Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(10): 17843–17858.
DOI: 10.3934/math.2022983
Received: 05 May 2022
Revised: 23 June 2022
Accepted: 13 July 2022
Published: 03 August 2022

Research article

On the Blaschke approach of Bertrand offsets of spacelike ruled surfaces

Maryam T. Aldossary[1] **and Rashad A. Abdel-Baky**[2,*]

[1] Department of Mathematics, College of Science, Imam Abdulrahman bin Faisal University, P.O. Box-1982, 31441 Damam, Saudi Arabia

[2] Department of Mathematics, Faculty of Science, University of Assiut, 71516 Assiut, Egypt

* **Correspondence:** Email: rbaky@live.com.

Abstract: In this paper using the Blaschke approach we generalized the Bertrand curves to spacelike ruled and developable surfaces. It is proved that, every spacelike ruled surface have a Bertrand offset if and only if an equation should be fulfilled among their dual integral invariants. Consequently, some new relationships and theorems for the developability of the Bertrand offsets of spacelike ruled surfaces are outlined.

Keywords: E. Study map; Bertrand offsets; striction curves
Mathematics Subject Classification: 53A04, 53A05, 53A17

1. Introduction

In Euclidean 3-space $\mathbb{E}^3$, the trajectories of oriented lines associated with a moving rigid body are generally ruled surfaces. The differential line geometry of ruled surfaces has been exceedingly utilized in Computer-Aided Manufacturing (CAM), Computer-Aided Geometric Design (CAGD), geometric modeling and kinematics [1–3]. Nowadays, both offsets surfaces and ruled surfaces have been studied in both of Euclidean and non-Euclidean spaces; for example, Ravani and Ku [4] defined the theory of Bertrand curves for Bertrand ruled surface-offsets depend on line geometry. They showed that a ruled surface can have an infinity of Bertrand offsets, in the same way as a planar curve can have an infinity of Bertrand mates. In view of this study, Küçük and Gürsoy gave several characterizations of Bertrand offsets of trajectory ruled surfaces in terms of the relationships among the projection areas for the spherical images of Bertrand offsets and their integral invariants [5] . In [6] Kasap and Kuruoglu gained the relationships between integral invariants of the pairs of the Bertrand ruled surface in Euclidean 3-space $\mathbb{E}^3$. In [7] Kasap and Kuruoglu initiated the work of Bertrand offsets of ruled surfaces in Minkowski 3-space. The involute-evolute offsets of ruled surface is defined by Kasap et al. in [8]. Orbay et al. in [9] initiated the study of Mannheim offsets of the ruled surface. Onder and Ugurlu obtained the relationships between invariants of Mannheim offsets of timelike ruled surfaces, and they gave the conditions for these surface offsets to be developable ruled surfaces [10]. These offset surfaces are defined using the geodesic Frenet frame which was given by [8] . Depend on the involute-evolute offsets of ruled surface in [10] Senturk and Yuce have calculated integral invariants of these offsets with respect to the geodesic Frenet frame [11]. Important contributions to the Bertrand offsets of these ruled surfaces have been studied in [12–18].

In this paper, a generalization of the well known theory of Bertrand curves is considered for spacelike ruled surfaces in Minkowski space $\mathbb{E}_1^3$. Using the E. Study map, two spacelike ruled surfaces which are offset in the sense of Bertrand are defined. Specially, we find how to construct the spacelike Bertrand offset from a spacelike ruled surface with vanishing dual geodesic curvature. Meanwhile, a spacelike developable surface can have a spacelike developable Bertrand offset if a linear equation holds between the curvature and torsion of its edge of regression.

2. Basic concepts

We begin with necessary concepts on the dual numbers, dual Lorentzian vectors and E. Study map (see [16–20]): An oriented (non-null) line in Minkowski 3-space $\mathbb{E}_1^3$ may be locate by a point $\boldsymbol{\alpha} \in L$ and a normalized direction non-null vector $\mathbf{x}$ of L, that is, $<\mathbf{x}, \mathbf{x}>= \pm 1$. A parametric equation of the line is:

$$L : \mathbf{y} = \boldsymbol{\alpha} + v\mathbf{x};\ v \in \mathbb{R}.$$

Then we define the moment of the non-null vector $\mathbf{x}$ with respect to a fixed origin point in $\mathbb{E}_1^3$ as:

$$\mathbf{x}^* = \mathbf{y} \times \mathbf{x} = \boldsymbol{\alpha} \times \mathbf{x}.$$

This means that $\mathbf{x}^*$ is the same for all choices of the points on L, and the non-null pair $(\mathbf{x}, \mathbf{x}^*) \in \mathbb{E}_1^3 \times \mathbb{E}_1^3$ satisfy the following relations:

$$<\mathbf{x}, \mathbf{x}>= \pm 1, \quad <\mathbf{x}^*, \mathbf{x}>=0. \tag{2.1}$$

The six components $x_i,\ x_i^*(i = 1, 2, 3)$ of $\mathbf{x}$ and $\mathbf{x}^*$are named the normalized Plűcker coordinates of the line L. Thus, the two non-null vectors $\mathbf{x}$ and $\mathbf{x}^*$ define the non-null oriented line L.

A dual number $\widehat{x}$ is a number $x + \varepsilon x^*$, where x, x^* in $\mathbb{R}$ and ε is a dual unit with the assets that $\varepsilon \neq 0$, and $\varepsilon^2 = 0$. Then the set:

$$\mathbb{D}^3 = \{\widehat{\mathbf{x}}:= \mathbf{x} + \varepsilon\mathbf{x}^* =(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)\},$$

with the Lorentzian inner product

$$<\widehat{\mathbf{x}}, \widehat{\mathbf{y}}>= \widehat{x}_1\widehat{y}_1 + \widehat{x}_2\widehat{y}_2 - \widehat{x}_3\widehat{y}_3,$$

forms the so named dual Lorentzian 3-space $\mathbb{D}_1^3$. This yields:

$$\left.\begin{array}{l} \widehat{\mathbf{f}}_1\times\widehat{\mathbf{f}}_2= -\widehat{\mathbf{f}}_3,\ \widehat{\mathbf{f}}_1 \times \widehat{\mathbf{f}}_3= -\widehat{\mathbf{f}}_2,\ \widehat{\mathbf{f}}_2\times\widehat{\mathbf{f}}_3=\widehat{\mathbf{f}}_1, \\ <\widehat{\mathbf{f}}_1, \widehat{\mathbf{f}}_1> = <\widehat{\mathbf{f}}_2,\widehat{\mathbf{f}}_2>= -<\widehat{\mathbf{f}}_3,\widehat{\mathbf{f}}_3>=1, \end{array}\right\},$$

where $\widehat{\mathbf{f}}_1$, $\widehat{\mathbf{f}}_2$, and $\widehat{\mathbf{f}}_3$, are the dual base at the origin point $\widehat{\mathbf{0}}(0, 0, 0)$ of the dual Lorentzian 3-space $\mathbb{D}_1^3$. Thereby a point $\widehat{x} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)^t$ has dual coordinates $\widehat{x}_i = (x_i + \varepsilon x_i^*) \in \mathbb{D}$. If $\widehat{\mathbf{x}}$=$\mathbf{x} + \varepsilon\mathbf{x}^*$ is a non-null dual vector the norm $\|\widehat{\mathbf{x}}\|$ of $\widehat{\mathbf{x}}$ is defined by

$$\begin{aligned}\|\widehat{\mathbf{x}}\| &= \sqrt{|<\widehat{\mathbf{x}},\widehat{\mathbf{x}}>|} = \sqrt{|<\mathbf{x},\mathbf{x}>|}+\varepsilon\frac{1}{2\sqrt{|<\mathbf{x},\mathbf{x}>|}}\frac{<\mathbf{x},\mathbf{x}>}{|<\mathbf{x},\mathbf{x}>|}2<\mathbf{x},\mathbf{x}^*> \\ &= \|\mathbf{x}\| + \varepsilon\frac{1}{\|\mathbf{x}\|}\frac{<\mathbf{x},\mathbf{x}>}{|<\mathbf{x},\mathbf{x}>|}<\mathbf{x},\mathbf{x}^*>.\end{aligned}$$

If $\mathbf{x}$ is spacelike vector, we have

$$\|\widehat{\mathbf{x}}\| = \|\mathbf{x}\| + \varepsilon\frac{1}{\|\mathbf{x}\|}<\mathbf{x},\mathbf{x}^*>= \|\mathbf{x}\|\left(1+\varepsilon\frac{1}{\|\mathbf{x}\|^2}<\mathbf{x},\mathbf{x}^*>\right).$$

If $\mathbf{x}$ is timelike vector, we have

$$\|\widehat{\mathbf{x}}\| = \|\mathbf{x}\| - \varepsilon\frac{1}{\|\mathbf{x}\|}<\mathbf{x},\mathbf{x}^*>= \|\mathbf{x}\|\left(1-\varepsilon\frac{1}{\|\mathbf{x}\|^2}<\mathbf{x},\mathbf{x}^*>\right).$$

Therefore, $\|\widehat{\mathbf{x}}\|$ is called a spacelike (resp. timelike) dual unit vector if $\|\widehat{\mathbf{x}}\|^2=1$(resp. $\|\widehat{\mathbf{x}}\|^2=-1$). The hyperbolic, and Lorentzian (de Sitter space) dual unit spheres with the center $\widehat{\mathbf{0}}$, respectively, are:

$$\mathbb{H}^2_+ = \left\{\widehat{\mathbf{x}}\in\mathbb{D}^3_1 \,|<\widehat{\mathbf{x}},\widehat{\mathbf{x}}>=\widehat{x}_1^2-\widehat{x}_2^2+\widehat{x}_3^2=-1\right\},$$

and

$$\mathbb{S}^2_1 = \left\{\widehat{\mathbf{x}}\in\mathbb{D}^3_1 \,|<\widehat{\mathbf{x}},\widehat{\mathbf{x}}>=\widehat{x}_1^2-\widehat{x}_2^2+\widehat{x}_3^2=1\right\}.$$

It is clear that

$$<\widehat{\mathbf{x}},\widehat{\mathbf{x}}>=\pm1 \Longleftrightarrow <\mathbf{x},\mathbf{x}>=\pm1, \quad <\mathbf{x}^*,\mathbf{x}>=0. \tag{2.2}$$

It follows that Eqs (2.1) and (2.2) are corresponding. Therefore, we have the following map (E. Study's map): The dual spheres are shaped as a pair of couple hyperboloids. The common asymptotic cone represents the set of null (lightlike) lines, the ring shaped hyperboloid represents the set of spacelike lines, and the oval shaped hyperboloid forms the set of timelike lines, opposite points of each hyperboloid represent the pair of opposite vectors on a line (see Figure 1). Therefore, a timelike ruled surface is then a spherical curve on $\mathbb{H}^2_+$, and a timelike (or spacelike) ruled surface is a regular curve on $\mathbb{S}^2_1$ [16–18].

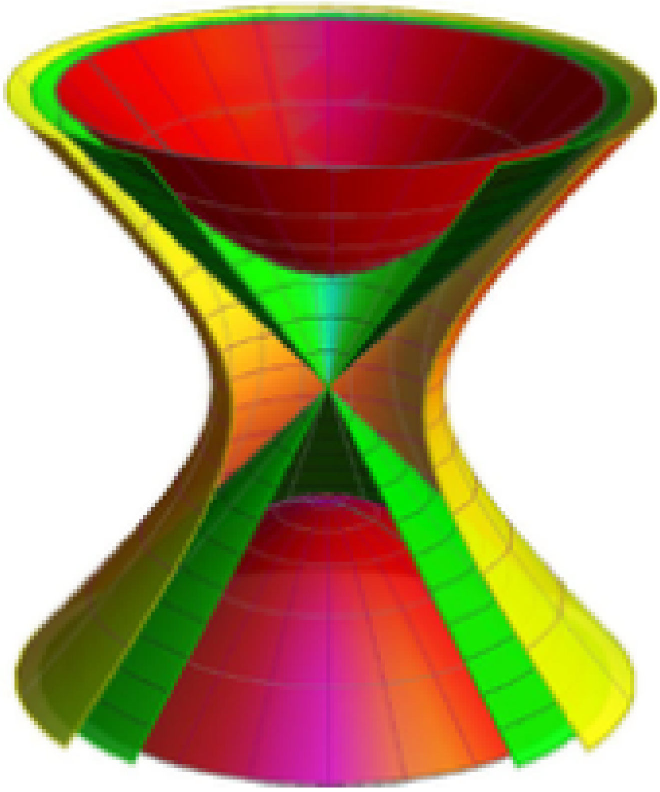

Figure 1. The dual hyperbolic and dual Lorentzian unit spheres.

3. The Blaschke approach

In this section, we define the Blaschke approach for ruled surfaces by bearing in mind the E. Study map. Therefore, based on the notations in Section 2, a smooth dual curve

$$t \in \mathbb{R} \mapsto \widehat{\mathbf{x}}(t) \in \mathbb{S}_1^2,$$

is a timelike (resp. spacelike) ruled surface in Minkowski 3-space $\mathbb{E}_1^3$. $\widehat{\mathbf{x}}(t)$ are identified with the rulings of the surface and from now on we do not discriminate among ruled surface and its parametrizing dual curve. It will be considered a spacelike ruled surface in our study, and let us denote this surface by (X). The vector

$$\widehat{\mathbf{t}}(t) = \mathbf{t} + \varepsilon \mathbf{t}^* = \frac{d\widehat{\mathbf{x}}(t)}{dt} \left\| \frac{d\widehat{\mathbf{x}}(t)}{dt} \right\|^{-1}$$

is timelike dual unit tangent vector on $\widehat{\mathbf{x}}(t)$. Introducing the spacelike dual unit vector $\widehat{\mathbf{g}}(t) = \mathbf{g}(t) + \varepsilon \mathbf{g}^*(t) = \widehat{\mathbf{x}} \times \widehat{\mathbf{t}}$ we have the moving frame $\{\widehat{\mathbf{x}}(t), \widehat{\mathbf{t}}(t), \widehat{\mathbf{g}}(t)\}$ on $\widehat{\mathbf{x}}(t)$, named Blaschke frame. Then,

$$\left.\begin{array}{c} <\widehat{\mathbf{x}}, \widehat{\mathbf{x}}> = -<\widehat{\mathbf{t}}, \widehat{\mathbf{t}}> = <\widehat{\mathbf{g}}, \widehat{\mathbf{g}}> = 1, \\ \widehat{\mathbf{g}} = \widehat{\mathbf{x}} \times \widehat{\mathbf{t}}, \widehat{\mathbf{x}} = \widehat{\mathbf{t}} \times \widehat{\mathbf{g}}, \widehat{\mathbf{t}} = \widehat{\mathbf{x}} \times \widehat{\mathbf{g}}, \end{array}\right\}$$

and the Blaschke formulae read [16]:

$$\frac{d}{dt}\begin{pmatrix}\widehat{\mathbf{x}}\\ \widehat{\mathbf{t}}\\ \widehat{\mathbf{g}}\end{pmatrix}=\begin{pmatrix}0 & \widehat{p} & 0\\ \widehat{p} & 0 & \widehat{q}\\ 0 & \widehat{q} & 0\end{pmatrix}\begin{pmatrix}\widehat{\mathbf{x}}\\ \widehat{\mathbf{t}}\\ \widehat{\mathbf{g}}\end{pmatrix}, \tag{3.1}$$

where

$$\widehat{p}(t)=p(t)+\varepsilon p^{*}(t)=\left\|\frac{d\widehat{\mathbf{x}}(t)}{dt}\right\|,\ \widehat{q}=q+\varepsilon q^{*}=-\det(\widehat{\mathbf{x}},\frac{d\widehat{\mathbf{x}}(t)}{dt},\frac{d^{2}\widehat{\mathbf{x}}(t)}{dt^{2}}),$$

are the Blaschke invariants of the timelike dual curve $\widehat{\mathbf{x}}(t)\in\mathbb{S}_{1}^{2}$. The dual unit vectors $\widehat{\mathbf{x}}$, $\widehat{\mathbf{t}}$, and $\widehat{\mathbf{g}}$ congruous to three concurrent mutually orthogonal oriented lines in $\mathbb{E}_{1}^{3}$ and they intersected at a point $\mathbf{c}$ on $\widehat{\mathbf{x}}$ named central point. The trajectory of the central points is the striction curve $\mathbf{c}(t)$ on (X).

The dual arc-length $\widehat{s}$ of $\widehat{\mathbf{x}}(t)$ is defined by

$$d\widehat{s}=ds+\varepsilon ds^{*}=\left\|\frac{d\widehat{\mathbf{x}}(t)}{dt}\right\|dt=\widehat{p}(t)dt. \tag{3.2}$$

Thus the Blaschke formulae become

$$\begin{pmatrix}\widehat{\mathbf{x}}'\\ \widehat{\mathbf{t}}'\\ \widehat{\mathbf{g}}'\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & \widehat{\gamma}\\ 0 & \widehat{\gamma} & 0\end{pmatrix}\begin{pmatrix}\widehat{\mathbf{x}}\\ \widehat{\mathbf{t}}\\ \widehat{\mathbf{g}}\end{pmatrix}=\widehat{\omega}\times\begin{pmatrix}\widehat{\mathbf{x}}\\ \widehat{\mathbf{t}}\\ \widehat{\mathbf{g}}\end{pmatrix};\ ('=\frac{d}{d\widehat{s}}), \tag{3.3}$$

where $\widehat{\omega}=\omega+\varepsilon\omega^{*}=\widehat{\chi}\widehat{\mathbf{x}}+\widehat{\mathbf{g}}$ is the Darboux vector, and $\widehat{\gamma}(\widehat{s}):=\gamma+\varepsilon\gamma^{*}$ is the dual geodesic curvature of $\widehat{\mathbf{x}}(\widehat{s})$ on $\mathbb{S}_{1}^{2}$. The tangent vector of $\mathbf{c}(s)$, is specified by [16]:

$$\frac{d\mathbf{c}}{ds}=\Gamma(s)\mathbf{x}+\mu(s)\mathbf{g}, \tag{3.4}$$

which is a spacelike curve. The functions $\gamma(s)$, $\Gamma(s)$ and $\mu(s)$ are the curvature (construction) functions of the ruled surface. These functions described as follows: γ is the geodesic curvature of the timelike spherical image curve $\mathbf{x}=\mathbf{x}(s)$; Γ defines the angle between the ruling of (X) and the tangent to the striction curve; and μ is the distribution parameter at the ruling. These functions provide an approach for constructing spacelike ruled surfaces by the equation

$$(X):\mathbf{y}(s,v)=\int_{0}^{s}(\Gamma(s)\mathbf{x}(s)+\mu(s)\mathbf{g}(s))\,ds+v\mathbf{x}(s). \tag{3.5}$$

The timelike unit normal vector field at any point is also given by

$$\mathbf{e}(s,v)=\frac{\frac{\partial\mathbf{y}(s,v)}{\partial s}\times\frac{\partial\mathbf{y}(s,v)}{\partial v}}{\left\|\frac{\partial\mathbf{y}(s,v)}{\partial s}\times\frac{\partial\mathbf{y}(s,v)}{\partial v}\right\|}=\pm\frac{\mu\mathbf{t}+v\mathbf{g}}{\sqrt{\mu^{2}-v^{2}}};\ |\mu|>|v|, \tag{3.6}$$

which is the timelike central normal $\mathbf{t}$ at the striction point $(v=0)$. So, we can find a hyperbolic rotation angel ϕ such that

$$\mathbf{e}(s,v)=\cosh\phi\mathbf{t}-\sinh\phi\mathbf{g}.$$

It is obvious that:

$$\tanh\phi = \frac{\nu}{\mu}.$$

This result is a Minkowski version of the well known Chasles Theorem [1–3].

The spacelike dual unit vector with the same sense as the Darboux vector $\widehat{\omega}$ is also given by

$$\widehat{\mathbf{b}}(\widehat{s}) := \mathbf{b} + \varepsilon\mathbf{b}^* = \frac{\widehat{\omega}}{\|\widehat{\omega}\|} = \frac{\widehat{\chi}}{\sqrt{\widehat{\gamma}^2+1}}\widehat{\mathbf{x}} + \frac{1}{\sqrt{\widehat{\gamma}^2+1}}\widehat{\mathbf{g}}.$$

It is obvious that $\widehat{\mathbf{b}}(\widehat{s})$ is the Disteli-axis of (X). Let $\widehat{\psi} = \psi + \varepsilon\psi^*$ be the spacelike dual angle (dual radius of curvature) between $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{x}}$. Then,

$$\widehat{\mathbf{b}}(\widehat{s}) = \cos\widehat{\psi}\widehat{\mathbf{x}} + \sin\widehat{\psi}\widehat{\mathbf{g}}, \text{ with } \cot\widehat{\psi} = \frac{\widehat{q}}{\widehat{p}}. \tag{3.7}$$

This dual angle is measured from the vector $\widehat{\mathbf{b}}(\widehat{s})$ to $\widehat{\mathbf{x}}$. In fact, it is important to consider the relations between the timelike dual curve $\widehat{\mathbf{x}}(t) \in \mathbb{S}_1^2$ and the dual curvature $\widehat{\kappa}(\widehat{s})$, as well as the dual torsion $\widehat{\tau}(\widehat{s})$. Therefore, the Serret-Frenet frame is made up of the set $\{\widehat{\mathbf{t}}(\widehat{s}), \widehat{\mathbf{n}}(\widehat{s}), \widehat{\mathbf{b}}(\widehat{s})\}$ where $\widehat{\mathbf{n}} = \widehat{\mathbf{b}}\times\widehat{\mathbf{t}}$ is the principal spacelike dual unit vector. Then, the relative orientation is given by:

$$\begin{pmatrix} \widehat{\mathbf{t}} \\ \widehat{\mathbf{n}} \\ \widehat{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\sin\widehat{\psi} & 0 & \cos\widehat{\psi} \\ \cos\widehat{\psi} & 0 & \sin\widehat{\psi} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{x}} \\ \widehat{\mathbf{t}} \\ \widehat{\mathbf{g}} \end{pmatrix}.$$

Similarly, we can describe the dual Serret-Frenet formulae

$$\begin{pmatrix} \widehat{\mathbf{t}}' \\ \widehat{\mathbf{n}}' \\ \widehat{\mathbf{b}}' \end{pmatrix} = \begin{pmatrix} 0 & \widehat{\kappa} & 0 \\ \widehat{\kappa} & 0 & \widehat{\tau} \\ 0 & -\widehat{\tau} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{t}} \\ \widehat{\mathbf{n}} \\ \widehat{\mathbf{b}} \end{pmatrix}, \tag{3.8}$$

where

$$\left.\begin{array}{l} \widehat{\gamma}(\widehat{s}) = \gamma + \varepsilon(\Gamma - \gamma\mu) = \cot\psi - \varepsilon\psi^*(1+\cot^2\psi), \\ \widehat{\kappa}(\widehat{s}) := \kappa + \varepsilon\kappa^* = \sqrt{1+\widehat{\gamma}^2} = \frac{1}{\sin\widehat{\psi}} = \frac{1}{\widehat{\rho}(\widehat{s})}, \\ \widehat{\tau}(\widehat{s}) := \tau + \varepsilon\tau^* = \pm\widehat{\psi}' = \pm\frac{\widehat{\gamma}'}{1+\widehat{\gamma}^2}. \end{array}\right\} \tag{3.9}$$

Proposition 1. If the dual geodesic curvature function $\widehat{\gamma}(\widehat{s})$ is constant, $\widehat{\mathbf{x}}(\widehat{s})$ is a timelike dual circle on $\mathbb{S}_1^2$.

Proof. From Eq (3.9) we can find that $\widehat{\gamma}(\widehat{s})$=constant yields that $\widehat{\tau}(\widehat{s}) = 0$, and $\widehat{\kappa}(\widehat{s})$ is constant, which implies $\widehat{\mathbf{x}}(\widehat{s})$ is a timelike dual circle on $\mathbb{S}_1^2$.

Definition 1. A non-developable spacelike ruled surface (X) is named as a constant Disteli-axis spacelike ruled surface if its dual geodesic curvature $\widehat{\gamma}(\widehat{s})$ is constant.

Via the E. Study map, the constant Disteli-axis spacelike ruled surface (X) is traced by a one-parameter helical motion with constant pitch h along the spacelike Disteli-axis $\widehat{\mathbf{b}}$, by the oriented spacelike line $\widehat{\mathbf{x}}$ located at a Lorentzian constant distance ψ^* and Lorentzian constant angle ψ relative to the spacelike Disteli-axis $\widehat{\mathbf{b}}$. In the special case, if $\widehat{\gamma}(\widehat{s}) = 0$, then $\widehat{\mathbf{x}}(\widehat{s})$ is a timelike great dual circle on $\mathbb{S}^2_1$, that is,

$$\widehat{c} = \{\widehat{\mathbf{x}} \in \mathbb{S}^2_1 \mid < \widehat{\mathbf{x}}, \widehat{\mathbf{b}} >= 0, \text{ with } \left\| \widehat{\mathbf{b}} \right\| = 1\}. \tag{3.10}$$

In this case, all the rulings of (X) intersected orthogonally with the spacelike Disteli-axis $\widehat{\mathbf{b}}$, that is, $\psi = \frac{\pi}{2}$, and $\psi^* = 0$. Thus, we have $\widehat{\gamma}(\widehat{s}) = 0 \Leftrightarrow (X)$ is a spacelike helicoidal surface. The constant Disteli-axis ruled, and the helicoidal surface are essential to the curvature theory of ruled surfaces. We will therefore inspect them in some detail later.

Notice that in Eq (3.4): (a) if $\mu = 0(\frac{d\mathbf{c}}{ds} \parallel \mathbf{x})$, then (X) is a spacelike tangential developable ruled surface. Namely, the Blaschke frame $\{\mathbf{x}(s), \mathbf{t}(s), \mathbf{g}(s)\}$ turn out to the usual Serret-Frenet frame and the striction curve $\mathbf{c}(s)$ turns out to be the edge of regression of (X). Let υ be arc length parameter of $\mathbf{c}(s)$ and $\{\mathbf{e}_1(\upsilon), \mathbf{e}_2(\upsilon), \mathbf{e}_3(\upsilon)\}$ is the usual Serret-Frenet frame of $\mathbf{c}$. Then,

$$\frac{d}{d\upsilon}\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix}\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix},$$

where $\kappa(\upsilon)$ and $\tau(\upsilon)$ are the natural curvature and torsion of the striction curve $\mathbf{c}(u)$, respectively;

$$\kappa(\upsilon) = \frac{1}{\Gamma(\upsilon)},\ \tau(\upsilon) = \frac{\gamma(\upsilon)}{\Gamma(\upsilon)}, \text{ with } \Gamma(\upsilon) \neq 0. \tag{3.11}$$

Therefore, the curvature function $\Gamma(\upsilon)$ is the radius of curvature of the spacelike striction curve $\mathbf{c}(\upsilon)$. We arrive therefore at the conclusion that the spacelike striction curve $\mathbf{c}(\upsilon)$ is the edge of regression of (X). In view of [21], we summarize this result in the following:

Theorem 1. Any spacelike ruled surface (X) with the curvature function

$$\Gamma(\upsilon) = b\sinh\theta - b\cosh\theta;\ \theta = \int_0^{\upsilon} \tau d\upsilon,$$

with real constants $(a, b) \neq (0, 0)$ is a spacelike tangential surface of a spacelike curve lies on a Lorentzian sphere of radius $\sqrt{b^2 - a^2}$.

Corollary 1. The curvature function $\kappa(\upsilon)$ and torsion function $\tau(\upsilon)$ of the spherical curve in Theorem 1, respectively, are:

$$\kappa(\upsilon) = \frac{1}{b\sinh\theta - b\cosh\theta}, \text{ and } \tau(\upsilon) = \frac{\gamma(\upsilon)}{b\sinh\theta - b\cosh\theta}.$$

(b) if $\Gamma(s) = 0$, then the striction curve is tangent to $\mathbf{g}$; it is normal to the ruling through $\mathbf{c}(s)$; $\mathbf{e}_1 = \mathbf{g}(s)$, $\mathbf{e}_2 = \mathbf{t}(s)$, $\mathbf{e}_3 = \mathbf{x}(s)$. In this case (X) a spacelike binormal ruled surface. Similarly, we find

$$\kappa(\upsilon) = \frac{\gamma(\upsilon)}{\mu(\upsilon)}, \text{ and } \tau(\upsilon) = \frac{1}{\mu(\upsilon)}, \text{ with } \mu(\upsilon) \neq 0. \tag{3.12}$$

Therefore, the curvature function $\mu(\upsilon)$ is the radius of torsion of the spacelike striction curve $\mathbf{c}(\upsilon)$ of the binormal surface. Hence, we summarize this result in the following:

Theorem 2. Any spacelike ruled surface (X) with the curvature function

$$\mu(\upsilon) = \gamma(\upsilon)\,(b\sinh\theta - b\cosh\theta)\,;\ \theta = \int_0^{\upsilon} \tau d\upsilon,$$

with real constants $(a, b) \neq (0, 0)$ is a spacelike binormal surface of a spacelike curve lies on a Lorentzian sphere of radius $\sqrt{b^2 - a^2}$.

Corollary 2. The curvature function $\kappa(\upsilon)$ and torsion function $\tau(\upsilon)$ of the spherical curve in Theorem 2, respectively, are:

$$\kappa(\upsilon) = \frac{(\upsilon)}{b\sinh\theta - b\cosh\theta}, \text{ and } \tau(\upsilon) = \frac{1}{\gamma(\upsilon)\,(b\sinh\theta - b\cosh\theta)}.$$

4. Bertrand offsets of spacelike ruled surfaces

In this section, we define the Bertrand offsets of spacelike ruled and developable surfaces, then a theory comparable to the theory of Bertrand curves can be developed for such surfaces.

Definition 2. Let (X) and $(\overline{X})$ be two non-developable spacelike ruled surfaces in $\mathbb{E}^3_1$. The surface $(\overline{X})$ is said to be Bertrand offsets of (X) if there exists a one-to-one correspondence between their rulings such that both surfaces have a common timelike central normal at the corresponding striction points.

Let us examine a non-developable spacelike ruled surface $(\overline{X})$ parametrized by

$$\overline{\widehat{\mathbf{x}}}(\widehat{s}) = \widehat{x}_1\widehat{\mathbf{x}} + \widehat{x}_2\widehat{\mathbf{t}} + \widehat{x}_3\widehat{\mathbf{g}}, \tag{4.1}$$

where $\widehat{x}_i = \widehat{x}_i(\widehat{s})$, $(i = 1,\ 2,\ 3)$ are its dual coordinate functions. Then

$$\widehat{x}_1^2 - \widehat{x}_2^2 + \widehat{x}_3^2 = 1. \tag{4.2}$$

Differentiating Eqs (4.1), (4.2) and with the aid of Eq (3.3), we find:

$$\left.\begin{aligned} &\overline{\widehat{\mathbf{x}}}' = (\widehat{x}_1' + \widehat{x}_2)\widehat{\mathbf{x}} + (\widehat{x}_2' + \widehat{x}_1 + \widehat{\gamma}\widehat{x}_3)\widehat{\mathbf{t}} + (\widehat{x}_3' + \widehat{\gamma}\widehat{x}_2)\widehat{\mathbf{g}}, \\ &\widehat{x}_1\widehat{x}_1' - \widehat{x}_2\widehat{x}_2' + \widehat{x}_3\widehat{x}_3' = 0. \end{aligned}\right\} \tag{4.3}$$

If we assume that the dual curves $\overline{\widehat{\mathbf{x}}} = \overline{\widehat{\mathbf{x}}}(\widehat{s})$ and $\widehat{x} = \widehat{x}(\widehat{s})$ are Bertrand offsets, that is, $\widehat{\mathbf{t}} = \overline{\widehat{\mathbf{t}}}$, then we have

$$(\widehat{x}_1' + \widehat{x}_2) = 0,\ (\widehat{x}_2' + \widehat{x}_1 + \widehat{\gamma}\widehat{x}_3) = <\overline{\widehat{\mathbf{x}}}', \widehat{\mathbf{t}}>,\ (\widehat{x}_3' + \widehat{\gamma}\widehat{x}_2) = 0. \tag{4.4}$$

Substituting Eq (4.4) into the second expression of Eq (4.3) and simplifying it, we obtain

$$\widehat{x}_2 = 0. \tag{4.5}$$

From Eqs (4.4) and (4.5) we get:

$$\widehat{x}_1' = 0,\ \widehat{x}_1 + \widehat{\gamma}\widehat{x}_3 = <\overline{\widehat{\mathbf{x}}}', \widehat{\mathbf{t}}>,\ \ \widehat{x}_3' = 0 \Rightarrow \widehat{x}_1 = \widehat{c}_1,\ \widehat{x}_3 = \widehat{c}_3 \in \mathbb{D}, \tag{4.6}$$

where $\widehat{c_1}$ and $\widehat{c_3}$ are dual constants of integrations. Therefore, we can find a constant spacelike dual angle $\widehat{\vartheta} = \vartheta + \varepsilon\vartheta^*$ such that $\widehat{c_1} = \cos\widehat{\vartheta}$ and $\widehat{c_3} = -\sin\widehat{\vartheta}$. Thus, as a result the following theorem can be given:

Theorem 3. The offset spacelike dual angle among the generating spacelike lines of a non-developable spacelike ruled surface and its spacelike Bertrand offset at corresponding central points remains constant.

It is clear from the above results that a spacelike ruled surface, in general, has a double infinity of spacelike Bertrand offsets. Each spacelike Bertrand offset can be traced by a constant linear offset $\vartheta^* \in \mathbb{R}$ and a constant angular offset $\vartheta \in [0, 2\pi]$. Any two spacelike surfaces of this family of spacelike ruled surfaces are reciprocal of one another; if $(\overline{X})$ is a spacelike Bertrand offset of (X), then (X) is also a spacelike Bertrand offset of $(\overline{X})$. Then, Eq (4.1) become:

$$\widehat{\overline{\mathbf{x}}}(\widehat{s})= \cos\widehat{\vartheta}\widehat{\mathbf{x}}(\widehat{s}) - \sin\widehat{\vartheta}\widehat{\mathbf{g}}(\widehat{s}). \tag{4.7}$$

Since $\widehat{\overline{\mathbf{t}}}(=\widehat{\mathbf{t}})$, at the congruent points on the two striction curves of (X) and $(\overline{X})$, we have:

$$\begin{pmatrix} \widehat{\overline{\mathbf{x}}}(\widehat{s}) \\ \widehat{\overline{\mathbf{t}}}(\widehat{s}) \\ \widehat{\overline{\mathbf{g}}}(\widehat{s}) \end{pmatrix} = \begin{pmatrix} \cos\widehat{\vartheta} & 0 & -\sin\widehat{\vartheta} \\ 0 & 1 & 0 \\ \sin\widehat{\vartheta} & 0 & \cos\widehat{\vartheta} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{x}}(\widehat{s}) \\ \widehat{\mathbf{t}}(\widehat{s}) \\ \widehat{\mathbf{g}}(\widehat{s}) \end{pmatrix}. \tag{4.8}$$

The above equation is precisely the same as its similar equation for Bertrand curves [4]. If $\vartheta = 0$ and $\vartheta = \pi/2$ then the Bertrand offsets are named to be oriented offsets and right offsets, respectively.

Now, we give some theorems and results between (X) and $(\overline{X})$. Let $\widehat{\overline{s}}$ be the arc length of $\widehat{\overline{\mathbf{x}}}(\widehat{s})$, then:

$$\frac{d}{d\widehat{\overline{s}}}\begin{pmatrix} \widehat{\overline{\mathbf{x}}}(\widehat{s}) \\ \widehat{\overline{\mathbf{t}}}(\widehat{s}) \\ \widehat{\overline{\mathbf{g}}}(\widehat{s}) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \widehat{\overline{\gamma}} \\ 0 & \widehat{\overline{\gamma}} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\overline{\mathbf{x}}}(\widehat{s}) \\ \widehat{\overline{\mathbf{t}}}(\widehat{s}) \\ \widehat{\overline{\mathbf{g}}}(\widehat{s}) \end{pmatrix}, \tag{4.9}$$

where

$$1 = \left(\cos\widehat{\vartheta} - \widehat{\gamma}\sin\widehat{\vartheta}\right)\frac{d\widehat{s}}{d\widehat{\overline{s}}}, \widehat{\overline{\gamma}}(\widehat{s}) = \left(\sin\widehat{\vartheta} + \widehat{\gamma}\cos\widehat{\vartheta}\right)\frac{d\widehat{s}}{d\widehat{\overline{s}}}. \tag{4.10}$$

By eliminating $\frac{d\widehat{s}}{d\widehat{\overline{s}}}$ we gain:

$$(\widehat{\overline{\gamma}} - \widehat{\gamma})\cos\widehat{\vartheta} - (1 + \widehat{\overline{\gamma}}\widehat{\gamma})\sin\widehat{\vartheta} = 0. \tag{4.11}$$

This is a new dual characterization of the Bertrand offsets of spacelike ruled surfaces in terms of their dual geodesic curvature.

Theorem 4. Two non-developable spacelike ruled surfaces (X) and $(\overline{X})$ are Bertrand offsets iff the Eq (4.11) is held.

Corollary 3. The Bertrand offset of a spacelike constant Disteli-axis ruled surface is a spacelike constant Disteli-axis ruled surface too.

Corollary 4. The Bertrand offset of a spacelike helicoidal surface, in general, does not have to be a helicoidal surface and can be a regular spacelike ruled surface.

Furthermore, for the spacelike ruled surface $(\overline{X})$, let $\overline{\mathbf{e}}(\overline{s},v)$ be the timelike unit normal vector of an arbitrary point. Then, as in Eq (3.6) , we have

$$\overline{\mathbf{e}}(\overline{s},v)=\frac{\overline{\mu}\,\bar{\mathbf{t}}-v\overline{\mathbf{g}}}{\sqrt{\overline{\mu}^2-v^2}}, \tag{4.12}$$

where $\overline{\mu}$ is the distribution parameter of $(\overline{X})$. It is clear from Eqs (3.6) and (4.12) that the timelike central normal to a spacelike ruled surface and its Bertrand offsets are not the same. This shows that the Bertrand offsets of a spacelike ruled surface are, generally, not parallel offsets. Now, it seems natural to put the following question: Under what condition (X) and $(\overline{X})$ are parallel offsets? The answer can be given as follows:

Theorem 5. Two non-developable spacelike ruled surfaces (X) and $(\overline{X})$ are parallel offsets iff $\mu=\overline{\mu}$, and each axis of the Blaschke frame of (X) is collinear with the corresponding axis of $(\overline{X})$.

Proof. Suppose (X) and $(\overline{X})$ are parallel offsets, or $\overline{\mathbf{e}}(\overline{s},v)\times\mathbf{e}(s,v)=\mathbf{0}$. Then, we have:

$$v^2\sin\vartheta\mathbf{t}+v(\mu\cos\vartheta-\overline{\mu})\mathbf{x}-v\sin\vartheta\mathbf{g}=\mathbf{0}. \tag{4.13}$$

This equation must hold true for any value $v\neq 0$, which leads to $\vartheta=0$ and $\mu=\overline{\mu}$. ■

As a result the following corollaries can be given:

Corollary 5. Two developable spacelike ruled surfaces (X) and $(\overline{X})$ are parallel offsets iff each axis of the Blaschke frame of (X) is collinear with the corresponding axis of $(\overline{X})$.

Corollary 6. A developable spacelike ruled surface and a non-developable spacelike surface can't be parallel offsets.

Example 1. In this example, we will construct the spacelike constant Disteli-axis ruled surface (X). Since $\widehat{\gamma}(\widehat{s})$ is constant, from Eqs (3.3) and (3.9) we obtain the ODE $\widehat{\mathbf{t}}''-\widehat{\kappa}^2\widehat{\mathbf{t}}=\mathbf{0}$. Without loss of the generalization, we may choose $\widehat{\mathbf{t}}(0)=(0,1,0)$. Then, the general solution of the ODE is:

$$\widehat{\mathbf{t}}(\widehat{s})=\left(\widehat{b}_1\sinh(\widehat{\kappa s}),\cosh(\widehat{\kappa s})+\widehat{b}_2\sinh(\widehat{\kappa s}),\widehat{b}_3\sinh(\widehat{\kappa s})\right),$$

for some dual constants $\widehat{b}_1$, $\widehat{b}_2$, and $\widehat{b}_3$. Since $\left\|\widehat{\mathbf{t}}\right\|^2=-1$, we get $\widehat{b}_2=0$, and $\widehat{b}_1^2+\widehat{b}_3^2=1$, it follows that $\widehat{\mathbf{x}}(\widehat{u})$ is given by

$$\widehat{\mathbf{x}}(\widehat{s})=\left(\widehat{b}_1\widehat{\rho}\cosh(\widehat{\kappa s})+\widehat{d}_1,\widehat{\rho}\sinh(\widehat{\kappa s}),\widehat{b}_3\widehat{\rho}\cosh(\widehat{\kappa s})+\widehat{d}_3\right),$$

for some dual constants $\widehat{d}_2$, and $\widehat{d}_3$ satisfying $\widehat{b}_1\widehat{d}_1+\widehat{b}_3\widehat{d}_3=0$, and $\widehat{d}_1^2+\widehat{d}_3^2=1-\widehat{\rho}^2$. We now change the dual coordinates by

$$\begin{pmatrix}\widetilde{x}_1\\ \widetilde{x}_2\\ \widetilde{x}_3\end{pmatrix}=\begin{pmatrix}\widehat{b}_1 & 0 & \widehat{b}_3\\ 0 & 1 & 0\\ -\widehat{b}_3 & 0 & \widehat{b}_1\end{pmatrix}\begin{pmatrix}\widehat{x}_1\\ \widehat{x}_2\\ \widehat{x}_3\end{pmatrix},$$

from which we obtain

$$\widehat{\mathbf{x}}(\widehat{s})=\left(\sin\widehat{\psi}\cosh(\widehat{\kappa s}),\sin\widehat{\psi}\sinh(\widehat{\kappa s}),\widehat{d}\right) \tag{4.14}$$

for a dual constant $\widehat{d}=\widehat{b}_1\widehat{d}_3-\widehat{b}_3\widehat{d}_1$, with $\widehat{d}=\pm\cos\widehat{\psi}$. Notice that $\widehat{\mathbf{x}}(\widehat{s})$ does not depend on the choice of the lower sign or upper sign of $\pm$. Therefore, through the paper we choice the upper sign, that is,

$$\widehat{\mathbf{x}}(\widehat{\varphi})=\left(\sin\widehat{\psi}\cosh\widehat{\varphi},\sin\widehat{\psi}\sinh\widehat{\varphi},\cos\widehat{\psi}\right), \tag{4.15}$$

where $\widehat{\varphi} = \widehat{\kappa s}$. It is a timelike dual spherical curve with the dual curvature $\widehat{\kappa} = \sqrt{\widehat{\gamma}^2 + 1}$ on the Lorentzian dual unit sphere $\mathbb{S}_1^2$. Let $\widehat{\varphi} = \varphi(1 + \varepsilon h)$; h denoting the constant pitch of the helical motion, and φ is the motion parameter, then Eq (29) parametrize a spacelike ruled surface. Thus, the Blaschke frame is as follows:

$$\begin{pmatrix} \widehat{\mathbf{x}}(\varphi) \\ \widehat{\mathbf{t}}(\varphi) \\ \widehat{\mathbf{g}}(\varphi) \end{pmatrix} = \begin{pmatrix} \sin\widehat{\psi}\cosh\widehat{\varphi} & \sin\widehat{\psi}\sinh\widehat{\varphi} & \cos\widehat{\psi} \\ \sinh\widehat{\varphi} & \cosh\widehat{\varphi} & 0 \\ -\cos\widehat{\psi}\cosh\widehat{\varphi} & -\cos\widehat{\psi}\sinh\widehat{\varphi} & \sin\widehat{\psi} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{f}}_1 \\ \widehat{\mathbf{f}}_2 \\ \widehat{\mathbf{f}}_3 \end{pmatrix}. \tag{4.16}$$

It is easily seen that

$$\left.\begin{aligned} \widehat{p}(\varphi) = (1 + \varepsilon h)\sin\widehat{\psi}, \widehat{q}(\varphi) = (1 + \varepsilon h)\cos\widehat{\psi}, \\ d\widehat{s} = \widehat{p}(\varphi)d\varphi, \widehat{\gamma}(\varphi) =: \tfrac{\widehat{q}(\varphi)}{\widehat{p}(\varphi)} = \cot\widehat{\psi}. \end{aligned}\right\} \tag{4.17}$$

From the real and dual parts of Eqs (31), we obtain

$$\mu = \psi^* \cot\psi + h, \text{ and } \Gamma = -\psi^* + h\cot\psi. \tag{4.18}$$

The spacelike Disteli-axis can be expressed as

$$\widehat{\mathbf{b}} = \cos\widehat{\psi}\widehat{\mathbf{x}} + \sin\widehat{\psi}\widehat{\mathbf{g}}. \tag{4.19}$$

It is easily seen from the Eqs (4.16) and (4.19) that $\widehat{\mathbf{b}} = \widehat{\mathbf{f}}_3$. This shows that the axis of the helical motion is the constant spacelike Disteli-axis $\widehat{\mathbf{b}}$. Furthermore, the equation of (X) in terms of the Plücker coordinates can be obtained as: If we separate $\widehat{\mathbf{x}}(\varphi)$ into real and dual parts we reach

$$\mathbf{x}(\varphi) = (\sin\psi\cosh\varphi, \sin\psi\sinh\varphi, \cos\psi), \tag{4.20}$$

and

$$\mathbf{x}^*(\varphi) = \begin{pmatrix} \widetilde{x}_1^* \\ \widetilde{x}_2^* \\ \widetilde{x}_3^* \end{pmatrix} = \begin{pmatrix} \varphi^*\sinh\varphi\sin\psi + \psi^*\cos\psi\cosh\varphi \\ \varphi^*\cosh\varphi\sin\psi + \psi^*\cos\psi\sinh\varphi \\ -\psi^*\sin\psi \end{pmatrix}. \tag{4.21}$$

Let $r(r_1, r_2, r_3)$ denote a point on $\widehat{\mathbf{x}}$. Since $\boldsymbol{\alpha}\times\mathbf{x} = \mathbf{x}^*$ we have the system of linear equations in r_1, r_2, and r_3:

$$\left.\begin{aligned} r_2\cos\psi - r_3\sin\psi\sinh\varphi = \widetilde{x}_1^*, \\ r_1\cos\psi - r_3\sin\psi\cosh\varphi = \widetilde{x}_2^*, \\ (r_1\sinh\varphi - r_2\cosh\varphi)\sin\psi = \widetilde{x}_3^*. \end{aligned}\right\}$$

The matrix of coefficients of unknowns r_1, r_2, and r_3 is

$$\begin{pmatrix} 0 & \cos\psi & -\sin\psi\sinh\varphi \\ \cos\psi & 0 & -\sin\psi\cosh\varphi \\ \sin\psi\sinh\varphi & -\sin\psi\cosh\varphi & 0 \end{pmatrix},$$

its rank is 2 with $s \neq 0$, and $\vartheta \neq p\pi$ (p is an integer). In addition the rank of the augmented matrix

$$\begin{pmatrix} 0 & -\sin\vartheta & -\cos\vartheta\sinh s & x_1^* \\ \sin\vartheta & 0 & -\cos\vartheta\cosh s & x_2^* \\ \cos\vartheta\sinh s & -\cos\vartheta\cosh s & 0 & x_3^* \end{pmatrix},$$

is 2. Hence this system has infinitely many solutions parametrized with

$$\begin{gathered} r_1 = \psi^* \sinh\varphi + (\varphi^* + r_3)\tan\psi\cosh\varphi, \\ r_2 = \psi^* \cosh\varphi + (\varphi^* + r_3)\tan\psi\sinh\varphi,, \\ r_1\sinh\varphi - r_2\cosh\varphi = -\varphi^*. \end{gathered} \tag{4.22}$$

Since r_3 is taken at random, then we may take $\varphi^* + r_3 = 0$. In this case, Eq (4.22) reduces to

$$r_1 = \psi^* \sinh\varphi,\ r_2 = \psi^* \cosh\varphi,\ r_3 = -h\varphi. \tag{4.23}$$

We now simply find the base curve as;

$$\mathbf{r}(\varphi) = (\psi^* \sinh\varphi, \psi^* \cosh\varphi, -h\varphi).$$

It can be show that $< \mathbf{r}', \mathbf{x}' >= 0$; $(' = \frac{d}{d\varphi})$ so the base curve of (X) is its striction curve. Thus, if $(x,\ y,\ z)$ are the coordinates of $\mathbf{y}$, then the spacelike ruled surface (X) is

$$(X): \ \mathbf{y}(\varphi, v) = \begin{pmatrix} \psi^* \sinh\varphi + v\sin\psi\cosh\varphi \\ \psi^* \cosh\varphi + v\sin\psi\sinh\varphi \\ -h\varphi + v\cos\psi \end{pmatrix}, \tag{4.24}$$

or

$$(X): -\frac{x^2}{\psi^{*2}} + \frac{y^2}{\psi^{*2}} + \frac{Z^2}{\chi^2} = 1, \tag{4.25}$$

where $\chi = \psi^* \cot\psi$, and $Z = z + h\varphi$. The constants h, ψ and ψ^* can control the shape of (X). So, (X) is a 3-parameter family of Lorentzian unit spheres. The intersection of each Lorentzian unit sphere and the corresponding spacelike plane $z = -h\varphi$ is a one-parameter family of Lorentzian circles $-x^2 + y^2 = \psi^{*2}$. Therefore, the envelope of (X) is a one-parameter family of Lorentzian cylinders. Take $\psi^* = 0$, $\psi = \frac{\pi}{2}$, and $h = 2$ for example, then the striction curve is the spacelike Disteli-axis, and the spacelike helicoidal surface is shown in Figure 2; $-2.5 \le v \le 2.5$, and $-1.7 \le \varphi \le 1.7$.

Example 2. In this example, we verify the idea of Corollary 4.

In view of Eqs (4.8), (4.11) and (4.16) we have: $\widehat{\gamma} = 0$ $(\psi = \frac{\pi}{2},\ \psi^* = 0) \Leftrightarrow \overline{\widehat{\gamma}} = \cot\widehat{\vartheta}$, and

$$\overline{\widehat{\mathbf{x}}}(\widehat{\varphi}) = \left(\cos\widehat{\vartheta}\cosh\widehat{\varphi}, \cos\widehat{\vartheta}\sinh\widehat{\varphi}, -\sin\widehat{\vartheta}\right). \tag{4.26}$$

The equation of the striction curve of $(\overline{X})$, in terms of (X), can therefore be written as:

$$\overline{\mathbf{r}}(\varphi) := \mathbf{r}(\varphi) + \vartheta^* \mathbf{t}(\varphi) = (0, 0, -h\varphi) + \vartheta^* (\sinh\varphi, \cosh\varphi, 0). \tag{4.27}$$

By a similar procedure, we have

$$\left.\begin{aligned} \overline{x} &= \vartheta^* \sinh\varphi + v\cos\vartheta\cosh\varphi, \\ \overline{y} &= \vartheta^* \cosh\varphi + v\cos\vartheta\sinh\varphi, \\ \overline{z} &= -h\varphi - v\sin\vartheta, \end{aligned}\right\}$$

or

$$(\overline{X}): -\frac{\overline{x}^2}{\vartheta^{*2}} + \frac{\overline{y}^2}{\vartheta^{*2}} + \frac{\overline{Z}^2}{\overline{\chi}^2} = 1, \tag{4.28}$$

where $\overline{\chi} = \vartheta^* \cot\vartheta$, and $Z = z + h\varphi$. The constants h, ϑ and ϑ^* can control the shape of the surface $(\overline{X})$. Hence, $(\overline{X})$ has the same geometrical properties as in Eq (4.25). Take $\vartheta = \frac{\pi}{3}$, $\vartheta^* = 1$and $h = 2$ for example, the spacelike Bertrand offset is shown Figure 3, where $-3 \leq v \leq 3$, and $0 \leq \varphi \leq 2\pi$. The graph of the spacelike helicoidal surface (X) with its Bertrand offset $(\overline{X})$ is shown in Figure 4.

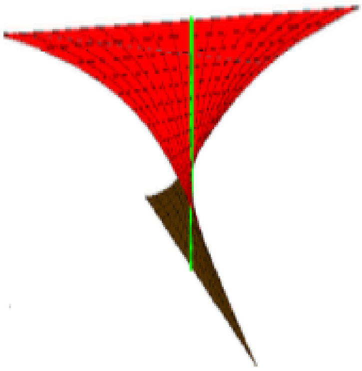

Figure 2. Spacelike helicoidal surface.

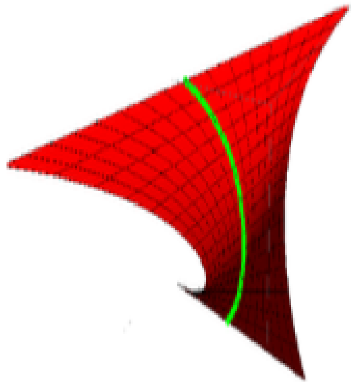

Figure 3. Spacelike Bertrand offset.

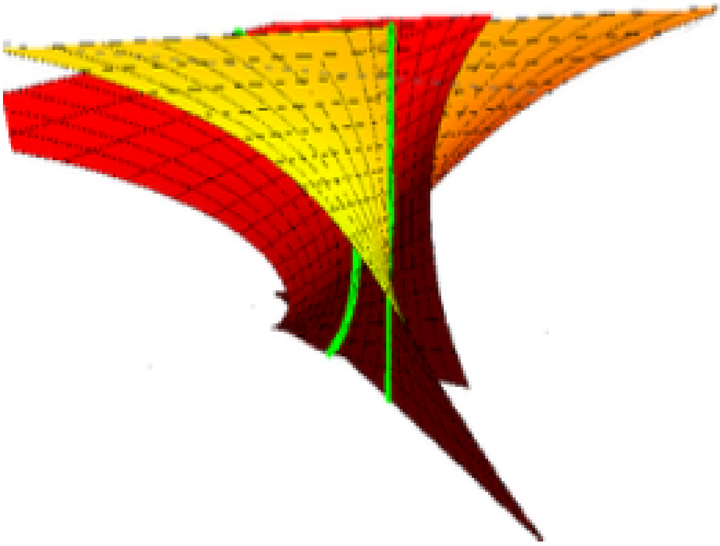

Figure 4. (X) and its Bertrand offset $(\overline{X})$.

4.1. Properties of the striction curves

Now, we study the relationships among the striction curves of (X) and $(\overline{X})$. Therefore, with the using of Definition 2, the striction curve $\overline{\mathbf{c}}(\overline{s})$ of $(\overline{X})$ is

$$\overline{\mathbf{c}}(\overline{s}) = \mathbf{c}(s) + \vartheta\mathbf{t}(s), \tag{4.29}$$

from which we obtain

$$\frac{d\overline{\mathbf{c}}(\overline{s})}{d\overline{s}}\frac{d\overline{s}}{ds} = (\Gamma + \vartheta^*)\mathbf{x} + (\mu + \gamma\vartheta^*)\mathbf{g}, \tag{4.30}$$

whereas, as in Eq (4.3) , is:

$$\frac{d\overline{\mathbf{c}}(\overline{s})}{d\overline{s}} = \overline{\Gamma}(\overline{s})\overline{\mathbf{x}}(\overline{s}) + \overline{\mu}(\overline{s})\overline{\mathbf{g}}(\overline{s}). \tag{4.31}$$

It follows from the Eqs (4.30) and (4.31) that:

$$\frac{d\overline{s}}{ds} = \frac{\Gamma + \vartheta^*}{\overline{\Gamma}\cos\vartheta + \overline{\mu}\sin\vartheta} = \frac{\mu + \gamma\vartheta^*}{-\overline{\Gamma}\sin\vartheta + \overline{\mu}\cos\vartheta}. \tag{4.32}$$

Hence, the following cases can be obtained:
Case (1) If (X) is a spacelike tangential surface, that is, $\mu = 0$. In this case, from Eq (4.32) , it follows that

$$\overline{\mu} = \overline{\Gamma}\frac{\gamma\vartheta^*\cos\vartheta + (\Gamma + \vartheta^*)\sin\vartheta}{\gamma\vartheta^*\sin\vartheta - (\Gamma + \vartheta^*)\cos\vartheta}. \tag{4.33}$$

Thus the Bertrand offset of a spacelike tangential is not spacelike tangential, that is, $\overline{\mu}(s) \neq 0$. If $(\overline{X})$ is also a spacelike tangential, that is, $\overline{\mu}(s) = 0$. In this case, we can obtain the relationships

$$(1 + \vartheta^*\kappa(\upsilon))\sin\vartheta + \vartheta^*\tau(\upsilon)\cos\vartheta = 0. \tag{4.34}$$

Theorem 6. If (X) and $(\overline{X})$ are two spacelike tangential Bertrand offsets then their striction curves are spacelike Bertrand curves.

Furthermore, from Eq (48) the offset distance ϑ^* is

$$\vartheta^* = -\frac{\sin\vartheta}{\kappa(\upsilon)\sin\vartheta + \tau(\upsilon)\cos\vartheta}. \tag{4.35}$$

Hence, the following corollaries may be given:
Corollary 7. If (X) and $(\overline{X})$ are two spacelike tangential Bertrand oriented offsets ($\vartheta = 0$) then their generators are coincident, that is, $\vartheta^* = 0$, and for $\tau(\upsilon) \neq 0$ they are coincident.
Corollary 8. If (X) and $(\overline{X})$ are two spacelike tangential Bertrand right offsets ($\vartheta = \frac{\pi}{2}$) then edge of regression of (X) is a spacelike curve with constant curvature, that is, $\kappa(\upsilon) = 1/\vartheta^*$.
Case (2) If (X) is a spacelike binormal ruled surface of its spacelike striction curve, that is, $\Gamma = 0$. In this case, from Eq (4.32) , it follows that

$$\overline{\Gamma} = \overline{\mu}\frac{\vartheta^*\cos\vartheta - (\gamma\vartheta^* + \mu)\sin\vartheta}{\vartheta^*\sin\vartheta + (\gamma\vartheta^* + \mu)\cos\vartheta}. \tag{4.36}$$

Thus the Bertrand offset of a spacelike binormal is not spacelike binormal, that is, $\overline{\Gamma}(s) \neq 0$. Furthermore, if the spacelike Bertrand offset $(\overline{X})$ is also a spacelike binormal, then we have:

$$\tau(\upsilon)\vartheta^*\cos\vartheta - (\kappa(\upsilon)\vartheta^* + 1)\sin\vartheta = 0. \tag{4.37}$$

Theorem 7. If (X) and $(\overline{X})$ are two spacelike binormal Bertrand offsets then their striction curves are spacelike Bertrand curves.

From Eq (4.37) , we also have:

$$\vartheta^* = -\frac{\sin\vartheta}{\tau(\upsilon)\cos\vartheta - \kappa(\upsilon)\sin\vartheta}.$$

Hence, we can have all the corresponding corollaries for spacelike tangential; we omit the details here.

5. Conclusions

In this paper, we study Bertrand surface offsets by considering the E. Study map of lines in terms of dual numbers in Minkowski 3-space $\mathbb{E}_1^3$. We obtain relationships between the invariants of Bertrand offsets of spacelike ruled surfaces. Furthermore, we obtain some theorems and results characterizing Bertrand offsets of spacelike ruled surface offsets. For future work we suggest that Gaussian and mean curvatures of these Bertrand offsets can be calculated, when the Weingarten map for the Bertrand offsets spacelike ruled surfaces is defined.

Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

1. O. Bottema, B. Roth, *Theoretical kinematics,* North-Holland Press, New York, 1979.
2. A. Karger, J. Novak, *Space kinematics and Lie groups*, Gordon and Breach Science Publishers, New York, 1985.
3. H. Pottman, J. Wallner, *Computational line geometry,* Springer-Verlag, Berlin, Heidelberg, 2001.
4. B. Ravani, T. S. Ku, Bertrand offsets of ruled and developable surfaces, *Comput. Aided Design,* **23** (1991), 145–152. https://doi.org/10.1016/0010-4485(91)90005-H
5. A. Küçük, O. Gürsoy, On the invariants of Bertrand trajectory surface offsets, *AMC,* 2003, 11–23. https://doi.org/10.1002/0471141755.ph1107s23
6. E. Kasap, N. Kuruoglu, Integral invariants of the pairs of the Bertrand ruled surface, *Bull. Pure Appl. Sci. Sect. E Math.*, **21** (2002), 37–44.
7. E. Kasap, N. Kuruoglu, The Bertrand offsets of ruled surfaces in $\mathbb{R}_1^3$, *Acta Math. Vietnam.*, **31** (2006), 39–48.
8. E. Kasap, S. Yuce, N. Kuruoglu, The involute-evolute offsets of ruled surfaces, *Iran. J. Sci. Technol. A*, **33** (2009), 195–201.
9. K. Orbay, E. Kasap, I. Aydemir, Mannheim offsets of ruled surfaces, *Math. Probl. Eng.*, 2009, 160917.
10. M. Onder, H. H. Ugurlu, Frenet frames and invariants of timelike ruled surfaces, *Ain Shams Eng. J.,* **4** (2013), 507–513. https://doi.org/10.1016/j.asej.2012.10.003
11. G. Y. Sentrk, S. Yuce, Properties of integral invariants of the involute-evolute offsets of ruled surfaces, *Int. J. Pure Appl. Math.,* **102** (2015), 75.
12. M. T. Aldossary, R. A. Abdel-Baky, On the Bertrand offsets for ruled and developable surfaces, *Boll. Unione Mat. Ital.*, **8** (2015), 53–64. https://doi.org/10.1007/s40574-015-0025-1
13. G. Y. Sentrk, S. Yuce, E. Kasap, Integral invariants of Mannheim offsets of ruled surfaces, *Appl. Math. E-Notes,* **16** (2016), 198–209.

14. G. Y. Sentrk, S. Yuce, Bertrand offsets of ruled surfaces with Darboux frame, *Results Math.*, **72** (2017), 1151–1159. https://doi.org/10.1007/s00025-016-0571-6
15. G. Y. Sentrk, S. Yuce, On the evolute offsets of ruled surfaces usin the Darboux frame, *Commun. Fac. Sci. Univ.*, **68** (2019), 1256–1264.
16. R. A Abdel-Baky, Y. Unluturk, A new construction of timelike ruled surfaces with constant Disteli-axis, *Honam Math. J.,* **42** (2020), 551–568 .
17. N. S. Alluhaibi, R. A Abdel-Baky, M. F Naghi, On the Bertrand offsets of timelike ruled surfaces in Minkowski 3-space, *Symmetry*, **14** (2022), 673. https://doi.org/10.3390/sym14040673
18. R. A Abdel-Baky, F. Mofarreh, A study on the Bertrand offsets of timelike ruled surfaces in Minkowski 3-space, *Symmetry,* **14** (2022), 783. https://doi.org/10.3390/sym14040783
19. B. O'Neil, *Semi-Riemannian Geometry geometry, with applications to relativity,* Academic Press, New York, 1983.
20. J. Walfare, *Curves and surfaces in Minkowski space*, Ph.D. Thesis, K. U. Leuven, Faculty of Science, Leuven, 1995.
21. U. Pekmen, S. Pasali, Some characterizations of Lorentzian spherical spacelike curves, *Math. Morav*, **3** (1999), 31–37.

AIMS Press