AIMS Mathematics

AIMS Mathematics, 7 (10): 17829–17842.
DOI: 10.3934/math.2022982
Received: 15 April 2022
Revised: 26 June 2022
Accepted: 05 July 2022
Published: 03 August 2022

http://www.aimspress.com/journal/Math

Research article

An efficient algorithm for the numerical evaluation of pseudo differential operator with error estimation

Amit K. Pandey[1], Manoj P. Tripathi[2], Harendra Singh[3,*], Pentyala S. Rao[1], Devendra Kumar[4], D. Baleanu[5,6]

[1] Department of Mathematics and Computing, Indian Institute of Technology (ISM) Dhanbad-826004, India
[2] Department of Mathematics, Udai Pratap Autonomous College, Varanasi-221002, India
[3] Department of Mathematics, Post-Graduate College, Ghazipur-233001, Uttar Pradesh, India
[4] Department of Mathematics, University of Rajasthan, Jaipur-302004, Rajasthan, India
[5] Department of Mathematics, Faculty of Arts and Sciences, Cankaya University, Eskisehir Yolu 29. Km, Yukarıyurtcu Mahallesi Mimar Sinan Caddesi No: 406790, Etimesgut, Turkey
[6] Institute of Space Sciences, Magurele-Bucharest, Romania

* **Correspondence:** Email: harendra059@gmail.com.

Abstract: In this paper we introduce an efficient and new numerical algorithm for evaluating a pseudo differential operator. The proposed algorithm is time saving and fruitful. The theoretical as well as numerical error estimation of the algorithm is established, together with its stability analysis. We have provided numerical illustrations and established that the numerical findings echo the analytical findings. The proposed technique has a convergence rate of order three. CPU time of computation is also listed. Trueness of numerical findings are validated using figures.

Keywords: pseudo differential operator; numerical algorithm; Hat functions
Mathematics Subject Classification: 33F05, 35S99, 65K99

1. Introduction

Let H_μ denotes the test function space consisting of all complex valued infinitely differentiable function $\varphi(x)$ defined on I=$(0,\infty)$ satisfying, $\gamma_{m,k}^{\mu}(\varphi)$=Sup $\left|x^m(x^{-1}\frac{d}{dx})^k(x^{-\mu-\frac{1}{2}}\varphi(x)\ \ )\right| < \infty$,

$\forall m,k \in N_0$ ($N_0 = N \cup \{0\}$), and $H_\mu{}'$ is the dual of test function space H_μ. The Hankel transformation was extended to distributions belonging to $H_\mu{}'$ by Zemanian [1] as

$$(h_\mu\varphi)(x)=\int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy)\varphi(y)dy.$$

L. Schwartz's [2] systematic study of the Fourier Transform of a distribution in $\Im'(\mathbb{R}^n)$ has been exploited by many author's to study pseudo differential operators, see for instance Zaidman [3]. A pseudo differential operator (PDO) T is defined by means of a symbol $\sigma(x,\xi)$ which is a function of $x,\xi \in \mathbb{R}^n$ (sometimes restricted to $\xi \neq 0$), and by the formal rule $T(e^{ix\cdot\xi}) = \sigma(x,\xi)e^{ix\cdot\xi}$ which is reminiscent of amplitude modulation in radio detection. In most cases, the relation between the operator T and the symbol $\sigma(x,\xi)$ is linearized. Using the Fourier transform, we write every $f \in \vartheta$ ($\mathbb{R}^n$) as a superposition of functions $e^{ix\cdot\xi}$.

$$f(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi\,. \tag{1.1}$$

By linearity, we get

$$Tf(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}\sigma(x,\xi)\,\hat{f}(\xi)\,d\xi\,. \tag{1.2}$$

Equation (1.2) makes sense when $\sigma(x,\xi) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and induces the study of the PDO' s through the Fourier transform.

Using the Zemanian theory of the Hankel transform , Singh and Pandey [4] extended it to study the PDO $(-x^{-1}D)^\nu$, for $\nu \in \mathbb{R}$ and $x \in \mathrm{I}$, I=$(0,\infty)$, $D = \frac{d}{dx}$ and proposed that the pseudo differential operator $(-x^{-1}D)^\nu$ is an automorphism on a certain Frechet space F consisting of complex valued C^∞ functions defined on I=$(0,\infty)$. They also deduced that $(-x^{-1}D)^\nu$ is almost inverse of the Hankel Transform h_μ in the sense that,

$$[h_\nu o\ (-x^{-1}D)^\nu](\varphi) = h_0(\varphi),\ \varphi \in \mathrm{F}.$$

Further, Pathak et al. [5] also used the Zemanian theory to study a certain class of pseudo differential operators which would encompass the theory of Bessel differential operators as a special case. Two new pseudo differential operators associated with Bessel operators were also developed by Pathak & Upadhya [6]. These developments in the field are purely analytical and are based on the distribution theory. In 2015, Tripathi et al. [7] developed an algorithm for numerical evaluation of the Hankel transform of order $\nu > -1$, and it was established that the PDO is the inverse of the Hankel transform h_ν [4]. So, the numerical evaluation of a PDO can be achieved via numerical evaluation of the Hankel transform, and this motivated us for the present work. Some other applications of functions approximations using orthogonal polynomials can be found in [8–10]. In this paper. we are for the first time providing a numerical algorithm for the evaluation of the pseudo differential operator.

2. Preliminaries

In this section we introduce some basic preliminaries. We define the extended hat functions and some basic properties. Suppose that F is the space of all functions $\varphi(x)$ where $\varphi(x)$ is a C^∞ function, such that $x \in I = (0,\infty)$, and

$$\varphi(x) = \sum_{i=0}^{k} a_i x^{2i} + o(x^{2k}) \tag{2.1}$$

near the origin and is rapidly decreasing as $x \to \infty$.

For $-1/2 < \nu$, ν^{th} order Hankel transform h_ν is defined on F [4] by

$$\Phi(y) = [\, h_\nu \varphi(x)](y) = \int_0^\infty \varphi(x) \mathrm{J}_\nu(xy)\, dm(x), \tag{2.2}$$

where

$$dm(x) = m'(x)dx = \left[2^\nu \Gamma(\nu+1)\right]^{-1} x^{2\nu+1} dx\,, \tag{2.3}$$

$$\mathrm{J}_\nu(x) = 2^\nu \Gamma(\nu+1)\, x^{-\nu} J_\nu(x)\,, \tag{2.4}$$

and $J_\nu(x)$ is the ν^{th} order Bessel function.

The inversion formula for Eq (2.2) is given by [11–13]:

$$\varphi(x) = \int_0^\infty \Phi(y) \mathrm{J}_\nu(xy)\, dm(y). \tag{2.5}$$

The hat functions are members of $C[0,1]$ with the shape of hats in a two dimensional plane. When the closed unit interval $[0,1]$ is meshed in to n collocation points $0,\ h,\ 2h,\ 3h,\ \ldots\ldots\ldots,\ nh = 1$, then the hat function's family of first $(n+1)$ functions is given as follows [14]:

$$\psi_0(t) = \begin{cases} \dfrac{h-t}{h}\,, & 0 \le t < h, \\ 0\,, & \text{otherwise}, \end{cases} \tag{2.6}$$

$$\psi_i(t) = \begin{cases} \dfrac{t-(i-1)h}{h}\,, & (i\text{-}1)h \le t < ih, \\ \dfrac{(i+1)h-t}{h}\,, & ih \le t < (i+1)h, \qquad i = 1,\ 2,\ 3\ldots., n-1, \\ 0\,, & \text{otherwise}, \end{cases} \tag{2.7}$$

$$\psi_n(t) = \begin{cases} \dfrac{t-(1-h)}{h}\,, & 1\text{-}\ h \le t \le 1, \\ 0\,, & \text{otherwise}. \end{cases} \tag{2.8}$$

It is evident that the value of the i^{th} hat function $\psi_i(t)$ at the $(k+1)^{th}$ collocation point kh is given by

$$\psi_i(kh) = \begin{cases} 1\,, & i = k, \\ 0\,, & i \ne k. \end{cases} \tag{2.9}$$

3. Numerical evaluation

This section deals with numerical evaluation of the PDO $\left(-x^{-1}D\right)^{\nu}$. We use the hat basis function to approximate the various Hankel transforms appearing in the formulation of the PDO and propose an algorithm for the numerical approximation of the PDO. To derive the algorithm, we first assume that the effective domain space of input signal $\varphi(x)$ is limited to a finite region $0 \le x \le T$. From the physical point of view, this assumption seems reasonable due to the fact that the input signal $\varphi(x)$ representing a physical field either is zero or has an infinitely long decaying tail outside a disc of finite radius T. Therefore, in many practical applications, either the input signal $\varphi(x)$ has a compact support, or for a given $\varepsilon > 0$ there exists a positive real T such that $\left|\int_T^{\infty} x\varphi(x) J_\nu(xy)dx\right| < \varepsilon$, which is the case if $\varphi(x) = o(x^{\lambda})$, where $\lambda < -3/2$ as $x \to \infty$.

In [4] it has been shown that the PDO $\left(-x^{-1}D\right)^{\nu}$, $\nu \neq 1/2$, is almost an inverse of h_ν in the sense that,

$$[h_\nu o\ (-x^{-1}D)^{\nu}](\varphi) = h_0(\varphi),\ \varphi \in F. \tag{3.1}$$

So,

$$\begin{aligned}(-x^{-1}D)^{\nu}(\varphi) &= [h_\nu^{-1} o\ h_0]\varphi \\ &= h_\nu^{-1}(h_0(\varphi)) \\ &= h_\nu^{-1}\Phi(y),\quad \Phi(y) = h_0(\varphi).\end{aligned} \tag{3.2}$$

Using Eqs (2.2)–(2.4),

$$\begin{aligned}\Phi(y) &= [\,h_0\varphi(x)](y) \\ &= \int_0^{\infty} x\varphi(x)\ J_0(xy)dx.\end{aligned} \tag{3.3}$$

Using inversion formula (2.5), Eq (3.2) becomes

$$\begin{aligned}(-x^{-1}D)^{\nu}(\varphi) &= \int_0^{\infty} \Phi(y) J_\nu(xy)dm(y) \\ &= x^{-\nu}\int_0^{\infty} y^{\nu+1}\Phi(y)J_\nu(xy)dy.\end{aligned} \tag{3.4}$$

Now, we use two level approximation in (3.3) and (3.4) as follows:

$$\begin{aligned}(-x^{-1}D)^{\nu}\varphi(x) &= x^{-\nu}\int_0^{\infty} y^{\nu+1}\Phi(y)J_\nu(xy)dy \\ &\cong x^{-\nu}\int_0^{T} y^{\nu+1}\Phi(y)J_\nu(xy)dy \\ &= x^{-\nu}\int_0^{1} y^{\nu+1}\Phi(y)J_\nu(xy)dy \ \ \text{(By scaling).}\end{aligned} \tag{3.5}$$

$$\Phi(\mathrm{y}) = \int_0^\infty x\varphi(x)J_0(xy)dx \cong \int_0^T x\varphi(x)J_0(xy)dx$$

$$= \int_0^1 x\varphi(x)J_0(xy)dx. \text{ (By scaling)} \tag{3.6}$$

The computation of the integral is not an easy one due to the involvement of the rapidly oscillating function $J_0(xy)$ in the integrand of $\int_0^1 x\varphi(x)J_0(xy)dx$. So, we approximate $J_0(xy)$ through the hat basis functions $\psi_i(x)$ as,

$$J_0(xy) \cong \sum_{i=0}^{q} c_i\psi_i(x), \text{ where } c_i = J_0(ihy),\ h = 1/q, \tag{3.7}$$

and q is the number of collocation points in the interval $[0,1]$. With this approximation, Eq (3.6) becomes

$$\Phi(y) \cong \int_0^1 x\varphi(x)\left(\sum_{i=0}^{q} c_i\psi_i(x)\right)dx = \sum_{i=0}^{q} J_0(yih)\int_0^1 x\varphi(x)\psi_i(x)dx. \tag{3.8}$$

Using the support of the hat basis functions, Eq (3.8) is written as

$$\begin{aligned}\Phi(y) = J_0(0) \int_0^h x\varphi(x)\psi_0(x)dx \\ + \sum_{i=1}^{q-1} J_0(yih)\int_{(i-1)h}^{(i+1)h} x\varphi(x)\psi_i(x)\,dx + J_0(yqh)\int_{1-h}^{1} x\varphi(x)\psi_q(x)\,dx.\end{aligned} \tag{3.9}$$

The value of $\Phi(y)$ computed from Eq (3.9) is used to find the approximate numerical value of the PDO from Eq (3.5), as follows.

We approximate $J_\nu(xy)$ as

$$J_\nu(xy) \cong \sum_{j=0}^{m} d_j\psi_j(y), \text{ where } d_j = J_\nu(jh'x),\ h' = 1/m, \tag{3.10}$$

and m is the number of collocation points. Substituting $J_\nu(xy)$ in Eq (3.5), we obtain $\left(-x^{-1}D\right)^\nu(\varphi(x))$ $\cong\ x^{-\nu}\int_0^1 \Phi(y)y^{\nu+1}\left(\sum_{j=0}^{m} d_j\psi_j(y)\right)dy$. Hence, we get the following algorithm for the numerical computation of the PDO:

$$\begin{aligned}\left(-x^{-1}D\right)^\nu(\varphi(x)) &\cong x^{-\nu}\left(\sum_{j=0}^{m} d_j\int_0^1 \Phi(y)y^{\nu+1}\psi_j(y)dy\right) \\ &= x^{-\nu}\left(J_\nu(0)\int_0^{h'} \Phi(y)y^{\nu+1}\psi_0(y)dy + \sum_{j=1}^{m-1} J_\nu(xjh')\int_{(j-1)h'}^{(j+1)h'} \Phi(y)y^{\nu+1}\psi_j(y)dy\right. \\ &\quad \left. + J_\nu(xqh')\int_{1-h'}^{1} \Phi(y)y^{\nu+1}\psi_m(y)dy\right),\end{aligned} \tag{3.11}$$

where $\Phi(y)$ is calculated using Eq (3.9).

4. Error analysis

In this section, we provide error estimation of the proposed scheme. We use the following notations:

$$J_{0q}(xy) \equiv \sum_{j=0}^{q} J_0(yjh)\psi_j(x), \quad h = 1/q\,, \tag{4.1}$$

$$J_{\nu m}(xy) \equiv \sum_{i=0}^{m} J_\nu(xih')\psi_i(y), \quad h' = 1/m, \tag{4.2}$$

$$\Phi_q(y) \equiv \int_0^1 x\varphi(x) J_{0q}(xy)dx\,, \tag{4.3}$$

$$\left(-x^{-1}D\right)_{qm}^{\nu} (\varphi(x)) \equiv x^{-\nu}\int_0^1 y^{\nu+1}\Phi_q(y) J_{\nu m}(xy)dy. \tag{4.4}$$

Theorem 4.1. If $J_\nu(xy)$ is approximated by a set of first $(m+1)$ hat functions $\psi_0(y), \psi_1(y), ..., \psi_m(y)$ as Eq (3.10), and $J_0(xy)$ is approximated by a set of first $(q+1)$ hat functions $\psi_0(x),\ \psi_1(x), ..., \psi_q(x)$ as Eq (3.7), then

(i) $\left|J_\nu(xih') - J_{\nu m}(xih')\right| = 0, \quad for\ \ i = 0,1,2,...,m.$

(ii) $\left|J_\nu(xy) - J_{\nu m}(xy)\right| \le \dfrac{x^2}{2m^2} + O\left(\dfrac{x^3}{m^3}\right),$ for $ih' < y < (i+1)h',\ \ i = 0,1,2,...,m-1.$

(iii) $\left|\Phi(y) - \Phi_q(y)\right| \le \dfrac{My^2}{4q^2} + O\left(\dfrac{y^3}{n^3}\right)$, where $\left|\varphi(x)\right| \le M$.

Proof.

(i) From Eqs (4.2) and (2.9), the value of $J_{\nu m}(xy)$ at the i^{th} collocation point $y = ih',\ \ i = 0,1,2,...,m$, is given by

$$\begin{aligned} J_{\nu m}(xih') &= \sum_{j=0}^{m} J_\nu(xjh')\psi_j(ih') \\ &= J_\nu(xih')\psi_i(ih') + J_\nu(xih' + ih')\psi_{i+1}(ih') = J_\nu(xih')\,. \end{aligned}$$

So,

$$\left|J_\nu(xih') - J_{\nu m}(xih')\right| = 0, \quad for\ \ i = 0,1,2,...,m.$$

(ii) For $ih' < y < (i+1)h', \quad i = 0,1,2,...,m-1$, $j = 0,1,2,...,n-1$, and then from Eq (4.2), we have

$$\begin{aligned} J_{\nu m}(xy) &= J_\nu(xih')\psi_i(y) + J_\nu(xih' + xh')\psi_{i+1}(y) \\ &= J_\nu(xih')\left(\frac{(i+1)h' - y}{h'}\right) + J_\nu(xih' + xh')\left(\frac{y - ih'}{h'}\right) \text{ (using Eqs (2.6–2.8))} \\ &= J_\nu(xih') - xih'\left(\frac{J_\nu(xih' + xh') - J_\nu(xh')}{xh'}\right) + xy\left(\frac{J_\nu(xih' + xh') - J_\nu(xh')}{xh'}\right) \end{aligned}$$

$$= J_\nu(xih') + (xy - xih')\left(\frac{J_\nu(xih' + xh') - J_\nu(xih')}{xh'}\right). \tag{4.5}$$

As $h \to 0,$ from Eq (4.5), we obtain

$$J_{\nu m}(xy) \cong J_\nu(xih') + (xy - xih')J_\nu{}'(xih'), \tag{4.6}$$

where $J_\nu{}'(xih')$ denotes the derivative of J_ν with respect to xy at $y = ih'$.

By expanding $J_\nu(xy)$ in the form of Taylor series, in the powers of $(xy - xih')$, we have

$$J_\nu(xy) = \sum_{k=0}^{\infty} \frac{(xy - xih')^k}{k!} J_\nu{}^{(k)}(xih'), \tag{4.7}$$

where $J_\nu{}^{(k)}$, denotes the k^{th} order derivative of J_ν with respect to xy at $y = ih'$. Using Eqs (4.7) and (4.6), we have

$$\begin{aligned} J_\nu(xy) - J_{\nu m}(xy) &= \sum_{k=2}^{\infty} \frac{(xy - xih')^k}{k!} J_\nu{}^{(k)}(xih') \\ &= \frac{(xy - xih')^2}{2} J_\nu{}''(xih') + O(xy - xih')^3. \end{aligned} \tag{4.8}$$

Since $(xy - xih') < xh'$ and $mh' = 1$, from Eq (4.8), we get

$$\left|J_\nu(xy) - J_{\nu n}(xy)\right| \le \frac{x^2}{2m^2}\left|J_\nu''(xih')\right| + O\left(\frac{x^3}{m^3}\right). \tag{4.9}$$

Since $J_\nu' = \frac{1}{2}(J_{\nu-1} - J_{\nu+1})$, we have $J_\nu'' = \frac{1}{4}(J_{\nu-2} - 2J_\nu + J_{\nu+1})$, and thus from Eq (4.9) we have

$$\begin{aligned} \left|J_\nu(xy) - J_{\nu n}(xy)\right| &\le \frac{x^2}{8m^2}\left|J_{\nu-2}(xih') - 2J_\nu(xih') + J_{\nu+1}(xih')\right| + O\left(\frac{x^3}{m^3}\right) \\ &\le \frac{x^2}{2m^2} + O\left(\frac{x^3}{m^3}\right) (\text{as } \left|J_\nu(xih')\right| \le 1). \end{aligned} \tag{4.10}$$

(iii) From Eqs (3.6) and (4.3), we get

$$\left|\Phi(y) - \Phi_q(y)\right| = \left|\int_0^1 x\varphi(x)\left(J_0(xy) - J_{0q}(xy)\right)dx\right| \le \sum_{i=0}^{q-1}\int_{ih}^{(i+1)h} x\left|\varphi(x)\right|\left|J_0(xy) - J_{0q}(xy)\right|dx.$$

Replacing h' by h and interchanging the role of x and y, from Theorem 4.1(**ii**), we get

$$\begin{aligned} \left|\Phi(y) - \Phi_q(y)\right| &\le \sum_{i=0}^{q-1}\int_{ih}^{(i+1)h} x\left|\varphi(x)\right|\left|\frac{y^2}{2q^2} + O\left(\frac{y^3}{q^3}\right)\right|dx \\ &\le M\left|\frac{y^2}{2q^2} + O\left(\frac{y^3}{q^3}\right)\right| \sum_{i=0}^{q-1}\int_{ih}^{(i+1)h} x\,dx = M\left(\frac{y^2}{4q^2} + O\left(\frac{y^3}{q^3}\right)\right). \end{aligned} \tag{4.11}$$

This proves the third part of the theorem.

Theorem 4.2. With the postulates stated in Theorem 4.1, the upper bound for the absolute error $\varepsilon_{qm}(x)$ between $\left(-x^{-1}D\right)^{\nu}(\varphi(x))$ and $\left(-x^{-1}D\right)_{qm}^{\nu}(\varphi(x))$ is given by

$$\varepsilon_{qm}(x) \le Mx^{-\nu}\left[\left(\frac{1}{4q^2(\nu+3)}+O\left(\frac{1}{q^3}\right)\right)+\frac{1}{2(\nu+2)}\left(\frac{x^2}{2m^2}+O\left(\frac{x^3}{m^3}\right)\right)\right].$$

Proof. From Eqs (3.5) and (4.4),

$$\left(-x^{-1}D\right)^{\nu}(\varphi(x)) - \left(-x^{-1}D\right)_{qm}^{\nu}(\varphi(x)) = x^{-\nu}\left[\int_0^1 y^{\nu+1}\left(\Phi(y)J_\nu(xy)-\Phi_q(y)J_{\nu m}(xy)\right)dy\right]$$

$$= x^{-\nu}\left[\int_0^1 y^{\nu+1}\left(\Phi(y)J_\nu(xy)-\Phi_q(y)J_\nu(xy)+\Phi_q(y)J_\nu(xy)-\Phi_q(y)J_{\nu m}(xy)\right)dy\right].$$

Thus,

$$\varepsilon_{qm}(x) = \left|\left(-x^{-1}D\right)^{\nu}(\varphi(x))-\left(-x^{-1}D\right)_{qm}^{\nu}(\varphi(x))\right|$$

$$\le x^{-\nu}\left[\int_0^1 y^{\nu+1}\left|\Phi(y)-\Phi_q(y)\right|\left|J_\nu(xy)\right|dy+\int_0^1 y^{\nu+1}\left|\Phi_q(y)\right|\left|J_\nu(xy)-J_{\nu m}(xy)\right|dy\right]$$

$$= x^{-\nu}\left[\int_0^1 y^{\nu+1}\left|\Phi(y)-\Phi_q(y)\right|\left|J_\nu(xy)\right|dy+\sum_{i=0}^{m-1}\int_{ih'}^{(i+1)h'} y^{\nu+1}\left|\Phi_q(y)\right|\left|J_\nu(xy)-J_{\nu m}(xy)\right|dy\right]. \quad (4.12)$$

Further from Eqs (4.3) and (4.1),

$$\left|\Phi_q(y)\right| = \left|\int_0^1 x\varphi(x)J_{0q}(xy)dx\right| \le \int_0^1 x\left|\varphi(x)\right|\left|J_{0q}(xy)\right|dx$$

$$\le M\sum_{i=0}^{q}\left|J_0(yih)\right|\int_0^1 x\psi_i(x)dx \le M\sum_{i=0}^{q}\int_0^1 x\psi_i(x)dx \quad (\text{as } \left|J_0(yih)\right|\le 1)$$

$$= M\left[\int_0^h x\psi_0(x)dx+\sum_{i=1}^{q-1}\int_{(i-1)h}^{(i+1)h} x\psi_i(x)dx+\int_{(q-1)h}^{1} x\psi_q(x)dx\right]$$

$$= M\left[\int_0^h x\left(\frac{h-x}{h}\right)dx+\sum_{i=1}^{q-1}\left(\int_{(i-1)h}^{ih} x\left(\frac{x-(i-1)h}{h}\right)dx+\int_{ih}^{(i+1)h} x\left(\frac{(i+1)h-x}{h}\right)dx\right)\right.$$

$$\left.+\int_{(q-1)h}^{1} x\left(\frac{x-(q-1)h}{h}\right)dx\right]$$

$$= Mh^2\left(\frac{1}{6}+\sum_{i=1}^{q-1}i+\frac{3q-1}{6}\right) = \frac{1}{2}Mq^2h^2 = \frac{1}{2}M\,. \quad (4.13)$$

Using Eqs (4.10), (4.11) and (4.13), the inequality in Eq (4.12) becomes

$$\varepsilon_{qm}(x) \le x^{-\nu}\left[\int_0^1 M\left(\frac{y^2}{4q^2}+O\left(\frac{y^3}{q^3}\right)\right)y^{\nu+1}dy+\frac{1}{2}M\left(\frac{x^2}{2m^2}+O\left(\frac{x^3}{m^3}\right)\right)\sum_{i=0}^{m-1}\int_{ih'}^{(i+1)h'} y^{\nu+1}dy\right]$$
$$= Mx^{-\nu}\left[\left(\frac{1}{4q^2(\nu+3)}+O\left(\frac{1}{q^3}\right)\right)+\frac{1}{2(\nu+2)}\left(\frac{x^2}{2m^2}+O\left(\frac{x^3}{m^3}\right)\right)\right].$$

5. Stability analysis

In this section we have analyzed the stability of the proposed algorithm under the influence of random noise. If the data function $\varphi(x)$ is perturbed by adding a random noise α, and the perturbed data function is denoted by $\varphi^{\alpha}(x)$, then $\varphi^{\alpha}(x) = \varphi(x)+\alpha\theta$, where θ is a uniform random variable in $[-1,1]$. So, we have $\left|\varphi^{\alpha}(x)-\varphi(x)\right|<\alpha$.

Using the approximation scheme developed in Sections 3 and 4, we have obtained Eq (4.4) as $\left(-x^{-1}D\right)^{\nu}_{qm}(\varphi(x)) \equiv x^{-\nu}\int_0^1 y^{\nu+1}\Phi_q(y)\,J_{\nu m}(xy)\,dy$, where $J_{\nu m}$ and Φ_q are given by Eqs (4.2) and (4.3), respectively. Now, Eq (4.4) can be used to obtain the pseudo-differential operator of the noisy data function $\varphi^{\alpha}(x)$ as

$$\left(-x^{-1}D\right)^{\nu}_{qm}(\varphi^{\alpha}(x)) \equiv x^{-\nu}\int_0^1 y^{\nu+1}\Phi^{\alpha}{}_q(y)J_{\nu m}(xy)dy\,, \tag{5.1}$$

where

$$\Phi^{\alpha}{}_q(y)=\int_0^1 x\varphi^{\alpha}(x)J_{0q}(xy)dx\,, \text{ (from Eq (4.3)).} \tag{5.2}$$

So, we have

$$\left|\left(-x^{-1}D\right)^{\nu}_{qm}(\varphi^{\alpha}(x))-\left(-x^{-1}D\right)^{\nu}_{qm}(\varphi(x))\right|=\left|x^{-\nu}\int_0^1 y^{\nu+1}\Phi^{\alpha}{}_q(y)J_{\nu m}(xy)\,dy-x^{-\nu}\int_0^1 y^{\nu+1}\Phi_q(y)J_{\nu m}(xy)\,dy\right|$$
$$=\left|x^{-\nu}\int_0^1 y^{\nu+1}(\Phi^{\alpha}{}_q(y)-\Phi_q(y))J_{\nu m}(xy)\,dy\right|$$
$$\le\left|x^{-\nu}\right|\int_0^1\left|y^{\nu+1}\right|\left|\Phi^{\alpha}{}_q(y)-\Phi_q(y)\right|\left|J_{\nu m}(xy)\right|dy$$
$$\le\left|x^{-\nu}\right|\int_0^1\left|\Phi^{\alpha}{}_q(y)-\Phi_q(y)\right|dy\,. \tag{5.3}$$

Now, replacing $\varphi^{\alpha}(x) = \varphi(x)+\alpha\theta$ in Eq (5.2), we have

$$\Phi^{\alpha}{}_q(y)=\int_0^1 x\varphi^{\alpha}(x)J_{0q}(xy)\,dx = \int_0^1 x\varphi(x)\,J_{0q}(xy)\,dx+\alpha\theta\int_0^1 x\,J_{0q}(xy)\,dx\,. \tag{5.4}$$

Using Eq (5.1), Eq (5.4) can be rewritten as $\Phi^{\alpha}{}_q(y)-\Phi_q(y) = \alpha\theta\int_0^1 x\,J_{0q}(xy)\,dx$. So, we have

$$\Phi^{\alpha}{}_q(y)-\Phi_q(y) = \alpha\theta\int_0^1 x\, J_{0q}(xy)\, dx$$

$$= \alpha\theta\int_0^1 x\sum_{j=0}^{q} J_0(yjh)\ \psi\, j(h)\, dx$$

$$= \alpha\theta\left[J_0(0)\int_0^h x\psi_0(x)dx+\sum_{j=1}^{(q-1)h} J_0(yjh)\int_{(j-1)h}^{(j+1)h} x\psi_j(x)\, dx + J_0(yqh)\int_{(1-h)}^{1} x\psi_q(x)\, dx\right]$$

$$= \alpha\theta\left[J_0(0)\int_0^h \frac{x(h-x)}{h}dx+\sum_{j=1}^{(q-1)} J_0(yjh)\left[\int_{(j-1)h}^{jh} \frac{x\left(x-(j-1)h\right)}{h}dx+\int_{jh}^{(j+1)h}\frac{x\left((j+1)h-x\right)}{h}dx\right]\right]$$

(using definition of hat functions)

$$= \alpha\theta\left[\frac{h^2}{6}+\sum_{j=1}^{(q-1)} J_0(yjh)\left[h^2\left(\frac{j}{2}-\frac{1}{6}\right)+h^2\left(\frac{j}{2}+\frac{1}{6}\right)\right]+J_0(y)\left(\frac{h}{2}-\frac{h^2}{6}\right)\right].$$

Therefore,

$$\left|\Phi^{\alpha}{}_q(y)-\Phi_q(y)\right| = \left|\alpha\theta\left[\frac{h^2}{6}+\sum_{j=1}^{(q-1)} J_0(yjh)\left[h^2\left(\frac{j}{2}-\frac{1}{6}\right)+h^2\left(\frac{j}{2}+\frac{1}{6}\right)\right]+J_0(y)\left(\frac{h}{2}-\frac{h^2}{6}\right)\right]\right| \le \frac{|\alpha|}{2}.$$

Using the above upper bound for $\left|\Phi^{\alpha}{}_q(y)-\Phi_q(y)\right|$, the inequality in Eq (5.3) reduces to

$$\left|\left(-x^{-1}D\right)^{\nu}_{qm}(\varphi^{\alpha}(x))-\left(-x^{-1}D\right)^{\nu}_{qm}(\varphi(x))\right| \le \left|x^{-\nu}\right|\left|\int_0^1 \frac{\alpha}{2}dy = \frac{\alpha}{2}\right|x^{-\nu}\left|.\right. \tag{5.5}$$

The above analysis leads to the following theorem.

Theorem 5.1. When the input data $\varphi(x)$ is corrupted with a random noise α, the proposed algorithm reduces the noise at least by a factor of $\frac{\alpha}{2}\left|x^{-\nu}\right|$ in the numerically approximated PDO $\left(-x^{-1}D\right)^{\nu}_{qm}$.

6. Results discussion

We use the algorithm developed to compute the numerical evaluation of the pseudo differential operator for $\left(-x^{-1}D\right)^{\nu} e^{-x^2}$ and $\left(-x^{-1}D\right)^{\nu} (x^2 e^{-x^2/2}),\ \nu\in\mathbb{R}$. All the computations have been done using MATLAB-7.0. For evaluating $\left(-x^{-1}D\right)^{\nu} e^{-x^2}$ numerically, we take $T=7$, $\nu=1/2$ and $q=m=100$. In Figure 1, a comparison between the exact PDO [15] $\left(-x^{-1}D\right)^{1/2} e^{-x^2}$ and approximate value of the PDO $\left(-x^{-1}D\right)^{1/2}_{qm} e^{-x^2}$ is shown. Figure 2 shows the absolute error $\varepsilon_{qm}(x)$, for $q=m=100$, between the approximated and the exact PDO $\left(-x^{-1}D\right)^{1/2}\left(e^{-x^2}\right)$. The elapsed time (CPU time) for this example is 9.797 seconds. For this example, we have also calculated the absolute errors $\varepsilon_{qm}(x_i),\ i=1,2,3,4,5$ for different values of m at randomly selected different node points $x_1=0.1,\ x_2=1.6,\ x_3=3.1,\ x_4=4.6,\ x_5=6.1$ and listed them in Table 1.

Table 1. Absolute errors $\varepsilon_{qm}(x_i)$, for different values of m and $q=m$ at node points $x_1=0.1$, $x_2=1.6$, $x_3=3.1$, $x_4=4.6$, $x_5=6.1$.

m	$\varepsilon_{qm}(x_1)$	$\varepsilon_{qm}(x_2)$	$\varepsilon_{qm}(x_3)$	$\varepsilon_{qm}(x_4)$	$\varepsilon_{qm}(x_5)$
10	0.0560	0.0068	4.9765×10^{-5}	7.0761×10^{-6}	2.4117×10^{-5}
20	0.0136	0.0017	2.7442×10^{-5}	$6.3599\ 10^{-7}$	8.5926×10^{-6}
50	0.0017	2.1862×10^{-4}	1.5035×10^{-5}	9.6113×10^{-6}	1.4369×10^{-5}
100	5.1915×10^{-5}	9.5177×10^{-6}	1.1446×10^{-5}	1.2587×10^{-5}	1.4984×10^{-5}
500	6.0178×10^{-4}	5.7430×10^{-5}	7.8500×10^{-6}	1.4771×10^{-5}	1.4840×10^{-5}

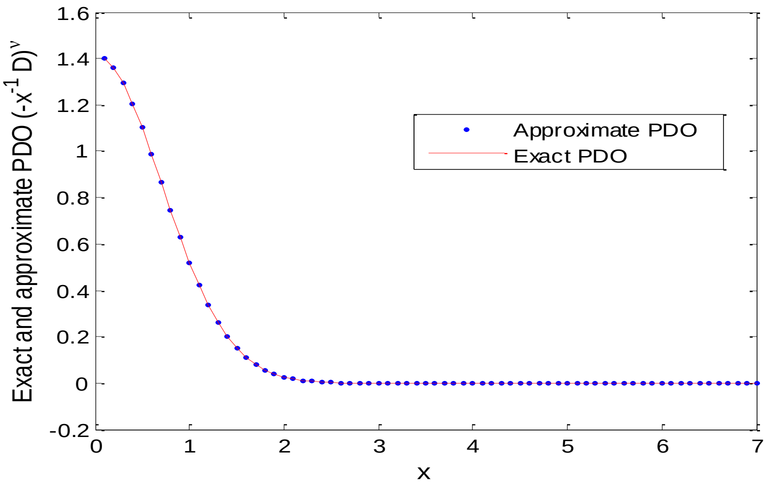

Figure 1. Exact (solid red) and approximate (blue-dashed) values of $\left(-x^{-1}D\right)^{1/2} e^{-x^2}$.

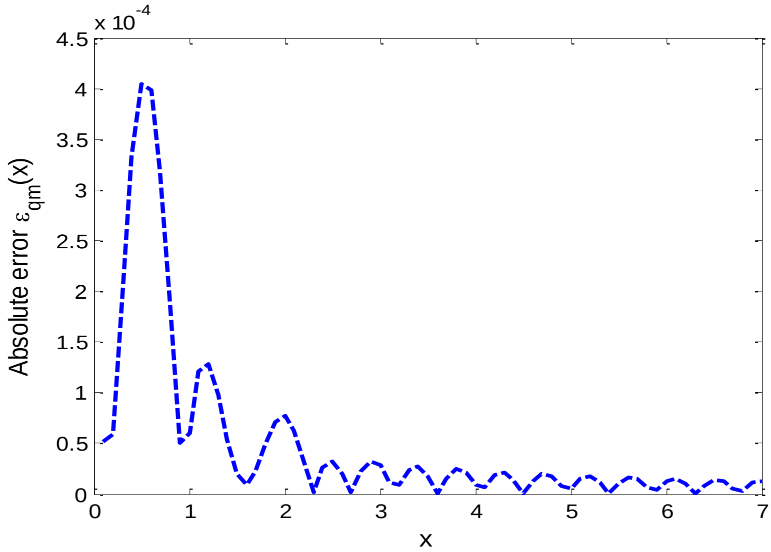

Figure 2. Absolute error $\varepsilon_{qm}(x)$, for $q=m=100$.

For numerical evaluation of the PDO $\left(-x^{-1}D\right)^{-1/3}(x^2e^{-x^2/2})$, we take $T=6$, $\nu=-1/3$ and $q=m=100$. Figure 3 compares the exact value of $\left(-x^{-1}D\right)^{-1/3}(x^2e^{-x^2/2})$, i.e., $(x^2-2\nu)e^{-x^2/2}$, $\nu=-1/3$, and the approximate value of the PDO, i.e., $\left(-x^{-1}D\right)_{qm}^{-1/3}(x^2e^{-x^2/2})$. Figure 4 presents the absolute error $\varepsilon_{qm}(x)$, for $q=m=1000$, between the exact PDO $\left(-x^{-1}D\right)^{-1/3}(x^2e^{-x^2/2})$ and approximate PDO $\left(-x^{-1}D\right)_{qm}^{-1/3}(x^2e^{-x^2/2})$. The elapsed time (CPU time) for this example is 10.64 seconds.

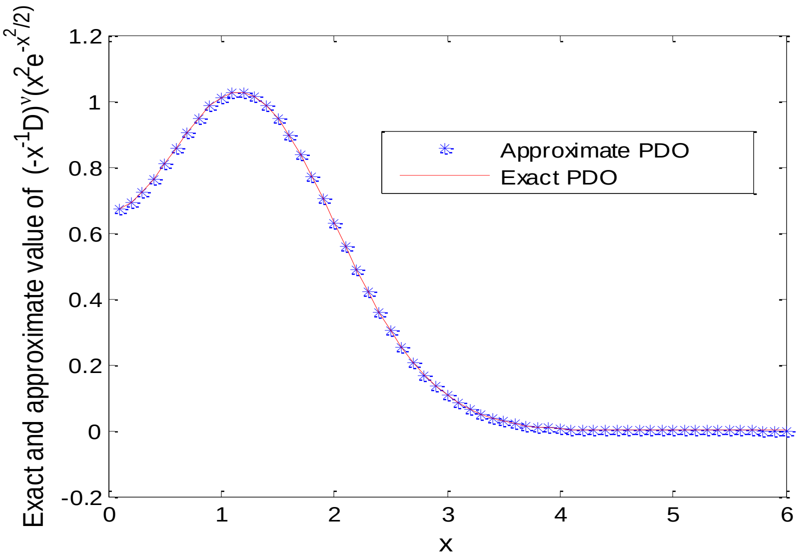

Figure 3. Exact (solid red) and approximate (blue - starred) values of $\left(-x^{-1}D\right)^{1/2}(x^2e^{-x^2/2})$.

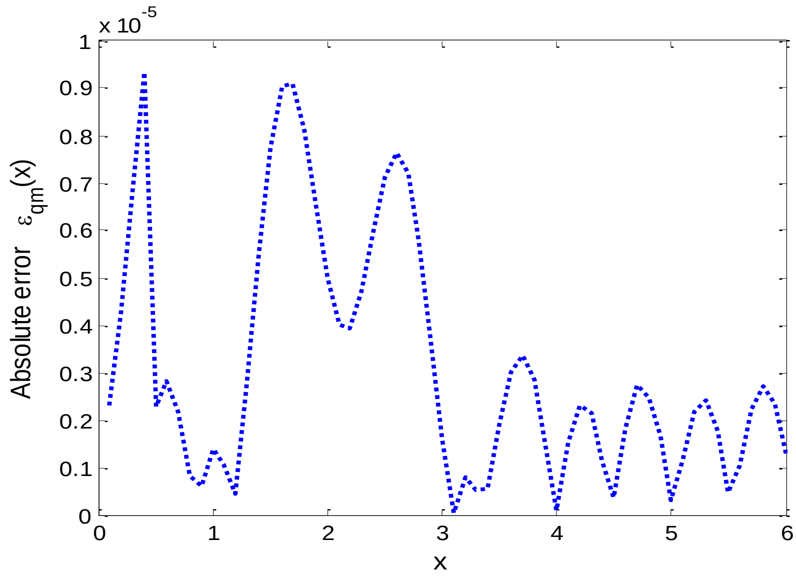

Figure 4. Absolute error $\varepsilon_{qm}(x)$, for $q=m=1000$.

7. Conclusions

In the present manuscript, the algorithm developed for the numerical evaluation of $\left(-x^{-1}D\right)^{\nu}$, $\nu \in \mathbb{R}$, successfully corresponds to the analytical findings. The error estimation of the algorithm is also given in the paper, which is accompanied with stability analysis. Since it is the first time that numerical evaluation of the pseudo-differential operator has been achieved, we have nothing to compare with, but still we can say that theoretical finding of the proposed scheme together with the error estimation is very effective. From the numerical discussion section we can see that the proposed scheme is also time saving because a very little span of time is required for computation. In the future, we can use different orthogonal polynomials and wavelets using the same scheme for better results.

Conflict of interest

The authors declare no conflicts of interest regarding this article.

References

1. A. H. Zemanian, *Distribution theory and transform analysis*, McGraw-Hill, McGraw-Hill, New York, 1965.
2. L. Schwartz, *Theorie des distributions*, Hermann, Paris, 1997.
3. S. Zaidman, *Distribution and pseudo-differential operators*, Longman, Essex, England, 1991.
4. O. P. Singh, J. N. Pandey, The Fourier-Bessel series representation of the pseudo differential operator $\left(-x^{-1}D\right)^{\nu}$, *Proc. Amer. Math. Soc.*, **115** (1992), 969–976. https://doi.org/10.1090/S0002-9939-1992-1107924-6
5. R. S. Pathak, P. K. Pandey, A class of pseudo-differential operators associated with Bessel operators, *J. Math. Anal. Appl.*, **196** (1995), 736–747. https://doi.org/10.1006/jmaa.1995.1437
6. R. S. Pathak, S. K. Upadhyay, Pseudo-differential operators involving Hankel transforms, *J. Math. Anal. Appl.*, **213** (1997), 133–147. https://doi.org/10.1006/jmaa.1997.5495
7. M. P. Tripathi, O. P. Singh, A Hankel transform approach to inverse quasi-static steady-state thermal stresses in a thick circular plate, *Int. J. Appl. Comput. Math.*, **2** (2016), 609–624. https://doi.org/10.1007/s40819-015-0081-3
8. H. Singh, Numerical simulation for fractional delay differential equations, *Int. J. Dyn. Control*, **9** (2020), 463–474. https://doi.org/10.1007/s40435-020-00671-6
9. H. Singh, A. M. Wazwaz, Computational method for reaction diffusion-model arising in a spherical catalyst, *Int. J. Appl. Comput. Math.*, **7** (2021), 65. https://doi.org/10.1007/s40819-021-00993-9
10. H. Singh, A. K. Singh, R. K. Pandey, D. Kumar, J. Singh, An efficient computational approach for fractional Bratu's equation arising in electrospinning process, *Math. Methods Appl. Sci.*, **44** (2021), 10225–10238. https://doi.org/10.1002/mma.7401
11. L. S. Dube, J. N. Pandey, On the Hankel transformation of distributions, *Tohoku Math. J.*, **27** (1975), 337–354. https://doi.org/10.2748/tmj/1203529246
12. W. Y. Lee, On Schwartz's Hankel transformation of certain spaces of distributions, *SIAM J. Math. Anal.*, **6** (1975), 427–432. https://doi.org/10.1137/0506037

13. A. L. Schwartz, An inversion theorem for Hankel transforms, *Proc. Amer. Math. Soc.*, **22** (1969), 713–717. https://doi.org/10.2307/2037465
14. E. Babolian, M. Mordad, A numerical method for solving systems of linear and nonlinear integral equations of the second kind by hat basis functions, *Comput. Math. Appl.*, **62** (2011), 187–198. https://doi.org/10.1016/j.camwa.2011.04.066
15. O. P. Singh, Orthogonal expansions of certain pseudo differential operator, *Int. J. Math. Sci.*, **3** (2004), 129–142.