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Research article

Nonlocal fuzzy fractional stochastic evolution equations with fractional Brownian motion of order (1,2)

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Abstract: In this manuscript, we formulate the system of fuzzy stochastic fractional evolution equations (FSFEEs) driven by fractional Brownian motion. We find the results about the existence-uniqueness of the formulated system by using the Lipschitizian conditions. By using these conditions we have also investigated the exponential stability of the solution for the above system driven by fractional Brownian motion. Finally, the applications in financial mathematics are presented and the use of financial mathematics in the fractional Black and Scholes model is also discussed. An example is propounded to show the applicability of our results.

Keywords: fuzzy set theory; fuzzy stochastic processes; fuzzy stochastic differential equation; fractional Brownian motion
Mathematics Subject Classification: 34A07, 34A08, 60G22

1. Introduction

In modern-world systems where phenomena related to randomness and fuzziness as two types of uncertainty, such as economics and finance, the FSDEs are utilized. There are several articles on FSDEs, each of which takes a different approach. The fuzzy stochastic It$\hat{o}$ integral was defined by the author in [1]. The fuzzy *Itô* stochastic integral was driven by fuzzy non-anticipating stochastic processes and the Wiener process in [2–5]. To construct a fuzzy random variable, the method involves embedding a crisp It$\hat{o}$ stochastic integral into fuzzy space.

There appears to be some difficulty in the mode of a variety of modern-world systems, such as trying to characterize physical systems and differing viewpoints on their properties. The fuzzy set theory will be utilized to resolve this issue [6]. It can handle linguistic claims like "big" and "less" mathematically using this approach. A fuzzy set provides the ability to examine fuzzy differential equations (FDEs) in representing a variety of phenomena, including imprecision. For example, FSDEs could be used to explore a wide range of economic and technical problems involving two types of uncertainty: Randomness and fuzziness.

Fei et al. investigated the existence-uniqueness of solutions to FSDEs under Lipschitizian conditions in citation [7]. Jafari et al. investigate FSDEs generated by fractional Brownian motion in [8]. In [9], Jialn Zhu et al. demonstrate the existence of solutions to SDEs using fractional Brownian motion. Analytical solutions of multi-time scale FSDEs driven by fractional Brownian motion were investigated by Ding and Nieto [10]. Vas'kovskii et al. [11] show that p^{th} moments, $p \geq 1$, of strong mixed-type SDE solutions are driven by standard Brownian motion and an fractional Brownian motion with two types of uncertainty: Randomness and fuzziness.

Because some research has been done on the topic of existence-uniqueness of solutions to SDEs and FSDEs interrupted by Brownian motions or semimartingales ([4, 12–14]), a form of FFSDE driven by an fractional Brownian motion has not been studied. Abbas et al. [15, 16] worked on a partial differential equation. Agarwal et al. [17, 18] investigated and explore the idea of a solution for FDEs with uncertainty, as well as the results of several FFDEs and optimum control nonlocal evolution equations. In [19–21], Zhou et al. published various important publications on the stability study of such SFDEs. [22] investigates the existence-uniqueness results for FSDEs with local martingales under Lipschitzian conditions. Uluçay et al. [23, 24] and BAKBAK [25] worked on intuitionistic trapezoidal fuzzy multi-numbers. Arhrrabi et al. worked on the existence and stability of solution of FFSDEs with fractional Brownian motions. Niazi et al. [26], Iqbal et al. [27], Shafqat et al. [28], Abuasbeh et al. [29] and Alnahdi [30] existence-uniqueness of the FFEE were investigated. Arhrrabi et al. [31] worked on the existence and stability of solution to FFSDEs with fractional Brownian motion. By the motivation of the above paper, we worked on the existence and stability of solution of FFSEEs with fractional Brownian motions for order (1,2) by using nonlocal conditions,

$$\begin{aligned} &{}_{0}^{c}D_{\nu}^{\gamma}Y(\nu) = f(\nu, Y(\nu)) + \mathrm{g}(\nu, Y(\nu))d\mathcal{B}_{H}(\nu),\ \nu \in [0, T], \\ &Y(0) + m(Y) = Y_0, \\ &Y'(0) = Y_1. \end{aligned} \tag{1.1}$$

There has been a recent interest in input noises lacking independent increments and exhibiting long-range dependence and self-similarity qualities, which has been motivated by some applications in hydrology, telecommunications, queueing theory, and mathematical finance. When the covariances of a stationary time series converge to zero like a power function and diverge so slowly that their sums diverge, this is known as long-range dependence. The self-similarity property denotes distribution invariance when the scale is changed appropriately. Fractional Brownian motion is a generalization of classical Brownian motion, is one of the simplest stochastic processes that are Gaussian, self-similar, and exhibit stationary increments. When the Hurst parameter is more than 1/2, the fractional Brownian motion exhibits long-range dependency, as we will see later. In this note, we look at some of the features of fractional Brownian motion and discuss various strategies for constructing a stochastic calculus for this process. We'll also go through some turbulence and mathematical finance applications. The remaining of this paper is as follows. In Section 2, we discuss outlines that are the most important features. Existence is discussed in Section 3. The uniqueness of solutions to FFSDEs is demonstrated. In addition, Section 4 investigates the stability of solutions. Finally, in Section 5, a conclusion is given.

2. Preliminaries

The notations, definitions, and background material that will be used throughout the text are introduced in this section. The family of nonempty convex and compact subsets of $\mathbf{R^m}$ is called $\mathcal{M}(\mathbf{R^m})$. In $\mathcal{M}(\mathbf{R^m})$, the distance $D_{\mathcal{H}}$ is defined

$$D_{\mathcal{H}}(\mathcal{N},\mathcal{K}) = \max\left(\sup_{n\in\mathcal{N}}\inf_{k\in\mathcal{K}}\|n-k\|, \sup_{k\in\mathcal{K}}\inf_{n\in\mathcal{N}}\|n-k\|\right), \mathcal{N},\mathcal{K}\in\mathcal{M}(\mathbf{R^m}).$$

We represent $\mathcal{N}(\Omega,\mathcal{A},\mathcal{M}(\mathbf{R^m}))$ the family of $\mathcal{A}$-measurable multifunction, taking value in $\mathcal{M}(\mathbf{R^m})$.

Definition 2.1. [22, 32] A multifunction $\mathcal{G}\in\mathcal{N}(\Omega,\mathcal{A},\wp,\mathbf{R}^+)$ is called $\mathcal{L}^p$-integrably bounded if $\exists h\in\mathcal{L}^p(\Omega,\mathcal{A},\wp;\mathbf{R}^+)$ such that $|||\mathcal{G}|||\leq h\wp$-a.e, where

$$|||\mathcal{B}||| = D_{\mathcal{H}}(\mathcal{B},\hat{0}) = \sup_{b\in\mathcal{B}}\|b\|,\ for\ \mathcal{B}\in\mathcal{M}(\mathbf{R^m}).$$

We show

$$\mathcal{L}^p(\Omega,\mathcal{A},\wp,\mathcal{N}(\mathbf{R^m})) = \{\mathcal{G}\in\mathcal{N},(\Omega,\mathcal{A},\mathcal{M}(\mathbf{R^m})) : |||\mathcal{G}|||\in\mathcal{L}^p(\Omega,\mathcal{A},\wp;\mathbf{R}^+)\}.$$

Suppose E represent the set of the fuzzy $x:\mathbf{R^m}\to[0,1]$ such that $[x]^\alpha\in\mathcal{M}(\mathbf{R^m})$, for every $\alpha\in[0,1]$, where $[x]^\alpha=\{a\in R: x(a)\geq\alpha\}$, for $\alpha\in(0,1]$, and $[x]^0 = cl\{a\in\mathbf{R^m}: x(a)>0\}$. Suppose the metric be $D_\infty(x,y) = \underset{\alpha\in[0,1]}{sup}\ D_{\mathcal{H}}([x]^\alpha,[y]^\alpha)$ in $E, a\in\mathbf{R^m}$; we have $D_\infty(x+z,y+z) = d_\infty(x,y), D_\infty(x+y,z+\omega)\leq D_\infty(x,z)+D_\infty(y,\omega)$, and $D_\infty(ax,ay)=|a|D_\infty(x,y)$.

Definition 2.2. [33] Assume $f:[c,d]\to\mathbf{E^m}$ be fuzzy RL integral of f is given by

$$(\mathrm{g}^\alpha_{c^+}f)(u) = \frac{1}{\Gamma(\alpha)}\int_c^u (u-v)^{\alpha-1}f(v)dv.$$

Definition 2.3. [33] Assume $Df\in C([c,d],\mathbf{E^m})\cap\mathcal{L}([c,d],\mathbf{E^m})$. The fuzzy fractional Caputo differentiability of f is defined by

$$^cD^\alpha_{c^+}f(u) = \mathrm{g}^{1-\alpha}_{c^+}(Df)(u) = \frac{1}{1-\alpha}\int_c u(u-v)^{-\alpha}(Df)(v)dv. \tag{2.1}$$

The Henry-Gronwall inequality [34] is defined, and it can be applied to prove our result.

Lemma 2.1. *Assume $f,\mathrm{g}:[0,T]\to\mathbf{R}^+$ be continuous functions. If g is nondecreasing and there exists constants $\mathcal{M}\geq 0$ and $\alpha>0$ as*

$$f(u)\leq \mathrm{g}(u)+\mathcal{M}\int_0^u(u-v)^{\alpha-1}f(v)dv,\ u\in[0,T],$$

then

$$f(u)\leq \mathrm{g}(u)+\int_0^u\Big[\sum_{m-1}^{\infty}\frac{\mathcal{M}\Gamma(\alpha)^m}{\Gamma(m\alpha)}(u-v)^{n\alpha-1}\mathrm{g}(v)\Big]dv,\ u\in[0,T].$$

The following inequality is modified into $\mathrm{g}(u)=b$ if $\mathrm{g}(u)=b$ that is constant on [0,T],

$$f(u)\leq bE_\alpha(\mathcal{M}\Gamma(\alpha)u^\alpha),\ u\in[0,T],$$

where E_α is given by

$$E_\alpha(z) = \sum_{m=0}^{\infty}\frac{z^m}{\Gamma(m\alpha+1)}.$$

Remark 2.2. *[34] For all $u \in [0,T], \exists N^*_{\mathcal{K}} > 0$ does not depend on b that is $f(u) \leq \mathcal{K}^*_k b$.*

Definition 2.4. [32, 33] A function $f : \Omega \to \mathbf{E^m}$ is said fuzzy random variable if $[f]^\alpha$ is an $\mathcal{A}$-measurable random variable $\forall\alpha \in [0,1]$. A fuzzy random variable $f : \Omega \to \mathbf{E^m}$ is said $\mathcal{L}^p$-integrably bounded, $p \geq 1$, if $[f]^\alpha \in \mathcal{L}^p(\Omega, \mathcal{A}, \wp; \mathcal{K}(\mathbf{R^m})), \forall\alpha \in [0,1]$.

The set of all fuzzy random variables is $\mathcal{L}^p(\Omega, \mathcal{A}, \wp, \mathbf{E^m})$, and they are $\mathcal{L}^p$-integrally bounded.

We used [35] to explain the concept of an fractional Brownian motion.

Suppose us define a sequence of partitions of [a,b] by $\{\varphi_n, n \in \mathcal{K}\}$ such that $|\varphi_n| \to 0$ as $m \to \infty$. If in $L^2(\Omega, \mathcal{A}, \wp)$, $\sum_{i=0}^{n-1} \phi(v_i^{(n)})(\mathcal{B}^{\mathcal{H}}(v_{v+1}^{(n)} - \mathcal{B}^{\mathcal{H}}(v_i^{(n)}))$ converge to the same limit for all this sequences $\{\varphi_n, n \in \mathcal{K}\}$, then this limit is said a Stratonovich-type stochastic integral and noted by $\int_a^b \phi(s) D\mathcal{B}^{\mathcal{H}}(s)$. Suppose $\jmath = [0,T]$, where $0 < T < \infty$.

Definition 2.5. [32, 33] A function $f : \ell \times \Omega \to \mathbf{E^m}$ is called fuzzy stochastic process; if $\forall v \in \ell, f(v,.) = f(v) : \Omega \to \mathbf{E^m}$ is a fuzzy random variable.

A fuzzy stochastic process f is continuous; if $f(.,v); \ell \to \mathbf{E^m}$ are continuous, and it is $\{A_v^{\mathcal{H}}\}_{v\in\ell}$-adapted if for every $\alpha \in [0,1]$ and for all $v \in \ell, [f(v)]^\alpha : \Omega \to \mathcal{K}(\mathbf{R^m})$ is $A_v^{\mathcal{H}}$-measurable.

Definition 2.6. [32,33] The function f is called measurable if $[f] : \ell \times \Omega \to \mathcal{M}(\mathbf{R^m})$ is $\mathcal{B}(\ell) \otimes A$-measurable, for all $\alpha \in [0,1]$.

The function $f : \ell \times \Omega \to \mathbf{E^m}$ is said to be non-anticipating if it is $\{\mathcal{A}_v^{\mathcal{H}}\}_{v\in\ell}$-adapted and measurable.

Remark 2.3. *The process x is non anticipating if and only if x is measurable with respect to*

$$\mathcal{K} = \{\mathcal{A} \in \mathcal{B}(\ell) \otimes \mathcal{A} : \mathcal{A}^u \in \mathcal{A}_u^{\mathcal{H}}, u \in \ell\},$$

where, for $u \in \ell, \mathcal{A}^u = \{v : (u,v) \in \mathcal{A}\}$.

Definition 2.7. [32,33] A fuzzy process $f : \ell \times \Omega \to \mathbf{E^m}$ is said $\mathcal{L}^p$-integrally bounded if

$$\exists h \in \mathcal{L}^p(\ell \times \Omega, \mathcal{K}; \mathbf{R^m}) / D_\infty(f(s,v), \hat{0}) \leq h(s,v).$$

The set of all $\mathcal{L}^p$-integrally bounded and non-anticipating fuzzy stochastic processes is denoted by $\mathcal{L}^p(\ell \times \Omega, \mathcal{K}; \mathbf{E^m})$.

Proposition 2.4. *[5] For $f \in \mathcal{L}^p(\ell \times \Omega, \mathcal{N}; \mathbf{E^m})$ and $p \geq 1$, we have*

$$\ell \times \Omega \in (v,v) \to \int_0^v f(s,v)ds \in \mathcal{L}^p(\ell \times \Omega, \mathcal{N}; \mathbf{E^m})$$

and D_∞-continuous.

Proposition 2.5. *[5] For $f, \mathrm{g} \in \mathcal{L}^p(\ell \times \Omega, \mathcal{K}; \mathbf{E^m})$ and $p \geq 1$, we have*

$$E \sup_{a\in[0,1]} D_\infty^p\Big(\int_0^a f(u)du, \int_0^a \mathrm{g}(u)du\Big) \leq v^{p-1} \int_0^v ED_\infty^p(f(u), \mathrm{g}(u))du.$$

Proposition 2.6. *[33] Suppose $\varphi : \ell \to \mathbf{R^m}$; then, for $v \in \ell$,*

$$\sup_{a\in[0,v]} E\|\int_0^a \varphi(s) d\mathcal{B}^H(s)\|^2 \leq \|\varphi(s)\|^2 ds.$$

Let us define the embedding of $\mathbf{R^m}$ to $\mathbf{E^m}$ as $\langle . \rangle : \mathbf{R^m} \to \mathbf{E^m}$:

Proposition 2.7. *[5] Suppose the function* $\varphi : \ell \to \mathbf{R^m}$ *satisfies* $\int_0^T \|\varphi(v)\|^2 dv < \infty$. *Then,*

(i) *The fuzzy stochastic Itô integral* $\langle \int_0^v \varphi(u) d\mathcal{B}^{\mathcal{H}}(u) \rangle \in \mathcal{L}^2(\ell \times \Omega, \mathcal{K}; \mathbf{E^m})$.
(ii) *For* $x \in \mathcal{L}^2(\ell \times \Omega, \mathcal{K}; E)$, *we have, for* $u \leq v \in \ell$,

$$
\begin{aligned}
& D_\infty\Big(\int_0^v x(\omega_1) d\omega_1 + \int_0^v \varphi(\Omega_2) d\mathcal{B}^{\mathcal{H}}(\Omega_2), \int_0^u x(\omega_1) d\omega_1 + \int_0^u \varphi(\omega_2) d\mathcal{B}^{\mathcal{H}}(\omega_2) \Big) \\
& = D_\infty\Big(\int_u^v x(\omega_1) d\omega_1 + \int_u^v \varphi(\omega_2) d\mathcal{B}^{\mathcal{H}}(\omega_2), \hat{0} \Big).
\end{aligned}
$$

3. Main results

We now look into the FFSDEs that are generated by a fractional Brownian motion given by

$$
\begin{aligned}
& {}_0^c D_v^\gamma Y(v) = f(v, Y(v)) + \langle \mathrm{g}(v, Y(v)) d\mathcal{B}_{\mathcal{H}}(v) \rangle, \ v \in [0, T], \\
& Y(0) + m(Y) = Y_0, \\
& Y'(0) = Y_1,
\end{aligned}
$$

where

$$
f, \mathrm{g} : \ell \times \Omega \times \mathbf{E^m} \to \mathbf{E^m}, \ \ Y_0, Y_1 : \Omega \to \mathbf{E^m},
$$

and $\{\mathcal{B}_{\mathcal{H}}(s)\}_{s \in \ell}$ is FBM defined on $(\Omega, \mathcal{A}, \{\mathcal{A}_s^{\mathcal{H}}\}_{s \in \ell}, \wp)$ with Hirst index $\mathcal{H} \in (\frac{3}{2}, 2)$.

Definition 3.1. If the following conditions hold, a process $Y : \ell \times \Omega \to \mathbf{E^m}$ is considered to be a solution to Eq (1.1):

(i) $Y \in \mathcal{L}^2(\ell \times \Omega, \mathcal{N}; \mathbf{E^m})$.
(ii) Y is d_∞-continuous.
(iii) We have

$$
Y(v) = C_q(v)(Y_0 - m(Y)) + \mathcal{K}_q(v) Y_1 + \frac{1}{\Gamma(\alpha)} \int_0^v \frac{f(s, Y(s))}{(v - s)^{1-\alpha}} ds + \Big\langle \frac{1}{\Gamma(\alpha)} \int_0^v \frac{\mathrm{g}(s, Y(s))}{(v - s)^{1-\alpha}} d\mathcal{B}_{\mathcal{H}}(s) \Big\rangle. \quad (3.1)
$$

We will assume that all through this paper, $f : (\ell \times \Omega) \times \mathbf{E^m} \to \mathbf{E^m}$ is $\mathcal{B}_{d_s} \otimes \mathcal{K} | \mathcal{B}_{d_\infty}$-measurable. Let's start with some assumptions.

($\mathbf{J_1}$) If Y_0 is $\mathcal{A}_0$-measurable,

$$
E d_\infty^2(Y_0, \check{0}) < \infty. \quad (3.2)
$$

($\mathbf{J_2}$) For $f(s, \check{0})$ and $\mathrm{g}(s, \check{0})$,

$$
\max\{d_\infty^2(f(s, \check{0}), \mathrm{g}(s, \check{0}))\} \leq c. \quad (3.3)
$$

($\mathbf{J_3}$) For all $Z, \omega \in \mathbf{E^m}$,

$$
d_\infty^2(f(s, Z), f(s, \omega)) \leq c d_\infty^2(Z, \omega), \quad (3.4)
$$

and

$$
d_\infty^2(\mathrm{g}(s, Z), \mathrm{g}(s, \omega)) \leq c d_\infty^2(Z, \omega), \quad (3.5)
$$

in ($\mathbf{J_2}$), c is equal to one. Let's start with the main theorem in this section.

Theorem 3.1. *On the basis of assumptions* ($\mathbf{J_1}$), ($\mathbf{J_2}$) *and* $Y_0 \in L^2(\Omega, \mathcal{A}_0, \wp, \mathbf{E^m})$*, the main Eq (1.1) has unique solution* $Y(\nu)$*.*

Proof. To show that there is a solution to the problem, the method of successive approximations will be employed (1.1). As a result, define the following sequence $Y_n : \ell \times \Omega \to \mathbf{E^m}$:

$$Y_0(\nu) = Y_0$$

and for $n = 1, ...,$

$$Y_n(\nu) = C_q(\nu)(Y_0 - m(Y)) + \mathcal{K}_q(\nu)Y_1 + \frac{1}{\Gamma(\alpha)}\int_0^\nu \frac{f(s, Y_{n-1}(s))}{(\nu - s)^{1-\alpha}}ds + \left\langle \int_0^\nu \frac{g(s, Y_{n-1}(s))}{(\nu - s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\right\rangle.$$

It is obvious, $Y_n(s)$ are in $\mathcal{L}^2(\ell \times \Omega, \mathcal{K}; \mathbf{E^m})$ and d_∞-continuous. Certainly, we obtain $Y_0 \in \mathcal{L}^2(\ell \times \Omega, \mathcal{K}; \mathbf{E^m})$ and Y_0 is d_∞-continuous.

Assume that $k \in \mathcal{K}$ and $\nu \in \ell$, $\mathcal{M}_n = \sup_{0\le u\le \nu} Ed^2_\infty(Y_n(u), Y_{n-1}(u))$. From Propositions 2.5 and 2.6 and ($\mathbf{J_1}$)–($\mathbf{J_3}$), we get that

$$\begin{aligned}
\mathcal{M}_1(\nu) &= \sup_{0\le u\le \nu} Ed^2_\infty\Big(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s, Y_0)}{(u - s)^{1-\alpha}}ds + \langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{g(s, Y_0)}{(u - s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\rangle, \check{0}\Big) \\
&\le 2\sup_{0\le u\le \nu}\Big[Ed^2_\infty\Big(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s, Y_0)}{(u - s)^{1-\alpha}}ds, \check{0}\Big) + 2\sup_{0\le u\le \nu}\Big[Ed^2_\infty\Big(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{g(s, Y_0)}{(u - s)^{1-\alpha}}ds, \check{0}\Big)\Big] \\
&\le 2\sup_{0\le u\le \nu}\Big[2Ed^2_\infty\Big(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s, Y_0)}{(u - s)^{1-\alpha}}ds, \frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s, \check{0})}{(u - s)^{1-\alpha}}ds\Big)\Big] \\
&\quad +4\sup_{0\le u\le \nu} Ed^2_\infty\Big(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s, \check{0})}{(u - s)^{1-\alpha}}, \check{0}\Big) + 2\sup_{0\le u\le \nu}\Big[2Ed^2_\infty\Big(\Big\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{g(s, Y_0)}{(u - s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\Big\rangle, \\
&\quad \Big\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{g(s, \check{0})}{(u - s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\Big\rangle\Big)\Big] + 4\sup_{0\le u\le \nu} Ed^2_\infty\Big(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{g(s, \check{0})}{(u - s)^{1-\alpha}}, \check{0}\Big) \\
&\le \frac{4T}{\Gamma(\alpha)}\int_0^\nu \frac{Ed^2_\alpha(f(s, Y_0), f(s, \check{0}))}{(\nu - s)^{1-\alpha}}ds + \frac{4T}{\Gamma(\alpha)}\int_0^\nu \frac{Ed^2_\infty f(s, \check{0})}{(\nu - s)^{1-\alpha}}ds \\
&\quad \Big\langle\frac{4T}{\Gamma(\alpha)}\int_0^\nu \frac{Ed^2_\alpha(g(s, Y_0), g(s, \check{0}))}{(\nu - s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\Big\rangle + \Big\langle\frac{4T}{\Gamma(\alpha)}\int_0^\nu \frac{Ed^2_\infty g(s, \check{0})}{(\nu - s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\Big\rangle \\
&\le \frac{4Tc\nu^\alpha}{\Gamma(\alpha + 1)}d^2_\infty(Y_0, \check{0}) + \frac{4T\nu^\alpha c}{\Gamma(\alpha + 1)} + \frac{4Tc\nu^\alpha}{\Gamma(\alpha + 1)}d^2_\infty(Y_0, \check{0}) + \frac{4T\nu^\alpha c}{\Gamma(\alpha + 1)} \\
&:= \frac{l_1\nu^\alpha}{\Gamma(\alpha + 1)},
\end{aligned}$$

where $l_1 = 2(4cTd^2_\infty(Y_0, \check{0}) + 4Tc)$. Moreover, similarly, we have

$$\begin{aligned}
\mathcal{M}_{n+1}(\nu) &= \sup_{0\le u\le \nu} Ed^2_\infty\Big(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s, Y_n(s))}{(u - s)^{1-\alpha}}ds + \Big\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{g(s, Y_n(s))}{(u - s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\Big\rangle, \\
&\quad \frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s, Y_{n-1}(s))}{(u - s)^{1-\alpha}}ds + \Big\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{g(s, Y_n(s))}{(u - s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\Big\rangle\Big) \\
&\le 2\sup_{0\le u\le \nu} Ed^2_\infty\frac{1}{\Gamma(\alpha)}\Big(\int_0^u \frac{f(s, Y_n(s))}{(u - s)^{1-\alpha}}ds, \int_0^u \frac{f(s, Y_{n-1}(s))}{(u - s)^{1-\alpha}}ds\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\Big\langle 2\sup_{0\le u\le v} Ed_\infty^2 \frac{1}{\Gamma(\alpha)}\Big(\int_0^u \frac{\mathrm{g}(s,Y_n(s))}{(u-s)^{1-\alpha}} d\mathcal{B}_\mathcal{H}(s)\Big\rangle, \Big\langle \int_0^u \frac{\mathrm{g}(s,Y_{n-1}(s))}{(u-s)^{1-\alpha}} d\mathcal{B}_\mathcal{H}(s)\Big\rangle\Big)\\
\le\ & \frac{2v}{\Gamma(\alpha)}\int_0^v (v-s)^{\alpha-1} Ed_\infty^2(f(s,Y_n(s)), f(s,Y_{n-1}(s)))ds\\
&+\Big\langle \frac{2v}{\Gamma(\alpha)}\int_0^v (v-s)^{\alpha-1} Ed_\infty^2(\mathrm{g}(s,Y_n(s)), \mathrm{g}(s,Y_{n-1}(s)))d\mathcal{B}_\mathcal{H}(s)\Big\rangle\\
\le\ & \frac{4vc}{\Gamma(\alpha)}\int_0^v (v-s)^{\alpha-1} \sup_{u\in[0,s]} Ed_\infty^2((Y_n(s)), Y_{n-1}(s))ds\\
\le\ & \frac{4v}{\Gamma(\alpha)}\int_0^v (v-s)^{\alpha-1}\mathcal{K}_n(s)ds. \qquad (3.6)
\end{aligned}
$$

Then, we have

$$\mathcal{K}_n(v) \le \frac{l_1}{l_2}\frac{(l_2 v^\alpha)^n}{n!\Gamma(\alpha+1)},\ \forall\, v\in\ell, k\in\mathcal{K},$$

where $l_2 = 4Tc$.

As a result of Chebyshev's inequality and (3.6),

$$\wp\Big(\sup_{u\in\ell} d_\infty^2(Y_n(u), Y_{n-1}(u))\Big) > \frac{1}{4^n} \le \frac{l_1}{l_2}\frac{(4l_2T^\alpha)^n}{n!\Gamma(\alpha+1)}.$$

According to Borel-Cantelli lemma, series $\sum_{n\ge1}(4l_2T^\alpha)^n/n!$ converges,

$$\wp\Big(\sup_{u\in\ell} d_\infty(Y_n(u), Y_{n-1}(u)) > \frac{1}{2^n}\Big) = 0.$$

As a consequence, the sequence $\{X_n(.,v)\}$ is uniformly convergent to $\bar{Y}(.,v) : \ell \to \mathbf{R^m}$ for $v \in \Omega_c$, where $\Omega_c \in \mathcal{A}$ and $\wp(\Omega_c) = 1$. Therefore,

$$\lim_{n\to\infty}\sup_{v\in\ell} Ed_\infty^2(Y_n(v), \bar{Y}(v)) = 0. \qquad (3.7)$$

Let us define $Y : \ell\times\Omega \to \mathbf{E^m}$ as follows:

$$Y(.,v) = \begin{cases} \tilde{X}(.,v),\ if\ v\in\Omega_c,\\ freely\ chosen,\ if\ v\in\frac{\Omega}{\Omega_c}.\end{cases} \qquad (3.8)$$

We can observe that, for each $1\le\alpha\le2$ and $v\in\ell$, we have

$$\lim_{u\to\infty} d_H([Y_n,(.,v)]^\alpha, [Y_{n-1}(.,v)]^\alpha) = 0.$$

Then, $[Y(v,.)]^\alpha : \Omega \to \mathcal{K}(\mathbf{R^m})$ is $\mathcal{A}_1$-measurable. Therefore, Y is non-anticipating. By (3.7),

$$\lim_{n\to\infty}\sup_{v\in\ell} Ed_\infty^2(Y_n(v), Y(v)) = 0, \qquad (3.9)$$

which demonstrates that $\exists\lambda > 0$ is independent of $n\in\mathcal{N}$, i.e.

$$\sup_{v\in\ell} Ed_\infty^2(Y_n(v), Y(v)) \le \lambda. \qquad (3.10)$$

We have $Y_n(\nu) \in \mathcal{L}^2(\Omega, \mathcal{A}, \wp; \mathbf{E^m})$ because $Y_n \in \mathcal{L}^2(\ell \times \Omega, \mathcal{K}; \mathbf{E^m})$. Furthermore, we show that $Y \in \mathcal{L}^2(\ell \times \Omega, \mathcal{K}; \mathbf{E^m})$.

Let us represent $k \in \mathcal{K}$ and $\nu \in \ell$,

$$\varphi_n(\nu) = \sup_{0\le u\le \nu} Ed^2_\infty(Y_0, \check{0}). \tag{3.11}$$

Then by applying (3.11) on (1.1), we obtain

$$\begin{aligned}
\varphi_n(\nu) &\le 3Ed^2_\infty(Y_0, \check{0}) + 3 \sup_{0\le u\le \nu}\Big(\frac{1}{\Gamma(\alpha)}\int_0^u (u-s)^{\alpha-1} f(s, Y_{n-1}(s))ds, \check{0}\Big),\\
\varphi_n(\nu) &\le 3Ed^2_\infty(Y_0, \check{0}) + 3 \sup_{0\le u\le \nu}\Big(\Big\langle\frac{1}{\Gamma(\alpha)}\int_0^u (u-s)^{\alpha-1} \mathrm{g}(s, Y_{n-1}(s))d\mathcal{B}_{\mathcal{H}}(s)\Big\rangle, \check{0}\Big).
\end{aligned}$$

We may deduce $(\mathbf{J_1})$–$(\mathbf{J_3})$ from triangle inequality and Propositions 2.5 and 2.6,

$$\begin{aligned}
\varphi_n(\nu) &\le 3Ed^2_\infty(Y_0, \check{0}) + \frac{6\nu}{\Gamma(\alpha)}\int_0^\nu (\nu-s)^{\alpha-1}\{Ed^2_\infty(f(s, Y_{n-1}(s)), f(s,\check{0})) + Ed^2_\infty(f(s,\check{0}),\check{0})\}ds\\
&\quad + \frac{6\nu}{\Gamma(\alpha)}\int_0^\nu (\nu-s)^{\alpha-1}\{Ed^2_\infty(\mathrm{g}(s, Y_{n-1}(s)), \mathrm{g}(s,\check{0})) + Ed^2_\infty(\mathrm{g}(s,\check{0}),\check{0})\}d\mathcal{B}_{\mathcal{H}}(s)\\
&\le 3Ed^2_\infty(Y_0, \check{0}) + \frac{6c\nu}{\Gamma(\alpha)}\int_0^\nu (\nu-s)^{\alpha-1} Ed^2_\infty(Y_{n-1}(s), \check{0})ds + \frac{6c\nu^{\alpha+1}}{\Gamma(\alpha+1)}\\
&\quad + \frac{6c\nu}{\Gamma(\alpha)}\int_0^\nu (\nu-s)^{\alpha-1} Ed^2_\infty(Y_{n-1}(s), \check{0})ds + \frac{6c\nu^{\alpha+1}}{\Gamma(\alpha+1)}.
\end{aligned}$$

We obtain

$$\varphi_n(\nu) \le \mathcal{A}_1 + \mathcal{A}_2\int_0^\nu (\nu-s)^{\alpha-1}\varphi_{n-1}(s)ds,$$

$\mathcal{A}_1 = 3Ed^2_\infty(Y_0, \check{0}) + (12c\nu^{\alpha+1}/\Gamma(\alpha+1)))$ and $\mathcal{A}_2 = 12c\nu/\Gamma(\alpha)$. There exists constant $\mathcal{M}_{\mathcal{A}_2} > 0$ that is independent of $\mathcal{A}_1$, thus according to Lemma 2.1 and Remark 2.2,

$$\varphi_n(\nu) \le \mathcal{M}_{\mathcal{A}_2}\mathcal{A}_1. \tag{3.12}$$

Due to $(\mathbf{J_1})$, (3.10) and (3.12), we get

$$\sup_{0\le s\le \nu} Ed^2_\infty(Y(s), \check{0}) \le 2\sup_{0\le s\le \nu} Ed^2_\infty(Y(s), Y_n(s)) + 2\sup_{0\le s\le \nu} Ed^2_\infty(Y_n(s), \check{0}) \le 2\lambda + 2\mathcal{M}_{\mathcal{A}_1}\mathcal{A}_1 < \infty,$$

which implies

$$\int_0^T Ed^0_\infty(Y(s), \check{0})ds \le T\sup_{\nu\in\ell} Ed^2_\infty(Y(\nu), \check{0}) < \infty.$$

As a consequence, $Y \in \mathcal{L}^2(\ell \times \Omega, \mathcal{K}; \mathbf{E^m})$.

Consequently,

$$\begin{aligned}
\sup_{\nu\in\ell} Ed^2_\infty\Big(&Y(\nu), C_q(\nu)(Y_0 - m(Y)) + \mathcal{K}_q(\nu)Y_1 + \frac{1}{\Gamma(\alpha)}\int_0^\nu \frac{f(s, Y(s))}{(\nu-s)^{1-\alpha}}ds\\
&+ \Big\langle\frac{1}{\Gamma(\alpha)}\int_0^\nu \frac{\mathrm{g}(s, Y(s))}{(\nu-s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\Big\rangle\Big) = 0.
\end{aligned} \tag{3.13}$$

Indeed, we observe

$$\begin{aligned}
&\sup_{\nu\in\ell} Ed_\infty^2\Big(Y(\nu), C_q(\nu)(Y_0 - m(Y)) \\
&+ \mathcal{K}_q(\nu)Y_1 + \frac{1}{\Gamma(\alpha)}\int_0^\nu \frac{f(s,Y(s))}{(\nu-s)^{1-\alpha}}ds + \Big\langle \frac{1}{\Gamma(\alpha)}\int_0^\nu \frac{\mathrm{g}(s,Y(s))}{(\nu-s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\Big\rangle\Big) \\
\le&\Big[\sup_{\nu\in\ell} Ed_\infty^2(Y(\nu), Y_n(\nu)) + \sup_{\nu\in\ell} Ed_\infty^2\Big(X_n(\nu), C_q(\nu)(Y_0 - m(Y)) + \mathcal{K}_q(\nu)Y_1 \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^\nu \frac{f(s,X_{n-1}(s))}{(\nu-s)^{1-\alpha}}ds \\
&+ \Big\langle \frac{1}{\Gamma(\alpha)}\int_0^\nu \frac{\mathrm{g}(s,Y_{n-1}(s))}{(\nu-s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\Big\rangle\Big) + \sup_{\nu\in\ell} Ed_\infty^2\Big(Y_0 + \frac{1}{\Gamma(\alpha)}\int_0^\nu \frac{f(s,Y_{n-1}(s))}{(\nu-s)^{1-\alpha}}ds \\
&+ \Big\langle \frac{1}{\Gamma(\alpha)}\int_0^\nu \frac{\mathrm{g}(s,Y_{n-1}(s))}{(\nu-s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\Big\rangle\Big).C_q(\nu)(Y_0 - m(Y)) + \mathcal{K}_q(\nu)Y_1 \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^\nu \frac{f(s,Y(s))}{(\nu-s)^{1-\alpha}}ds + \Big\langle \frac{1}{\Gamma(\alpha)}\int_0^\nu \frac{\mathrm{g}(s,Y(s))}{(\nu-s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\Big\rangle\Big] \\
&:= I_1 + I_2 + I_3,
\end{aligned} \tag{3.14}$$

where $\lim_{n\to\infty} I_1 = 0$ and $I_2 = 0$. For I_3, by using Propositions 2.5 and 2.6, ($\mathbf{J_3}$), and (3.9), we have

$$\lim_{n\to\infty} I_3 \le \lim_{n\to\infty}\Big(\frac{T^{\alpha+1}c}{\Gamma(\alpha+1)}\sup_{\nu\in\ell} Ed_\infty^2(Y(u), Y_{n-1}(u)du)\Big) = 0.$$

As a result, we get (3.13), implying that (3.1) is true. As a result of Definition 3.1, $Y(\nu)$ is solution to equation.

For uniqueness of solution Y, assume $Y, Z : \ell\times\Omega \to \mathbf{E^m}$ are solutions to Eq (1.1). We represent $\mathcal{M}(\nu) := \sup_{\nu\in\ell} Ed_\infty^2(Y(\nu), Z(\nu))$. As a result, for each $\nu\in\ell$, we get

$$\mathcal{K}(\nu) \le \frac{\nu c}{\Gamma(\alpha)}\int_0^\nu \frac{Ed_\infty^2(Y(s), Z(s))}{(\nu-s)^{1-\alpha}}ds \le \int_0^\nu \frac{K(s)}{(\nu-s)^{1-\alpha}}ds.$$

As a result, by Lemma 2.1, $\nu\in\ell, \mathcal{M}(\nu) = 0$, which implies

$$\sup_{\nu\in\ell} d_\infty(Y(\nu), Z(\nu)) = 0.$$

□

4. Stability result

Using the Henry-Gronwall inequality, we investigate the solution's stability with respect to initial values in this section. Let's use Y and Z to represent the solutions to the below FFSDEs:

$$\begin{aligned}
&{}_0^cD_\nu^\gamma Y(\nu) = f(\nu, Y(\nu)) + \langle \mathrm{g}(\nu, Y(\nu))d\mathcal{B}_H(\nu)\rangle,\ \nu\in[0,T], \\
&Y(0) + m(Y) = Y_0, \\
&Y'(0) = Y_1,
\end{aligned} \tag{4.1}$$

$$\begin{aligned}&{}_0^cD_\nu^\gamma Z(\nu) = f(\nu, Z(\nu)) + \langle \mathrm{g}(\nu, Z(\nu))d\mathcal{B}_H(\nu)\rangle,\ \nu \in [0,T],\\ &Z(0) + m(Z) = Z_0,\\ &Z'(0) = Z_1,\end{aligned} \tag{4.2}$$

respectively.

Proposition 4.1. *Assume* $Y_0, Z_0 \in \mathcal{L}^2(\Omega, \mathcal{A}_0, \wp, \mathbf{E^m})$ *and* $f : \ell \times \Omega \times \mathbf{E^m} \to \mathbf{E^m}, g : \ell \to \mathbf{R^m}$ *satisfy* $(\mathbf{J_1})$–$(\mathbf{J_3})$. *Therefore,*

$$\sup_{0\le u\le \nu} Ed_\infty^2(Y(u), Z(u)) \le \lambda_0 N_\lambda, \tag{4.3}$$

where $\lambda_0 = 2Ed_\infty^2(Y_0, Z_0)$ *and* $\lambda_1 = 2Tc/\Gamma(\alpha)$. *Particularly,* $Y(\nu) = Z(\nu)$ *if* $Y_0 = Z_0$.

Proof. Assume that the solutions to Eqs (4.1) and (4.2) are $Y, Z : \ell \times \Omega \to \mathbf{E^m}$. Let

$$\mathcal{M}(\nu) := E \mathop{sup}_{0\le u\le \nu} d_\infty^2(Y(u), Z(u))$$

be the condition. We get $(\mathbf{J_3})$ as a result of Propositions 2.5 and 2.6.

$$\begin{aligned}\mathcal{M}(\nu) &\le 2Ed_\infty^2(Y_0, Z_0) + \frac{2}{\Gamma(\alpha)}\sup_{u\in[0,\nu]} Ed_\infty^2\Big(\int_0^u \frac{f(s, Y(s))}{(\nu-s)^{1-\alpha}}ds, \int_0^u \frac{f(s, Z(s))}{(\nu-s)^{1-\alpha}}ds\Big)\\ &\le 2Ed_\infty^2(Y_0, Z_0) + \frac{2Tc}{\Gamma(\alpha)}\int_0^\nu \frac{Ed_\infty^2(Y(s), Z(s))}{(\nu-s)^{1-\alpha}}ds\\ &\le 2Ed_\infty^2(Y_0, Z_0) + \frac{2Tc}{\Gamma(\alpha)}\int_0^\nu \sup_{u\in(0,s)} Ed_\infty^2(Y(u), Z(u))(\nu-s)^{\alpha-1}ds\\ &\le 2Ed_\infty^2(Y_0, Z_0) + \frac{2Tc}{\Gamma(\alpha)}\int_0^\nu \frac{M(s)}{(\nu-s)^{1-\alpha}}ds\\ &:= \lambda_0 + \lambda_1\int_0^\nu \frac{\mathcal{M}(s)}{(\nu-s)^{1-\alpha}}ds.\end{aligned}$$

However, according to Lemma 2.1 and Remark 2.2, a constant $\mathcal{K}_{\lambda_1} > 0$ exists that is independent of λ_0,

$$\mathcal{M}(\nu) \le \lambda_0\mathcal{K}_{\lambda_1}, \forall \nu \in \ell.$$

Then, $\lambda_0 = 0$ if $X_0 = Z_0$. As a result, we have $Y(\nu) = Z(\nu)$.

Furthermore, we consider the solution exponential stability to FFSDEs that disturbed an fractional Brownian motion in terms of f and g. Assume Y and Y_n signify solutions to FFSDEs as below:

$$\begin{aligned}&{}_0^cD_\nu^\gamma Y(\nu) = f(\nu, Y(\nu)) + \langle \mathrm{g}(\nu, Y(\nu))d\mathcal{B}_{\mathcal{H}}(\nu)\rangle,\ \nu \in [0,T],\\ &Y(0) + m(Y) = Y_0,\\ &Y'(0) = Y_1,\end{aligned} \tag{4.4}$$

$$\begin{aligned}&{}_0^cD_\nu^\gamma Y_n(\nu) = f(\nu, Y_n(\nu)) + \langle \mathrm{g}(\nu, Y_n(\nu))d\mathcal{B}_{\mathcal{H}}(\nu)\rangle,\ \nu \in [0,T],\\ &Y_n(0) + m(Y_n) = Y_0,\\ &Y_n'(0) = Y_1,\end{aligned} \tag{4.5}$$

respectively. □

Proposition 4.2. *Suppose that* $Y_0 \in \mathcal{L}^2(\Omega, \mathcal{A}_0, \wp, \mathbf{E^m})$ *and* $f, f_n : \ell \times \mathbf{E^m} \to \mathbf{E^m}$ $\mathfrak{g}, \mathfrak{g}_n \to \mathbf{R^m}(k \in \mathcal{K})$ *fulfill* $(\mathbf{J_1})$–$(\mathbf{J_3})$*. Moreover, consider the following:*

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{\Gamma(\alpha)}\int_0^{\nu}(\nu-s)^{\alpha-1}Ed_\infty^2((\nu,Y),f_n(\nu,Y)(\nu,Y))d\nu\right) &= 0,\\ \lim_{n\to\infty}\left(\frac{1}{\Gamma(\alpha)}\int_0^{\nu}(\nu-s)^{\alpha-1}Ed_\infty^2((\nu,Y),\mathfrak{g}_n(\nu,Y)(\nu,Y))d\mathcal{B}_{\mathcal{H}}(\nu)\right) &= 0.\end{aligned}$$

Proof. The solutions Y and Y_n are unique and exist, according to Theorem 3.1. We may derive from Propositions 2.6 and 2.7 that for any $\nu \in \ell$,

$$\begin{aligned}\sup_{0\le u\le\nu} Ed_\infty^2(Y(u),Y_n(u)) \le{}& 2\sup_{0\le u\le\nu} Ed_\infty^2\Big(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s,Y(s))}{(u-s)^{1-\alpha}}ds, \Gamma(\alpha)\int_0^u \frac{f_n(s,Y_n(s))}{(u-s)^{1-\alpha}}ds\Big)\\ &+2\sup_{0\le u\le\nu} Ed_\infty^2\Big(\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{\mathfrak{g}(s,Y(s))}{(u-s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\rangle, \langle\Gamma(\alpha)\int_0^u \frac{\mathfrak{g}_n(s,Y_n(s))}{(u-s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\rangle\Big)\\ &\le q4\sup_{0\le u\le\nu} Ed_\infty^2\Big(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f_n(s,Y(s))}{(u-s)^{1-\alpha}}ds, \frac{1}{\Gamma(\alpha)}\int_0^u \frac{f_n(s,Y_n(s))}{(u-s)^{1-\alpha}}ds\Big)\\ &+4\sup_{0\le u\le\nu} Ed_\infty^2\Big(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s,Y(s))}{(u-s)^{1-\alpha}}ds, \frac{1}{\Gamma(\alpha)}\int_0^u \frac{f_n(s,Y(s))}{(u-s)^{1-\alpha}}ds\Big)\\ &q4\sup_{0\le u\le\nu} Ed_\infty^2\Big(\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{\mathfrak{g}_n(s,Y(s))}{(u-s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\rangle, \langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{\mathfrak{g}_n(s,Y_n(s))}{(u-s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\rangle\Big)\\ &+4\sup_{0\le u\le\nu} Ed_\infty^2\Big(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{\mathfrak{g}(s,Y(s))}{(u-s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s), \frac{1}{\Gamma(\alpha)}\int_0^u \frac{\mathfrak{g}_n(s,Y(s))}{(u-s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\Big)\\ &\le \frac{4ct}{\Gamma(\alpha)}\int_0^{\nu}\frac{Ed_\infty^2(Y(s),Y_n(s))}{(\nu-s)^{1-\alpha}}ds + \frac{4\nu}{\Gamma(\alpha)}\int_0^{\nu}(\nu-s)^{\alpha-1}Ed_\infty^2(f(s,Y(s)),\end{aligned}$$

$$\begin{aligned}f_n(s,Y(s)))ds + \frac{4\nu}{\Gamma(\alpha)}\int_0^{\nu}(\nu-s)^{\alpha-1}Ed_\infty^2(\mathfrak{g}(s,Y(s)),\mathfrak{g}_n(s,Y(s)))d\mathcal{B}_{\mathcal{H}}(s)\\ \le \beta_1^n + \beta_2\int_0^{\nu}\frac{\underset{0\le u\le s}{sup}\ Ed_\infty^2(Y(s),Y_n(s))}{(\nu-s)^{1-\alpha}}ds,\end{aligned}$$

where

$$\beta_1^n := \frac{4T}{\Gamma(\alpha)}\int_0^{\nu}\frac{Ed_\infty^2(f(s,Y(s)),f_n(s,Y(s)))}{(\nu-s)^{1-\alpha}}ds + \Big\langle\frac{4T}{\Gamma(\alpha)}\int_0^{\nu}\frac{Ed_\infty^2(\mathfrak{g}(s,Y(s)),\mathfrak{g}_n(s,Y(s)))}{(\nu-s)^{1-\alpha}}d\mathcal{B}_{\mathcal{H}}(s)\Big\rangle,$$

$\beta_2 = 4cT/\Gamma(\alpha)$. As a result of Lemma 2.1 and Remark 2.3, $\exists\mathcal{N}_{\beta_2} > 0$ is independent of β_1^n that is

$$\sup_{u\in[0,\nu]} Ed_\infty^2(Y(u),Y_n(u)) \le \beta_1^n\mathcal{N}_{\beta_2}.$$

As a result of (4.4) and (4.5), we have $\lim_{n\to\infty}\beta_1^n = 0$. □

5. Application to financial mathematics

Fractional Brownian motion has been used to describe the behavior of asset prices and stock market volatility. This process is a good fit for describing these values because of its long-range dependence on self-similarity qualities. For a general discussion of the applications of fractional Brownian motion to model financial quantities, see Shiryaev [35]. Several writers have proposed a fractional Black and Scholes model to replace the traditional Black and Scholes model, which is memoryless and depends on the so-called fractional Black and Scholes model of geometric Brownian motion. The risky asset's market stock price is given by this model.

$$\mathbf{S}_\nu = \mathbf{S}_0 \exp\Big(\mu\nu + \sigma\mathcal{B}_\nu^{\mathcal{H}} - \frac{\sigma^2}{2}\nu^{2\mathcal{H}}\Big),$$

where $\mathcal{B}^{\mathcal{H}}$ is an FBM with the Hurst parameter $\mathcal{H}, \mu$ is the mean rate of return, and $\sigma > 0$ is the volatility, and at time ν, the price of non-risky assets is $e^{r\nu}$, where r is the interest rate.

6. Example

Assume the following FSDEs:

$${}_0^cD_\nu^\gamma Y(\nu) = Y(\nu) + \nu^2 + \nu + 4, \tag{6.1}$$

where $\nu \in [1, 2], f(\nu, Y(\nu)) = Y(\nu) + \nu^2 + \nu, \sigma(\nu, Y(\nu)) = 4, \frac{3}{2} < \beta < 2$.

It is easy to verify that f, σ satisfy the $\mathbf{J_1}$–$\mathbf{J_3}$. Define $\overline{f}(Y, Z)$ as follows:

$$\int_1^2 \overline{f}(Y, Z)(\nu^2 + \nu)d\nu = \int_1^2 \overline{f}(\nu, Y, Z)(\nu^2 + \nu)d\nu.$$

We can prove that $\overline{f}(Y, Z) = \frac{\nu^3}{3} + \frac{\nu^2}{2}$. Similarly, $\overline{\sigma}(Y, Z) = 1$. The averaging form of (6.1) can be written as

$${}_0^cD_\nu^\gamma Z(\nu) = Z(\nu)(\frac{\nu^3}{3} + \frac{\nu^2}{2}) + 4d\mathcal{B}_\nu^{\mathcal{H}}.$$

As ϵ approaches zero, the solutions $Y(\nu)$ and $Z(\nu)$ are equal in the sense of mean square, according to Theorem 3.1. As a result, the findings may be verified.

7. Conclusions

We show that under the Lipschitzian coefficient, solutions to FFSDEs exist and are unique. The stability of the solution to FFSDEs, on the other hand, is examined. The application of financial mathematics and the use of financial mathematics in the fractional Black and Scholes model is described. At the end of the manuscript, the example is also illustrated. In addition, future work may include expanding the concept introduced in this mission, adding observability, and generalizing other tasks. This is a fertile field with several research projects that can lead to a wide array of applications and theories. We plan to devote significant resources to this path.

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Conflict of interest

The authors declare no conflicts of interest.

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