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Research article

The non-linear Schrödinger equation associated with the soliton surfaces in Minkowski 3-space

Ayman Elsharkawy[1,*], Clemente Cesarano[2,*], Abdelrhman Tawfiq[3] and Abdul Aziz Ismail[1,4]

1 Mathematics Department, Faculty of Science, Tanta University, 31511 Tanta, Egypt
2 Section of Mathematics, International Telematic University Uninettuno, 001186 Roma, Italy
3 Mathematics Department, Faculty of Education, Ain-Shams University, 11566 Cairo, Egypt
4 Mechanical Engineering Department, College of Engineering and Islamic Architecture, Umm Al-Qura University, 5555 Makkah, Saudi Arabia. Email: aiismail@uqu.edu.sa

* **Correspondence:** Email: ayman_ramadan@science.tanta.edu.eg, c.cesarano@uninettunouniversity.net.

Abstract: The quasi frame is more efficient than the Frenet frame in investigating surfaces, and it is regarded a generalization frame of both the Frenet and Bishop frames. The geometry of quasi-Hasimoto surfaces in Minkowski 3-space $\mathbb{E}_1^3$ is investigated in this paper. For the three situations of non-lightlike curves, the geometric features of the quasi-Hasimoto surfaces in $\mathbb{E}_1^3$ are examined and the Gaussian and mean curvatures for each case are determined. The quasi-Hasimoto surfaces in $\mathbb{E}_1^3$ must satisfy a necessary and sufficient condition to be developable surfaces. As a result, the parameter curves of quasi-Hasimoto surfaces in $\mathbb{E}_1^3$ is described. Thus, the s-parameter and t-parameter curves of quasi-Hasimoto surfaces in $\mathbb{E}_1^3$ are said to be geodesics, asymptotic, and curvature lines under necessary and sufficient circumstances are proved. Finally, quasi curves and associated quasi-Hasimoto surface correspondences are discussed.

Keywords: Minkowski space; Hasimoto surfaces; smoke ring equation; Gaussian and mean curvatures
Mathematics Subject Classification: 35Q51, 51B20, 53A35, 76B47

1. Introduction

In 1972, Hasimoto was interested in studying a thin, isolated vortex filament and its approximation of self-induced motion in an incompressible fluid. The position vector of the vortex filament is given by $\Gamma = \Gamma(s, t)$ in [8]. The relation holds for the vortex filament or smoke ring equation

$$\Gamma_t = \Gamma_s \times \Gamma_{ss}.$$

This relationship can also be used to investigate a dynamical system on space curves in Euclidean space. It is possible to demonstrate that the absence of form change in vortex motions corresponds to travelling wave solutions of the Non-linear Schrödinger equations (NLS) [16]. The Hasimoto or NLS surface is the NLS equation linked with the soliton surface.

The binormal motion of the curves was used to assess Hasimoto surfaces. In [18], the binormal motion of constant curvature and torsion curves is addressed. In [17], the intrinsic geometry of the nonlinear Schrödinger equation in $\mathbb{E}^3$ is addressed. In [5], In a generic intrinsic geometric setting containing a normal congruence, a nonlinear heat system and nonlinear Schrödinger equation repulsive type for timelike curves were developed. in $\mathbb{E}_1^3$. The motion of timelike surfaces was investigated in [6], which corresponds to a NLS equation of repulsive type in timelike geodesic coordinates.

In 2012, Hasimoto surfaces in 3-dimensional Euclidean space were studied in [1]. In 2014, Hasimoto surfaces in 3-dimensional Minkowski space according to the Frenet frame were studied in [17]. In 2019, the geometry of Hasimoto surfaces in Euclidean 3-space according to Bishop frame was discussed in [12]. In 2021, the Hasimoto surfaces according to Galilean space were discussed in [3], bright and dark solitons of a weakly non-local Schrodinger equation incorporating the non-linearity of the parabolic law in [10], the harmonic evolute surface of Hasimoto surfaces is discussed in [13] and approximate solutions for inextensible Heisenberg antiferromagnetic flow and solitonic magnetic flux surfaces along the normal direction in Minkowski space in [14]. In 2022, geometry of quasi-vortex filament equation solutions in Euclidean three-space $\mathbb{E}^3$ in [7] and the optical solitons of a high-order nonlinear Schrodinger equation with nonlinear dispersions and the Kerr effect in [9].

In this paper, the geometry of quasi-Hasimoto surfaces in $\mathbb{E}_1^3$ is investigated. This study is structured as follows: In Section 2, background information about Minkowski 3-space and a summary of the quasi frame in $\mathbb{E}_1^3$ are provided. In Section 3, we explore the geometric features of quasi-Hasimoto surfaces in $\mathbb{E}_1^3$ for the three situations of non-lightlike curves and determine their Gaussian and mean curvatures. Furthermore, we provide a necessary and sufficient condition for quasi-Hasimoto surfaces in $\mathbb{E}_1^3$ to be developable surfaces and demonstrate that the quasi frame is superior to the Frenet frame for studying surfaces. In Section 4, we characterize the parameter curves of quasi-Hasimoto surfaces in $\mathbb{E}_1^3$. In addition, we provide sufficient and necessary criteria for the s-parameter and t-parameter curves of quasi-Hasimoto surfaces in $\mathbb{E}_1^3$ to be geodesic, asymptotic, and curvature lines. In Section 5 and by using a Mathematica software, we present the quasi-curves and their accompanying quasi-Hasimoto surfaces as models to ensure the accuracy of estimated results. Each model is arbitrarily chosen to satisfy the criteria of the instance we wish to depict, and the associated quasi-Hasimoto surface is evolved using the introduced calculations. The selected curves simulate the real-world situation that we wish to convey. This study omits the actual circumstance and focuses on the approach used to create the simulation. Finally, we provide a summary of all the subjects discussed in the paper and demonstrate the significance of this work.

2. Preliminaries

A Cartesian three-dimensional space $\mathbb{R}^3$ with a Lorentzian inner product is defined as

$$g(\alpha,\beta) = -\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ is called the Minkowski 3-space $\mathbb{E}^3_1$.

Lorentzian inner product classifies the vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ as

- if $g(\alpha, \alpha) > 0$ or $\alpha = 0$, then α is called spacelike,
- if $g(\alpha, \alpha) < 0$, then α is called timelike,
- if $g(\alpha, \alpha) = 0$ and $\alpha \neq 0$, then α is called lightlike or null.

Any curve ξ in $\mathbb{E}^3_1$ is said to be spacelike, timelike or lightlike if and only if the tangent vector field t of a curve ξ is spacelike, timelike or lightlike, respectively. for more details, see [11, 15].

According to any unit speed curve $\xi(s)$ in $\mathbb{E}^3_1$, i.e., $g(\xi_s(s), \xi_s(s)) = \pm 1$, parameterized by its arc-length s, the quasi frame is a general frame of both Frenet and Bishop frames, consists of three orthogonal vector fields $\{\mathbf{t}(s), \mathbf{n}_q(s), \mathbf{b}_q(s)\}$, where $\mathbf{t}(s)$, $\mathbf{n}_q(s)$ and $\mathbf{b}_q(s)$ are the tangent, quasi-normal and quasi-binormal vector fields, respectively, and defined by

$$\mathbf{t}(s) = \frac{\xi'_s(s)}{\| \xi'_s(s) \|}, \quad \mathbf{n}_q(s) = \frac{\mathbf{t}(s) \times \mathbf{k}(s)}{\| \mathbf{t}(s) \times \mathbf{k}(s) \|}, \quad \mathbf{b}_q(s) = \mathbf{t}(s) \times \mathbf{n}_q(s), \tag{2.1}$$

where $\mathbf{k}(s)$ is the projection vector. In our study, we choose the projection vector $\mathbf{k}(s) = (0, 0, 1)$. In case of the tangent vector $\mathbf{t}(s)$ parallels to $\mathbf{k}(s)$, then the quasi frame becomes singular, and in this case we only change our choice to the vector $\mathbf{k}(s)$ to $\mathbf{k}(s) = (0, 1, 0)$ or $\mathbf{k}(s) = (1, 0, 0)$. The matrix expression for the relationship between quasi and Frenet vector fields is

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix},$$

where θ is an Euclidean angle between the principal normal $\mathbf{n}(s)$ and the quasi-normal $\mathbf{n}_q(s)$. And the relation between quasi and Frenet curvatures are

$$K_1 = \kappa \cos\theta, \quad K_2 = -\kappa \sin\theta, \quad K_3 = d\theta + \tau,$$

where κ and τ are the curvature and torsion of Frenet frame. And $\{K_i | i = 1, 2, 3\}$ are the first, second and third curvatures of the quasi frame, respectively.

Every quasi curve admits a quasi frame filed $\{\mathbf{t}(s), \mathbf{n}_q(s), \mathbf{b}_q(s)\}$ which is orthogonal filed along $\xi(s)$ satisfying the quasi-equations

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{pmatrix}_s = \begin{pmatrix} 0 & \epsilon_1 K_1 & \epsilon_2 K_2 \\ \epsilon_3 K_1 & 0 & \epsilon_4 K_3 \\ \epsilon_5 K_2 & \epsilon_6 K_3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{pmatrix}, \tag{2.2}$$

According to Eq (2.2)

- If $\epsilon_2 = 1$ and $\epsilon_i = -1$ $(\forall i = 1, 3, 4, 5, 6)$, then the curve $\xi(s)$ is a spacelike quasi curve with a timelike

quasi-normal,
• If $\epsilon_i = 1$ $(\forall i = 1, 4, 6)$ and $\epsilon_j = -1$ $(\forall j = 2, 3, 5)$, then the curve $\xi(s)$ is a spacelike quasi curve with a timelike quasi-binormal,
• If $\epsilon_i = 1$ $(\forall i = 1, 2, 3, 5, 6)$ and $\epsilon_4 = -1$, then the curve $\xi(s)$ is a timelike quasi curve.

If $K_3 = 0$ or $K_2 = 0$ we get Bishop or Frenet equations, respectively, which means the quasi frame and equations are more general than both Bishop and Frenet.

3. The geometric characteristics of quasi-Hasimoto surfaces in $\mathbb{E}_1^3$

In this section, we investigate the geometric characteristics of quasi-Hasimoto surfaces for the three cases of non-lightlike curves in $\mathbb{E}_1^3$, discuss the conditions under which quasi-Hasimoto surfaces become developable, and compare our results with those of previous studies of Hasimoto surfaces according to other frames to ensure that the quasi frame and equations are more efficient and general.

The vortex filament or smoke ring equation depicts the movement of a thin vortex in a thin, viscous fluid as a curve in $\mathbb{E}_1^3$.

$$\Gamma_t = \Gamma_s \times \Gamma_{ss}. \tag{3.1}$$

This connection also applies to a dynamical system on the space of curves in $\mathbb{E}_1^3$.

Proposition 3.1. *Let $\Gamma = \Gamma(s,t)$ be a vortex filament solution in $\mathbb{E}_1^3$ and $\Gamma(s,0)$ is a vortex filament solution parameterized by arc-length, then $\Gamma = \Gamma(s,t)$ is a vortex filament solution for all t.*

Proof. It is sufficient to show $g_t(\Gamma_s, \Gamma_s) = 0$ for all solutions of Eq (3.1)

$$\begin{aligned} g_t(\Gamma_s, \Gamma_s) &= 2g((\Gamma_t)_s, \Gamma_s) \\ &= 2g((\Gamma_s \times \Gamma_{ss})_s, \Gamma_s) \\ &= 2g(\Gamma_{ss} \times \Gamma_{ss} + \Gamma_s \times \Gamma_{sss}, \Gamma_s) = 0. \end{aligned}$$

□

Now, we demonstrate the geometric interpretation of Eq (3.1) on the space of curves in $\mathbb{E}_1^3$
• The motion fulfilling Equation (3.1) yields a spacelike quasi-Hasimoto surface if $\Gamma = \Gamma(s,t)$ is a spacelike quasi curve with timelike quasi-normal for all t,
• The motion fulfilling Equation (3.1) yields a timelike quasi-Hasimoto surface if $\Gamma = \Gamma(s,t)$ is a spacelike quasi curve with timelike quasi-binormal for all t,
• The motion fulfilling Equation (3.1) yields a timelike quasi-Hasimoto surface if $\Gamma = \Gamma(s,t)$ is a timelike quasi curve for all t.

First two cases are relevant to *non-linear heat system*; see [2]

$$q_t = q_{ss} + q^2\Gamma, \quad \Gamma_t = -\Gamma_{ss} - \Gamma^2 q.$$

For the third case, by a solution of *repulsive type non linear Schröinger equation*

$$iq_t = -q_{ss} + 2|q|^2 q,$$

the Hasimoto surface has been determined in $\mathbb{E}_1^3$. By the motion of a timelike quasi-geodesic coordinates corresponds to NLS-equation, a timelike quasi-Hasimoto surfaces generated, was deeply discussed according to Frenet frame in Minkowski 3-space in [6].

Definition 3.1. *We define $\Gamma = \Gamma(s, t)$ is a quasi-Hasimoto surface of*
• ***type 1*** *if Γ is a unit speed spacelike quasi-curve with a timelike quasi-normal vector field for all t,*
• ***type 2*** *if Γ is a a unit speed spacelike quasi-curve with a timelike quasi-binormal vector field for all t,*
• ***type 3*** *if Γ is a unit speed timelike quasi-curve for all t.*

Theorem 3.1. *Let $\Gamma = \Gamma(s, t)$ be of **type 1**, therefore the subsequent conditions are true*

$$i.\begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n}_q \\ \boldsymbol{b}_q \end{pmatrix}_s = \begin{pmatrix} 0 & -K_1 & K_2 \\ -K_1 & 0 & -K_3 \\ -K_2 & -K_3 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n}_q \\ \boldsymbol{b}_q \end{pmatrix},$$

$$ii.\begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n}_q \\ \boldsymbol{b}_q \end{pmatrix}_t = \begin{pmatrix} 0 & \alpha & \beta \\ \alpha & 0 & \gamma \\ -\beta & \gamma & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n}_q \\ \boldsymbol{b}_q \end{pmatrix},$$

where $\{\boldsymbol{t}, \boldsymbol{n}_q, \boldsymbol{b}_q\}$ is the quasi frame field, $\{K_i | i = 1, 2, 3\}$ are the curvature functions of the curve Γ for all t, and

$$\alpha = -K_{2s} - K_1 K_3,$$
$$\beta = K_{1s} + K_2 K_3,$$
$$\gamma = \frac{1}{K^*}\Big[- K_1^2 K_3^2 + K_2(K_2 K_3^2 + 2K_3 K_{1s} + K_{2ss} - K_{1t}) + K_1(-2K_3 K_{2s} - K_{1ss} + K_{2t})\Big],$$

where $K^* = K_1^2 - K_2^2$.

Proof. i. This is clearly obtained from Eq (2.2) under choice $\epsilon_2 = 1$ and $\epsilon_i = -1$ ($\forall i = 1, 3, 4, 5, 6$).

ii. There are random functions α, β, and γ such that α, β, and γ satisfy

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{pmatrix}_t = \begin{pmatrix} 0 & \alpha & \beta \\ \alpha & 0 & \gamma \\ -\beta & \gamma & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{pmatrix}.$$

We need to find these functions in terms of $\{K_i | i = 1, 2, 3\}$ as a solution to the curve $\Gamma = \Gamma(s, t)$ of vortex filament for all t. By using compatibility condition $\mathbf{t}_{ts} = \mathbf{t}_{st}$ and $(\mathbf{n}_q)_{ts} = (\mathbf{n}_q)_{st}$, we get

$$\alpha_s = -K_{1t} + \gamma K_2 + \beta K_3, \tag{3.2a}$$
$$\beta_s = K_{2t} - \gamma K_1 + \alpha K_3, \tag{3.2b}$$
$$\gamma_s = -K_{3t} - \beta K_1 - \alpha K_2. \tag{3.2c}$$

Now, we suppose that the velocity of the curve is

$$\Gamma_t = \lambda \mathbf{t} + \mu \mathbf{n}_q + \nu \mathbf{b}_q.$$

By using the compatibility criterion $\Gamma_{ts} = \Gamma_{st}$, We obtain the subsequent

$$0 = \lambda_s - \mu K_1 - \nu K_2, \tag{3.3a}$$
$$\alpha = \mu_s - \lambda K_1 - \nu K_3, \tag{3.3b}$$

$$\beta = \nu_s + \lambda K_2 - \mu K_3. \tag{3.3c}$$

By multiplying the (3.2a) by K_2, (3.2b) by K_1 and add them, we get

$$\gamma = \frac{1}{K^*}\Big[\alpha_s K_2 + \beta_s K_1 + K_{1t}K_2 - K_{2t}K_1 - K_3(\beta K_2 + \alpha K_1)\Big], \tag{3.4}$$

where $K^* = K_1^2 - K_2^2$. For the solution of vortex filament, the velocity vector is

$$\begin{aligned}\Gamma_t = \Gamma_s \times \Gamma_{ss} &= \mathbf{t} \times (-K_1\mathbf{n}_q + K_2\mathbf{b}_q)\\ &= -K_2\mathbf{n}_q + K_1\mathbf{b}_q.\end{aligned}$$

Here, $(\lambda, \mu, \nu) \to (0, -K_2, K_1)$, then by substituting into Eqs (3.3b), (3.3c) and (3.4), we get

$$\begin{aligned}&\alpha = -K_{2s} - K_1K_3,\\ &\beta = K_{1s} + K_2K_3,\\ &\gamma = \frac{1}{K^*}\Big[- K_1^2K_3^2 + K_2(K_2K_3^2 + 2K_3K_{1s} + K_{2ss} - K_{1t}) + K_1(-2K_3K_{2s} - K_{1ss} + K_{2t})\Big].\end{aligned}$$

□

Corollary 3.1. *Let $\Gamma = \Gamma(s,t)$ be of* ***type 1****, then the quasi-Gaussian K_q and quasi-mean H_q curvatures of Γ are provided by:*

$$\begin{aligned}K_q &= \frac{K_2^2\left(K_{1s}^2 + K_1K_{1ss}\right) + K_1K_2\left(K_1K_{2ss} - 2K_{1s}K_{2s}\right) + K_1^2\left(K_{2s}^2 - K_1K_{1ss}\right) - K_2^3K_{2ss}}{K^2\left(K_1^2 - K_2^2\right)},\\ H_q &= \frac{K_1\left(2K_3K_{2s} + K_{1ss}\right) + K_2\left(-2K_3K_{1s} - K_{2ss} + K_2^3 - K_3^2K_2\right) + K_1^4 + \left(K_3^2 - 2K_2^2\right)K_1^2}{2K\left(K_1^2 - K_2^2\right)},\end{aligned}$$

where $K = \parallel \Gamma_s \times \Gamma_t \parallel = \sqrt{|-K_1^2 + K_2^2|}$.

Proof. The E, F and G coefficients of the first fundamental form are

$$\begin{aligned}E &= g(\Gamma_s, \Gamma_s) = 1,\\ F &= g(\Gamma_s, \Gamma_t) = 0,\\ G &= g(\Gamma_t, \Gamma_t) = K_1^2 - K_2^2.\end{aligned}$$

The normal vector field of the surface is

$$N = \frac{\Gamma_s \times \Gamma_t}{\parallel \Gamma_s \times \Gamma_t \parallel} = \frac{\mathbf{t} \times (-K_2\mathbf{n}_q + K_2\mathbf{b}_q)}{\parallel \mathbf{t} \times (-K_2\mathbf{n}_q + K_2\mathbf{b}_q) \parallel} = \frac{1}{K}[-K_1\mathbf{n}_q + K_2\mathbf{b}_q],$$

where $K = \parallel \Gamma_s \times \Gamma_t \parallel = \sqrt{|-K_1^2 + K_2^2|}$.

The second fundamental form coefficients are

$$e = g(\Gamma_{ss}, N) = \frac{1}{K}[-K_1^2 + K_2^2],$$
$$f = g(\Gamma_{st}, N) = \frac{1}{K}[\alpha K_1 + \beta K_2],$$
$$g = g(\Gamma_{tt}, N) = \frac{1}{K}[K_2 K_{1t} - K_1 K_{2t} + \gamma(K_1^2 - K_2^2)].$$

The quasi-Gaussian K_q and quasi-mean H_q are determined by

$$K_q = g(n_q, n_q)\frac{eg - f^2}{EG - F^2},$$
$$H_q = g(n_q, n_q)\frac{Eg - 2fF + Ge}{2(EG - F^2)}.$$

□

Corollary 3.2. *If we put $K_3 = 0$ in Corollary (3.1), then the quasi-Gaussian curvature does not change, while the quasi-mean curvature is denoted by*

$$H_q = \frac{K_1 K_{1ss} + K_2(K_2^3 - K_{2ss}) + K_1^4 - 2K_2^2 K_1^2}{2K(K_1^2 - K_2^2)}.$$

Which are the results according to the Bishop frame.

Theorem 3.2. *Let $\Gamma = \Gamma(s,t)$ be of **type 2**, therefore the subsequent conditions are true*

$$i. \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n}_q \\ \boldsymbol{b}_q \end{pmatrix}_s = \begin{pmatrix} 0 & K_1 & -K_2 \\ -K_1 & 0 & K_3 \\ -K_2 & K_3 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n}_q \\ \boldsymbol{b}_q \end{pmatrix},$$
$$ii. \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n}_q \\ \boldsymbol{b}_q \end{pmatrix}_t = \begin{pmatrix} 0 & \delta & \zeta \\ -\delta & 0 & \eta \\ \zeta & \eta & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n}_q \\ \boldsymbol{b}_q \end{pmatrix},$$

where

$$\delta = -K_{2s} + K_1 K_3,$$
$$\zeta = -K_2 K_3 + K_{1s},$$
$$\eta = \frac{-1}{K^*}\Big[- K_2\left(-2K_3 K_{1s} + K_{2ss} + K_{1t} + K_2 K_3^2\right) + K_1\left(-2K_3 K_{2s} + K_{1ss} + K_{2t}\right) + K_1^2 K_3^2\Big],$$

where $K^* = K_1^2 - K_2^2$.

Corollary 3.3. *Let $\Gamma = \Gamma(s,t)$ be of **type 2**, then the quasi-Gaussian K_q and quasi-mean H_q curvatures of Γ are provided by:*

$$K_q = \frac{K_2^2\left(K_{1s}^2 + K_1 K_{1ss}\right) + K_1 K_2\left(K_1 K_{2ss} - 2K_{1s} K_{2s}\right) + K_1^2\left(K_{2s}^2 - K_1 K_{1ss}\right) - K_2^3 K_{2ss}}{K^2\left(K_1^2 - K_2^2\right)},$$
$$H_q = \frac{K_1\left(2K_3 K_{2s} - K_{1ss}\right) + K_2\left(-2K_3 K_{1s} + K_{2ss} + K_2^3 + K_3^2 K_2\right) + K_1^4 - \left(2K_2^2 + K_3^2\right) K_1^2}{2K\left(K_1^2 - K_2^2\right)},$$

where $K = \| \Gamma_s \times \Gamma_t \| = \sqrt{|K_1^2 - K_2^2|}$.

Corollary 3.4. *If we put* $K_3 = 0$ *in Corollary (3.3), then the quasi-Gaussian curvature does not change, while the quasi-mean curvature is denoted by*

$$H_q = \frac{1}{2K(K_1^2 - K_2^2)^2}\Big[K_2K_1^2(-K_{2ss} - 2K_{1t} + 3K_2^3) + K_2^2K_1(K_{1ss} + 2K_{2t}) + K_1^3K_{1ss} - K_2^3(K_{2ss} + K_2^3) + K_1^6 - 3K_2^2K_1^4\Big].$$

Which are the results according to the Bishop frame.

Theorem 3.3. *Let* $\Gamma = \Gamma(s,t)$ *be of* ***type 3****, therefore the subsequent conditions are true*

$$i. \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n}_q \\ \boldsymbol{b}_q \end{pmatrix}_s = \begin{pmatrix} 0 & K_1 & K_2 \\ K_1 & 0 & -K_3 \\ K_2 & K_3 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n}_q \\ \boldsymbol{b}_q \end{pmatrix},$$

$$ii. \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n}_q \\ \boldsymbol{b}_q \end{pmatrix}_t = \begin{pmatrix} 0 & \phi & \psi \\ \phi & 0 & \xi \\ \psi & -\xi & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n}_q \\ \boldsymbol{b}_q \end{pmatrix},$$

where

$$\phi = -K_{2s} + K_1K_3,$$
$$\psi = K_2K_3 + K_{1s},$$
$$\xi = \frac{-1}{K^*}\Big[K_2\left(2K_3K_{1s} - K_{2ss} - K_{1t} + K_2K_3^2\right) + K_1\left(2K_3K_{2s} + 2K_2K_{3s} + K_{1ss} - K_{2t}\right) - K_1^2K_3^2\Big],$$

where $K^* = K_1^2 - K_2^2$.

Corollary 3.5. *Let* $\Gamma = \Gamma(s,t)$ *be of* ***type 3****, then its quasi-Gaussian* K_q *and quasi-mean* H_q *curvatures are provided by*

$$K_q = \frac{1}{K^2}\Big[\frac{1}{K_1^2 + K_2^2}\Big(-K_1K_{2s} + K_2(K_{1s} + K_2K_3) + K_3K_1^2\Big)^2 - K_2K_{1t} + K_1K_{2t} - \frac{1}{K_1^2 - K_2^2}\Big((K_1^2 + K_2^2)\Big(K_2(-K_3(2K_{1s} + K_2K_3) + K_{2ss} + K_{1t}) + K_1^2K_3^2 - K_1(2K_3K_{2s} + 2K_2K_{3s} + K_{1ss} - K_{2t})\Big)\Big)\Big],$$

$$H_q = \frac{-1}{2K(K_1^2 + K_2^2)}\Big[\frac{1}{K_1^2 - K_2^2}\Big((K_1^2 + K_2^2)\Big(K_2(-K_3(2K_{1s} + K_2K_3) + K_{2ss} + K_{1t}) - K_1(2K_3K_{2s} + 2K_2K_{3s} + K_{1ss} - K_{2t}) + K_1^2K_3^2\Big)\Big) + K_2K_{1t} - K_1K_{2t} - (K_1^2 + K_2^2)^2\Big],$$

where $K = \| \Gamma_s \times \Gamma_t \| = \sqrt{|K_1^2 + K_2^2|}$.

Corollary 3.6. *If we put $K_3 = 0$ in Corollary (3.5), then the quasi-Gaussian curvature and quasi-mean curvature are determined by*

$$K_q = \frac{\frac{(K_2K_{1s}-K_1K_{2s})^2}{K_1^2+K_2^2} - \frac{(K_1^2+K_2^2)(K_2(K_{2ss}+K_{1t})-K_1(K_{1ss}-K_{2t}))}{K_1^2-K_2^2} - K_2K_{1t} + K_1K_{2t}}{K^2},$$
$$H_q = \frac{-K_1K_{1ss} + K_2(K_{2ss} + K_2^3) + K_1^4 + 2K_2^2K_1^2}{2K(K_1^2 + K_2^2)}.$$

Which are the results according to the Bishop frame.

Quasi-Hasimoto surface with parameterization $\Gamma = \Gamma(s,t)$ in $\mathbb{E}_1^3$ is a developable surface if it can be flattened onto a plane without distortion, i.e., the quasi-Gaussian curvature K_q is zero.

Corollary 3.7. *The quasi-Hasimoto surface parametrized by $\Gamma = \Gamma(s,t)$ in $\mathbb{E}_1^3$ is developable if and only if*
*• for **type 1** and **type 2**:*

$$K_2^2(K_{1s}^2 + K_1K_{1ss}) + K_1K_2(K_1K_{2ss} - 2K_{1s}K_{2s}) + K_1^2(K_{2s}^2 - K_1K_{1ss}) - K_2^3K_{2ss} = 0.$$

*• for **type 3**:*

$$\frac{1}{K_1^2 + K_2^2}\Big(-K_1K_{2s} + K_2(K_{1s} + K_2K_3) + K_3K_1^2\Big)^2 - K_2K_{1t} + K_1K_{2t}$$
$$-\frac{1}{K_1^2 - K_2^2}\Big((K_1^2 + K_2^2)\Big(K_2(-K_3(2K_{1s} + K_2K_3) + K_{2ss} + K_{1t}) + K_1^2K_3^2$$
$$-K_1(2K_3K_{2s} + 2K_2K_{3s} + K_{1ss} - K_{2t})\Big)\Big) = 0.$$

Proof. The proof comes directly form the results of Corollaries (3.1), (3.3) and (3.5). □

Corollary 3.8. *If we put $K_3 = 0$ in Corollary (3.7), then the quasi-Hasimoto surface is developable if and only if*
*• for **type 1** and **type 2**, the same result as Corollary (3.7).*
*• for **type 3***

$$\frac{(K_2K_{1s} - K_1K_{2s})^2}{K_1^2 + K_2^2} - \frac{(K_1^2 + K_2^2)(K_2(K_{2ss} + K_{1t}) - K_1(K_{1ss} - K_{2t}))}{K_1^2 - K_2^2} - K_2K_{1t} + K_1K_{2t} = 0.$$

Which are the results according to the Bishop frame.

4. Characterizing the parameter curves of a quasi-Hashimoto surface in $\mathbb{E}_1^3$

In this section, we characterize the parametric curves of quasi-Hasimoto surfaces in $\mathbb{E}_1^3$. Then we provide sufficient and necessary criteria for the s-parameter and t-parameter curves of quasi-Hasimoto surfaces in $\mathbb{E}_1^3$ to be geodesics, asymptotics, and lines of curvature.

Definition 4.1. *The quasi-Hasimoto surface with parameterization $\Gamma = \Gamma(s,t)$ in $\mathbb{E}^3_1$, for s-parameter curves of the surface, is said to be*
• *geodesic if the second derivative of position vector with respect to s, Γ_{ss}, takes the same direction of the normal to the surface N i.e. the quasi-geodesic curvature is zero, $(K_q)_g = 0$,*
• *asymptotics if the normal curvature K_n is equal to zero i.e. $K_n = g(\Gamma_{ss}, N) = 0$,*
• *lines of curvatures if $g(\Gamma_{st}, N) = g(\Gamma_s, \Gamma_t) = 0$.*

Definition 4.2. *The quasi-Hasimoto surface with parameterization $\Gamma = \Gamma(s,t)$ in $\mathbb{E}^3_1$, for t-parameter curves of the surface, is said to be*
• *geodesic if the triple scalar product of the second derivative of the curve with respect to t, Γ_{tt}, the normal to the surface N, and the tangent to the surface with respect to* ***t***, *Γ_t, is equal to zero i.e., the quasi-geodesic curvature is zero, $(K_q)_g = g(\Gamma_{tt}, N \times \boldsymbol{t}) = 0$*
• *asymptotic if the normal curvature K_n is equal to zero i.e. $K_n = g(\Gamma_{tt}, N) = 0$.*

Definition 4.3. *In quasi-Hasimoto surface with parameterization $\Gamma = \Gamma(s,t)$ in $\mathbb{E}^3_1$, the family of all s and t-parameter curves is denoted by Λ and Ω, respectively.*

Theorem 4.1. *Suppose $\Gamma = \Gamma(s,t)$ is a quasi-Hasimoto surface in $\mathbb{E}^3_1$, then the followings conditions are satisfied*
i. All curves in Λ are geodesics,
ii. All curves in Ω are geodesics if and only if $K_1K_{1t} = \eta_i K_2 K_{2t}$, where $\{K_i | i = 1,2,3\}$ are the curvature functions of the curve Γ for all t, and $\eta_i = \pm 1$.

Proof. Suppose $\Gamma = \Gamma(s,t)$ be of **type 1**.
i. By Theorem (3.1) and its results, we know that

$$\Gamma_{ss} = \mathbf{t}_s = -K_1\mathbf{n}_q + K_2\mathbf{b}_q.$$

And the normal to the surface Γ is

$$N = \frac{1}{K}[-K_1\mathbf{n}_q + K_2\mathbf{b}_q],$$

where $K = \| \Gamma_s \times \Gamma_t \| = \sqrt{|-K_1^2 + K_2^2|}$. Thus, Γ_{ss} is parallel to the normal of the surface which means all curves in Λ are geodesics.

ii. By Theorem (3.1) and its results, we know that

$$\Gamma_{tt} = (-\alpha K_2 - \beta K_1)\mathbf{t} + (-K_{2t} + \gamma K_1)\mathbf{n}_q + (-\gamma K_2 + K_{1t})\mathbf{b}_q,$$

where the values of functions α, β and γ are given as above. Then $g(\Gamma_{tt}, N \times t) = 0$ if and only if $K_1K_{1t} = K_2K_{2t}$.

The proof is similar if Γ is a quasi-Hasimoto surface of **type 2** and **type 3**. □

Corollary 4.1. *According to Theorem (4.1), if we put $K_3 = 0$, the results according to quasi frame is the same as Bishop frame in $\mathbb{E}^3_1$.*

Theorem 4.2. *Let $\Gamma = \Gamma(s,t)$ be a quasi-Hasimoto surface in $\mathbb{E}^3_1$, then the followings conditions are satisfied:*

i. All curves in Λ are asymptotics of the surface if and only if $K = 0$,
ii. All curves in Ω are asymptotics of the surface if and only if

- Γ *of **type 1**, then*

$$-K_1(2K_3K_{2s} + K_{1ss}) + K_2(2K_3K_{1s} + K_{2ss} + K_2K_3^2) - K_1^2K_3^2 = 0.$$

- Γ *of **type 2**, then*

$$K_2K_1^2(-2K_3K_{1s} + K_{2ss} + 2K_{1t}) + K_2^2K_1(2K_3K_{2s} - K_{1ss} - 2K_{2t}) \\ + K_1^3(2K_3K_{2s} - K_{1ss}) + K_2^3(-2K_3K_{1s} + K_{2ss} + K_2K_3^2) - K_3^2K_1^4 = 0.$$

- Γ *of **type 3**, then*

$$-K_1(2K_3K_{2s} + 2K_2K_{3s} + K_{1ss}) + K_2(K_{2ss} - K_3(2K_{1s} + K_2K_3)) + K_1^2K_3^2 = 0.$$

Proof. Let $\Gamma = \Gamma(s,t)$ be of **type 1**.

i. By Theorem (3.1) and its results, we know that

$$\Gamma_{ss} = \mathbf{t}_s = -K_1\mathbf{n}_q + K_2\mathbf{b}_q.$$

And the normal to the surface Γ is

$$N = \frac{1}{K}[-K_1\mathbf{n}_q + K_2\mathbf{b}_q],$$

where $K = \| \Gamma_s \times \Gamma_t \| = \sqrt{|-K_1^2 + K_2^2|}$. Then $g(\Gamma_{ss}, N) = 0 \Leftrightarrow K = 0$.

ii. By Theorem (3.3) and its results, we know that

$$\Gamma_{tt} = (-\alpha K_2 - \beta K_1)\mathbf{t} + (-K_{2t} + \gamma K_1)\mathbf{n}_q + (-\gamma K_2 + K_{1t})\mathbf{b}_q,$$

where the values of functions α, β and γ are given as above. Then $g(\Gamma_{tt} \times N, \Gamma_t) = 0 \Leftrightarrow$

$$-K_1(2K_3K_{2s} + K_{1\mathrm{ss}}) + K_2(2K_3K_{1s} + K_{2\mathrm{ss}} + K_2K_3^2) - K_1^2K_3^2 = 0.$$

The proof is similar if Γ is a quasi-Hasimoto surface of **type 2** and **type 3**. □

Corollary 4.2. *According to Theorem (4.2), if we put $K_3 = 0$, then the followings conditions are satisfied*
i. All curves in Λ are asymptotic of the surface if and only if $K = 0$,
ii. All curves in Ω are asymptotic of the surface if and only if

- Γ *of **type 1** and **type 2**, then*

$$K_1K_{1ss} = K_2K_{2ss}.$$

- Γ *of **type 3**, then*

$$K_2K_1^2(K_{2ss} + 2K_{1t}) + K_2^2K_1(-K_{1ss} - 2K_{2t}) - K_1^3K_{1ss} + K_2^3K_{2ss} = 0.$$

Which are the results according to the Bishop frame.

Corollary 4.3. *If all curves in Λ are asymptotics, then the all curves in Ω are also asymptotics if and only if $K_2 = 0$.*

Corollary 4.4. *All curves in Λ and Ω of a quasi-Hasimoto surface $\Gamma = \Gamma(s,t)$ in $\mathbb{E}_1^3$ are lines of curvature if and only if*

$$K_2(\varepsilon_i K_2 K_3 + K_{1s}) + K_1(\varepsilon_i K_1 K_3 - K_{2s}) = 0,$$

where $\varepsilon_i = \pm 1$.

Proof. For quasi-Hasimoto surface in $\mathbb{E}_1^3$, we know $f = F = 0$ if and only if

$$K_2(\varepsilon_i K_2 K_3 + K_{1s}) + K_1(\varepsilon_i K_1 K_3 - K_{2s}) = 0.$$

□

Corollary 4.5. *If we put $K_3 = 0$ in Corollary (4.4), then the all curves in Λ and Ω of a quasi-Hasimoto surface $\Gamma = \Gamma(s,t)$ in $\mathbb{E}_1^3$ are lines of curvature if and only if $K_2 K_{1s} = K_1 K_{2s}$. Which are the results according to the Bishop frame.*

Corollary 4.6. *In whole results of this paper, if we put $K_2 = 0$, then we get the same results of Hasimoto surface according to Frenet-Serret frame; Check [4].*

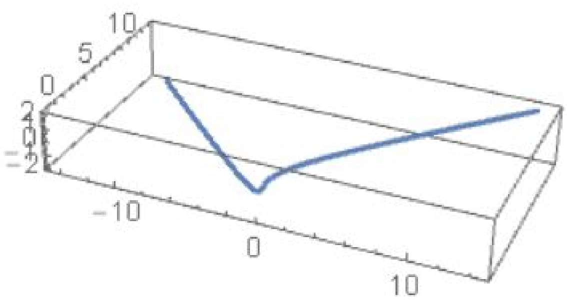
5. Application

In this section, we introduce some quasi-curves and their correspondence quasi-Hasimoto surfaces. Each model is arbitrarily chosen to satisfy the criteria of the instance we wish to depict, and the associated quasi-Hasimoto surface is evolved using the introduced calculations. The selected curves simulate the real-world situation that we wish to convey. This study omits the actual circumstance and focuses on the approach used to create the simulation.

Based on Theorem (3.1), we introduce the curve of **type 1** as

$$\Gamma(s) = \frac{1}{5}\Big(\cosh(\sqrt{5}\, s),\ \sqrt{20}\, s, \sinh(\sqrt{5}\, s)\Big),$$

and its correspondence spacelike quasi-Hasimoto surface (See Figure 1).



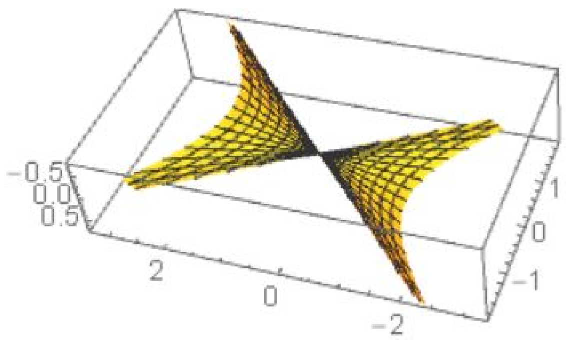


Figure 1. The unit speed spacelike quasi curve with timelike quasi-normal vector field and its correspondence spacelike quasi-Hasimoto surface.

Based on Theorem (3.2), we introduce the curve of **type 2** as

$$\Gamma(s) = \frac{1}{3}\Big(2\ \sinh(\sqrt{3}\ s),\ \sqrt{3}\ s, 2\ \cosh(\sqrt{3}\ s)\Big),$$

and its correspondence timelike quasi-Hasimoto surface (See Figure 2).

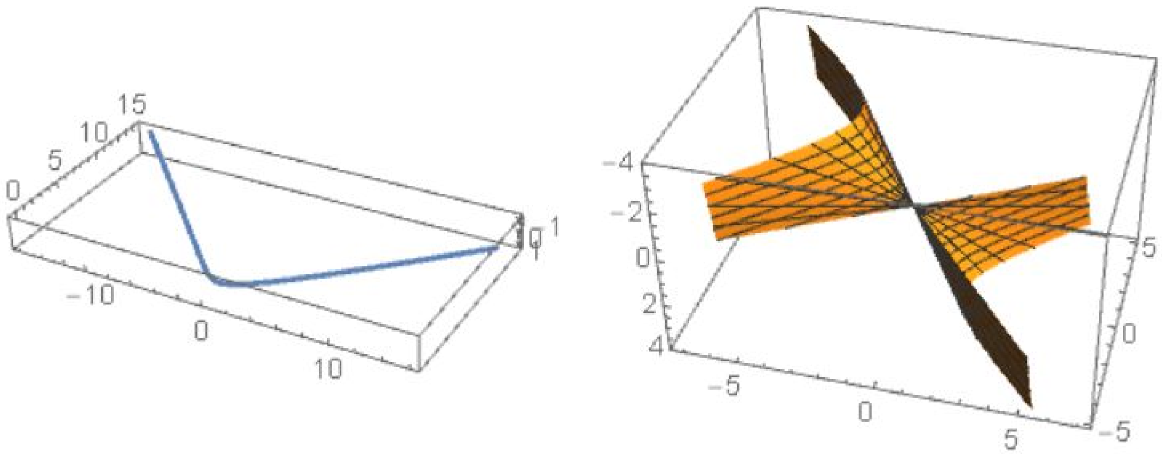


Figure 2. The unit speed spacelike quasi curve with timelike quasi-binormal vector field and its correspondence timelike quasi-Hasimoto surface.

Based on Theorem (3.3), we introduce the curve of **type 3** as

$$\Gamma(s) = \frac{1}{3}\Big(\sqrt{12}\ s, \cos(\sqrt{3}\ s), \sin(\sqrt{3}\ s)\Big),$$

and its correspondence timelike quasi-Hasimoto surface (See Figure 3).

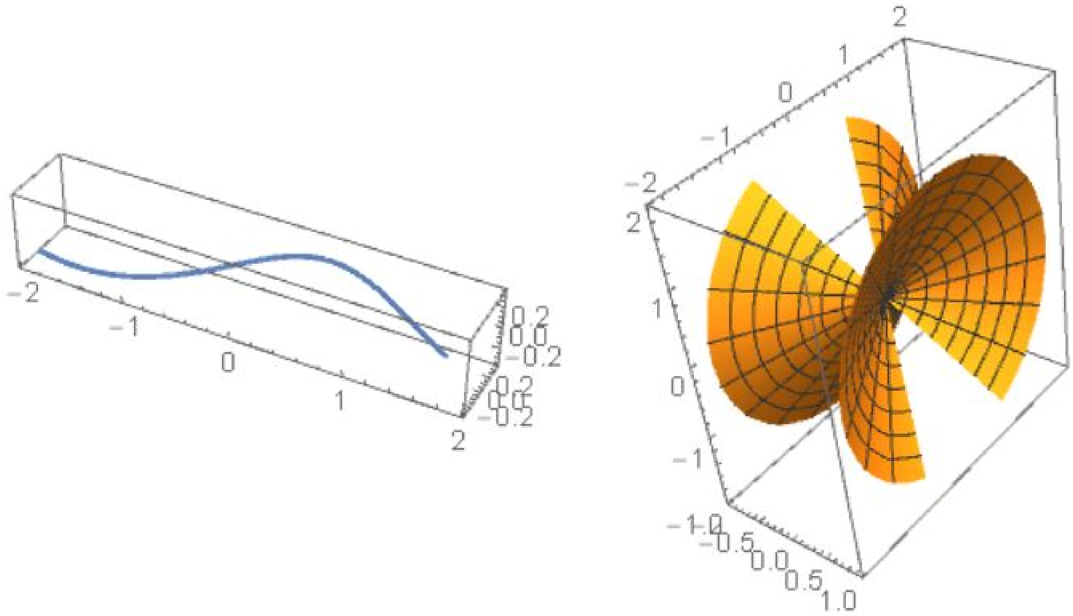


Figure 3. The unit speed timelike quasi curve and its correspondence timelike quasi-Hasimoto surface.

6. Conclusions

The quasi-frame and equations are more efficient and general than Frenet and Bishop. In the case of Frenet, the quasi is defined at all points. In the case of Bishop, the quasi gives more accuracy and easier in computation.

In this paper, we investigated the geometry of quasi-Hasimoto surfaces in Minkowski 3-space $\mathbb{E}_1^3$. For the three situations of non-lightlike curves, we examined the geometric features of the quasi-Hasimoto surfaces in $\mathbb{E}_1^3$ and determined the Gaussian and mean curvatures for each case. we showed the necessary and sufficient condition of the quasi-Hasimoto surfaces in $\mathbb{E}_1^3$ to be developable surfaces. As a result, we described the parameter curves of quasi-Hasimoto surfaces in $\mathbb{E}_1^3$. Thus, we discussed the necessary and sufficient conditions of s-parameter and t-parameter curves of quasi-Hasimoto surfaces in $\mathbb{E}_1^3$ to be geodesics, asymptotic, and curvature lines. Finally, we discussed quasi curves and associated quasi-Hasimoto surface correspondences.

This studying is more general, efficient and a new contribution to the field. Especially, for the pure binormal motion of curves, in the future, it may be needed for some specific applications in studying surfaces as Hasimoto, Razzaboni, etc., and in many areas of science.

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Conflict of interest

The authors state that they have no known competing financial interests or personal ties that could appear to have influenced the research reported in this study.

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