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Research article

On triple correlation sums of Fourier coefficients of cusp forms

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Abstract: Let p be a prime. In this paper, we study the sum

$$\sum_{m\geq 1}\sum_{n\geq 1} a_n\lambda_g(m)\lambda_f(m+pn)\,U\Big(\frac{m}{X}\Big)V\Big(\frac{n}{H}\Big)$$

for any newforms $g \in \mathcal{B}_k(1)$ (or $\mathcal{B}^*_\lambda(1)$) and $f \in \mathcal{B}_k(p)$ (or $\mathcal{B}^*_\lambda(p)$), with the aim of determining the explicit dependence on the level, where $\mathbf{a} = \{a_n \in \mathbb{C}\}$ is an arbitrary complex sequence. As a result, we prove a uniform bound with respect to the level parameter p, and present that this type of sum is non-trivial for any given $H, X \geq 2$.

Keywords: automorphic forms; Fourier coefficients; triple correlation sums
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1. Introduction

A basic but important problem in number theory is the triple correlation sums problem, which concerns the non-trivial bounds for

$$\sum_{h\leq H}\sum_{n\leq X} a(n)b(l_1n+l_2h)c(h) \quad \text{or} \quad \sum_{h\leq H}\sum_{n\leq X} a(n)b(n+l_1h)c(n+l_2h).$$

Here, $a(n), b(n)$ and $c(n)$ are three arithmetic functions, $H, X \geq 2$ and $l_1, l_2 \in \mathbb{Z}$. These type of sums play the vital roles of their own in many topics, such as the moments of L-functions (or zeta-functions), subconvexity, the Gauss circle problem and the Quantum Unique Ergodicity (QUE) conjecture, etc (see for instance [1, 5, 7–9, 11–13, 15, 19] and the references therein). In the case of all the arithmetic functions being the divisor functions, Browning [4] found that

$$\sum_{h\le H, n\le X} a(n)b(n+h)c(n+2h) = \frac{11}{8}\Upsilon(h)\prod_p\left(1-\frac{1}{p}\right)^2\left(1+\frac{2}{p}\right)HX\log^3 X + o\Big(HX\log^3 X\Big)$$

for certain function $\Upsilon(h)$, provided that $H \ge X^{3/4+\varepsilon}$. It is remarkable that Blomer [2] used the spectral decomposition for partially smoothed triple correlation sums to prove that

$$\sum_{h\ge1}\sum_{n\le X} W\left(\frac{h}{H}\right)\tau(n)a(n+h)\tau(n+2h) = H\widehat{W}(1)\sum_{n\le X}a(n)\sum_{d\ge1}\frac{S(2n,0;d)}{d^2} \times(\log n + 2\gamma - 2\log d)^2 + O\left(\left(\frac{H^2}{\sqrt{X}} + HX^{\frac14} + \sqrt{XH} + \frac{X}{\sqrt{H}}\right)\|\mathbf{a}\|_2\right).$$

Here, the gamma constant $\gamma \approx 0.57721$, W is a bump function supported on $[1/2, 5/2]$, satisfying that $x^jW^{(j)}(x) \ll 1$ for any $j \in \mathbb{N}_+$; while, $\widehat{W}$ is the Mellin transform of W and

$$\|\mathbf{a}\|_2 = \sqrt{\sum_{n\le X}|a^2(n)|}$$

is the ℓ^2-norm. Notice that, the parameter H is reduced to $H \ge X^{1/3+\varepsilon}$. Let $k, k' \in 2\mathbb{N}_+$. Let $f_1 \in \mathcal{B}_k^*(1)$ and $f_2 \in \mathcal{B}_{k'}^*(1)$ be two Hecke newforms on GL_2, with λ_{f_1} and λ_{f_2} being their n-th Hecke eigenvalues, respectively (see §2 for definitions). Subsequently, Lin [20] claimed that

$$\sum_{h\ge1}\sum_{n\le X} W\left(\frac{h}{H}\right)\lambda_{f_1}(n)a(n+h)\lambda_{f_2}(n+2h) \ll X^{\varepsilon}\left(\sqrt{XH} + \frac{X}{\sqrt{H}}\right)\|\mathbf{a}\|_2,$$

which is non-trivial, provided that $H \ge X^{2/3+\varepsilon}$. Recently, Singh [28], however, were able to attain that, for any $f_1, f_2, f_3 \in \mathcal{B}_k^*(1)$ (or $\mathcal{B}_\lambda^*(1)$) and some constant $\eta > 0$,

$$\sum_{h\ge1}\sum_{n\ge1} W_1\left(\frac{h}{H}\right)W_2\left(\frac{n}{X}\right)\lambda_{f_1}(n)\lambda_{f_2}(n+h)\lambda_{f_3}(n+2h) \ll X^{1-\eta+\varepsilon}H,$$

where W_1, W_2 are any bump functions compactly supported on $[1/2, 5/2]$ with bounded derivatives. Here, for any $N \in \mathbb{N}^+$, $\mathcal{B}_\lambda^*(N)$ denotes the collection of the primitive newforms of Laplacian eigenvalue λ on $\Gamma_0(N)$ (see §2 for backgrounds). Until now, the best result is due to Lü-Xi [21, 22] who achieved that

$$\sum_{h\ge1}\sum_{n\le X} W\left(\frac{h}{H}\right)a(n)b(n+h)\lambda_{f_1}(n+2h) \ll X^{\varepsilon}\Delta_1(X,H)\,\|\mathbf{a}\|_2\,\|\mathbf{b}\|_2,$$

which allows one to take $H \ge X^{2/5+\varepsilon}$. Here, the definition of $\Delta_1(X,H)$ can be referred to [22, Theorem 3.1]. More recently, Hulse et al. [14] successfully attained

$$\sum_{h\ge1}\sum_{n\ge1}\lambda_{g_1}(n)\lambda_{g_2}(h)\lambda_{g_3}(2n-h)\exp\left(-\frac{h}{H}-\frac{n}{X}\right) \ll X^{\kappa-1+\vartheta+\frac12+\varepsilon}H^{\frac{\kappa-1}{2}-\vartheta+\frac12+\varepsilon}, \tag{1.1}$$

where $\vartheta < 7/64$ denotes the currently best record for the Generalized Ramanujan Conjecture. Here, $\lambda_{g_1}(n)$ (resp. $\lambda_{g_1}(n)$ and $\lambda_{g_1}(n)$) denote the n-th *non-normalized* coefficients of holomorphic cusp forms g_1 (resp. g_2 and g_3), each of weight $\kappa \geq 2$ and level $M \geq 2$.

In the present paper, we aim to consider the level aspect for the triple correlation sums. It is noticeable that, just lately, Munshi [26] obtained that, whenever $X^{1/3+\varepsilon} \leq p \leq X$, one has the inequality

$$\sum_{n\geq 1} \lambda_f(n)\lambda_f(n+pm) \ll p^{\frac{1}{4}} X^{\frac{3}{4}+\varepsilon} \tag{1.2}$$

for any $f \in \mathcal{B}_k^*(p)$ and fixed integer m such that $|m| \leq X/p$. In this paper, we would like to go further obtaining the following quantitive estimate.

Theorem 1.1. *Let $X, H \geq 2$, and p be a prime such that $p \leq X$. Let U, V be two smooth weight functions supported* $[1/2, 5/2]$ *with bounded derivatives. Then, for any sequence $\mathbf{a} = \{a_n \in \mathbb{C}\}$ and any newforms $g \in \mathcal{B}_k^*(1)$ (or $\mathcal{B}_\lambda^*(1)$) and $f \in \mathcal{B}_k^*(p)$ (or $\mathcal{B}_\lambda^*(p)$), we have*

$$\sum_{m\geq 1}\sum_{n\geq 1} a_n \lambda_g(m)\lambda_f(m+pn)\, U\Big(\frac{m}{X}\Big)V\Big(\frac{n}{H}\Big) \ll X^\varepsilon \max\Big(\sqrt{XHp}, X\Big)\|\mathbf{a}\|_2, \tag{1.3}$$

where the implied constant depends only on the weight k (or the spectral parameter λ) and ε.

Remark 1.2. *Our main result* (1.3) *is non-trivial for any given parameters X and H; particularly, for any automorphic cusp form π of any rank N, $N \geq 2$, with $\lambda_\pi(n)$ being its n-th normoailzed Fourier coefficient, we find*

$$\sum_{m\geq 1}\sum_{n\geq 1} \lambda_\pi(n)\lambda_g(m)\lambda_f(m+pn)\, U\Big(\frac{m}{X}\Big)V\Big(\frac{n}{H}\Big) \ll X^\varepsilon \max\Big(H\sqrt{Xp}, X\sqrt{H}\Big)$$

by the Rankin-Selberg's bound, which says that $\sum_{n\leq X}|\lambda_\pi(n)|^2 \ll_{\pi,\varepsilon} X^{1+\varepsilon}$ (see for instance [6, Remark 12.1.8]).

Remark 1.3. *The merits that comes from* (1.3) *is that the implied constant does not depend on the level parameter anymore. One may verify that our result* (1.3)*, however, is exhibited to be a strengthened upper-bound whenever $H \geq \sqrt{X/p}$, compared with an application of Munshi's estimate* (1.2)*. Indeed, Munshi's estimate implies the upper-bound $\ll p^{1/4}X^{3/4+\varepsilon}\sqrt{H}\|\mathbf{a}\|_2$ for the triple sum above. Moreover, one might save roughly a magnitude of $\sqrt{H}$ in the interesting case of $pH \asymp X$; in the average sense, the main result improves upon the estimate due to Munshi. One, on the other hand, wanders whether or not the non-trivial bounds for the scenarios where the cusp forms f, g being of higher rank could be achieved; we shall plan to investigates this problem in the future work.*

Notations. *Throughout the paper, ε always denotes an arbitrarily small positive constant. $n \sim X$ means that $X < n \leq 2X$. For any integers m, n, (m, n) means the great common divisor of m, n. Finally, $\mu(n)$ denotes the Möbius function of n.*

2. Preliminaries

2.1. Automorphic forms

For any $k \in 2\mathbb{N}_+$ and $N \in \mathbb{N}_+$, let us denote by $\mathcal{S}_k(N)$ the vector space of the normalized holomorphic cusp forms on $\Gamma_0(N)$ of weight k and trivial nebentypus. Whenever $f \in \mathcal{S}_k(N)$, one has

$$f(z) = \sum_{n\geq 1} \lambda_f(n) n^{\frac{k-1}{2}} e(nz)$$

for $\mathrm{Im}(z) > 0$. We also denote by $\mathcal{S}_\lambda(N)$ the vector space of the normalized Maaß forms on $\Gamma_0(N)$ of weight 0, Laplacian eigenvalue $\lambda = 1/4 + r^2$ ($r \in \mathbb{R}$) and trivial nebentypus. For any $f \in \mathcal{S}_\lambda(N)$, there exists the following Fourier expansion

$$f(z) = 2\sqrt{|y|} \sum_{n\neq 0} \lambda_f(n) K_{ir}(2\pi|ny|) e(nx),$$

where $z = x + iy$. The set of the *primitive forms* $\mathcal{B}_k^*(N)$ (or $\mathcal{B}_\lambda^*(N)$) consists of common eigenfunctions of all the Hecke operators T_n for any $n \geq 1$. Regarding the individual bounds for $\lambda_f(n)$, we have

$$\lambda_f(n) \ll (nN)^\varepsilon, \tag{2.1}$$

whenever $f \in \mathcal{B}_k^*(N)$ (or $\mathcal{B}_\lambda^*(N)$) .

We will need the Voronoi summation formula in the analysis; see [18, Theorem A.4].

Lemma 2.1. *Let k, N and the form f be as before. For any $a, q \in \mathbb{N}_+$ such that $(a, q) = 1$, we set $N_2 := N/(N, q)$. Let h be a bump function of bounded derivatives. Then, there exists a constant ϱ of modulus one and a newform $f^\star \in \mathcal{B}_k^*(N)$ (or $\mathcal{B}_\lambda^*(N)$) such that*

$$\begin{aligned}\sum_{n\geq 1} \lambda_f(n) e\left(\frac{an}{q}\right) h\left(\frac{n}{X}\right) =& \frac{2\pi\varrho}{q\sqrt{N_2}} \sum_{n\geq 1} \lambda_{f^\star}(n) e\left(-\frac{\overline{aN_2}n}{q}\right) \mathscr{H}^\flat\left(\frac{nX}{q^2N_2}; h\right) \\ &+ \frac{2\pi\varrho}{q\sqrt{N_2}} \sum_{n\geq 1} \lambda_{f^\star}(n) e\left(\frac{\overline{aN_2}n}{q}\right) \mathscr{H}^\natural\left(\frac{nX}{q^2N_2}; h\right),\end{aligned}$$

where

$$\mathscr{H}^\flat(x; h) = \int_0^\infty h(\xi) J_f\left(4\pi\sqrt{x\xi}\right) \mathrm{d}\xi, \quad \text{and} \quad \mathscr{H}^\natural(x; h) = \int_0^\infty h(\xi) K_f\left(4\pi\sqrt{x\xi}\right) \mathrm{d}\xi.$$

Here, if f is holomorphic

$$J_f(x) = 2\pi i^k J_{k-1}(x), \quad K_f(x) = 0;$$

while, if f is a Maaß form

$$J_f(x) = \frac{-\pi}{\sin(\pi i r)} (J_{2ir}(x) - J_{-2ir}(x)), \quad K_f(x) = 4\cosh(\pi r) K_{2ir}(x).$$

For any $s \in \mathbb{R}$, one may write

$$J_{k-1}(s) = s^{-\frac{1}{2}}(F_k^+(s)e(x) + F_k^-(s)e(-s)) \tag{2.2}$$

for some smooth functions $F^{\pm}$ satisfying that

$$s^j F_k^{\pm(j)}(s) \ll_{k,j} \frac{s}{(1+s)^{\frac{3}{2}}}$$

for any $j \geq 0$; the resource might be referred to [29, Section 6.5] if $s < 1$ and [29, Section 3.4] if $s \geq 1$. One thus sees that, for $\mathscr{H}^{\flat}$, the n-variable is essentially truncated at $n \ll q^2 N_2/X^{1-\varepsilon}$, by repeated integration by parts. Furthermore, notice that, by Appendix of [3],

$$K_{2ir}(s) \ll_{r,\varepsilon} \begin{cases} s^{-\frac{1}{2}}\exp(-s), & s > 1+\pi|r|, \\ (1+|r|)^{\varepsilon}, & 0 < s \leq 1+\pi|r|; \end{cases} \tag{2.3}$$

one will find the n-variable enjoys the analogous truncation range for $\mathscr{H}^{\natural}$ with that for $\mathscr{H}^{\flat}$.

2.2. The Wilton-type bound

Now, let us recall the following Wilton-type bound; the resource, however, may be referred to [17], together with [10, 27].

Lemma 2.2. *Let $X \geq 2$ and W be a smooth function, compactly supported on $[1/2, 5/2]$ such that $x^j W^{(j)}(x) \ll 1$ for any $j \in \mathbb{N}_+$. For any newform $f \in \mathcal{B}_k^*(N)$ (or $\mathcal{B}_\lambda^*(N)$) and $\alpha \in \mathbb{R}$, we thus have*

$$\sum_{n\geq 1} \lambda_f(n) e(n\alpha) W\left(\frac{n}{X}\right) \ll \sqrt{X} N^{\frac{1}{3}+\varepsilon}, \tag{2.4}$$

where the implied $\ll$-constant depends merely on k (or λ) and ε.

2.3. The delta method

As a variant of the circle method, the δ-symbol method plays a focal role in number theory. We will now briefly recall a version of the circle method; see for instance [16, Chapter 20].

Lemma 2.3. *Fix $X, Q \geq 1$. For any $n \leq X$, one has*

$$\delta(n) = \frac{1}{Q}\sum_{q\leq Q}\frac{1}{q}\sum_{\substack{a \bmod q\\(a,q)=1}} e\left(\frac{an}{q}\right)\int_{\mathbb{R}} g(q,\tau)\, e\left(\frac{n\tau}{qQ}\right)\mathrm{d}\tau,$$

where

$$g(q,\tau) = 1 + h(q,\tau) \quad \text{with} \quad h(q,\tau) = O\left(\frac{1}{qQ}\left(|\tau| + \frac{q}{Q}\right)\right)^A,$$

$$\tau^j \frac{\partial^j}{\partial \tau^j} g(q,\tau) \ll \log Q \min\left(\frac{Q}{q}, \frac{1}{|\tau|}\right),$$

and $g(q,\tau) \ll |\tau|^{-A}$ for any sufficiently large A. In particular, the effective range of the τ-integral is $[-X^{\varepsilon}, X^{\varepsilon}]$.

2.4. Some estimates involving Kloosterman sums

We will have a need of the following lemmas which will be applied in §3.

Lemma 2.4. *Let $Q \geq 2$. Let $F(x, y)$ be a smooth bump function supported on $[1/2, 5/2] \times [1/2, 5/2]$, which satisfies that*

$$X^i Y^j \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} F\Big(\frac{x}{X}, \frac{y}{Y}\Big) \ll_{i,j} 1$$

for any integers $i, j \in \mathbb{N}_+$ and any $X, Y \geq 1$. Then, for any $c \in \mathbb{Z}$, sequence $\mathbf{a} = \{a_n \in \mathbb{C}\}$ and newform $f \in \mathcal{B}_k^*(1)$ (or $\mathcal{B}_\lambda^*(1)$), there holds that

$$\sum_{q\geq 1} \sum_{n \geq 1} a_n\, S(n, c; q)\, F\Big(\frac{n}{X}, \frac{q}{Q}\Big) \ll X^{\varepsilon} \left(\sqrt{X} Q + \mathbf{1}_{Q^2 > X}\, Q^2 \right) \|\mathbf{a}\|_2, \tag{2.5}$$

where the symbol $\mathbf{1}_{\mathcal{P}}$ equals 1 if the assertion $\mathcal{P}$ is true, and 0 otherwise.

Proof. First, via the Cauchy-Schwarz inequality, we might evaluate the double sum as

$$\leq \left(\sum_{q_1, q_2 \geq 1} \sum_{n \geq 1} S(n, c; q_1) \overline{S(n, c; q_1)} F\Big(\frac{n}{X}, \frac{q_1}{Q}\Big) F\Big(\frac{n}{X}, \frac{q_2}{Q}\Big) \right)^{\frac{1}{2}} \|\mathbf{a}\|_2.$$

It thus follows from the Weil bound that the non-generic terms $q_1 = q_2$ shall contribute a upper-bound $\ll X^{1+\varepsilon} Q^2$ to the parentheses above, which gives the term $\sqrt{X} Q \|\mathbf{a}\|_2$ on the RHS of (2.5); while, for the generic terms $q_1 \neq q_2$, if one writes $q_1 = \widehat{q_1}\delta$, $q_2 = \widehat{q_2}\delta$ with $\delta = (q_1, q_2)$ satisfying that $(\delta, \widehat{q_1}) = 1$, Poisson summation formula with the modulus $\widehat{q_1}\widehat{q_2}\delta$ thus might produce the following bound for the triple sum that

$$\ll \sum_{\delta \leq Q} \sum_{\widehat{q_1}, \widehat{q_2} \leq Q/\delta} \sup_{0 < |l| \ll \widehat{q_1}\widehat{q_2}\delta / X^{1-\varepsilon}} \left| \sum_{\alpha \bmod \widehat{q_1}\widehat{q_2}\delta} S(\alpha, c; \widehat{q_1}\delta) \overline{S(\alpha, c; \widehat{q_2}\delta)}\, e\left(\frac{\alpha l}{\widehat{q_1}\widehat{q_2}\delta} \right) \right|.$$

Notice, here, the inner-most sum vanishes, if $l = 0$, and it is necessary that $Q^2 > X$ as well. At this point, on applying Chinese remainder theorem, the sum over α turns out to be

$$\widehat{q_1}\widehat{q_2}\delta\, e\left(-\frac{c\overline{\delta l} \cdot \widehat{q_2}}{\widehat{q_1}} \right) \sideset{}{^*}\sum_{s \bmod \delta} e\left(\frac{\overline{a\widehat{q_1} s}}{\delta} - \frac{\widehat{q_1} c \cdot \overline{(\widehat{q_2} s + l)}}{\widehat{q_2}\delta} \right)$$

with $\bar{\delta}\delta \equiv 1 \bmod \widehat{q_1}$, $\bar{l}l \equiv 1 \bmod \widehat{q_1}$, $\overline{\widehat{q_1}}\widehat{q_1} \equiv 1 \bmod \delta$ and $\overline{\widehat{q_2}s + l}(\widehat{q_2}s + l) \equiv 1 \bmod \widehat{q_2}\delta$; trivially evaluating everything thus exactly leads to the term $Q^2 \|\mathbf{a}\|_2$ in (2.5). □

Lemma 2.5. *Let the parameters X, Q, c, the form f and the sequence $\mathbf{a} = \{a_n \in \mathbb{C}\}$ be as in Lemma 2.4. Let $W(x, y, z)$ be a smooth bump function supported on $[1/2, 5/2] \times [1/2, 5/2] \times [1/2, 5/2]$, with the partial derivatives satisfying*

$$X^i Y^j Z^k \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} \frac{\partial^k}{\partial z^k} W\Big(\frac{x}{X}, \frac{y}{Y}, \frac{z}{Z}\Big) \ll_{i,j,k} 1$$

for ever integers $i, j, k \in \mathbb{N}_+$ and any $X, Y, Z \geq 1$. There thus holds that

$$\sum_{q \geq 1} \sum_{n \geq 1} a_n \sum_{m \geq 1} \frac{\lambda_f(m)}{\sqrt{m}} S(m - np, c; q)\, W\Big(\frac{m}{X}, \frac{n}{H}, \frac{q}{Q}\Big) \ll X^{\varepsilon} \left(\sqrt{HQ} + Q^2 \right) \|\mathbf{a}\|_2. \tag{2.6}$$

Proof. To show the lemma, the initial procedure is to invoke the Cauchy-Schwarz inequality; we are thus led to evaluating

$$\sum_{q_1,q_2\geq 1}\sum_{n\geq 1}\sum_{m_1,m_2\geq 1}\frac{\lambda_f(m_1)\overline{\lambda_f(m_2)}}{\sqrt{m_1m_2}}S(m_1-np,c;q_1)\overline{S(m_2-np,c;q_2)}\,W\Big(\frac{m_1}{X},\frac{n}{H},\frac{q_1}{Q}\Big)W\Big(\frac{m_2}{X},\frac{n}{H},\frac{q_2}{Q}\Big). \quad (2.7)$$

(1) First, let us begin with considering the generic terms $q_1 = q_2 = q$, say. In this moment, Poisson is applicable, which yields an alternative form for (2.7):

$$H\sum_{l_1\in\mathbb{Z}}\sum_{q\sim Q}\frac{1}{q}\sum_{m_1,m_2\geq 1}\frac{\lambda_f(m_1)\overline{\lambda_f(m_2)}}{\sqrt{m_1m_2}}\mathcal{Y}_0(l_1,m_1,m_2,p,c;q)\,\mathscr{I}_0(l_1,m_1,m_2),$$

where

$$\mathcal{Y}_0(l,m_1,m_2,p,c;q)=\sum_{\alpha \bmod q}S(m_1-\alpha p,c;q)\overline{S(m_2-\alpha p,c;q)}\,e\left(-\frac{\alpha l}{q}\right),$$

and

$$\mathscr{I}_0(l,m_1,m_2)=\int_{\mathbb{R}}W\Big(\frac{m_1}{X},\xi,\frac{q}{Q}\Big)W\Big(\frac{m_2}{X},\xi,\frac{q}{Q}\Big)e\left(\frac{lH\xi}{q}\right)\mathrm{d}\xi.$$

Notice that the exponential sum modulo q asymptotically equals

$$q\,e\left(-\frac{m_1\overline{p}l}{q}\right)\sideset{}{^*}\sum_{\alpha \bmod q}e\left(\frac{(m_1-m_2)\alpha+n\cdot\overline{\alpha-\overline{p}l}-n\overline{\alpha}}{q}\right),$$

where we have employed the relation involving Ramanujan sum that

$$S(n,0;q)=\sum_{ab=q}\mu(a)\sum_{\beta \bmod q}e\left(\frac{\beta n}{b}\right). \quad (2.8)$$

Upon combining with Lemma 2.2, one thus sees that the zero-frequency shall contribute a bound by $\ll HQX^{\varepsilon}$ for any $\varepsilon > 0$. While, on the other hand, if $Q > H$, one may find the non-zero frequencies will be indispensable to contribute a magnitude to (2.7). It can be demonstrated that, in this situation, the contribution, however, is estimated as $\ll Q^2X^{\varepsilon}$; this gives totally a quantity by

$$\ll HQX^{\varepsilon}+\mathbf{1}_{Q>H}Q^2X^{\varepsilon}. \quad (2.9)$$

(2) Now, we are left with the non-generic case where $q_1 \neq q_2$. One writes $q_1 = q_1'h$, $q_2 = q_2'h$, with $(q_1, q_2) = h$ and $(q_1', q_2') = 1$. Notice that h is co-prime with one of factors q_1', q_2'; without loss of generality, one assumes that $(h, q_1') = 1$. The expression in (2.7) thus becomes

$$\sum_{h\ll Q}\sum_{q_1,q_2\leq Q/h}\sum_{n\geq 1}\sum_{m_1,m_2\geq 1}\frac{\lambda_f(m_1)\overline{\lambda_f(m_2)}}{\sqrt{m_1m_2}}S(m_1-np,c;q_1'h)$$
$$\times\overline{S(m_2-np,c;q_2'h)}\,W\Big(\frac{m_1}{X},\frac{n}{H},\frac{q_1'h}{Q}\Big)W\Big(\frac{m_2}{X},\frac{n}{H},\frac{q_2'h}{Q}\Big). \quad (2.10)$$

Upon exploiting the Poisson twice, we thus arrive at

$$H\sum_{l_2\in\mathbb{Z}}\sum_{h\ll Q}\sum_{q_1',q_2'\le Q/h}\frac{1}{q_1'q_2'h}\sum_{m_1,m_2\ge 1}\frac{\lambda_f(m_1)\overline{\lambda_f(m_2)}}{\sqrt{m_1m_2}}\mathcal{Y}^{\dagger}(l_2,m_1,m_2,p,c;q_1',q_2',h)\,\mathscr{I}^{\dagger}(l_2,m_1,m_2,q_1',q_2'h),$$

where the exponential sum $\mathcal{Y}^{\dagger}$ and the resulting integral $\mathscr{I}^{\dagger}$ are defined as

$$\mathcal{Y}^{\dagger}(l,m_1,m_2,p,c;q_1',q_2',h)=\sum_{\alpha \bmod q_1'q_2'h} S(m_1-\alpha p,c;q_1'h)\,\overline{S(m_2-\alpha p,c;q_2'h)}\,e\left(-\frac{\alpha l}{q_1'q_2'h}\right),$$

and

$$\mathscr{I}^{\dagger}(l,m_1,m_2,q_1',q_2',h)=\int_{\mathbb{R}}W\left(\frac{m_1}{X},\xi,\frac{q_1'h}{Q}\right)W\left(\frac{m_2}{X},\xi,\frac{q_2'h}{Q}\right)e\left(\frac{lH\xi}{q_1'q_2'h}\right)\mathrm{d}\xi.$$

Here, one finds that the zero-frequency $l_2=0$ does exist anymore. Indeed, upon recalling that $(h,q_1')=1$, one writes $\alpha=q_1'\overline{q_1'}x+q_2'h\overline{q_2'}y$, with $x \bmod q_2'h$ and $y \bmod q_1'$ such that $(x,q_2'h)$ and $(y,q_1')=1$; applying Chinese remainder theorem and (2.8), the sum over α thus essentially turns out to be

$$q_1'q_2'h\,e\left(\frac{m_1\overline{q_2'}pl+nq_2'p\overline{h}l}{q_1'}+\frac{m_2\overline{q_1'}pl}{q_2'h}\right)\sideset{}{^*}\sum_{\alpha \bmod h}e\left(\frac{(m_1-m_2)\overline{q_1'}q_2'\alpha-nq_1'\cdot\overline{q_2'\alpha-\overline{p}l}+nq_2'\overline{q_1'}\overline{\alpha}}{q_2'h}\right).$$

Via Lemma 2.2, it thus follows that the display (2.10) is dominated by $\ll_{\varepsilon} X^{\varepsilon}Q^4$, upon opening the Kloosterman sum above. This, together with (2.9), shows the desired estimates in the parentheses of (2.6). □

3. Proof of Theorem 1.1

In this part, let us focus on the proof of Theorem 1.1. We shall first manage to separate the variables n,m by applying Lemma 2.3; in this paper, we shall employ a vital trick, that is, the 'conductor lowering mechanism' (see [23, 24] or the survey [25]). One may see that actually there holds the following

$$\delta(n)=\frac{1}{pQ}\sum_{q\le Q}\frac{1}{q}\sum_{\substack{a \bmod qp\\(a,q)=1}}e\left(\frac{an}{qp}\right)\int_{\mathbb{R}}g(q,\tau)\,e\left(\frac{n\tau}{qQp}\right)\mathrm{d}\tau; \tag{3.1}$$

while, the parameter Q shall be taken as $Q=\sqrt{X/p}$. Now, for three smooth functions U,V,R, supported $[1/2,5/2]$ with bounded derivatives, we shall detect the shift $l=m+pn$ via (3.1), which yields an alternative form for the triple sum in (1.3) as follows

$$\begin{aligned}\mathcal{S}(X,p,H)=\frac{X\sqrt{H}}{pQ}\int_{\mathbb{R}}\sum_{q\le Q}\frac{g(q,\tau)}{q}\sum_{\substack{\gamma \bmod pq\\(\gamma,q)=1}}\sum_{l\ge1}\frac{a_l}{\sqrt{l}}e\left(-\frac{\gamma l}{q}\right)V_{\tau}\left(\frac{l}{H}\right)\sum_{m\ge1}\frac{\lambda_g(m)}{\sqrt{m}}\\ \times e\left(\frac{m\gamma}{pq}\right)U_{\tau}\left(\frac{m}{X}\right)\sum_{n\ge1}\frac{\lambda_f(n)}{\sqrt{n}}e\left(-\frac{n\gamma}{pq}\right)R_{\tau}\left(\frac{n}{X}\right)\mathrm{d}\tau,\end{aligned}\tag{3.2}$$

where

$$U_\tau(m) = U(m)e\left(\frac{mX\tau}{pqQ}\right), \quad R_\tau(n) = R(n)e\left(-\frac{nX\tau}{pqQ}\right), \quad V_\tau(u) = V(u)e\left(-\frac{uH\tau}{qQ}\right).$$

We shall proceed to distinguish whether $(\gamma, p) = 1$ or not in the analysis, so that we are led to three parts, i.e., the non-degenerate term $\mathcal{S}^{\text{Non-de.}}$, the degenerate term $\mathcal{S}^{\text{Deg.}}$ and the error term $\mathcal{S}^{\text{Err.}}$, which are respectively given by

$$\begin{aligned}\mathcal{S}^{\text{Non-de.}}(X, p, H) =& \frac{X\sqrt{H}}{pQ}\int_{\mathbb{R}} \sum_{\substack{q\le Q\\(q,p)=1}} \frac{g(q,\tau)}{q} \sideset{}{^*}\sum_{\gamma \bmod pq} \sum_{l\ge1} \frac{a_l}{\sqrt{l}} e\left(-\frac{l\gamma}{pq}\right) V_\tau\left(\frac{l}{H}\right)\\ &\times \sum_{m\ge1} \frac{\lambda_g(m)}{\sqrt{m}} e\left(\frac{m\gamma}{pq}\right) U_\tau\left(\frac{m}{X}\right) \sum_{n\ge1} \frac{\lambda_f(n)}{\sqrt{n}} e\left(-\frac{n\gamma}{q}\right) R_\tau\left(\frac{n}{X}\right) \mathrm{d}\tau,\end{aligned} \tag{3.3}$$

$$\begin{aligned}\mathcal{S}^{\text{Deg.}}(X, p, H) =& \frac{X\sqrt{H}}{pQ}\int_{\mathbb{R}} \sum_{\substack{q\le Q\\(q,p)=1}} \frac{g(q,\tau)}{q} \sideset{}{^*}\sum_{\gamma \bmod q} \sum_{l\ge1} \frac{a_l}{\sqrt{u}} e\left(-\frac{lp\gamma}{q}\right) V_\tau\left(\frac{l}{H}\right)\\ &\times \sum_{m\ge1} \frac{\lambda_\pi(1,m)}{\sqrt{m}} e\left(\frac{m\gamma}{q}\right) U_\tau\left(\frac{m}{X}\right) \sum_{n\ge1} \frac{\lambda_f(n)}{\sqrt{n}} e\left(-\frac{n\gamma}{q}\right) R_\tau\left(\frac{n}{X}\right) \mathrm{d}\tau,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\mathcal{S}^{\text{Err.}}(X, p, H) =& \frac{X\sqrt{H}}{pQ}\int_{\mathbb{R}} \sum_{\substack{q\le Q\\p\mid q}} \frac{g(q,\tau)}{q} \sideset{}{^*}\sum_{\gamma \bmod pq} \sum_{l\ge1} \frac{a_l}{\sqrt{l}} e\left(-\frac{l\gamma}{pq}\right) V_\tau\left(\frac{l}{H}\right)\\ &\times \sum_{m\ge1} \frac{\lambda_g(m)}{\sqrt{m}} e\left(\frac{m\gamma}{pq}\right) U_\tau\left(\frac{m}{X}\right) \sum_{n\ge1} \frac{\lambda_f(n)}{\sqrt{n}} e\left(-\frac{n\gamma}{q}\right) R_\tau\left(\frac{n}{X}\right) \mathrm{d}\tau.\end{aligned} \tag{3.5}$$

One might see that, here, it suffices to consider $\mathcal{S}^{\text{Non-de.}}$; the same argument works for $\mathcal{S}^{\text{Err.}}$ which serves as a noisy term and for which we save more. We shall now now begin with $\mathcal{S}^{\text{Non-de.}}$; the analysis of the term $\mathcal{S}^{\text{Deg.}}$ will be postponed to the end of this paper.

3.1. Treatment of $\mathcal{S}^{\text{Non-de.}}$

In this part, let us concentrate on the analysis of $\mathcal{S}^{\text{Non-de.}}$. For any $\iota, \nu, \upsilon, \rho \in \mathbb{R}$, write

$$\mathscr{W}_\tau(\iota, \nu, \upsilon, \rho) = U_\tau(\iota) R_\tau(\nu) V_\tau(\upsilon) \eta_Q(\rho Q),$$

where η_Q is a smooth bump function supported on $[1Q/2, 5Q/2]$, satisfying that $\eta_Q^{(j)}(x) \ll 1$ for any $j \in \mathbb{N}_+$. One might find that the quantity we are focusing on is the following

$$\frac{X\sqrt{H}}{pQ} \sup_{\tau \ll X^\varepsilon} \sup_{Q \leq \mathcal{Q}} \sum_{\substack{q \geq 1 \\ (q,p)=1}} \frac{g(q,\tau)}{q} \sideset{}{^*}\sum_{\gamma \bmod pq} \sum_{u \geq 1} \frac{a_l}{\sqrt{l}} e\left(-\frac{l\gamma}{q}\right) \sum_{m \geq 1} \frac{\lambda_g(m)}{\sqrt{m}} \times e\left(\frac{m\gamma}{pq}\right) \sum_{n \geq 1} \frac{\lambda_f(n)}{\sqrt{n}} e\left(-\frac{n\gamma}{pq}\right) \mathscr{W}_\tau\left(\frac{m}{X}, \frac{n}{X}, \frac{l}{H}, \frac{q}{Q}\right). \tag{3.6}$$

We intend to invoking the Voronoi formula, Lemma 2.1; we thus arrive at

$$\frac{X\sqrt{H}}{pQ} \sup_{\tau \ll X^\varepsilon} \sup_{Q \leq \mathcal{Q}} \sum_{\substack{q \geq 1 \\ (q,p)=1}} \frac{g(q,\tau)}{q} \sum_{l \geq 1} \frac{a_l}{\sqrt{l}} \sum_{m \geq 1} \frac{\lambda_g(m)}{\sqrt{m}} \sum_{n \geq 1} \frac{\lambda_f(n)}{\sqrt{n}} \times S(n-m, pl; pq) \left\{ \widehat{\mathscr{W}_\tau^\flat}\left(\frac{mX}{pQ^2}, \frac{nX}{pQ^2}, \frac{l}{H}, \frac{q}{Q}\right) + \widehat{\mathscr{W}_\tau^\natural}\left(\frac{mX}{pQ^2}, \frac{nX}{pQ^2}, \frac{l}{H}, \frac{q}{Q}\right) \right\}, \tag{3.7}$$

where, for any $\star, * \in \{\flat, \natural\}$, each integral $\widehat{\mathscr{W}_\tau^{\star,*}}$ is defined as

$$\widehat{\mathscr{W}_\tau^{\star,*}}(\iota, \nu, \upsilon, \rho) = \eta_Q(\rho Q) V_\tau(\upsilon) \mathscr{H}^\star\left(\frac{Q^2\iota}{q^2}; R_\tau\right) \mathscr{H}^*\left(\frac{Q^2\nu}{q^2}; U_\tau\right).$$

It is remarkable that, here, from (2.2) and (2.3), we have the identical crude estimate that $\widehat{\mathscr{W}_\tau^{\star,*}} \ll X^\varepsilon$ for any $\varepsilon > 0$; in this sense, one sees that it suffices to deal simply with $\widehat{\mathscr{W}_\tau^{\flat,\flat}}$, upon noticing that the argument of the other terms (i.e., $\widehat{\mathscr{W}_\tau^{\flat,\natural}}$, $\widehat{\mathscr{W}_\tau^{\natural,\flat}}$ and $\widehat{\mathscr{W}_\tau^{\natural,\natural}}$) can follow similarly with it. One, however, on the other hand, sees that the inner-most sum modulo pq can be converted into

$$pS(\overline{p}(n-m), l; q)$$

with $n \equiv m \bmod p$. Now, if one writes $n = m + pk$ with $k \ll X^\varepsilon$, we find that (3.7) is no more than

$$\ll \frac{X\sqrt{H}}{Q} \sup_{\tau \ll X^\varepsilon} \sup_{Q \leq \mathcal{Q}} \sum_{m \ll pX^\varepsilon} \sum_{k \ll X^\varepsilon} \frac{\lambda_f(m)\lambda_f(pk+m)}{\sqrt{m(pk+m)}} \times \sum_{\substack{q \geq 1 \\ (q,p)=1}} \frac{g(q,\tau)}{q} \sum_{l \geq 1} \frac{a_l}{\sqrt{l}} S(l,k;q) \widehat{\mathscr{W}_\tau^{\flat,\flat}}\left(\frac{mX}{pQ^2}, \frac{nX}{pQ^2}, \frac{l}{H}, \frac{q}{Q}\right).$$

At this point, an application of Lemma 2.3 shows that the RHS of the expression above is bounded by

$$\begin{aligned} &\ll \frac{X^{1+\varepsilon}}{Q} \sup_{\tau \ll X^\varepsilon} \sup_{Q \leq \mathcal{Q}} \left(\sqrt{H} + \mathbf{1}_{Q^2 > H} Q\right) \|\mathbf{a}\|_2 \\ &\ll X^\varepsilon \max\left(\sqrt{XHp}, X\right) \|\mathbf{a}\|_2, \end{aligned} \tag{3.8}$$

upon recalling the value of $\mathcal{Q}$.

3.2. Treatment of $\mathcal{S}^{\text{Deg.}}$

Now, let us have a look at the multiple-sum $\mathcal{S}^{\text{Deg.}}$. One might verify that $\mathcal{S}^{\text{Deg.}}$ is of the form

$$\frac{X\sqrt{H}}{pQ}\sup_{\tau\ll X^{\varepsilon}}\sup_{Q\leq Q}\sum_{\substack{q\geq 1\\(q,p)=1}}\frac{g(q,\tau)}{q}\sideset{}{^*}\sum_{\gamma \bmod q}\sum_{l\geq 1}\frac{a_l}{\sqrt{u}}e\left(-\frac{lp\gamma}{q}\right)\sum_{m\geq 1}\frac{\lambda_g(m)}{\sqrt{m}}$$
$$\times e\left(\frac{m\gamma}{q}\right)\sum_{n\geq 1}\frac{\lambda_f(n)}{\sqrt{n}}e\left(-\frac{n\gamma}{q}\right)\mathscr{W}_{\tau}\left(\frac{n}{X},\frac{m}{X},\frac{u}{H},\frac{q}{Q}\right) \tag{3.9}$$

with $\mathscr{W}_\tau$ being as before. We will now proceed by appealing to the Voronoi formula, Lemma 2.1, again to transform the sums over n into the dualized form, so that we infer that the expression above should be controlled by

$$\frac{X\sqrt{H}}{pQ}\sup_{\tau\ll X^{\varepsilon}}\sup_{Q\leq Q}\sup_{n\ll pQ^2X^{-1+\varepsilon}}\left|\sum_{\substack{q\geq 1\\(q,p)=1}}\frac{g(q,\tau)}{q}\sum_{l\geq 1}\frac{a_l}{\sqrt{l}}\sum_{m\geq 1}\frac{\lambda_g(m)}{\sqrt{m}}\right.$$
$$\left.\times S(m-lp,n;q)\left\{\widetilde{\mathscr{W}_{\tau}^{\flat}}\left(\frac{m}{X},\frac{nX}{pQ^2},\frac{l}{H},\frac{q}{Q}\right)+\widetilde{\mathscr{W}_{\tau}^{\natural}}\left(\frac{m}{X},\frac{nX}{pQ^2},\frac{l}{H},\frac{q}{Q}\right)\right\}\right|,$$

where, for $\star\in\{\flat,\natural\}$, each integral transform $\widetilde{\mathscr{W}_{\tau}^{\star}}$ is given by

$$\widetilde{\mathscr{W}_{\tau}^{\star}}(\iota,\nu,\upsilon,\rho)=\eta_Q(\rho Q)V_{\tau}(\upsilon)\,U_{\tau}(\iota)\,\mathscr{H}^{\star}\left(\frac{Q^2\nu}{q^2};R_{\tau}\right).$$

Via Lemma 2.5, one thus deduces

$$\mathcal{S}^{\text{Deg.}}(X,p,H)\ll\frac{X^{1+\varepsilon}}{pQ}\sup_{\tau\ll X^{\varepsilon}}\sup_{Q\leq Q}\left(\sqrt{H}+Q\right)\|\mathbf{a}\|_2$$
$$\ll X^{\varepsilon}\left(\sqrt{\frac{XH}{p}}+\frac{X}{p}\right)\|\mathbf{a}\|_2.$$

This leads to the estimates we would like to prove in Theorem 1.1, upon combining with (3.8).

4. Conclusions

In this paper, we investigate the triple correlations sums of Fourier coefficient of newforms on GL_2, with the levels aspects being explicitly determined; our method is flexible enough to deal with the Maaß new forms. It is also remarkable that more recently, the authors are able to establish a sharp bound in the scenario where one of the froms f,g in (1.3) is a Maaß cuspidal form on GL_3 (not necessarily self-dual) with the trivial level.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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