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Research article

Almost sure convergence theorems for arrays under sub-linear expectations

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Abstract: In this work, inspired by the extended negatively dependent arrays, we want to obtain a limit theorem on almost sure convergence relying on non-additive probabilities. Meanwhile, we offer two appropriate upper integration conditions as an application, allowing us to derive deterministic bounds based on logarithm. Furthermore, these results extend the limit theorems in classical probability space.

Keywords: sub-linear expectation; almost sure convergence; extended negative dependence; law of the logarithm; weakly mean dominated
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1. Introduction

There is no doubt that the theory of limits of probability occupies a prominent place in classical probability theory. They were widely utilized in various sectors, including statistics, finance, and economics. The classical limit theory of probability considers only additive probabilities and additive expectations. It is mainly applicable to deterministic models. Many phenomena arising from quantum mechanics and risk measures are uncertain. They cannot be measured by additive probabilities and expectations, highlighting the limitations of applying the limit theory of probability space. So, can there be a new metric for the portrayal of uncertain phenomena? Having been prompted by the exigencies of modeling uncertainty in practice, academician Peng [1, 2] constructed a general theoretical framework for sub-linear expectations under general function space. This theoretical framework breaks away from the traditional linear probability space. It allows the stochasticity and risk generated by models of uncertainty to be depicted in terms of capacity and non-linear expectation. It's apparent that such a new expectation bridges the gap of the narrow scope of application of probability theory and open up a new horizon for the development and application of limit theory.

As a nascent theoretical system, it naturally arouses strong research interest among scholars. So far, in terms of the definition, properties, and research tools, academics have continuously made significant efforts to improve the content and have achieved many excellent works under the sub-linear expectations. In Peng's framework, we can see that plenty limit theorems have recently progressively emerged, containing the celebrated (weighted) central limit theory (see Peng [3], Li and Shi [4], Zhang and Chen [5], Li [6], Liu and Zhang [7]), strong law of large numbers (SLLN) (see Chen [8], Wu and Jiang [9], Huang and Wu [10], Zhan and Wu [11], Ma and Wu [12]), weak LLN (see Chen et al. [13], Hu [14]), Marcinkiewicz-Zygmund LLN (see Hu [15]), Marcinkiewicz's SLLN (see Zhang and Lin [16]). It is worth mentioning that Zhang [17–19] obtained exponential inequalities, Rosenthal inequalities and strong limit theorems under the sub-linear expectations, and his results provide robust tools for scholars to continue to explore the convergence of diverse categories of series under the sub-linear expectations. Recently, Wu [20] established strong limit theorems, Ding [21] investigated a general form of precise asymptotics for complete convergence. In the setting of non-additive probabilities, Guo and Zhang [22] gave the definition of m-dependent sequence of random variables, and created moderate deviation principle, Feng [23] introduced pseudo-independence and investigated the logarithmic law of weighted sums, Xu and Zhang [24] made a suitable innovation in the conditions of theorems by relaxing the conditions to obtain a law of logarithm, Liu and Zhang [25] presented the concept of a strictly stationary sequence, and acquired the law of the iterated logarithm (LIL), Wu and Lu [26], and Zhang [27] studied the Chover's, Chung's LIL, respectively, Guo et al. [28] obtained the Hartman-Wintner LIL. By analyzing the results of the existing studies, it is not unexpected to observe that the intense interest of many scholars in sub-linear expectations has yielded numerous rich results that enrich the content of non-linear expectations. Moreover, we can see that many classical limit theories under the sub-linear expectations are derived from probability space, and the extension of probability space theory has emerged as a new research trend. Limit theorems for sub-linear expectation space are fraught with more unknowns and challenges than those for probability space, because the expectation and capacity do not have additive properties.

Encouraged by the non-linear expectations, we focus on an array of END random variables and establish almost sure convergence theorems. Zhang [19] defined END in Peng's framework. Since then, the scope of research on random variables has been expanded. The problems considered have become more and more relevant to real-life situations, providing a reliable aid to model uncertainty problems. This paper aims to promote the results of Da Silva [29] from linear to non-linear space and obtain an almost sure convergence theorem. Furthermore, we give two conditions on upper integrals based on the theorem and yield deterministic bounds for END random variables.

The following is the outline for this paper. Section 2 briefly summarizes some of the properties, definitions, and descriptions of the special notations involved in this work. Not only that, but we have also enumerated some significant lemmas that will play a key role in our subsequent proofs. In Section 3, we establish an almost sure convergence theorem based on some conditions and present corollaries relied on different upper integration conditions to obtain deterministic bounds. The last section is a detailed proof of the theorem and corollaries of the third section.

2. Preliminaries

We use the framework and notions of Peng [1]. Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ

satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq c(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \ \forall \ \mathbf{x}, \mathbf{y} \in \mathbb{R}_n,$$

for some $c > 0$, $m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of random variables. In this case we denote $X \in \mathcal{H}$.

Definition 2.1. (Peng [1]) A sub-linear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\hat{\mathbb{E}} : \mathcal{H} \to \bar{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have
(a) Monotonicity: If $X \geq Y$ then $\hat{\mathbb{E}}(X) \geq \hat{\mathbb{E}}(Y)$;
(b) Constant preserving: $\hat{\mathbb{E}}(c) = c$;
(c) Sub-additivity: $\hat{\mathbb{E}}(X + Y) \leq \hat{\mathbb{E}}(X) + \hat{\mathbb{E}}(Y)$; whenever $\hat{\mathbb{E}}(X) + \hat{\mathbb{E}}(Y)$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;
(d) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}(X), \lambda \geq 0$.

Here $\bar{\mathbb{R}} := [-\infty, \infty]$. The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space.

Give a sub-linear expectation $\hat{\mathbb{E}}$, let us denote the conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ by

$$\hat{\varepsilon}(X) := -\hat{\mathbb{E}}(-X), \ \forall X \in \mathcal{H}.$$

From the definition, it is easily shown that for all $X, Y \in \mathcal{H}$

$$\hat{\varepsilon}(X) \leq \hat{\mathbb{E}}(X), \ \hat{\mathbb{E}}(X + c) = \hat{\mathbb{E}}(X) + c,$$
$$|\hat{\mathbb{E}}(X - Y)| \leq \hat{\mathbb{E}}(|X - Y|) \text{ and } \hat{\mathbb{E}}(X - Y) \geq \hat{\mathbb{E}}(X) - \hat{\mathbb{E}}(Y).$$

If $\hat{\mathbb{E}}(Y) = \hat{\varepsilon}(Y)$, then $\hat{\mathbb{E}}(X + aY) = \hat{\mathbb{E}}(X) + a\hat{\mathbb{E}}(Y)$ for any $a \in \mathbb{R}$.

Next, we consider the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\emptyset) = 0, \ V(\Omega) = 1 \text{ and } V(A) \leq V(B) \ \ \forall A \subseteq B, A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$. In the sub-linear space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, we denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\hat{\mathbb{E}}(\xi); I(A) \leq \xi, \xi \in \mathcal{H}\}, \ \mathcal{V}(A) := 1 - \mathbb{V}(A^c), \ \forall A \in \mathcal{F},$$

where $\mathbb{V}(A^c)$ is the complement set of A. It is obvious that $\mathbb{V}$ is sub-additive, and

$$\mathcal{V}(A) \leq \mathbb{V}(A), \ \forall A \in \mathcal{F}; \ \mathbb{V}(A) = \hat{\mathbb{E}}(I(A)), \ \mathcal{V}(A) = \hat{\varepsilon}(I(A)), \text{ if } I(A) \in \mathcal{H}.$$

Property 2.1. *For all $B \in \mathcal{F}$, if $\eta \leq I(B) \leq \xi$, $\eta, \xi \in \mathcal{H}$, then*

$$\hat{\mathbb{E}}(\eta) \leq \mathbb{V}(B) \leq \hat{\mathbb{E}}(\xi). \tag{2.1}$$

Remark 2.1. *From* (2.1)*, for all $X \in \mathcal{H}$, $y > 0$, $\gamma > 0$, it emerges that $\mathbb{V}(|X| \geq y) \leq \hat{\mathbb{E}}(|X|^\gamma)/y^\gamma$, which is the well-known Markov's inequality.*

Remark 2.2. *Mathematical expectation corresponds to the integral in $(\Omega, \mathcal{A}, P)$, where the integral depends on a probability. In $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, capacity is an alternative to probability, so what is the relationship between the capacity and integral? The following is the definition of the upper integral.*

Definition 2.2. For all $|X| \in \mathcal{H}$, define

$$C_{\mathbb{V}}(|X|) := \int_0^{\infty} \mathbb{V}\left(|X| > x\right) \mathrm{d}x.$$

From the above definition, we cannot help but think of the definition of mathematical expectation in probability space, $E(|X|) := \int_0^{\infty} P\left(|X| > x\right) \mathrm{d}x$. In $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, $\hat{\mathbb{E}}(|X|)$ and $C_{\mathbb{V}}(|X|)$ are not related in the general situations. From Zhang [17], we can learn that $\hat{\mathbb{E}}(|X|) \leq C_{\mathbb{V}}(|X|)$ if one of the following three circumstances is satisfied: (i) $\hat{\mathbb{E}}$ is countably sub-additive; (ii) $\hat{\mathbb{E}}(|X|-d)I(|X| > d) \to 0$, as $d \to \infty$; (iii) $|X|$ is bounded.

Definition 2.3. (Zhang [19] extended negative dependence (END)) In $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, r.v. $\{X_n; n \geq 1\}$ is known as being upper (lower) END, if for a certain controlling constant $M \geq 1$ that makes

$$\hat{\mathbb{E}}\left(\prod_{i=1}^{n} \varphi_i\left(X_i\right)\right) \leq M \prod_{i=1}^{n} \hat{\mathbb{E}}\left(\varphi_i\left(X_i\right)\right), \quad \forall n \geq 2,$$

as long as $\varphi_i(x)$ is non-negative, $\varphi_i(x) \in C_{b,Lip}(\mathbb{R}), i \geq 1$, are all non-decreasing (resp. all non-increasing).

What is clear is that suppose $\{X_n; n \geq 1\}$ is a END r.v. and for $i \geq 1$, $f_i(x) \in C_{l,Lip}(\mathbb{R})$ are all non-decreasing (resp. all non-increasing), so $\{f_n(X_n); n \geq 1\}$ is as well an END r.v. sequence.

Remark 2.3. *The definition of the independence can be found in Peng [1]. Zhang [18] introduced negative dependence (ND), this is the first extension of the study of r.v. in* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *since Peng first introduced the definition of independence. Immediately following, Zhang [19] presented the concept of END, which further expanded the scope of the study of random variables, and it is based on this relatively broad random variable that this paper is based on.*

Definition 2.4. (Zhang [17]) (i) A sub-linear expectation $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ is called to be countably sub-additive if it satisfies

$$\hat{\mathbb{E}}(X) \leq \sum_{n=1}^{\infty} \hat{\mathbb{E}}(X_n), \text{ whenever } X \leq \sum_{n=1}^{\infty} X_n, \ X, X_n \in \mathcal{H}, \ X \geq 0, X_n \geq 0, \ n \geq 1.$$

(ii) A function $V : \mathcal{F} \to [0, 1]$ is called to be countably sub-additive if

$$V\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} V\left(A_n\right), \ \forall A_n \in \mathcal{F}.$$

Wu and Jiang [9] described in detail the almost sure convergence under $\hat{\mathbb{E}}$. Moreover, Wu and Lu [26] gave an example to elaborate on $X_n \to X$ a.s. $\mathcal{V}$ cannot derive $X_n \to X$ a.s. $\mathbb{V}$.

Under sub-linear expectations, since expectation and capacity are uncertain, almost sure convergence is different from the traditional probability space, so many of the criteria that hold in probability space do not necessarily apply to sub-linear expectations. Therefore, it is more challenging to study the almost sure convergence theorems under sub-linear expectations.

Our main purpose in defining a function $g(x) \in C_{l,Lip}(\mathbb{R})$ is to modify the indicator function so that a function like $I(|x| \leq a)$ remains continuous, which we define as follows.

For $0 < \mu < 1$, suppose that the even function $g(x) \in C_{l,Lip}(\mathbb{R})$ and $g(x) \downarrow$ in $x > 0$, such that $0 \leq g(x) \leq 1$ for all x and $g(x) = 1$ if $|x| \leq \mu$, $g(x) = 0$ if $|x| \geq 1$. Then

$$I(|x| \leq \mu) \leq g(x) \leq I(|x| \leq 1), I(|x| > 1) \leq 1 - g(x) \leq I(|x| > \mu). \tag{2.2}$$

In the whole process of this work, c refers to a positive constant that varies according to location. $a_n \ll b_n$ denotes that there is a constant $c > 0$ such that $a_n \leq cb_n$ for sufficiently large n. $I(\cdot)$ denotes indicator function. $\log n := \ln\{\max(n, e)\}$. $a_n \sim b_n$ means $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

The following auxiliary tools are required to prove our results.

According to the definition of capacity, we know that capacity is sub-additive, but not necessarily countably sub-additive. When Borel-Cantelli's lemma from probability space is generalized to sub-linear expectations, the first section of the lemma requires the addition of the condition that the capacity is countably sub-additive.

Lemma 2.1. *(Zhang [17] Borel-Cantelli's lemma) Assume that $\{A_m; m \geq 1\}$ represents a sequence of occurrences in $\mathcal{F}$, and V is countably sub-additive. As long as $\sum_{m=1}^{\infty} V(A_m) < \infty$, concluding $V\left(\bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} A_i\right) = 0$.*

Lemma 2.2. *(Zhong and Wu [30]) Assume $X \in \mathcal{H}, p > 0, \beta > 0$, and the slowly varying function is represented by $l(x)$. Then,*

$$C_{\mathbb{V}}\left(|X|^p l\left(|X|^{1/\beta}\right)\right) < \infty \Leftrightarrow \sum_{n=1}^{\infty} n^{\beta p-1} l(n) \mathbb{V}\left(|X| > cn^{\beta}\right) < \infty, for\ any\ c > 0. \tag{2.3}$$

Lemma 2.3. *(Zhang [19]) Let $\{X_k; k \geq 1\}$ be a sequence of upper END r.v. in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, and $\hat{\mathbb{E}}(X_k) \leq 0$. Then for every $x, y > 0$,*

$$\mathbb{V}(S_n > x) \leq \mathbb{V}\left(\max_{1\leq k\leq n} X_k > y\right) + M \exp\left\{-\frac{x^2}{2(xy + B_n)}\left(1 + \frac{2}{3}\log\left(1 + \frac{xy}{B_n}\right)\right)\right\}, \tag{2.4}$$

where $B_n = \sum_{k=1}^{n} \hat{\mathbb{E}}(X_k^2)$.

3. Main results

Theorem 3.1. *It is assumed that $\mathbb{V}$ and $\hat{\mathbb{E}}$ are countably sub-additive. Suppose that $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ is an array of row-wise upper END r.v., where there are a r.v. X and a constant c_0, fulfilling*

$$\frac{1}{n}\sum_{k=1}^{n} \hat{\mathbb{E}}(h(|X_{nk}|)) \leq c_0 \hat{\mathbb{E}}(h(|X|)), for\ n \geq 1, h \in C_{l,Lip}(\mathbb{R})\ and\ h \geq 0. \tag{3.1}$$

Let $\{c_n, n \geq 1\}$ and $\{d_n, n \geq 1\}$ both be positive increasing constants sequences, with $\sup_{n\geq 1} c_{4n}/(nc_n) < \infty$ and $n/\sqrt{d_n \log n} \uparrow$. If

$$\sum_{k=1}^{n} \hat{\mathbb{E}}(X_{nk}^2) \leq d_n/4, \tag{3.2}$$

$$a := \limsup_{n\to\infty} 2c_n \sqrt{\log n/d_n} < \infty, \tag{3.3}$$

$$\sum_{n=1}^{\infty} \frac{nc_n}{\sqrt{d_n \log n}} \mathbb{V}\left(|X| > \mu^2 c_n\right) < \infty, \tag{3.4}$$

where μ is the same as in (2.2).

Then

$$\limsup_{n\to\infty} \frac{1}{\sqrt{d_n \log n}} \sum_{k=1}^{n} \left(X_{nk} - \hat{\mathbb{E}}(X_{nk})\right) \le a + \sqrt{2+a^2} \quad \text{a.s. } \mathbb{V}. \tag{3.5}$$

Further, if $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ is lower END, then

$$\liminf_{n\to\infty} \frac{1}{\sqrt{d_n \log n}} \sum_{k=1}^{n} (X_{nk} - \hat{\varepsilon}(X_{nk})) \ge -a - \sqrt{2+a^2} \quad \text{a.s. } \mathbb{V}. \tag{3.6}$$

In particular, if $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ is END and $\hat{\mathbb{E}}(X_{nk}) = \hat{\varepsilon}(X_{nk})$, then

$$\limsup_{n\to\infty} \frac{1}{\sqrt{d_n \log n}} \left| \sum_{k=1}^{n} \left(X_{nk} - \hat{\mathbb{E}}(X_{nk})\right) \right| \le a + \sqrt{2+a^2} \quad \text{a.s. } \mathbb{V}. \tag{3.7}$$

Remark 3.1. *The conclusion of Theorem 3.1 is very widespread. For c_n and d_n, we can choose distinct values depending on the corresponding requirements, thus yielding distinct outcomes. For instance, when $c_n = \sqrt{d_n/4\log n}$, $d_n = 4c_0 n\hat{\mathbb{E}}(X^2)$, condition* (3.4) *turns into*

$$\hat{\mathbb{E}}(X^2) < \infty, \; C_{\mathbb{V}}(X^4 \log |X|) < \infty. \tag{3.8}$$

Corollary 3.1. *If $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of row-wise END r.v.. Suppose that both $\hat{\mathbb{E}}$ and $\mathbb{V}$ are countably sub-additive. Conditions* (3.1) *and* (3.8) *also hold, then*

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n \log n}} \sum_{k=1}^{n} \left(X_{nk} - \hat{\mathbb{E}}(X_{nk})\right) \le 2(1+\sqrt{3})\sqrt{c_0 \hat{\mathbb{E}}(X^2)} \quad \text{a.s. } \mathbb{V}, \tag{3.9}$$

and

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n \log n}} \sum_{k=1}^{n} (X_{nk} - \hat{\varepsilon}(X_{nk})) \ge -2(1+\sqrt{3})\sqrt{c_0 \hat{\mathbb{E}}(X^2)} \quad \text{a.s. } \mathbb{V}. \tag{3.10}$$

Particularly, if $\hat{\mathbb{E}}(X_{nk}) = \hat{\varepsilon}(X_{nk})$, then

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n \log n}} \left| \sum_{k=1}^{n} \left(X_{nk} - \hat{\mathbb{E}}(X_{nk})\right) \right| \le 2(1+\sqrt{3})\sqrt{c_0 \hat{\mathbb{E}}(X^2)} \quad \text{a.s. } \mathbb{V}, \tag{3.11}$$

where c_0 is the same as in (3.1).

Remark 3.2. *Furthermore, taking $d_n = c_0 n \log\log n/(2\log n)$, $c_n = \frac{1}{4}\sqrt{d_n/\log n}$ in Theorem 3.1, condition* (3.4) *turns into*

$$\hat{\mathbb{E}}(X^2) < \infty, \; C_{\mathbb{V}}\left(\frac{|X|^4 (\log |X|)^3}{(\log\log |X|)^2}\right) < \infty. \tag{3.12}$$

Thus, we can obtain the following outcome.

Corollary 3.2. *If* $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ *be an array of row-wise END r.v.. Suppose that both* $\hat{\mathbb{E}}$ *and* $\mathbb{V}$ *are countably sub-additive. Conditions* (3.1), (3.12) *are also satisfied. Supposed that* (3.2) *holds with* $d_n = c_0 n \log\log n/(2\log n)$*, then*

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n\log\log n}} \sum_{k=1}^{n} \left(X_{nk} - \hat{\mathbb{E}}(X_{nk})\right) \le \sqrt{c_0} \quad \text{a.s. } \mathbb{V}, \tag{3.13}$$

and

$$\liminf_{n\to\infty} \frac{1}{\sqrt{2n\log\log n}} \sum_{k=1}^{n} (X_{nk} - \hat{\varepsilon}(X_{nk})) \ge -\sqrt{c_0} \quad \text{a.s. } \mathbb{V}. \tag{3.14}$$

Particularly, if $\hat{\mathbb{E}}(X_{nk}) = \hat{\varepsilon}(X_{nk})$*, then*

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n\log\log n}} \left| \sum_{k=1}^{n} \left(X_{nk} - \hat{\mathbb{E}}(X_{nk})\right) \right| \le \sqrt{c_0} \quad \text{a.s. } \mathbb{V}, \tag{3.15}$$

where c_0 *is the same as in* (3.1).

Remark 3.3. *The result of Corollary 3.2 is similar to the Hartman-Wintner LIL in probability space.*

Remark 3.4. *Our Theorem 3.1 and Corollary 3.1 extend the findings of Da Silva [29] from* $(\Omega, \mathcal{A}, P)$ *to the* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$.

Remark 3.5. (3.1) *is very close to the interpretation of weak average dominance in* $(\Omega, \mathcal{A}, P)$*. This definition is weaker than the definition of stochastic domination. In* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$*, stochastic domination is usually defined as that* $\hat{\mathbb{E}}(f(|X_{nk}|)) \le c\hat{\mathbb{E}}(f(|X|))$*, for* $n \ge 1, 1 \le k \le n, 0 \le f \in C_{l,Lip}(\mathbb{R})$*, which is a weaker condition than the identically distributed (Peng [1]). Because of this, condition* (3.1) *is relatively weak.*

4. Proof of main result

Proof of Theorem 3.1. We first prove (3.5). For fixed $n \ge 1$, and for any $1 \le k \le n$, denote

$$\begin{aligned} Y_{nk} &:= -c_n I(X_{nk} < -c_n) + X_{nk} I(|X_{nk}| \le c_n) + c_n I(X_{nk} > c_n), \\ Y'_{nk} &:= X_{nk} - Y_{nk} = (X_{nk} + c_n) I(X_{nk} < -c_n) + (X_{nk} - c_n) I(X_{nk} > c_n). \end{aligned} \tag{4.1}$$

Noting that

$$\begin{aligned} \frac{1}{\sqrt{d_n\log n}} \sum_{k=1}^{n} \left(X_{nk} - \hat{\mathbb{E}}(X_{nk})\right) =& \frac{1}{\sqrt{d_n\log n}} \sum_{k=1}^{n} \left(Y_{nk} - \hat{\mathbb{E}}(Y_{nk})\right) + \frac{1}{\sqrt{d_n\log n}} \sum_{k=1}^{n} Y'_{nk} \\ &+ \frac{1}{\sqrt{d_n\log n}} \sum_{k=1}^{n} \left(\hat{\mathbb{E}}(Y_{nk}) - \hat{\mathbb{E}}(X_{nk})\right) \\ :=& I_1 + I_2 + I_3. \end{aligned}$$

Accordingly, to prove (3.5), simply verify the following

$$\limsup_{n\to\infty} I_1 \le a + \sqrt{2+a^2} \ \text{ a.s. } \mathbb{V}, \tag{4.2}$$

$$\limsup_{n\to\infty} I_2 \le 0 \ \text{ a.s. } \mathbb{V}, \tag{4.3}$$

$$\lim_{n\to\infty} I_3 = 0. \tag{4.4}$$

First, we prove (4.2). For arbitrary $\varepsilon > 0$, $\left\{Y_{nk} - \hat{\mathbb{E}}(Y_{nk}), 1 \le k \le n, n \ge 1\right\}$ satisfies the requirements of Lemma 2.3, taking $x = \sqrt{d_n \log n}\left(a + \sqrt{2+a^2} + \varepsilon\right)$, $y = 2c_n$ in (2.4), we obtain

$$\begin{aligned}
&\mathbb{V}\left(S_n > \sqrt{d_n \log n}\left(a + \sqrt{2+a^2} + \varepsilon\right)\right) \\
\le &\mathbb{V}\left(\max_{1\le k\le n} X_k > 2c_n\right) \\
&+ M \exp\left\{-\frac{\left(\sqrt{d_n \log n}\left(a + \sqrt{2+a^2} + \varepsilon\right)\right)^2}{2\left(\sqrt{d_n \log n}\left(a + \sqrt{2+a^2} + \varepsilon\right)2c_n + B_n\right)}\left(1 + \frac{2}{3}\log\left(1 + \frac{\sqrt{d_n \log n}\left(a + \sqrt{2+a^2} + \varepsilon\right)2c_n}{B_n}\right)\right)\right\}.
\end{aligned} \tag{4.5}$$

By $|Y_{nk}| = \min\{|X_{nk}|, c_n\}$, it is easy to get that

$$|Y_{nk} - \hat{\mathbb{E}}(Y_{nk})| \le |Y_{nk}| + |\hat{\mathbb{E}}(Y_{nk})| \le |Y_{nk}| + \hat{\mathbb{E}}(|Y_{nk}|) \le 2c_n,$$

and $\mathbb{V}\left(\max_{1\le k\le n}(Y_{nk} - \hat{\mathbb{E}}(Y_{nk})) > 2c_n\right) = 0$. By C_r inequality, we get $\hat{\mathbb{E}}(Y_{nk} - \hat{\mathbb{E}}(Y_{nk}))^2 \le 2(\hat{\mathbb{E}}(|Y_{nk}|^2) + |\hat{\mathbb{E}}(Y_{nk})|^2) \le 4\hat{\mathbb{E}}(|Y_{nk}|^2)$, and by (3.2), so it's not hard to find that

$$\sum_{k=1}^{n} \hat{\mathbb{E}}\left(Y_{nk} - \hat{\mathbb{E}}(Y_{nk})\right)^2 \le 4\sum_{k=1}^{n} \hat{\mathbb{E}}(|Y_{nk}|^2) \le 4\sum_{k=1}^{n} \hat{\mathbb{E}}(|X_{nk}|^2) \le d_n.$$

From (4.5), we have access to

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathbb{V}\left\{\sum_{k=1}^{n}\left(Y_{nk} - \hat{\mathbb{E}}(Y_{nk})\right) > \sqrt{d_n \log n}\left(a + \sqrt{2+a^2} + \varepsilon\right)\right\} \\
&\le M\sum_{n=1}^{\infty} \exp\left\{-\frac{d_n \log n\left(a + \sqrt{2+a^2} + \varepsilon\right)^2}{2\left(2c_n\sqrt{d_n \log n}\left(a + \sqrt{2+a^2} + \varepsilon\right) + d_n\right)}\left(1 + \frac{2}{3}\log\left(1 + \frac{\sqrt{d_n \log n}\left(a + \sqrt{2+a^2} + \varepsilon\right)2c_n}{d_n}\right)\right)\right\} \\
&\le M\sum_{n=1}^{\infty} \exp\left\{-\frac{\left(a + \sqrt{2+a^2} + \varepsilon\right)^2}{2\left(\frac{2c_n\sqrt{\log n}\left(a+\sqrt{2+a^2}+\varepsilon\right) + \sqrt{d_n}}{\sqrt{d_n}}\right)}\log n\right\} \\
&= M\sum_{n=1}^{\infty} \exp\left\{-\frac{\left(a + \sqrt{2+a^2} + \varepsilon\right)^2}{2\left(2c_n\sqrt{\frac{\log n}{d_n}}\left(a + \sqrt{2+a^2} + \varepsilon\right) + 1\right)}\log n\right\}.
\end{aligned} \tag{4.6}$$

Note that

$$
\begin{aligned}
r :=& \liminf_{n\to\infty} \frac{\left(a+\sqrt{2+a^2}+\varepsilon\right)^2}{2\left(2c_n\sqrt{\frac{\log n}{d_n}}\left(a+\sqrt{2+a^2}+\varepsilon\right)+1\right)}\\
=&\frac{\left(a+\sqrt{2+a^2}+\varepsilon\right)^2}{2\left(\limsup_{n\to\infty} 2c_n\sqrt{\frac{\log n}{d_n}}\left(a+\sqrt{2+a^2}+\varepsilon\right)+1\right)}\\
=&\frac{\left(a+\sqrt{2+a^2}+\varepsilon\right)^2}{2\left(a\left(a+\sqrt{2+a^2}+\varepsilon\right)+1\right)}\\
=&\frac{2a^2+2a\sqrt{2+a^2}+2a\varepsilon+2+\varepsilon^2+2\sqrt{2+a^2}\varepsilon}{2a^2+2a\sqrt{2+a^2}+2a\varepsilon+2}\\
>&1.
\end{aligned}
$$

Thus, by (4.6), taking $1 < r_1 < r$, we have

$$
\sum_{n=1}^{\infty}\mathbb{V}\left\{\sum_{k=1}^{n}\left(Y_{nk}-\hat{\mathbb{E}}(Y_{nk})\right) > \sqrt{d_n\log n}\left(a+\sqrt{2+a^2}+\varepsilon\right)\right\} \ll \sum_{n=1}^{\infty}\exp(\log n^{-r_1}) = \sum_{n=1}^{\infty} n^{-r_1} < \infty.
$$

By Borel-Cantelli's lemma, we get

$$
\mathbb{V}\left(\sum_{k=1}^{n}\left(Y_{nk}-\hat{\mathbb{E}}(Y_{nk})\right) > \sqrt{d_n\log n}\left(a+\sqrt{2+a^2}+\varepsilon\right); \text{i.o.}\right) = 0.
$$

It means

$$
\limsup_{n\to\infty} I_1 \le a+\sqrt{2+a^2} \ \text{ a.s. } \ \mathbb{V}. \tag{4.7}
$$

Next, we prove (4.4). For any n, it is always possible to find an i satisfying $2^i \le n < 2^{i+1}$. By (3.4), and $n/\sqrt{d_n\log n}\uparrow$, we get

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{nc_n}{\sqrt{d_n\log n}}\mathbb{V}\left(|X|>\mu^2c_n\right) =& \sum_{i=0}^{\infty}\sum_{2^i\le n<2^{i+1}}\frac{nc_n}{\sqrt{d_n\log n}}\mathbb{V}\left(|X|>\mu^2c_n\right)\\
\ge& \sum_{i=1}^{\infty}\sum_{2^i\le n<2^{i+1}}\frac{2^i}{\sqrt{d_{2^i}\log 2^i}}c_{2^i}\mathbb{V}\left(|X|>\mu^2c_{2^{i+1}}\right)\\
=& \sum_{i=1}^{\infty}\frac{(2^i)^2}{\sqrt{d_{2^i}\log 2^i}}c_{2^i}\mathbb{V}\left(|X|>\mu^2c_{2^{i+1}}\right).
\end{aligned}
$$

Thus, (3.4) implies

$$
\sum_{i=1}^{\infty}\frac{(2^i)^2}{\sqrt{d_{2^i}\log 2^i}}c_{2^i}\mathbb{V}\left(|X|>\mu^2c_{2^{i+1}}\right) < \infty. \tag{4.8}
$$

For $j \geq 1$, we assume that $g_j(x) \in C_{l,Lip}(\mathbb{R})$ be an even function, and for all $x, g_j(x) \in [0,1]$, satisfying

$$g_j(x) = \begin{cases} 1, & c_{2^{j-1}} < |x| \leq c_{2^j}, \\ 0, & |x| \leq \mu c_{2^{j-1}} \text{ or } |x| > (1+\mu)c_{2^j}, \end{cases}$$

where μ is the identical to that in (2.2).

This shows

$$\begin{aligned} I\left(c_{2^{j-1}} < |X| \leq c_{2^j}\right) \leq & g_j(|X|) \leq I\left(\mu c_{2^{j-1}} < |X| \leq (1+\mu)c_{2^j}\right), \\ |X|^l g\left(\frac{|X|}{c_{2^i}}\right) \leq & c_1^l + \sum_{j=1}^{i} |X|^l g_j(|X|), \quad \forall l > 0, \end{aligned} \tag{4.9}$$

and

$$1 - g\left(\frac{|X|}{c_{2^i}}\right) \leq I\left(|X| > \mu c_{2^i}\right) \leq I\left(\frac{|X|}{\mu} > c_{2^{i-1}}\right) = \sum_{j=i}^{\infty} I\left(c_{2^{j-1}} < \frac{|X|}{\mu} \leq c_{2^j}\right) \leq \sum_{j=i}^{\infty} g_j\left(\frac{|X|}{\mu}\right). \tag{4.10}$$

According to the (4.1), (2.2), and the definition of Y'_{nk} can be known that

$$\begin{aligned} |Y'_{nk}| &\leq |X_{nk} + c_n| I(X_{nk} < -c_n) + |X_{nk} - c_n| I(X_{nk} > c_n) \\ &= (|X_{nk}| - c_n) I(|X_{nk}| > c_n) \\ &\leq |X_{nk}| \left(1 - g\left(\frac{|X_{nk}|}{c_n}\right)\right). \end{aligned} \tag{4.11}$$

Combined with (2.1), (3.1), (4.9), (4.10), (4.11), $g(x) \downarrow$ in $x > 0$, $n/\sqrt{d_n \log n} \uparrow$, $2^i \leq n < 2^{i+1}$, $\sup_{n \geq 1} c_{4n}/(nc_n) < \infty$, and $\hat{\mathbb{E}}$ is countably sub-additive, we can obtain

$$\begin{aligned} |I_3| \leq & \frac{1}{\sqrt{d_n \log n}} \sum_{k=1}^{n} \left| \hat{\mathbb{E}}(Y_{nk}) - \hat{\mathbb{E}}(X_{nk}) \right| \\ \leq & \frac{1}{\sqrt{d_n \log n}} \sum_{k=1}^{n} \hat{\mathbb{E}}\left(|X_{nk} - Y_{nk}|\right) \\ = & \frac{1}{\sqrt{d_n \log n}} \sum_{k=1}^{n} \hat{\mathbb{E}}\left(\left|Y'_{nk}\right|\right) \\ \leq & \frac{1}{\sqrt{d_n \log n}} \sum_{k=1}^{n} \hat{\mathbb{E}}\left(|X_{nk}| \left(1 - g\left(\frac{|X_{nk}|}{c_n}\right)\right)\right) \\ \ll & \frac{1}{\sqrt{d_n \log n}} n \hat{\mathbb{E}}\left(|X| \left(1 - g\left(\frac{|X|}{c_n}\right)\right)\right) \\ \leq & \frac{2^{i+1}}{\sqrt{d_{2^{i+1}} \log 2^{i+1}}} \hat{\mathbb{E}}\left(|X| \left(1 - g\left(\frac{|X|}{c_{2^i}}\right)\right)\right) \\ \leq & \frac{2^{i+1}}{\sqrt{d_{2^{i+1}} \log 2^{i+1}}} \sum_{j=i}^{\infty} \hat{\mathbb{E}}\left(|X| g_j\left(\frac{|X|}{\mu}\right)\right) \end{aligned}$$

$$\begin{aligned}
&\ll \frac{2^{i+1}}{\sqrt{d_{2^{i+1}}\log 2^{i+1}}}\sum_{j=i}^{\infty} c_{2^j}\mathbb{V}\left(|X|>\mu^2 c_{2^{j-1}}\right)\\
&\ll \sum_{j=i}^{\infty}\frac{c_{2^j}}{2^{j-2}c_{2^{j-2}}}\frac{(2^{j-2})^2}{\sqrt{d_{2^{j-2}}\log 2^{j-2}}}c_{2^{j-2}}\mathbb{V}\left(|X|>\mu^2 c_{2^{j-1}}\right)\\
&\ll \sum_{j=i}^{\infty}\frac{(2^{j-2})^2}{\sqrt{d_{2^{j-2}}\log 2^{j-2}}}c_{2^{j-2}}\mathbb{V}\left(|X|>\mu^2 c_{2^{j-1}}\right).
\end{aligned}\tag{4.12}$$

Combining with (4.8), we obtain

$$\sum_{j=3}^{\infty}\frac{(2^{j-2})^2}{\sqrt{d_{2^{j-2}}\log 2^{j-2}}}c_{2^{j-2}}\mathbb{V}\left(|X|>\mu^2 c_{2^{j-1}}\right)=\sum_{j=1}^{\infty}\frac{(2^{j})^2}{\sqrt{d_{2^{j}}\log 2^{j}}}c_{2^{j}}\mathbb{V}\left(|X|>\mu^2 c_{2^{j+1}}\right)<\infty. \tag{4.13}$$

So, we have

$$I_3\to 0,\ \text{as}\ \ n\ \to\infty. \tag{4.14}$$

Now, we shall demonstrate (4.3). By (3.1), (4.8), (4.11), (4.13), $n/\sqrt{d_n\log n}\uparrow$, $\sup_{n\geq 1}c_{4n}/(nc_n)<\infty$, Markov's inequality, and $\hat{\mathbb{E}}$ is countably sub-additive, we get that for every $\varepsilon>0$,

$$\begin{aligned}
&\sum_{n=2}^{\infty}\mathbb{V}\left(\frac{1}{\sqrt{d_n\log n}}\sum_{k=1}^{n}Y'_{nk}>\varepsilon\right)\\
&\ll \sum_{n=2}^{\infty}\frac{1}{\sqrt{d_n\log n}}\sum_{k=1}^{n}\hat{\mathbb{E}}\left(\left|Y'_{nk}\right|\right)\\
&\leq \sum_{n=2}^{\infty}\frac{1}{\sqrt{d_n\log n}}\sum_{k=1}^{n}\hat{\mathbb{E}}\left(|X_{nk}|\left(1-g\left(\frac{|X_{nk}|}{c_n}\right)\right)\right)\\
&\ll \sum_{n=2}^{\infty}\frac{1}{\sqrt{d_n\log n}}n\hat{\mathbb{E}}\left(|X|\left(1-g\left(\frac{|X|}{c_n}\right)\right)\right)\\
&\leq \sum_{i=1}^{\infty}\sum_{2^i\leq n<2^{i+1}}\frac{1}{\sqrt{d_{2^{i+1}}\log 2^{i+1}}}2^{i+1}\hat{\mathbb{E}}\left(|X|\left(1-g\left(\frac{|X|}{c_{2^i}}\right)\right)\right)\\
&\ll \sum_{i=1}^{\infty}\frac{(2^i)^2}{\sqrt{d_{2^{i+1}}\log 2^{i+1}}}\sum_{j=i}^{\infty}\hat{\mathbb{E}}\left(|X|g_j\left(\frac{|X|}{\mu}\right)\right)\\
&\leq \sum_{j=3}^{\infty}\hat{\mathbb{E}}\left(|X|g_j\left(\frac{|X|}{\mu}\right)\right)\sum_{i=1}^{j}\frac{(2^i)^2}{\sqrt{d_{2^i}\log 2^i}}\\
&\ll \sum_{j=3}^{\infty}\frac{(2^j)^2}{\sqrt{d_{2^j}\log 2^j}}c_{2^j}\mathbb{V}\left(|X|>\mu^2 c_{2^{j-1}}\right)\\
&\ll \sum_{j=3}^{\infty}\frac{c_{2^j}}{2^{j-2}c_{2^{j-2}}}\frac{(2^{j-2})^2}{\sqrt{d_{2^{j-2}}\log 2^{j-2}}}c_{2^{j-2}}\mathbb{V}\left(|X|>\mu^2 c_{2^{j-1}}\right)
\end{aligned}$$

$$\ll \sum_{j=3}^{\infty} \frac{(2^{j-2})^2}{\sqrt{d_{2^{j-2}} \log 2^{j-2}}} c_{2^{j-2}} \mathbb{V}\left(|X| > \mu^2 c_{2^{j-1}}\right)$$
$$< \infty.$$

Thus, by Borel-Cantelli's lemma, it is fairly straightforward to obtain

$$\limsup_{n\to\infty} I_2 \leq 0 \quad \text{a.s. } \mathbb{V}. \tag{4.15}$$

Together with (4.7) and (4.14) hold, yields the (3.5).

Further considerations, if $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ is lower END, then $\{-X_{nk}, 1 \leq k \leq n, n \geq 1\}$ is upper END. In (3.5), replace $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ with $\{-X_{nk}, 1 \leq k \leq n, n \geq 1\}$, from $\hat{\varepsilon}(X_{nk}) := -\hat{\mathbb{E}}(-X_{nk})$, for which we have

$$a + \sqrt{2 + a^2} \geq \limsup_{n\to\infty} \frac{1}{\sqrt{d_n \log n}} \sum_{k=1}^{n} \left(-X_{nk} - \hat{\mathbb{E}}(-X_{nk})\right) = -\liminf_{n\to\infty} \frac{1}{\sqrt{d_n \log n}} \sum_{k=1}^{n} (X_{nk} - \hat{\varepsilon}(X_{nk})) \text{ a.s. } \mathbb{V}. \tag{4.16}$$

From this it is clear that

$$\liminf_{n\to\infty} \frac{1}{\sqrt{d_n \log n}} \sum_{k=1}^{n} (X_{nk} - \hat{\varepsilon}(X_{nk})) \geq -a - \sqrt{2 + a^2} \quad \text{a.s. } \mathbb{V}.$$

That is to say (3.6) also holds.

In particular, if $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ is END and $\hat{\mathbb{E}}(X_{nk}) = \hat{\varepsilon}(X_{nk})$, then (3.7) holds from (3.5) and (3.6).

Consequently, we complete the proof of Theorem 3.1.

Proof of Corollary 3.1. Taking $d_n = 4c_0 n\hat{\mathbb{E}}(X^2)$, by (3.1), we obtain

$$\sum_{k=1}^{n} \hat{\mathbb{E}}(X_{nk}^2) \leq c_0 n\hat{\mathbb{E}}(X^2) = d_n/4. \tag{4.17}$$

Thus, the condition of (3.2) holds. Putting $c_n = \sqrt{d_n/(4\log n)}$, by condition (3.3) in Theorem 3.1, we have

$$\begin{aligned} a &= \limsup_{n\to\infty} 2c_n \sqrt{\frac{\log n}{d_n}} \\ &= \limsup_{n\to\infty} 2\sqrt{\frac{d_n}{4\log n}} \times \sqrt{\frac{\log n}{d_n}} \\ &= 1 < \infty. \end{aligned}$$

Taking $p = 4, \beta = 1$ and $l(|X|) = \log|X|$ in (2.3) in Lemma 2.2. For any $c > 0$, we can gain that

$$C_{\mathbb{V}}\left(|X|^4 \log|X|\right) < \infty \Leftrightarrow \sum_{n=2}^{\infty} n^3 \log n \mathbb{V}(|X| > cn) < \infty,$$

$$\Leftrightarrow \int_2^{\infty} y^3 \log y \mathbb{V}\left(|X| > cy\right) \mathrm{d}y < \infty. \tag{4.18}$$

Next, by $c_n = \sqrt{\frac{d_n}{4\log n}} = \sqrt{\frac{c_0 n\hat{\mathbb{E}}(X^2)}{\log n}}$, we consider that

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{nc_n}{\sqrt{d_n \log n}} \mathbb{V}\left(|X| > \mu^2 c_n\right) \\
&= \sum_{n=2}^{\infty} n\sqrt{\frac{d_n}{4\log n}} \times \frac{1}{\sqrt{d_n \log n}} \mathbb{V}\left(|X| > \mu^2 c_n\right) \\
&= \sum_{n=2}^{\infty} \frac{n}{2\log n} \mathbb{V}\left(|X| > \mu^2 c_n\right) \\
&\sim \int_2^{\infty} \frac{x}{2\log x} \mathbb{V}\left(|X| > \mu^2 c_x\right) \mathrm{d}x \qquad \left(c_x = \sqrt{c_0 x\hat{\mathbb{E}}(X^2)/\log x}\right) \\
&= \int_2^{\infty} \frac{x}{2\log x} \mathbb{V}\left(|X| > c\sqrt{x/\log x}\right) \mathrm{d}x. \qquad (4.19)
\end{aligned}$$

Let $y = \sqrt{x/\log x}$, then $x \to \infty \Leftrightarrow y \to \infty$.
Note that

$$y^2 = \frac{x}{\log x}. \tag{4.20}$$

Taking the logarithm on both sides of (4.20), we have

$$2\log y = \log x - \log\log x \sim \log x, \quad x \to \infty. \tag{4.21}$$

Take the derivative of both sides of (4.20) with respect to y, we get

$$2y = \frac{\log x - 1}{(\log x)^2} \cdot x'_y,$$

and combined with (4.21), we have

$$x'_y = \frac{2y(\log x)^2}{\log x - 1} \sim 2y\log x \sim 4y\log y, \quad y \to \infty. \tag{4.22}$$

Combined (4.19) and (4.20), we can get

$$h(y) := \frac{x}{2\log x} x'_y \sim 2y^3 \log y.$$

Hence, for $1 > 0$, there exists a constant $M_1 > 0$, such that when $y \geq M_1$, we have

$$h(y) \leq 4y^3 \log y.$$

By (3.8), (4.18), (4.19), and (4.20)–(4.22), we obtain

$$\int_2^{\infty} \frac{x}{2\log x} \mathbb{V}(|X| > c\sqrt{x/\log x}) \,\mathrm{d}x \quad \left(\text{let } y = \sqrt{x/\log x},\ b = \sqrt{2/\log 2}\right)$$

$$\leq \int_b^{M_1} h(y)\mathbb{V}(|X| > cy)\,\mathrm{d}y + \int_{M_1}^{\infty} 4y^3 \log y \mathbb{V}(|X| > cy)\,\mathrm{d}y$$
$$< \infty. \tag{4.23}$$

It follows that assumptions (3.4) in Theorem 3.1 is also fulfilled. Next, we verify that

$$\sup_{n\geq 1} \frac{c_{4n}}{nc_n} = \sup_{n\geq 1} \frac{\sqrt{\frac{c_0 4n\hat{\mathbb{E}}(X^2)}{\log 4n}}}{n\sqrt{\frac{c_0 n\hat{\mathbb{E}}(X^2)}{\log n}}} = \sup_{n\geq 1} \frac{2}{n}\sqrt{\frac{\log n}{\log 4n}} \leq 3 < \infty.$$

and

$$\frac{n}{\sqrt{d_n \log n}} = \frac{n}{\sqrt{4c_0 n\hat{\mathbb{E}}(X^2)\log n}} = c\sqrt{\frac{n}{\log n}},$$

apparently, $\frac{n}{\sqrt{d_n \log n}} \uparrow$, as $n \to \infty$.

Based on the above verification, it is clear that the conditions (3.2), (3.3) and (3.4) of Theorem 3.1 are all satisfied. Hence, putting $d_n = 4c_0 n\hat{\mathbb{E}}(X^2)$, $a = 1$ into (3.5), we can achieve

$$\limsup_{n\to\infty} \frac{1}{\sqrt{4c_0 n\hat{\mathbb{E}}(X^2)\log n}} \sum_{k=1}^{n} \left(X_{nk} - \hat{\mathbb{E}}(X_{nk})\right) \leq 1 + \sqrt{3} \quad \text{a.s. } \mathbb{V}.$$

That is,

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n\log n}} \sum_{k=1}^{n} \left(X_{nk} - \hat{\mathbb{E}}(X_{nk})\right) \leq 2(1+\sqrt{3})\sqrt{c_0\hat{\mathbb{E}}(X^2)} \quad \text{a.s. } \mathbb{V}.$$

Similarly, (3.10) and (3.11) also hold.

As a result, Corollary 3.1 is established.

Proof of Corollary 3.2. For $d_n = c_0 n \log\log n/(2\log n)$, $c_n = \frac{1}{4}\sqrt{d_n/\log n}$, and assumptions (3.1) and (3.2) both hold. Hence, we just need to verify the following

$$\begin{aligned} a &= \limsup_{n\to\infty} 2c_n\sqrt{\frac{\log n}{d_n}} \\ &= \limsup_{n\to\infty} 2\times\frac{1}{4}\sqrt{\frac{d_n}{\log n}}\times\sqrt{\frac{\log n}{d_n}} \\ &= \frac{1}{2} < \infty. \end{aligned}$$

Thus, (3.3) holds.

By (3.12), and similar considerations to those in (4.19), taking $p = 4, \beta = 1$ and $l(|X|) = \frac{(\log|X|)^3}{(\log\log|X|)^2}$ in (2.3) in Lemma 2.2. For any $c > 0$, we can gain that

$$C_{\mathbb{V}}\left(\frac{|X|^4(\log|X|)^3}{(\log\log|X|)^2}\right) < \infty \Leftrightarrow \sum_{n=3}^{\infty} \frac{n^3(\log n)^3}{(\log\log n)^2}\mathbb{V}(|X| > cn) < \infty,$$

$$\Leftrightarrow \int_3^{\infty} \frac{y^3(\log y)^3}{(\log\log y)^2}\mathbb{V}\left(|X| > cy\right) \mathrm{d}y < \infty. \tag{4.24}$$

Hence, by $c_n = \frac{1}{4}\sqrt{\frac{d_n}{\log n}} = \frac{1}{4}\sqrt{\frac{c_0 n\log\log n}{2\log^2 n}} = \frac{\sqrt{c_0 n\log\log n}}{4\sqrt{2}\log n}$, we have

$$\begin{aligned}
&\sum_{n=11}^{\infty} \frac{nc_n}{\sqrt{d_n \log n}}\mathbb{V}\left(|X| > \mu^2 c_n\right)\\
&= \sum_{n=11}^{\infty} \frac{n}{4}\sqrt{\frac{d_n}{\log n}} \times \frac{1}{\sqrt{d_n \log n}}\mathbb{V}\left(|X| > \mu^2 c_n\right)\\
&= \sum_{n=11}^{\infty} \frac{n}{4\log n}\mathbb{V}\left(|X| > \mu^2 \frac{\sqrt{c_0 n\log\log n}}{4\sqrt{2}\log n}\right)\\
&\sim \int_{11}^{\infty} \frac{x}{4\log x}\mathbb{V}\left(|X| > c\frac{\sqrt{x\log\log x}}{\log x}\right)\mathrm{d}x.
\end{aligned} \tag{4.25}$$

For (4.25), similar considerations to (4.23).

Let $y = \frac{\sqrt{x\log\log x}}{\log x}$, and denote

$$f(y) := \frac{x}{4\log x}x'_y = \frac{y^2\log x}{4\log\log x}\cdot x'_y \sim \frac{2y^2\log y}{4\log\log y}\cdot\frac{8y(\log y)^2}{\log\log y} = \frac{4y^3(\log y)^3}{(\log\log y)^2}.$$

Hence, for $1 > 0$, there exists a constant $M_2 > 0$, such that when $y \geq M_2$, we have

$$f(y) \leq \frac{8y^3(\log y)^3}{(\log\log y)^2}.$$

Combined (4.25), (4.24) and (3.12), we obtain

$$\begin{aligned}
&\int_{11}^{\infty} \frac{x}{4\log x}\mathbb{V}\left(|X| > c\frac{\sqrt{x\log\log x}}{\log x}\right)\mathrm{d}x \quad \left(\text{let } y = \frac{\sqrt{x\log\log x}}{\log x}\right)\\
&\leq \int_{b'}^{M_2} f(y)\mathbb{V}(|X| > cy)\,\mathrm{d}y + \int_{M_2}^{\infty}\frac{8y^3(\log y)^3}{(\log\log y)^2}\mathbb{V}(|X| > cy)\,\mathrm{d}y\\
&< \infty.
\end{aligned}$$

In this case, both c_n and d_n satisfy the conditions $\sup_{n\geq 1}\frac{c_{4n}}{nc_n} = \sup_{n\geq 1}\frac{\sqrt{c_0 4n\log\log 4n}}{4\sqrt{2}\log 4n} \times \frac{4\sqrt{2}\log n}{n\sqrt{c_0 n\log\log n}} < \infty$ and $\frac{n}{\sqrt{d_n\log n}} \uparrow$ in Theorem 3.1.

We have verified that both c_n and d_n satisfy the conditions of Theorem 3.1, so we can bring both $d_n = c_0 n\log\log n/(2\log n)$ and $a = 1/2$ into (3.5) to get that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{\frac{c_0 n\log\log n}{2\log n}}\log n}\sum_{k=1}^{n}\left(X_{nk} - \hat{\mathbb{E}}(X_{nk})\right) \leq \frac{1}{2} + \sqrt{2 + \left(\frac{1}{2}\right)^2} \quad \text{a.s. } \mathbb{V}. \tag{4.26}$$

That means

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n\log\log n}} \sum_{k=1}^{n} \left(X_{nk} - \hat{\mathbb{E}}(X_{nk})\right) \leq \sqrt{c_0} \quad \text{a.s. } \mathbb{V}.$$

In the (3.13), $-X_{nk}$ instead of X_{nk}, one can get (3.14), and (3.15) can be obtained through (3.13) and (3.14).

Thus, the proof is finished.

5. Conclusions

The key findings of this study emerge from the probability space. There have been numerous findings on the limit theory, and when compared to the limit theory of probability space, it is clear that studying limit theory under sub-linear expectations, and in particular almost sure convergence, appears to be more intractable of the fact that the expectations and capacities are no longer additive. Moreover, many rules that apply to probability space do not apply to sub-linear expectation space. Therefore, applying the appropriate auxiliary tools to conduct research properly becomes essential. In this paper, our research draws mainly on the notion of extended negative dependence proposed by Zhang [19]. Zhang [19] also constructed capacity inequalities, which provide a reliable aid to our proof process. The results of this paper are general compared to some of the existing results. Since the theorem's conclusion is relatively broad, it is possible to take values for c_n and d_n and thus obtain a fixed constant a and different corollaries. In future research work, we will further consider studying a wider range of random variables or arrays and continue to learn the strong limit theorem and consider more interesting outcomes.

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Conflict of interest

In this article, all authors disclaim any conflict of interest.

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