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Research article

On a non-Newtonian fluid type equation with variable diffusion coefficient

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Abstract: Since the non-Newtonian fluid type equations arise from a broad and in-depth background, many research achievements have been gained from 1980s. Different from the usual non-Newtonian fluid equation, there is a nonnegative variable diffusion in the equations considered in this paper. Such a variable diffusion reflects the characteristic of the medium which may not be homogenous. By giving a generalization of the Gronwall inequality, the stability and the uniqueness of weak solutions to the non-Newtonian fluid equation with variable diffusion are studied. Since the variable diffusion may be degenerate on the boundary $\partial\Omega$, it is found that a partial boundary value condition imposed on a submanifold of $\partial\Omega \times (0, T)$ is enough to ensure the well-posedness of weak solutions. The novelty is that the concept of the trace of $u(x, t)$ is generalized by a special way.

Keywords: the partial boundary value condition; variable diffusion; non-Newtonian fluid equation; submanifold; trace
Mathematics Subject Classification: 35B35, 35K20, 35K55

1. Introduction

The mathematical modelling of various physical processes, where spatial heterogeneity has a primary role, usually results in the derivation of nonlinear evolution equations with variable diffusion, or dispersion. As pointed out by Karachalios-Zographopoulos in [18], to name but a few, equations of such a type have been successfully applied to the heat propagation in heterogeneous materials [5, 12, 16, 17], the study of transport of electron temperature in a confined plasma [7], the propagation of varying amplitude waves in a nonlinear medium [24], the study of electromagnetic phenomena in nonhomogeneous superconductors [3, 13–15] and the dynamics of Josephson junctions [8, 9], the epidemiology and the growth and control of brain tumors [21]. It is not possible that all the characteristics of these applications explained in one equation due to its adaptability. In this paper, we focus on the non-Newtonian fluids equations. Applications of non-Newtonian fluids in wide range in many fields like fiber coating and crude oil extraction and many more have fascinated many researchers, one can refer to [22, 25] etc. In Newtonian fluids, the viscosity does not change, while in non-Newtonian fluids, the viscosity changes when under force/stress to either more liquid or more solid, and so the non-Newtonian fluids are fluids that describe the relationship between deformation rates and stress none linearly, and they do not follow Newton's law of viscosity.

Let us give three explicit equations derived from Newtonian flow or non-Newtonian flow. The first one is in the study of water infiltration through porous media, Darcy's linear relation

$$V = -K(\theta)\nabla\phi, \tag{1.1}$$

satisfactorily describes the flow conduction provided that the velocities are small. Here V represents the seepage velocity of water, θ is the volumetric moisture content, $K(\theta)$ is the hydraulic conductivity and ϕ is the total potential, which can be expressed as the sum of a hydrostatic potential $\psi(\theta)$ and a gravitational potential z

$$\phi = \psi(\theta) + z.$$

If it is assumed that infiltration takes place in a horizontal column of the medium, then the continuity equation has the form

$$\frac{\partial\theta}{\partial t} + \frac{\partial V}{\partial x} = 0.$$

Then we have

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}(D(\theta)|\theta_x|\theta_x), \tag{1.2}$$

with $D(\theta) = K(\theta)\psi'(\theta)$. Certainly, water is the most usual Newtonian fluid. Also, Eq (1.5) is called as a porous medium equation.

The second one is to consider a compressible fluid flow in a homogeneous isotropic rigid porous medium. Then the volumetric moisture content θ, the seepage velocity $\vec{V}$ and the density of the fluid are governed by the continuity equation

$$\theta\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\vec{V}) = 0. \tag{1.3}$$

For non-Newtonian fluid, according to Chapter 2 of [27], the linear Darcy's law is not longer valid, because the influence of many factors such as the molecular and ion effects needs to be concerned. Instead, one has the following nonlinear relation

$$\rho\vec{V} = -\lambda|\nabla P|^{\alpha-1}\nabla P, \tag{1.4}$$

where $\rho\vec{V}$ and P denote the momentum velocity and pressure respectively, $\lambda > 0$ and $\alpha > 0$ are some physical constants. Combing (1.3) with (1.4), one has

$$\theta\frac{\partial\rho}{\partial t} - \lambda\operatorname{div}(|\nabla P|^{\alpha-1}\nabla P) = 0. \tag{1.5}$$

The third one is to consider the flows in fractured media [20]. Let ε be the size ratio of the matrix blocks to the whole medium and let the width of the fracture planes and the porous block diameter be in the same order. If the permeability ratio of matrix blocks to fracture planes is of order $\varepsilon^{p_\varepsilon}$, where p_ε is a positive oscillating constant, then the nonlinear Darcy law combined with the continuity equation leads to the following equation

$$\omega^\varepsilon u_t^\varepsilon - \operatorname{div}(k^\varepsilon(x)|\nabla u^\varepsilon|^{p^\varepsilon-2}\nabla u^\varepsilon) = 0, \tag{1.6}$$

where u^ε is the density of the fluid, $\omega^\varepsilon, k^\varepsilon$ are the porosity and the permeability of the medium.

Equations (1.4)–(1.6) can be abstracted as

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) = 0, \tag{1.7}$$

and the quantity $p > 1$ is a characteristic of the medium, the media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics; if $p = 2$ they are Newtonian fluids. While $a(x)$ is only a nonnegative function, it reflects the characteristic of the medium which may not be homogenous. The most achievements before are focused on the case $a(x) = 1$,

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \tag{1.8}$$

which is simply called as the Non-Newtonian fluid equation usually. The existence, the uniqueness, the regularity and the long-time behaviors of weak solutions to this equation have been studied in [1, 2, 4, 19, 27, 34] etc.

In this paper, we consider the following non-Newtonian fluid type equation with a variable, nonnegative diffusion coefficient $a(x)$:

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) - \sum_{i=1}^{N} b_i(x,t)D_i u = f(u,x,t),\ (x,t) \in Q_T = \Omega \times (0,T), \tag{1.9}$$

where $D_i = \frac{\partial}{\partial x_i}$, $0 \le a(x) \in C(\overline{\Omega})$, $b_i(x,t) \in C^1(\overline{Q_T})$, and $f(u,x,t) \in C^1(\mathbb{R} \times \overline{Q_T})$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$. The most difference between Eq (1.9) and (1.8) is that the diffusion $a(x)$ in Eq (1.9) may degenerate at some points of $\overline{\Omega}$.

A special case of (1.9) is that $b_i(x,t) = b_i(x)$ and $a(x) \in C^1(\overline{\Omega})$ satisfies

$$a(x)|_{x\in\Omega} > 0 \ \text{ and } \ a(x)|_{x\in\partial\Omega} = 0. \tag{1.10}$$

In this case, the corresponding well-posed problem has been considered in [28, 31, 32] recently. In general, the initial value condition

$$u(x,0) = u_0(x), \ \ x \in \Omega, \tag{1.11}$$

is always needed, but instead of the usual boundary value condition

$$u(x,t) = 0, \ \ (x,t) \in \partial\Omega \times (0,T),$$

only a partial boundary value condition

$$u(x,t) = 0, \ \ (x,t) \in \Sigma_p \subseteq \partial\Omega \times (0,T), \tag{1.12}$$

is imposed, where $\Sigma_p = \Sigma_1 \times (0, T)$ and Σ_1 is a relatively open subset of $\partial\Omega$ in [28], even it can be an empty set sometime in [29–31]. In this paper, different [6, 10, 23, 28–31], since $b_i(x, t)$ depends on the time variable t, we find that Σ_p can not be expressed as a cylindrical space as $\Sigma_1 \times (0, T)$. Instead, the partial boundary value condition is only imposed on a submanifold of $\Sigma_p \subseteq \partial\Omega \times (0, T)$ (the details are given in (1.16) or (3.2) below).

Now, let us give the definition of weak solution and supply some other related backgrounds.

Definition 1.1. *A function $u(x, t)$* is said to be a weak solution of the initial-boundary value problem of Eq (1.9), if

$$u \in L^\infty(Q_T),\ \ u_t \in L^2(Q_T),\ \ a(x)|\nabla u|^p \in L^1(Q_T), \tag{1.13}$$

and for any function $g(s) \in C^1(\mathbb{R})$ with $g(0) = 0$, $\varphi_1 \in C_0^1(\Omega)$ and $\varphi_2 \in L^\infty(0, T; W^{1,p}_{loc}(\Omega))$, there holds

$$\begin{aligned}
&\iint_{Q_T} \left[u_t g(\varphi_1\varphi_2) + a(x)\, |\nabla u|^{p-2}\, \nabla u \cdot \nabla g(\varphi_1\varphi_2)\right] dxdt \\
&+ \sum_{i=1}^{N} \iint_{Q_T} u \left[b_{ix_i}(x, t) g(\varphi_1\varphi_2) + b_i(x, t) g_{x_i}(\varphi_1\varphi_2)\right] dxdt \\
=& \iint_{Q_T} f(u, x, t) g(\varphi_1\varphi_2) dxdt.
\end{aligned} \tag{1.14}$$

The initial value condition is satisfied in the sense of

$$\lim_{t\to 0} \int_\Omega |u(x, t) - u_0(x)| dx = 0. \tag{1.15}$$

Moreover, the partial boundary value condition is imposed as

$$u(x, t) = 0,\ \ (x, t) \in \Sigma = \left\{\partial\Omega \times (0, T) : \sum_{i=1}^{N} b_i(x, t) a_{x_i}(x) < 0\right\}. \tag{1.16}$$

If $b_i(x, t) = b_i(x)$, $f(u, x, t) = f(x, t) - c(x, t)u$, the existence of weak solution has been proved in [28]. In addition, if there is

$$\int_\Omega a(x)^{-\frac{1}{p-1}} dx < \infty, \tag{1.17}$$

then for a weak solution of Eq (1.9), we have

$$\int_\Omega |\nabla u| dx < \infty, \tag{1.18}$$

and the trace of $u(x, t)$ on the boundary $\partial\Omega$ can be defined in the classical sense [28,29]. If the inequality (1.17) is true and $b_i(x, t) = b_i(x)$, then a similar partial boundary value condition as (1.16) has been imposed in [28]. In this paper, we mainly consider the case of that the inequality (1.17) is not true, i.e.,

$$\int_\Omega a(x)^{-\frac{1}{p-1}} dx = \infty, \tag{1.19}$$

then we can not define the trace of $u(x,t)$ in the classical sense. So, the first dedication of this paper lies in that we give a generalization of the trace of $u(x,t)$ on the boundary $\partial\Omega$ in a special way. In details, inspired by [28, 32], we can define the trace of $u(x,t)$ on the boundary $\partial\Omega$ as

$$\operatorname{ess\,sup}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{(\Omega_\varepsilon\setminus\Omega_{2\varepsilon})\cap\Omega_{1t}} u^2\sum_{i=1}^{N} b_i(x,t)a_{x_i}(x)dx = 0, \tag{1.20}$$

where

$$\Omega_\varepsilon = \{x\in\Omega : a(x) > \varepsilon\},\ \Omega_{1t} = \left\{x\in\Omega : \sum_{i=1}^{N} b_i(x,t)a_{x_i}(x) \le 0\right\},$$

and

$$\operatorname{ess\,sup}\lim_{\lambda\to 0} f(\lambda) = \inf_{\delta>0}\{\operatorname{ess\,sup}\{f(\lambda) : |\lambda| < \delta\}\}$$

is the super limit. In what follows, we only simply denote $\operatorname{ess\,sup}\lim_{\lambda\to 0} f(\lambda)$ as $\lim_{\lambda\to 0} f(\lambda)$. The rationality of such a generalization of the classical trace will be specified later in this paper.

If $f(u,x,t)$ is a continuous function and is Lipchitz continuous about the variable u, then the existence of weak solutions of the initial-boundary value problem of Eq (1.9) can be proved in a similar way as those [28, 32], we don't prepare to prove the existence of weak solutions again. We mainly pay attention to the stability or the uniqueness of weak solutions by a generalized Gronwall inequality.

Theorem 1.2. Let $u(x,t)$ and $v(x,t)$ be two weak solutions of the initial-boundary value problem of Eq (1.9), and with the same homogeneous the partial boundary condition

$$u(x,t) = 0 = v(x,t),\ (x,t)\in\Sigma.$$

Here Σ has the form (1.16). If $2 > p > 1$, $a(x)$ satisfies (1.10), and

$$\int_\Omega a(x)^{-\frac{2}{p-1}}\left|\sum_{i=1}^{N} b_i(x,t)\right|^{\frac{2p}{p-1}} dx < \infty,\ t\in[0,T], \tag{1.21}$$

then

$$\int_\Omega |u(x,t)-v(x,t)|^2dx \le c\int_\Omega |u_0(x)-v_0(x)|^2dx,\ t\in[0,T). \tag{1.22}$$

Moreover, we can obtain a local stability of weak solutions as follows.

Theorem 1.3. Let $u(x,t)$ and $v(x,t)$ be two solutions of Eq (1.9) with the differential initial values $u_0(x)$ and $v_0(x)$ respectively, but no any boundary value condition is required. if $p > 1$, $a(x)$ satisfies (1.10), whether (1.17) or (1.19) is true, and

$$\int_\Omega \frac{\left|\sum_{i=1}^{N} a_{x_i}(x)b_i(x,t)\right|^2}{a(x)}dx \le c,\ t\in[0,T], \tag{1.23}$$

then

$$\int_\Omega a(x)|u(x,t)-v(x,t)|^2dx \le c\int_\Omega a(x)|u_0(x)-v_0(x)|^2dx,\ t\in[0,T). \tag{1.24}$$

Different from Theorem 1.2, in this theorem, there is not any boundary value condition imposed. Actually, the uniqueness of weak solution to Eq (1.9) can be obtained without conditions (1.21) and (1.23).

Theorem 1.4. Let $a(x) \geq 0$ satisfy (1.10), $p > 1$, $u(x,t)$ and $v(x,t)$ be two weak solutions of Eq (1.9) with the initial values $u_0(x) = v_0(x)$. If $a(x)$ satisfies (1.10) and one of the following assumptions is true.

i) $a(x)$ satisfies (1.17), $u(x,t)$ and $v(x,t)$ are with the same partial boundary value condition

$$u(x,t) = v(x,t) = 0, (x,t) \in \Sigma = \{(x,t) \in \partial\Omega \times (0,T) : \mathrm{div}(\vec{b}(x,t)) \neq 0\}, \tag{1.25}$$

in the sense of the classical trace. Here $\vec{b} = \{b_i\}$, $\mathrm{div}(\vec{b}(x,t)) = \sum_{i=1}^{N} \frac{\partial b_i(x,t)}{\partial x_i}$.

ii) Whether $a(x)$ satisfies (1.17) or (1.19), but

$$\mathrm{div}(\vec{b}(x,t)) = 0,\ (x,t) \in \partial\Omega \times (0,T). \tag{1.26}$$

Then

$$u(x,t) = v(x,t),\ (x,t) \in Q_T. \tag{1.27}$$

One can see that all theorems above admit the case of (1.19), so the generalization of classical trace to the general form (1.20) is the most novelty of this paper. One can refer to the appendix for more details. Certainly, some other restrictions on the convective coefficient $b_i(x,t)$, i.e., the inequalities (1.21), (1.23) and (3.1), are imposed. How to relieve these restrictions to obtain the same conclusions? This is a question worth discussing thoroughly.

2. The generalized Gronwall inequality

Let us review the classical Gronwall inequality.

Gronwall inequality: Let $x(t)$ and $c(t)$ be two nonnegative integral functions and $a(t)$ be a bounded function on $[0,T]$. If

$$x(t) \leq \int_0^t c(\tau)x(\tau)d\tau + a(t),\ t \in [0,T], \tag{2.1}$$

then

$$x(t) \leq \sup_{0\leq t\leq T} |a(t)| e^{\int_0^t c(\tau)d\tau}. \tag{2.2}$$

It is well-known that there are many applications of the Gronwall inequality in PDE, one can refer to [11,27] etc. In this paper, we find a generalization of the Gronwall inequality, and use it to prove the stability theorems of the degenerate parabolic equation (1.9).

Lemma 2.1. Let $x(t)$ and $c(t)$ be two nonnegative integral functions on $t \in [0,T]$, $a(t)$ be a bounded function. If there is a constant $0 < l \leq 1$ such that

$$x(t) \leq \left(\int_0^t c(\tau)x(\tau)d\tau\right)^l + a(t), \tag{2.3}$$

then

$$x(t) \leq \sup_{0\leq t\leq T} |a(t)| e^{c\int_0^T c(\tau)d\tau}, \tag{2.4}$$

where c is a constant depending on $\int_0^T x(\tau)d\tau$.

Proof. If $l = 1$, there is nothing to be proved. When $l < 1$, by (2.3), using the Young inequality, we have

$$\begin{aligned} x(t) \le & \left(\int_0^t c(\tau)x(\tau)d\tau \right)^l + a(t) \\ \le & l \int_0^t c(\tau)x(\tau)d\tau + 1 - l + a(t). \end{aligned}$$

By (2.1) and (2.2), we have (2.4). □

3. The proofs of theorems

For small $\eta > 0$, let

$$S_\eta(s) = \int_0^s h_\eta(\tau)d\tau, \ \ h_\eta(s) = \frac{2}{\eta}\left(1 - \frac{|s|}{\eta}\right)_+ .$$

Obviously, $h_\eta(s) \in C(\mathbb{R})$, and

$$\lim_{\eta\to 0} S_\eta(s) = \text{sgn}s, \ \ \lim_{\eta\to 0} sh_\eta(s) = 0. \tag{3.1}$$

Proof of Theorem 1.2. From the definition of weak solution, if $g(s) = s$, for any $\varphi_1 \in C_0^1(\Omega)$ and $\varphi_2 \in L^\infty\left(0, T; W^{1,p}_{loc}(\Omega)\right)$ we have

$$\begin{aligned} & \iint_{Q_T} \varphi_1\varphi_2 \frac{\partial(u - v)}{\partial t} dxdt \\ = & - \iint_{Q_T} a(x)\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \nabla(\varphi_1\varphi_2)dxdt \\ & - \sum_{i=1}^N \iint_{Q_T} (u - v)\left[b_{ix_i}(x, t)\varphi_1\varphi_2 + b_i(x, t)(\varphi_1\varphi_2)_{x_i}\right] dxdt \\ & + \iint_{Q_T} [f(u, x, t) - v(v, x, t)]\varphi_1\varphi_2 dxdt. \end{aligned} \tag{3.2}$$

Denote $\Omega_\varepsilon = \{x \in \Omega : a(x) > \varepsilon\}$. Let ξ be

$$\xi_\varepsilon(x) = \begin{cases} 1, & \text{if } x \in \Omega_{2\varepsilon}, \\ \frac{1}{\varepsilon}[a(x) - \varepsilon], & \text{if } x \in \Omega_\varepsilon \setminus \Omega_{2\varepsilon}, \\ 0, & \text{if } x \in \Omega \setminus \Omega_\varepsilon. \end{cases}$$

By a process of limit, we can choose

$$\varphi_1 = \xi_\varepsilon \text{ and } \varphi_2 = \chi_{[\tau,s]}(u - v)$$

in (3.2), where, $\chi_{[\tau,s]}$ is the characteristic function on $[\tau, s] \subset (0, T)$. Then

$$\begin{aligned}
&\frac{1}{2}\int_\Omega [u(x,s)-v(x,s]^2\xi_\varepsilon dx\\
=&\frac{1}{2}\int_\Omega [u(x,\tau)-v(x,\tau)]^2\xi_\varepsilon dx\\
&-\iint_{Q_{\tau s}} \xi_\varepsilon a(x)\left(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right)\nabla(u-v)dxdt\\
&-\iint_{Q_{\tau s}} (u-v)a(x)\left(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right)\nabla\xi_\varepsilon dxdt\\
&-\sum_{i=1}^N\iint_{Q_{\tau s}} (u-v)\left\{b_{ix_i}(x,t)(u-v)\xi_\varepsilon+b_i(x,t)[(u-v)\xi_\varepsilon]_{x_i}\right\}dxdt\\
&+\iint_{Q_{\tau s}} [f(u,x,t)-f(v,x,t)](u-v)\xi_\varepsilon dxdt,
\end{aligned} \tag{3.3}$$

where $Q_{\tau s} = \Omega \times [\tau, s]$.

A straightforward calculation leads to

$$\begin{aligned}
&\left|-\iint_{Q_{\tau s}} (u-v)a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\nabla\xi_\varepsilon dxdt\right|\\
\leq&\iint_{Q_{\tau s}} |u-v|a(x)\left(|\nabla u|^{p-1}+|\nabla v|^{p-1}\right)|\nabla\xi_\varepsilon|dxdt\\
\leq& c\int_\tau^s\int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}}\left[\frac{p-1}{p}a(x)\left(|\nabla u|^p+|\nabla v|^p\right)+\frac{1}{p}a(x)|\nabla\xi_\varepsilon|^p\right]dxdt\\
\leq& c\int_\tau^s\int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}}\left[\frac{p-1}{p}a(x)\left(|\nabla u|^p+|\nabla v|^p\right)+\frac{1}{p}a(x)\varepsilon^{-p}\right]dxdt.
\end{aligned} \tag{3.4}$$

Since $1 < p < 2$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}} a(x)\varepsilon^{-p}dx\\
=&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}} a(x)\varepsilon^{-(p-1)}dx\\
\leq&\lim_{\varepsilon\to 0}\frac{2}{\varepsilon}\int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}} a(x)^{2-p}dx\\
=&2\int_{\partial\Omega} a(x)^{2-p}d\Sigma\\
=&0.
\end{aligned}$$

By this inequality, one can see that the right-hand side of (3.4) tends to 0 as $\varepsilon \to 0$.

Noticing that

$$
\begin{aligned}
&\sum_{i=1}^{N}\iint_{Q_{\tau s}}(u-v)\left\{b_{ix_i}(x,t)(u-v)\xi_\varepsilon+b_i(x,t)[(u-v)\xi_\varepsilon]_{x_i}\right\}dxdt\\
=&\sum_{i=1}^{N}\iint_{Q_{\tau s}}(u-v)^2b_{ix_i}(x,t)\xi_\varepsilon dxdt\\
&+\sum_{i=1}^{N}\iint_{Q_{\tau s}}(u-v)^2b_i(x,t)\xi_{\varepsilon x_i}dxdt+\sum_{i=1}^{N}\iint_{Q_{\tau s}}(u-v)b_i(x,t)\xi_\varepsilon(u-v)_{x_i}dxdt,
\end{aligned}
\tag{3.5}
$$

if denoting

$$
\Omega_{1t}=\left\{x\in\Omega:\sum_{i=1}^{N}b_i(x,t)a_{x_i}(x)\leq 0\right\}\ \text{ and }\ \Omega_{2t}=\left\{x\in\Omega:\sum_{i=1}^{N}b_i(x,t)a_{x_i}(x)>0\right\},
$$

by the partial boundary value condition (1.16), we have

$$
\begin{aligned}
&-\lim_{\varepsilon\to0}\int_\tau^s\int_\Omega(u-v)^2\sum_{i=1}^{N}b_i(x,t)\xi_{\varepsilon x_i}dxdt\\
=&-\int_\tau^s\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}}(u-v)^2\sum_{i=1}^{N}b_i(x,t)a_{x_i}(x)dxdt\\
\leq&-\int_\tau^s\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_{(\Omega_\varepsilon\setminus\Omega_{2\varepsilon})\cap\Omega_{1t}}(u-v)^2\sum_{i=1}^{N}b_i(x,t)a_{x_i}(x)dxdt\\
=&-\int_\tau^s\int_{\Sigma_p}(u-v)^2\sum_{i=1}^{N}b_i(x,t)a_{x_i}(x)d\Sigma dt\\
=&\ 0.
\end{aligned}
\tag{3.6}
$$

Meanwhile, since $1 < p < 2$, we have $\frac{p}{p-1} > 2$. Due to that $u(x,t), v(x,t) \in L^\infty(Q_T)$, by (1.17), we have

$$\begin{aligned}
&\left|\int_\Omega (u-v)\sum_{i=1}^N b_i(x,t)\xi_\varepsilon (u-v)_{x_i} dx\right| \\
=&\left|\int_\Omega (u-v)\sum_{i=1}^N b_i(x,t)\xi_\varepsilon a(x)^{-\frac{1}{p}} a(x)^{\frac{1}{p}} (u-v)_{x_i} dx\right| \\
\le&\left(\int_\Omega \left|(u-v)\sum_{i=1}^N b_i(x,t) a(x)^{-\frac{1}{p}}\right|^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int_\Omega a(x)\left(|\nabla u|^p + |\nabla v|^p\right) dx\right)^{\frac{1}{p}} \\
\le c&\left(\int_\Omega \left|(u-v)\sum_{i=1}^N b_i(x,t) a(x)^{-\frac{1}{p}}\right|^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \\
= c&\left(\int_\Omega |u-v||u-v|^{\frac{p}{p-1}-1}\left|\sum_{i=1}^N b_i(x,t) a(x)^{-\frac{1}{p}}\right|^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \\
\le c&\left(\int_\Omega |u-v|\left|\sum_{i=1}^N b_i(x,t) a(x)^{-\frac{1}{p}}\right|^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \\
\le c&\left(\int_\Omega |u-v|^2 dx\right)^{\frac{p-1}{2p}} \left(\int_\Omega \left|\sum_{i=1}^N b_i(x,t) a(x)^{-\frac{1}{p}}\right|^{\frac{2p}{p-1}} dx\right)^{\frac{p-1}{2p}} \\
\le c&\left(\int_\Omega |u-v|^2 dx\right)^{\frac{p-1}{2p}}.
\end{aligned} \tag{3.7}$$

Naturally, we have

$$-\iint_{Q_{\tau s}} \xi_\varepsilon a(x)\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\nabla(u-v)dxdt \le 0. \tag{3.8}$$

Since $u(x,t), v(x,t) \in L^\infty(Q_T)$ and $f(u,x,t) \in C^1(\mathbb{R}\times\overline{Q_T})$, by (3.4), (3.6)–(3.8), from (3.3), we easily obtain

$$\int_\Omega [u(x,s)-v(x,s)]^2 dx \le \int_\Omega [u(x,\tau)-v(x,\tau)]^2 dx + c\left(\int_\tau^s\int_\Omega |u(x,t)-v(x,t)|^2 dxdt\right)^l,$$

where $l \le 1$.

By virtue of the generalized Gronwall Lemma 2.1, choosing $x(s) = \int_\Omega [u(x,s)-v(x,s)]^2 dx$, we have

$$\int_\Omega [u(x,s)-v(x,s)]^2 dx \le c(T)\int_\Omega [u(x,\tau)-v(x,\tau)]^2 dx,$$

and letting $\tau \to 0$, we arrive at the desire. □

Proof of Theorem 1.3. Let $u(x,t)$ and $v(x,t)$ be two solutions of Eq (1.9) with the initial values $u_0(x)$ and $v_0(x)$ respectively, but without any boundary value condition. From the definition of the weak solution, we can choose $\chi_{[\tau,s]}a(x)(u-v)$ as a test function, where $\chi_{[\tau,s]}$ is the characteristic function on $[\tau,s]\subset[0,T)$. Denoting $Q_{\tau s}=\Omega\times[\tau,s]$, then we have

$$\begin{aligned}
&\iint_{Q_{\tau s}} a(x)(u-v)\frac{\partial(u-v)}{\partial t}dxdt\\
=&-\iint_{Q_{\tau s}} a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\nabla[a(x)(u-v)]dxdt\\
&-\sum_{i=1}^{N}\iint_{Q_{\tau s}}(u-v)[b_{ix_i}(x,t)a(x)(u-v)+b_i(x,t)(a(x)(u-v))_{x_i}]dxdt\\
&+\iint_{Q_{\tau s}}[f(u,x,t)-f(v,x,t)]a(x)(u-v)dxdt.
\end{aligned}\tag{3.9}$$

In particular,

$$\begin{aligned}
&\iint_{Q_{\tau s}} a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\nabla[a(x)(u-v)]dxdt\\
=&\iint_{Q_{\tau s}} a(x)^2(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\nabla(u-v)dxdt\\
&+\iint_{Q_{\tau s}} a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)(u-v)\nabla a dxdt.
\end{aligned}\tag{3.10}$$

Clearly,

$$\iint_{Q_{\tau s}} a(x)^2(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\nabla(u-v)dxdt\geq 0.\tag{3.11}$$

For the second term on the right hand side of (3.10), we have

$$\begin{aligned}
&\left|\iint_{Q_{\tau s}}(u-v)a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\nabla a dxdt\right|\\
\leq&\iint_{Q_{\tau s}}|u-v|a(x)(|\nabla u|^{p-1}+|\nabla v|^{p-1})|\nabla a|dxdt\\
\leq&c\left(\int_\tau^s\int_\Omega a(x)(|\nabla u|^p+|\nabla v|^p)dxdt\right)^{\frac{p-1}{p}}\cdot\left(\int_\tau^s\int_\Omega a(x)|\nabla a|^p|u-v|^pdxdt\right)^{\frac{1}{p}}\\
\leq&c\left(\int_\tau^s\int_\Omega a(x)|\nabla a|^p|u-v|^pdxdt\right)^{\frac{1}{p}}\\
\leq&c\left(\int_\tau^s\int_\Omega a(x)|u-v|^pdxdt\right)^{\frac{1}{p}}.
\end{aligned}\tag{3.12}$$

If $p\geq 2$,

$$\left(\int_\tau^s\int_\Omega a(x)|u-v|^pdxdt\right)^{\frac{1}{p}}\leq c\left(\int_\tau^s\int_\Omega a(x)|u-v|^2dxdt\right)^{\frac{1}{p}}.\tag{3.13}$$

If $1 < p < 2$, by then the Hölder inequality

$$\left(\int_\tau^s \int_\Omega a(x)|u-v|^p dxdt\right)^{\frac{1}{p}} \leq c\left(\int_\tau^s \int_\Omega a(x)|u-v|^2 dxdt\right)^{\frac{1}{2}}. \tag{3.14}$$

Meanwhile, by (1.23),

$$\int_\Omega \frac{\left|\sum_{i=1}^N a_{x_i}(x)b_i(x,t)\right|^2}{a(x)} dx \leq c(T),$$

since $u(x,t), v(x,t) \in L^\infty(Q_T)$, we easily get that

$$\begin{aligned}
&\left|\iint_{Q_{\tau s}} \sum_{i=1}^N b_i(x,t)(u-v)[(u-v)a]_{x_i} dxdt\right| \\
=&\left|\iint_{Q_{\tau s}} \sum_{i=1}^N b_i(x,t)(u-v)^2 a_{x_i} dxdt + \iint_{Q_{\tau s}} \sum_{i=1}^N b_i(x,t)(u-v)(u-v)_{x_i} a(x) dxdt\right| \\
\leq& c\int_\tau^s \int_\Omega |u-v| \left|\sum_{i=1}^N a_{x_i} b_i(x,t)\right| dx \\
&+\left(\int_\tau^s \int_\Omega a^{(-\frac{1}{p})p'}(a(x)\sum_{i=1}^N b_i(x,t)|u-v|)^{p'} dxdt\right)^{\frac{1}{p'}} \\
&\cdot\left(\int_\tau^s \int_\Omega a(x)(|\nabla u|^p + |\nabla v|^p) dxdt\right)^{\frac{1}{p}} \\
\leq& c\left(\int_\tau^s \int_\Omega \left|\frac{\sum_{i=1}^N a_{x_i} b_i(x,t)}{\sqrt{a(x)}}\right|^2 dxdt\right)^{\frac{1}{2}} \left(\int_\tau^s \int_\Omega a(x)|u-v|^2 dxdt\right)^{\frac{1}{2}} \\
&+ c\left(\int_\tau^s \int_\Omega \left|\sum_{i=1}^N b_i(x,t)\right|^{\frac{p}{p-1}} dxdt\right)^{\frac{1}{p'}} \left(\int_\tau^s \int_\Omega a(x)(|\nabla u|^p + |\nabla v|^p) dxdt\right)^{\frac{1}{p}} \\
\leq& c\left(\int_\tau^s \int_\Omega a(x)|u-v|^2 dxdt\right)^{\frac{1}{2}} + c\left(\int_\tau^s \int_\Omega a(x)|u-v|^{p'} dxdt\right)^{\frac{1}{p'}}.
\end{aligned} \tag{3.15}$$

Here, $p' = \frac{p}{p-1}$ as usual.

At the same time, we have

$$\iint_{Q_{\tau s}} \left|\sum_{i=1}^N b_{ix_i}(x,t)(u-v)^2\right| a(x) dxdt \leq c\left(\int_\tau^s \int_\Omega a(x)|u-v|^2 dxdt\right)^{\frac{1}{2}}, \tag{3.16}$$

and since $|f(u,x,t) - f(v,x,t)| \leq c$, by that $u(x,t), v(x,t) \in L^\infty(Q_T)$, using the Hölder inequality, we have

$$\iint_{Q_{\tau s}} |[f(u,x,t) - f(v,x,t)](u-v)| a(x) dxdt \leq c\left(\int_\tau^s \int_\Omega a(x)|u-v|^2 dxdt\right)^{\frac{1}{2}}. \tag{3.17}$$

Also,

$$\begin{aligned}&\iint_{Q_{\tau s}}(u-v)a(x)\frac{\partial(u-v)}{\partial t}dxdt\\=&\int_\Omega a(x)[u(x,s)-v(x,s)]^2dx-\int_\Omega a(x)[u(x,\tau)-v(x,\tau)]^2dx.\end{aligned}\tag{3.18}$$

At last, by (3.10)–(3.18), we let $\lambda\to 0$ in (3.9). Then

$$\begin{aligned}&\int_\Omega a(x)[u(x,s)-v(x,s)]^2dx-\int_\Omega a(x)[u(x,\tau)-v(x,\tau)]^2dx\\\leq& c\left(\int_0^s\int_\Omega a(x)|u(x,t)-v(x,t)|^2dxdt\right)^q,\end{aligned}\tag{3.19}$$

where $q<1$. By (3.19), by the generalized Gronwall Lemma 2.1, choosing

$$x(s)=\int_\Omega a(x)\mid u(x,s)-v(x,s)\mid^2 dx,$$

we have

$$\int_\Omega a(x)\mid u(x,s)-v(x,s)\mid^2 dx\leq\int_\Omega a(x)\mid u(x,\tau)-v(x,\tau)\mid^2 dx.\tag{3.20}$$

Thus, by the arbitrary of τ, we have

$$\int_\Omega a(x)\mid u(x,s)-v(x,s)\mid^2 dx\leq\int_\Omega a(x)\mid u_0(x)-v_0(x)\mid^2 dx.$$

The proof is complete. □

In what follows, we study the uniqueness of weak solution to Eq (1.9).

Proof of Theorem 1.4. Let $u(x,t)$ and $v(x,t)$ be two weak solutions of Eq (2.1) with the same initial value $u_0(x)=v_0(x)$.

For any given small $\delta>0$, we denote $D_\delta=\{x\in\Omega:|w|=|u-v|>\delta\}$. Without loss of the generality, we may assume that there is a $\delta>0$ such that the measure $\mu(D_\delta)>0$. Let $\varphi_\lambda(\xi)$ be a even function. When $\xi\geq 0$, it its defined as

$$\varphi_\lambda(\xi)=\begin{cases}\frac{1}{1-\beta}\lambda^{\beta-1}-\frac{1}{1-\beta}\xi^{\beta-1}, & \text{if } \xi>\lambda,\\ 0, & \text{if } \xi\leq\lambda,\end{cases}\tag{3.21}$$

where λ is small enough satisfying $\delta>2\lambda>0$, $1>\beta>0$, and we define that

$$\Phi_\lambda(\xi)=\int_0^\xi\varphi_\lambda(\varsigma)d\varsigma.$$

By multiplying Eq (1.9) by $\varphi_\lambda(w) = \varphi_\lambda(u - v)$, integrating over $Q_t = \Omega \times (0, t)$, since $a(x) = 0$ when $x \in \partial\Omega$, using the condition (3.2) in i) or the condition (3.1) in ii), we have

$$\begin{aligned} 0 = & \int_0^t \int_\Omega \Big[w_t \varphi_\lambda(w) + a(x)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla\varphi_\lambda(w) \Big] dxdt \\ & + \sum_{i=1}^N \int_0^t \int_\Omega \frac{\partial b_i(x,t)}{\partial x_i} \Phi_\lambda(u-v)dxdt \\ & - \sum_{i=1}^N \int_0^t \int_\Omega (f(u,x,t) - f(v,x,t))\varphi_\lambda(u-v)dxdt. \end{aligned} \tag{3.22}$$

Let us analyse the three terms in the righthand side of this equality. First, the monotone inequality of the operator Δ yields

$$\begin{aligned} & \int_0^t \int_\Omega a(x)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u-v)\varphi_\lambda'(u-v)dxdt \\ & \geq \int_0^t \int_{D_\lambda} a(x)(u-v)^{2-\beta} 2^{-p} |\nabla w|^p \varphi_\lambda'(u-v)dxdt \\ & \geq 0. \end{aligned} \tag{3.23}$$

Secondly, since $u(x,t), v(x,t) \in L^\infty(Q_T)$,

$$\left| \sum_{i=1}^N \int_0^t \int_\Omega \frac{\partial b_i(x,t)}{\partial x_i} \Phi_\lambda(u-v)dxdt \right| + \left| \sum_{i=1}^N \int_0^t \int_\Omega (f(u,x,t) - f(v,x,t))\varphi_\lambda(u-v)dxdt \right| \leq c. \tag{3.24}$$

Thirdly, let $t_0 = \inf\{\tau \in (0,t] : w > \lambda\}$. Then

$$\begin{aligned} \int_0^t \int_{D_\lambda} w_t \varphi_\lambda(w)dxdt = & \int_{D_\lambda} \left(\int_0^{t_0} w_t \varphi_\lambda(w)dt + \int_{t_0}^t w_t \varphi_\lambda(w)dt \right) dx \\ \geq & \int_{D_\lambda} \int_\lambda^{w(x,t)} \varphi_\lambda(s)dsdx \\ \geq & \int_{D_\lambda} (w - 2\lambda)\varphi_\lambda(2\lambda)dx \\ \geq & (\delta - 2\lambda)\varphi_\lambda(2\lambda)\mu(D_\lambda). \end{aligned} \tag{3.25}$$

From (3.23)–(3.25), we have

$$(\delta - 2\lambda)\frac{1 - 2^{\beta-1}}{1-\beta}\lambda^{\beta-1} \leq c.$$

Letting $\lambda \to 0$, we get the contradiction. □

At the end of this section, we would like to point that, $\varphi_\lambda(w) = \varphi_\lambda(u - v)$ does not satisfy the request of the test function $g(\varphi_1\varphi_2)$, so in the proof of Theorem 1.4, we do not use the equality (1.14) in Definition 1.1. However, since the condition (3.2) in i) or the condition (3.1) in ii), all the boundary integrals disappears in (3.22), and the proof of Theorem 1.4 is true.

4. Conclusions

The non-Newtonian fluid equation with a variable diffusion is more applicable than the usual fluid equation. In the PDE theory, since the variable diffusion $a(x)$ may be degenerate on some points of $\overline{\Omega}$, one can not deduce the inequality

$$\iint_{Q_T} |\nabla u|^p dxdt < \infty,$$

and the weak solution matching up with the equation does not belong to $L^p(0,T;W_0^{1,p}(x))$. Such a fact makes the boundary value condition can not be imposed in the classical way. In this paper, a new kind of weak solution introduced, the trace is generalized in accordance with the weak solution defined, and it is found that a partial boundary value condition on a submanifold of $\partial\Omega \times (0,T)$ is enough to ensure the weak solution well-posedness. In fact, the definitions, the theorems and the methods used in this paper are able to be generalized to study the well-posed problem of the other degenerate parabolic equations such as

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\right) - \sum_{i=1}^{N} b_i(x,t)D_i u = f(u,x,t),\ (x,t) \in Q_T,$$

and

$$v_t - \operatorname{div}(|v|^{\alpha}|\nabla v|^{p-2}\nabla v) - \sum_{i=1}^{N} g^i(x,t,v)\frac{\partial v}{\partial x_i} = d(x,t,v),\ (x,t) \in Q_T,$$

where $\alpha > 0$ is a constant, $p(x,t) > 1$ is a continuous function.

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Conflict of interest

The authors declare no conflict of interest.

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A. Appendix: The generalizations of the trace

In this appendix, we give a simple discussion of the trace. Without loss the generality, we assume that $f(x) \geq 0$. It is well-known that, $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, and so the trace of $f(x) \in W^{1,p}(\Omega)$ on the boundary $\partial\Omega$

$$f(x) = 0, \ x \in \partial\Omega$$

is defined as the limit of a sequence $f_\varepsilon(x) \in C_0^\infty(\Omega)$ as

$$0 = \lim_{\varepsilon \to 0} f_\varepsilon(x)|_{x\in\partial\Omega}. \tag{A.1}$$

As above in this paper, we denote $\Omega_\lambda = \{x \in \Omega : d(x) > \lambda\}$ for a small enough positive number λ, and denote by $\mathbf{B}$ the closure of the set $C_0^\infty(Q_T)$ with respect to the norm

$$||u||_{\mathbf{B}} = \iint_{Q_T} a(x)\left(|u(x,t)|^p + |\nabla u(x,t)|^p\right) dxdt,\ u \in \mathbf{B}.$$

Yin-Wang [28] defined the trace of $u \in \mathbf{B}$, $u(x,t) = 0$ on Σ_2 as

$$\operatorname{ess}\lim_{\lambda\to 0} \int_{\left\{x\in\partial\Omega_\lambda : \sum_{i=1}^N b_i(x)n_i(x)<0\right\}} u^2 \sum_{i=1}^N b_i(x)n_i(x)d\sigma = 0. \tag{A.2}$$

One can see that if $a(x) = d(x)$, then $d_{x_i} = n_i$ with that $\vec{n} = \{n_i\}$ is the inner normal. Then the Definition (A.2) is just the same as that of (1.20). The reminder is to show that when $a(x)$ satisfies (1.10), $a(x) \neq d(x)$, the Definition (A.2) is equivalent to that of (1.20).

At first, since $a(x)$ satisfies (1.10) and $\int_\Omega a(x)|\nabla u|^p dx < \infty$, we know

$$\int_{\Omega_\lambda} |\nabla u|^p dx < \infty$$

and $u \in BV(\Omega_\lambda)$, the traces of

$$u^2 \sum_{i=1}^N b_i(x)n_i(x),\ x \in \partial\Omega_\lambda$$

and

$$u^2 \sum_{i=1}^N b_i(x)a_{x_i}(x),\ x \in \partial\Omega_\lambda$$

can be well defined in the sense of (A.1). Thus, the definition of (1.20) or the definition of (A.2) itself has a explicit meaning. In fact, recalling the BV function space $BV(\Omega)$, i.e. $\left|\frac{\partial f}{\partial x_i}\right|$ is a regular measure, and

$$BV(\Omega) = \left\{f(x) : \int_\Omega \left|\frac{\partial f}{\partial x_i}\right| < c, i = 1, 2, \cdots, N\right\}.$$

then $BV(\Omega)$ is a Banach space under the norm

$$||f||_{BV} = ||f||_{L^1} + \int_\Omega |Df|.$$

By the trace $f^+(y)$ of $f(x) \in BV(\Omega)$ on the boundary $\partial\Omega$ is defined as the limit of $f(x)$ along the normal (Lemma 1 of [26]). Thus, if we denote that

$$D_\lambda = \{x \in \Omega : d(x) = dist(x, \partial\Omega) > \lambda\},$$

then for a $f(x) \in BV(\Omega) \bigcap L^\infty(\Omega)$, we have

$$\lim_{\lambda\to 0} \frac{1}{\lambda} \int_{\Omega\setminus D_\lambda} f(x)dx = \int_{\partial\Omega} f^+(x)d\Sigma. \tag{A.3}$$

Secondly, for a smooth bounded domain Ω, there is a finite open cover such that

$$\Omega = \bigcup_{\alpha} B_\alpha, \quad \partial\Omega \subseteq \bigcup_{\alpha} \Sigma_\alpha,$$

where $\Sigma_\alpha = B_\alpha \bigcap \partial\Omega$. If $\Sigma_\alpha \neq \emptyset$, after using the flattening technique, we may assume that

$$B_1 = \left\{x \in \mathbb{R}^N : |x| < 1\}, \quad B_\alpha \bigcap \Omega = \{x \in \mathbb{R}^N : |x| < 1, x_N > 0\right\} = B^+(1),$$

$$\Sigma_\alpha = \left\{x \in \mathbb{R}^N : |x| < 1, x_N = 0\right\},$$

and

$$d(x, \partial\Omega) = x_N, \; x \in B_\alpha \bigcap \Omega,$$

and

$$d(x, \partial\Omega) = x_2, \; x \in B_\alpha.$$

Let $a(x) \in C^1(\overline{\Omega})$ satisfy

$$a(x) > 0, \; x \in \Omega, \; a(x) = 0, \; x \in \partial\Omega.$$

Since $a(x) > 0$ when $x \in B_1^+$, and there is a small enough δ_0 such that when $0 < x_2 < \delta$

$$\frac{\partial a(x)}{\partial x_2} \geq c(\delta_0) > 0, \; x \in \{B_1^+ : 0 < x_2 < \delta_0\}. \tag{A.4}$$

For small enough $\lambda < \delta_0$, we set

$$E_{x\lambda} = \{x \in B_1^+ : a(x) < \lambda\},$$

and denote the local coordinate representation B_α and Σ_α as above. By the coordinate transformation

$$y_1 = x_1, y_2 = x_2, \cdots, y_{N-1} = x_{N-1}, y_N = a(x) \tag{A.5}$$

the domain $E_{x\lambda}$ is transformed to $E_{y\lambda} = \Sigma_\alpha \times (0, \lambda)$.

In $E_{y\lambda}$, it is clear of that $d(y, \Sigma_\alpha) = y_N$. Since $f \in BV(E_{x\lambda})$ or $f \in BV(E_{y\lambda})$, as (A.3), we have

$$\begin{aligned} &\int_{\partial\Omega} f^+(y) d\Sigma \\ =&\lim_{\lambda \to 0} \frac{1}{\lambda} \int_{E_{y\lambda}} f(y) dy \\ =&\lim_{\lambda \to 0} \frac{1}{\lambda} \int_{E_{x\lambda}} f(x) \frac{\partial a}{\partial x_N} dx. \end{aligned} \tag{A.6}$$

When f is nonnegative and $f|_{\partial\Omega} = 0$, then the equality (A.6) can be re-calculated as

$$\begin{aligned} 0 =& \int_{\partial\Omega} f^+(y) d\Sigma \\ =&\lim_{\lambda \to 0} \frac{1}{\lambda} \int_{E_{y\lambda}} f(y) dy \\ =&\lim_{\lambda \to 0} \frac{1}{\lambda} \int_{E_{x\lambda}} f(x) \frac{\partial a}{\partial x_N} dx \\ \leq& M\left(\frac{\partial a}{\partial x_N}\right) \lim_{\lambda \to 0} \frac{1}{\lambda} \int_{E_{x\lambda}} f(x) dx \\ =& M\left(\frac{\partial a}{\partial x_N}\right) \int_{\Sigma_\alpha} f^+(x) d\Sigma = 0, \end{aligned} \tag{A.7}$$

where $M\left(\frac{\partial a}{\partial x_N}\right) = \sup_{x\in\Sigma_\alpha} \frac{\partial a}{\partial x_N}$. In other words, when $f \in BV(\Omega)$ is nonnegative and $f|_{\partial\Omega} = 0$, (A.7) implies that

$$\lim_{\lambda\to 0} \frac{1}{\lambda} \int_{E_{x\lambda}} f(x)dx = \int_{\Sigma_\alpha} f^+(x)d\Sigma = 0. \tag{A.8}$$

Thirdly, by (A.4), the coordinate transformation (A.5) has a inverse transformation

$$x_1 = y_1, x_2 = y_2, \cdots, x_{N-1} = y_{N-1}, x_N = a^{-1}(y). \tag{A.9}$$

Recalling the definition of (1.20), i.e.,

$$\text{ess sup} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{(\Omega_\varepsilon\setminus\Omega_{2\varepsilon})\cap\Omega_{1t}} u^2 \sum_{i=1}^{N} b_i(x,t)a_{x_i}(x)dx = 0,$$

without loss the generality, under the inverse transformation (A.9), we may assume that the domain $(\Omega_\varepsilon \setminus \Omega_{2\varepsilon}) \cap \Omega_{1t}$ is transformed to a domain $D_{\varepsilon y} \cap \Omega_{1t}$

Then, similar as the discussion of (A.7) and (A.8), the definition of (1.20) is equivalent to that

$$\text{ess sup} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{D_{\varepsilon y}\cap\Omega_{1t}} u^2 \sum_{i=1}^{N} b_i(y,t)n_{y_i}dy = 0.$$

From these point, one can see that the definition of (1.20) is just a reasonable version the definition of (A.2), which is the generalized definition of the trace of $u \in \mathbf{B}$ with degeneracy on the boundary. Actually, such a local analysis of the definition of the trace in $BV(\Omega)$ has appeared in our previous work [33]. Since [33] is unpublished till now, we re-give the details here.

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