$$

implying that

$$0 < G(c,\Gamma c,\Gamma^2 c) \leq \mu G(c,\Gamma c,\Gamma^2 c),$$

which is a contradiction. Hence, $\Gamma c = c$.
Similarly, for Case 2, we obtain

$$\begin{aligned} M(c,\Gamma c,\Gamma^2 c) = &\ G(c,\Gamma c,\Gamma^2 c)^{\lambda_1} \cdot G(c,\Gamma c,\Gamma^2 c)^{\lambda_2} \cdot G(\Gamma c,\Gamma^2 c,\Gamma^3 c)^{\lambda_3} \\ &\cdot G(\Gamma c,\Gamma^2 c,\Gamma^3 c)^{\lambda_4} \cdot G(\Gamma c,\Gamma^2 c,\Gamma^3 c)^{\lambda_5} \cdot G(\Gamma^2 c,\Gamma^3 c,\Gamma^4 c)^{\delta} \\ = &\ G(c,\Gamma c,\Gamma^2 c)^{\lambda_1} \cdot G(c,\Gamma c,\Gamma^2 c)^{\lambda_2} \cdot G(c,\Gamma c,\Gamma^2 c)^{\lambda_3} \\ &\cdot G(c,\Gamma c,\Gamma^2 c)^{\lambda_4} \cdot G(c,\Gamma c,\Gamma^2 c)^{\lambda_5} \cdot G(c,\Gamma c,\Gamma^2 c)^{\delta} \\ = &\ G(c,\Gamma c,\Gamma^2 c)^{(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\delta)} \\ = &\ G(c,\Gamma c,\Gamma^2 c), \end{aligned}$$

so that (3.8) becomes

$$0 < G(c, \Gamma c, \Gamma^2 c) \leq \mu G(c, \Gamma c, \Gamma^2 c),$$

which is also a contradiction. Hence, $\Gamma c = c$.
Therefore, c is a fixed point of Γ in Φ. □

Theorem 3.4. *If in addition to the hypotheses of Theorem 3.3, we assume supplementary that $\alpha(r, s, \Gamma s) \geq 1$ for any $r, s \in Fix(\Gamma)$, then the fixed point of Γ is unique.*

Proof. Let $v, c \in \Phi$ be any two fixed point of Γ with $v \neq c$. By replacing this in (3.1) and noting the additional hypothesis, we have:

$$G(c, v, \Gamma v) \leq \alpha(c, v, \Gamma v) G(\Gamma^2 c, \Gamma^2 v, \Gamma^3 v) \leq \mu M(c, v, \Gamma v). \tag{3.9}$$

By Case 1, we obtain

$$\begin{aligned} M(c, v, \Gamma v) = & \left[\lambda_1 G(c, v, \Gamma v)^q + \lambda_2 G(c, \Gamma c, \Gamma^2 c)^q + \lambda_3 G(v, \Gamma v, \Gamma^2 v)^q \right. \\ & \left. + \lambda_4 G(\Gamma c, \Gamma v, \Gamma^2 v)^q + \lambda_5 G(\Gamma c, \Gamma^2 c, \Gamma^3 c)^q + \delta G(\Gamma v, \Gamma^2 v, \Gamma^3 v)^q \right]^{\frac{1}{q}} \\ = & \left[\lambda_1 G(c, v, \Gamma v)^q + \lambda_2 G(c, c, c)^q + \lambda_3 G(v, v, v)^q \right. \\ & \left. + \lambda_4 G(c, v, \Gamma v)^q + \lambda_5 G(c, c, c)^q + \delta G(v, v, v)^q \right]^{\frac{1}{q}} \\ = & \left[(\lambda_1 + \lambda_4) G(c, v, \Gamma v)^q \right]^{\frac{1}{q}} \\ = & (\lambda_1 + \lambda_4)^{\frac{1}{q}} G(c, v, \Gamma v) \\ \leq & G(c, v, \Gamma v). \end{aligned}$$

Hence, (3.9) becomes

$$0 < G(c, v, \Gamma v) \leq \mu G(c, v, \Gamma v),$$

which is a contradiction. Therefore, $v = c$, and so the fixed point of Γ is unique. □

In the following result, we examine the existence of fixed point of Γ when the G-metric space (Φ, G) is regular.

Theorem 3.5. *Let (Φ, G) be a complete G-metric space and let $\Gamma : \Phi \longrightarrow \Phi$ be a hybrid-interpolative Reich-Istrăţescu-type (G-α-μ)-contraction for $q = 0$ such that $\lambda_2 > 0$ and $\lambda_5 > 0$. Suppose further that:*

(i) Γ is (G-α)-orbital admissible;
(ii) there exists $r_0 \in \Phi$ such that $\alpha(r_0, \Gamma r_0, \Gamma^2 r_0) \geq 1$;
(iii) (Φ, G) is regular with respect to α.

Then Γ has a fixed point.

Proof. In Theorem 3.2, we have established that for any $q \geq 0$, the sequence $\{r_n\}_{n\in\mathbb{N}}$ is G-Cauchy and by the completeness of the G-metric space (Φ, G), there exists a point c in Φ such that $r_n \to c$. To prove that c is a fixed point of Γ, suppose contrary that $\Gamma c \neq c$.

Assume $\{r_n\}_{n\in\mathbb{N}}$ is such that $r_n \neq r_m$ whenever $n \neq m$, for all $n, m \in \mathbb{N}$. Then there exists $n_0 \in \mathbb{N}$ such that r_n and c are distinct and not in $Fix(\Gamma)$ for all $n \geq n_0$. We will verify that c is a fixed point of Γ^3. Indeed, for all $n \geq n_0$,

$$\begin{aligned}
G(r_{n+2}, \Gamma^2 c, \Gamma^3 c) \leq\ & \alpha(r_n, c, \Gamma c) G(\Gamma^2 r_n, \Gamma^2 c, \Gamma^3 c) \leq \mu M(r_n, c, \Gamma c) \\
=\ & \mu G(r_n, c, \Gamma c)^{\lambda_1} \cdot G(r_n, \Gamma r_n, \Gamma^2 r_n)^{\lambda_2} \cdot G(c, \Gamma c, \Gamma^2 c)^{\lambda_3} \\
& \cdot G(\Gamma r_n, \Gamma c, \Gamma^2 c)^{\lambda_4} \cdot G(\Gamma r_n, \Gamma^2 r_n, \Gamma^3 r_n)^{\lambda_5} \cdot G(\Gamma c, \Gamma^2 c, \Gamma^3 c)^{\delta} \\
=\ & \mu G(r_n, c, \Gamma c)^{\lambda_1} \cdot G(r_n, r_{n+1}, r_{n+2})^{\lambda_2} \cdot G(c, \Gamma c, \Gamma^2 c)^{\lambda_3} \\
& \cdot G(r_{n+1}, \Gamma c, \Gamma^2 c)^{\lambda_4} \cdot G(r_{n+1}, r_{n+2}, r_{n+3})^{\lambda_5} \cdot G(\Gamma c, \Gamma^2 c, \Gamma^3 c)^{\delta}.
\end{aligned}$$

Since $\lambda_2 > 0$ and $\lambda_5 > 0$, then letting $n \to \infty$ and noting Proposition 1, it is verified that c is a fixed point of Γ^3. Hence, by (3.6) and (3.7), we obtain a contradiction. Therefore $\Gamma c = c$, implying that c is a fixed point of Γ. □

Example 3.6. Let $\Phi = [-1, 1]$ and let $\Gamma : \Phi \longrightarrow \Phi$ be a self-mapping on Φ defined by $\Gamma r = \frac{r}{2}$ for all $r \in \Phi$. Define $G : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ by

$$G(r, s, t) = |r - s| + |r - t| + |s - t| \quad \forall\ r, s, t \in \Phi.$$

Then (Φ, G) is a complete G-metric space. Define $\alpha : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ by

$$\alpha(r, s, t) = \begin{cases} 1, & \text{if} \quad r, s, t \in \{-1\} \cup [0, 1]; \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

Then obviously, Γ is a $(G\text{-}\alpha)$-orbital admissible and Γ is continuous for all $r \in \Phi$. Also, there exists $r_0 = \frac{1}{3} \in \Phi$ such that $\alpha(\frac{1}{3}, \Gamma(\frac{1}{3}), \Gamma^2(\frac{1}{3})) = \alpha(\frac{1}{3}, \frac{1}{6}, \frac{1}{12}) \geq 1$. Hence, conditions (i)-(iii) of Theorems 3.2 and 3.3 are satisfied.

To see that Γ is a hybrid-interpolative Reich-Istrăţescu-type $(G\text{-}\alpha\text{-}\mu)$-contraction, let $\mu = \frac{1}{2}$. Notice that $\alpha(r, s, \Gamma s) = 0$ for all $r, s \in (-1, 0)$. Hence, inequality (3.1) holds for all $r, s \in (-1, 0)$.

Now for $r, s \in \{-1, 1\}$, if $r = s$, then letting $\lambda_1 = \frac{1}{5}$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, $\lambda_5 = \delta = \frac{2}{5}$ and $q = 2$, we obtain

$$\begin{aligned}
\alpha(r, s, \Gamma s) G(\Gamma^2 r, \Gamma^2 s, \Gamma^3 s) &= \alpha\left(1, 1, \frac{1}{2}\right) G\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right) \\
&= \alpha\left(-1, -1, \frac{-1}{2}\right) G\left(\frac{-1}{4}, \frac{-1}{4}, \frac{-1}{8}\right) \\
&= \frac{1}{4} < \frac{2}{5} = \frac{1}{2}\left(\frac{4}{5}\right) \\
&= \frac{1}{2}\left(M\left(1, 1, \frac{1}{2}\right)\right) = \frac{1}{2}\left(M\left(-1, -1, \frac{-1}{2}\right)\right) \\
&= \mu M(r, s, \Gamma s).
\end{aligned}$$

Also, if $q = 0$, we have

$$\alpha(r, s, \Gamma s) G(\Gamma^2 r, \Gamma^2 s, \Gamma^3 s) = \frac{1}{4} < \frac{1}{2}\left(\frac{4}{5}\right) = \mu M(r, s, \Gamma s).$$

If $r \neq s$, then letting $\lambda_1 = \frac{3}{5}$, $\lambda_2 = \lambda_3 = \lambda_5 = 0$, $\lambda_4 = \delta = \frac{1}{5}$ and $q = 2$, we obtain

$$\begin{aligned}
\alpha(r,s,\Gamma s)G(\Gamma^2 r,\Gamma^2 s,\Gamma^3 s) &= \alpha\left(-1,1,\frac{1}{2}\right)G\left(\frac{-1}{4},\frac{1}{4},\frac{1}{8}\right)\\
&= \alpha\left(1,-1,\frac{-1}{2}\right)G\left(\frac{1}{4},\frac{-1}{4},\frac{-1}{8}\right)\\
&= 1 < \frac{8}{5} = \frac{1}{2}\left(\frac{16}{5}\right)\\
&= \frac{1}{2}\left(M\left(-1,1,\frac{1}{2}\right)\right) = \frac{1}{2}\left(M\left(1,-1,\frac{-1}{2}\right)\right)\\
&= \mu M(r,s,\Gamma s).
\end{aligned}$$

Also, for $q = 0$, we take $\lambda_1 = \frac{3}{5}$, $\lambda_2 = \lambda_5 = \delta = 0$, $\lambda_3 = \lambda_4 = \frac{1}{5}$. Then

$$\begin{aligned}
\alpha(r,s,\Gamma s)G(\Gamma^2 r,\Gamma^2 s,\Gamma^3 s) &= \alpha\left(-1,1,\frac{1}{2}\right)G\left(\frac{-1}{4},\frac{1}{4},\frac{1}{8}\right)\\
&= \alpha\left(1,-1,\frac{-1}{2}\right)G\left(\frac{1}{4},\frac{-1}{4},\frac{-1}{8}\right)\\
&= 1 < \frac{7}{5} = \frac{1}{2}\left(\frac{14}{5}\right)\\
&= \frac{1}{2}\left(M\left(-1,1,\frac{1}{2}\right)\right) = \frac{1}{2}\left(M\left(1,-1,\frac{-1}{2}\right)\right)\\
&= \mu M(r,s,\Gamma s).
\end{aligned}$$

Finally, for all $r, s \in (0,1)$, we take $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \delta = 0$. Then

$$\begin{aligned}
\alpha(r,s,\Gamma s)G(\Gamma^2 r,\Gamma^2 s,\Gamma^3 s) &= G(\Gamma^2 r,\Gamma^2 s,\Gamma^3 s)\\
&= \left|\frac{r}{4}-\frac{s}{4}\right| + \left|\frac{r}{4}-\frac{s}{8}\right| + \left|\frac{s}{4}-\frac{s}{8}\right|\\
&= \frac{1}{4}\left(|r-s| + \left|r-\frac{s}{2}\right| + \left|s-\frac{s}{2}\right|\right)\\
&= \frac{1}{4}G(r,s,\Gamma s)\\
&< \frac{1}{2}G(r,s,\Gamma s)\\
&= \mu M(r,s,\Gamma s)
\end{aligned}$$

for all $q \geq 0$.
Hence, inequality (3.1) is satisfied for all $r, s \in \Phi \backslash Fix(\Gamma)$. Therefore, Γ is a hybrid-interpolative Reich-Istrăţescu-type (G-α-μ)-contraction which satisfies all the assumptions of Theorems 3.2 and 3.3. The point $r = 0$ is the fixed point of Γ in Φ.

We now demonstrate that our result is independent and an improvement of the results of Karapınar et al. [7]. Let $\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be as given by Definition (2.12), $r_0 \in \Phi$ be such that $\alpha(r_0, \Gamma r_0) \geq 1$ and $d : \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be defined by

$$d(r,s) = |r-s| \quad \forall\ r,s \in \Phi.$$

Consider $r,s \in \{-1,1\}$ and take for Case 1, $r \neq s$, $\lambda_1 = \frac{1}{5}$, $\lambda_2 = \lambda_3 = \lambda_5 = 0$, $\lambda_4 = \frac{1}{2}$, $\delta = \frac{3}{10}$ and $q = 1$. Then inequality (3.1) becomes

$$\begin{aligned}
\alpha(r,s,\Gamma s)G(\Gamma^2 r,\Gamma^2 s,\Gamma^3 s) &= \alpha\left(-1,1,\frac{1}{2}\right)G\left(\frac{-1}{4},\frac{1}{4},\frac{1}{8}\right)\\
&= \alpha\left(1,-1,\frac{-1}{2}\right)G\left(\frac{1}{4},\frac{-1}{4},\frac{-1}{8}\right)\\
&= 1 \leq \frac{101}{100} = \frac{1}{2}\left(\frac{202}{100}\right)\\
&= \frac{1}{2}\left(M\left(-1,1,\frac{1}{2}\right)\right) = \frac{1}{2}\left(M\left(1,-1,\frac{-1}{2}\right)\right)\\
&= \mu M(r,s,\Gamma s),
\end{aligned}$$

while inequality (2.11) due to Karapınar et al. [7] yields

$$\begin{aligned}
\alpha(r,s)d(\Gamma^2 r,\Gamma^2 s) &= \alpha(-1,1)d\left(\frac{-1}{4},\frac{1}{4}\right) = \alpha(1,-1)d\left(\frac{1}{4},\frac{-1}{4}\right)\\
&= \frac{1}{2} > \frac{49}{100} = \frac{1}{2}\left(\frac{98}{100}\right)\\
&= \frac{1}{2}(M(-1,1)) = \frac{1}{2}(M(1,-1))\\
&= \mu M(r,s).
\end{aligned}$$

For Case 2, Karapınar et al. [7] have noted that their result is indeterminate for $r = s$, if either $\lambda_1 = 0$ or $\lambda_4 = 0$, since

$$[d(r,s)]^{\lambda_1} = [d(\Gamma r,\Gamma s)]^{\lambda_4} = 0^0.$$

Hence, they declared that r and s are distinct and that $0^0 = 1$, which in contrast, is unconventional. But our result is valid for all $r,s \in \Phi\backslash Fix(\Gamma)$. Therefore, hybrid-interpolative Reich-Istrăţescu-type (G-α-μ)-contraction is not hybrid-interpolative Reich-Istrăţescu-type contraction defined by Karapınar et al. [7], and so Theorems 12 and 17 due to Karapınar et al. [7] are not applicable to this example.

The following is an Istrăţescu-type (see [7]) consequence of our result.

Corollary 2. *Let (Φ, G) be a complete G-metric space and $\Gamma : \Phi \longrightarrow \Phi$ be a continuous self-mapping such that there exist $\eta, \lambda \in (0,1)$ with $\eta + \lambda < 1$, satisfying*

$$G(\Gamma^2 r,\Gamma^2 s,\Gamma^3 s) \leq \eta G(r,s,\Gamma s) + \lambda G(\Gamma r,\Gamma s,\Gamma^2 s)$$

for all $r,s \in \Phi$. Then Γ has a fixed point in Φ.

Proof. Let $\mu = \eta + \lambda \in (0,1)$. Consider Definition (3.1) and let $\alpha(r,s,\Gamma s) = 1$, $q = 1$, $\lambda_1 = \frac{\eta}{\mu}$, $\lambda_4 = \frac{\lambda}{\mu}$ and $\lambda_2 = \lambda_3 = \lambda_5 = \delta = 0$. Then for all $r,s \in \Phi$, we have

$$G(\Gamma^2 r,\Gamma^2 s,\Gamma^3 s) \leq \mu M(r,s,\Gamma s)$$

$$
\begin{aligned}
&= \mu\left[\frac{\eta}{\mu}G(r,s,\Gamma s)+\frac{\lambda}{\mu}G(\Gamma r,\Gamma s,\Gamma^2 s)\right]\\
&= \eta G(r,s,\Gamma s)+\lambda G(\Gamma r,\Gamma s,\Gamma^2 s),
\end{aligned}
\tag{3.11}
$$

implying that inequality (3.1) holds for all $r, s \in \Phi \backslash Fix(\Gamma)$.
Let $r_0 \in \Phi$ be arbitrary and define a sequence $\{r_n\}_{n\in\mathbb{N}}$ in Φ by $r_n = \Gamma^n r_0$. Then by (3.11), we have

$$
\begin{aligned}
G(r_{n+2}, r_{n+3}, r_{n+4}) &\leq \eta G(r_n, r_{n+1}, r_{n+2}) + \lambda G(r_{n+1}, r_{n+2}, r_{n+3})\\
&\leq \eta \max\{G(r_n, r_{n+1}, r_{n+2}), G(r_{n+1}, r_{n+2}, r_{n+3})\}\\
&\quad + \lambda \max\{G(r_n, r_{n+1}, r_{n+2}), G(r_{n+1}, r_{n+2}, r_{n+3})\}\\
&= (\eta+\lambda) \max\{G(r_n, r_{n+1}, r_{n+2}), G(r_{n+1}, r_{n+2}, r_{n+3})\}\\
&= \mu \max\{G(r_n, r_{n+1}, r_{n+2}), G(r_{n+1}, r_{n+2}, r_{n+3})\}.
\end{aligned}
$$

Hence, by Lemma 2.11, $\{r_n\}_{n\in\mathbb{N}}$ is a G-Cauchy sequence in (Φ, G) and since (Φ, G) is complete, then there exists $c \in \Phi$ such that $r_n \to c$. Since Γ is continuous, then we can conclude that c is a fixed point of Γ, that is, $\Gamma c = c$. □

In Definition (3.1), if we specialize the parameters λ_i $(i = 1, 2, ..., 5)$, δ and q, as well as let $\alpha(r, s, \Gamma s) = 1$ for all $r, s \in \Phi$ and $\Gamma s = t$, we obtain the following corollary, which is a consequence of Theorem 3.2.

Corollary 3. *Let (Φ, G) be a complete G-metric space and $\Gamma : \Phi \longrightarrow \Phi$ be a continuous self-mapping such that there exists $\mu \in (0, 1)$, satisfying*

$$
G(\Gamma^2 r, \Gamma^2 s, \Gamma^2 t) \leq \mu G(r, s, t) \tag{3.12}
$$

for all $r, s, t \in \Phi$. Then Γ has a fixed point in Φ.

4. Applications to solution of integral equation

In this section, an existence theorem for a solution of a class of integral equations is provided using Corollary 3. For similar results, we refer to [1, 4, 10, 17].
Consider the integral equation

$$
r(y) = \int_a^b \mathcal{L}(y, x) f(x, r(x))dx, \quad y \in [a, b]. \tag{4.1}
$$

Let $\Phi = C([a, b], \mathbb{R})$ be the set of all continuous real-valued functions. Define $G : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ by

$$
\begin{aligned}
G(r, s, t) = \max_{y\in[a,b]} |r(y) - s(y)| + \max_{y\in[a,b]} |r(y) - t(y)| + \max_{y\in[a,b]} |s(y) - t(y)|&\\
\forall r, s, t \in \Phi, \quad y \in [a, b].&
\end{aligned}
\tag{4.2}
$$

Then, (Φ, G) is a complete G-metric space.
Define a function $\Gamma : \Phi \longrightarrow \Phi$ as follows:

$$
\Gamma r(y) = \int_a^b \mathcal{L}(y, x) f(x, r(x))dx, \quad y \in [a, b]. \tag{4.3}
$$

Then a point u^* is said to be a fixed point of Γ if and only if u^* is a solution to (4.1).
Now, we study existence conditions of the integral equation (4.1) under the following hypotheses.

Theorem 4.1. *Assume that the following conditions are satisfied:*

(C_1) $\mathcal{L} : [a,b] \times [a,b] \longrightarrow \mathbb{R}_+$ *and* $f : [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ *are continuous;*
(C_2) *for all* $r, s \in \Phi$, $x \in [a,b]$, *we have* $|f(x, r(x)) - f(x, s(x))| \leq |r(x) - s(x)|$*;*
(C_3) $\max\limits_{y\in[a,b]} \int_a^b \mathcal{L}(y,x)dx \leq \lambda$ *for some* $\lambda < 1$.

Then, the integral equation (4.1) has a solution u^* *in* Φ.

Proof. Observe that for any $r, s \in \Phi$, using (4.3) and the above hypotheses, we obtain

$$\begin{aligned}
|\Gamma r(y) - \Gamma s(y)| &= \left| \int_a^b \left[\mathcal{L}(y,x) f(x, r(x)) - \mathcal{L}(y,x) f(x, s(x)) \right] dx \right| \\
&\leq \int_a^b \mathcal{L}(y,x) \left| f(x, r(x)) - f(x, s(x)) \right| dx \\
&\leq \int_a^b \mathcal{L}(y,x) \left| r(x) - s(x) \right| dx \\
&\leq \int_a^b \mathcal{L}(y,x) \max_{x\in[a,b]} \left| r(x) - y(x) \right| dx \\
&\leq \lambda \max_{y\in[a,b]} \left| r(y) - s(y) \right|,
\end{aligned}$$

so that

$$\begin{aligned}
|\Gamma^2 r(y) - \Gamma^2 s(y)| &\leq \lambda \max_{y\in[a,b]} |\Gamma r(y) - \Gamma s(y)| \\
&\leq \lambda \max_{y\in[a,b]} \left[\lambda \max_{y\in[a,b]} |r(y) - s(y)| \right] \\
&\leq \lambda^2 \max_{y\in[a,b]} |r(y) - s(y)|.
\end{aligned}$$

Using this in (4.2), we have

$$\begin{aligned}
G(\Gamma^2 r, \Gamma^2 s, \Gamma^2 t) &= \max_{y\in[a,b]} |\Gamma^2 r - \Gamma^2 s| + \max_{y\in[a,b]} |\Gamma^2 r - \Gamma^2 t| + \max_{y\in[a,b]} |\Gamma^2 s - \Gamma^2 t| \\
&\leq \lambda^2 \max_{y\in[a,b]} |r - s| + \lambda^2 \max_{y\in[a,b]} |r - t| + \lambda^2 \max_{y\in[a,b]} |s - t| \\
&= \lambda^2 \left(\max_{y\in[a,b]} |r - s| + \max_{y\in[a,b]} |r - t| + \max_{y\in[a,b]} |s - t| \right) \\
&= \mu G(r, s, t),
\end{aligned}$$

where $\mu = \lambda^2 < 1$.
Hence, all the hypotheses of Corollary 3 are verified, implying that there exists a solution u^* in Φ of the integral equation (4.1).
Conversely, if u^* is a solution of (4.1), then u^* is also a solution of (4.3), so that $\Gamma u^* = u^*$, that is, u^* is a fixed point of Γ. □

Remark 1.

(i) We can deduce many other corollaries by particularizing some of the parameters in Definition (3.1).
(ii) None of the results presented in this work is expressible in the form $G(s, r, r)$ or $G(s, s, r)$. Hence, they cannot be obtained from their corresponding versions in metric spaces.

5. Conclusions

A generalization of metric space was introduced by Mustafa and Sims [9], namely G-metric space and several fixed point results were studied in that space. However, Jleli and Samet [5] as well as Samet et al. [16] established that most fixed point theorems obtained in G-metric space are direct consequences of their analogues in metric space. Contrary to the above observation, a new family of contraction, called hybrid-interpolative Reich-Istrăţescu-type (G-α-μ)-contraction is introduced in this manuscript and some fixed point theorems that cannot be deduced from their corresponding ones in metric space are proved. The main distinction of this class of contraction is that its contractive inequality is expressible in a number of ways with respect to multiple parameters. Consequently, a few corollaries, including some recently announced results in the literature are highlighted and analyzed. Nontrivial comparative examples are constructed to validate the assumptions of our obtained theorems. Furthermore, one of our obtained corollaries is applied to set up novel existence conditions for solution of a class of integral equations.

Conflict of interest

The authors declare that they have no competing interests.

References

1. E. Ameer, H. Aydi, M. Arshad, H. Alsamir, M. S. Noorani, Hybrid multivalued type contraction mappings in αk-complete partial b-metric spaces and applications, *Symmetry*, **11** (2019), 86. https://doi.org/10.3390/sym11010086
2. A. H. Ansari, S. Chandok, N. Hussain, Z. Mustafa, M. M. M. Jaradat, Some common fixed point theorems for weakly α-admissible pairs in g-metric spaces with auxiliary functions, *J. Math. Anal.*, **8** (2017), 80–107.
3. J. Chen, C. X. Zhu, L. Zhu, A note on some fixed point theorems on g-metric spaces, *J. Appl. Anal. Comput.*, **11** (2021), 101–112. https://doi.org/10.11948/20190125
4. K. Javed, F. Uddin, H. Aydi, A. Mukheimer, M. Arshad, Ordered-theoretic fixed point results in fuzzy b-metric spaces with an application, *J. Math.*, **2021** (2021), 6663707. https://doi.org/10.1155/2021/6663707
5. M. Jleli, B. Samet, Remarks on g-metric spaces and fixed point theorems, *Fixed Point Theory Appl.*, **2012** (2012), 210. https://doi.org/10.1186/1687-1812-2012-210
6. E. Karapınar, R. P. Agarwal, Further fixed point results on g-metric space, *Fixed Point Theory Appl.*, **2013** (2013), 154. https://doi.org/10.1186/1687-1812-2013-154

7. E. Karapinar, A. Fulga, N. Shahzad, A. F. Roldán López de Hierro, Solving integral equations by means of fixed point theory, *J. Funct. Space.*, **2022** (2022), 7667499. https://doi.org/10.1155/2022/7667499

8. Z. Mustafa, A new structure for generalised metric spaces with applications to fixed point theory, PhD Thesis, The University of Newcastle, 2005.

9. Z. Mustafa, B. Sims, A new approach to generalised metric spaces, *J. Nonlinear Convex Anal.*, **7** (2006), 289–297.

10. Z. Mustafa, M. Arshad, S. U. KhZan, J. Ahmad, M. M. M. Jaradat, Common fixed points for multivalued mappings in g-metric spaces with applications, *J. Nonlinear Sci. Appl.*, **10** (2017), 2550–2564. https://doi.org/10.22436/jnsa.010.05.23

11. Z. Mustafa, H. Obiedat, F. Awawdeh, Some fixed point theorem for mapping on complete g-metric spaces, *Fixed Point Theory Appl.*, **2008** (2008), 189870. https://doi.org/10.1155/2008/189870

12. V. Parvaneh, M. R. Haddadi, H. Aydi, On best proximity point results for some type of mappings, *J. Funct. Space.*, **2020** (2020), 6298138. https://doi.org/10.1155/2020/6298138

13. P. Patle, D. Patel, H. Aydi, S. Radenović, On H^+ type multivalued contractions and applications in symmetric and probabilistic spaces, *Mathematics*, **7** (2019), 144. https://doi.org/10.3390/math7020144

14. O. Popescu, Some new fixed point theorems for α-geraghty contraction type maps in metric spaces, *Fixed Point Theory Appl.*, **2014** (2014), 190. https://doi.org/10.1186/1687-1812-2014-190

15. H. Qawaqneh, M. D. Noorani, W. Shatanawi, H. Aydi, H. Alsamir, Fixed point results for multi-valued contractions in b-metric spaces and an application, *Mathematics*, **7** (2019), 132. https://doi.org/10.3390/math7020132

16. B. Samet, C. Vetro, F. Vetro, Remarks on g-metric spaces, *Int. J. Anal.*, **2013** (2013), 917158. http://doi.org/10.1155/2013/917158

17. S. S. Mohammed, M. Alansari, A. Azam, S. Kanwal, Fixed points of (φ, F)-weak contractions on metric-like spaces with applications to integral equations on time scales. *Bol. Soc. Mat. Mex.*, **27** (2021), 39. https://doi.org/10.1007/s40590-021-00347-x