AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(10): 19325–19343.
DOI: 10.3934/math.20221061
Received: 17 July 2022
Revised: 20 August 2022
Accepted: 23 August 2022
Published: 31 August 2022

Research article

On the solution of nonlinear fractional-order shock wave equation via analytical method

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Abstract: In this study, we propose a method to study fractional-order shock wave equations and wave equations arising from the motion of gases. The fractional derivative is taken in Caputo manner. The approaches we used are the combined form of the Yang transform (YT) together with the homotopy perturbation method (HPM) called homotopy perturbation Yang transform method (HPYTM) and also Yang transform (YT) with the Adomian decomposition method called Yang transform decomposition method (YTDM). The HPYTM is a combination of the Yang transform, the homotopy perturbation method and He's polynomials, whereas the YTDM is a combination of the Yang transform, the decomposition method and the Adomian polynomials. Adomian and He's polynomials are excellent tools for handling nonlinear terms. The manipulation of the recurrence relation, which generates the series solutions in a limited number of iterations, is the essential innovation we describe in this study. We give several graphical behaviors of the exact and analytical results, absolute error graphs, and tables that highly agree with one another to demonstrate the reliability of the suggested methodologies. The results we obtained by implementing the proposed approaches indicate that it is easy to implement and computationally very attractive.

Keywords: homotopy perturbation method; Adomian decomposition method; Yang transform; fractional-order nonlinear shock wave equation; Caputo operator
Mathematics Subject Classification: 34A34, 35A20, 35A22, 44A10, 33B15

1. Introduction

Fractional differential equations have gotten attention and prominence due to their numerous applications in engineering and science. These equations, for instance, are increasingly being utilized to explain phenomena in many different physical processes, including fluid mechanics, biology, electromagnetic, signal processing, acoustics and many more. In these and other applications, the non-locality of fractional differential equations is their main benefit. The integer order differential operator is typically considered a local operator, whereas the fractional order differential operator is non-local. This shows that the next state of a system depends on both its present state and all of its previous states. This is more practical, which is one of the reasons fractional calculus is growing in popularity [1–9].

Finding solutions to fractional differential equations has therefore received a lot of interest. A fractional differential equation is typically challenging to solve exactly. Researchers were interested in numerical approaches, the perturbation method among them. However, there are significant drawbacks to perturbation methods. For instance, the approximate solution necessitates a number of small parameters, which is problematic because most nonlinear problems do not have any small parameters. Even if an adequate selection of minor parameters occasionally results in the optimal solution, inappropriate selections frequently have negative consequences for the solutions. J. H. He, a Chinese researcher, invented the homotopy perturbation method (HPM) and presented it in 1998 [10, 11].

The Adomian decomposition method [12, 13] is a powerful approach for solving linear and nonlinear, homogeneous and nonhomogeneous partial and differential equations, and integro-differential equations of integer and non-integer order, which gives us convergent series from exact solutions. Several mathematicians used the homotopy perturbation method for handling nonlinear equations arising in engineering and science [14–17]. Many authors have focused their attention recently on examining the solutions to both linear and nonlinear partial differential equations using a variety of techniques such as Elzaki transform decomposition method [18, 19], homotopy perturbation transform technique [20, 21], homotopy analysis transform method [22, 23], q-homotopy analysis transform method [24], iterative Laplace transform method [25], variational iteration method [26, 27] and many others among them.

The mathematical theory of shock waves describes the first-order quasi-linear hyperbolic system of partial differential equations (PDEs) surface of discontinuity's characteristics. In the latter half of the 19th century, concerns with the compressible fluids and motion of gases gave rise to the mathematical theory of shock waves. The research of Earnshaw [28], Riemann [29], Rankine [30] and Hugoniot [31] laid the groundwork for it. One imagines a medium devoid of dissipation, viscosity and thermal conductivity as an idealization of actual gases and fluids. There may be discontinuities in the distribution of all flow parameters during motion in such idealized media (the temperature, the pressure, the velocity, the density, etc.). The collection of flow parameter discontinuity spots may be highly intricate when this set has piecewise-smooth surfaces of discontinuity made up of discontinuity points of the first kind; only the most elementary basic cases have been carefully taken into account. In the general scenario, discontinuous two-dimensional surfaces move in the three-dimensional space $\mathbb{R}^3$ as time passes. Conservation laws are among the characteristics shared by all continuum physics theories. By connecting the values of the primary vector field $\mathbb{J}$ to the flux f, constitutive relations that characterize the specific medium in question are added to the laws. By connecting the principal vector field $\mathbb{J}$ values to the flux f, constitutive relations that characterize the specific medium in question are added to the laws.

Since we are assuming that these relations are represented by smooth forms in this scenario, the conservation laws lead to nonlinear hyperbolic PDEs, which are expressed in their most basic form as

$$\mathbb{J}_{\vartheta}(\kappa,\vartheta) + f(\mathbb{J}(\kappa,\vartheta))_{\kappa} = 0, \quad \kappa \in \mathbb{R},\ \vartheta > 0, \tag{1.1}$$

subjected to the initial condition

$$\mathbb{J}(\kappa,0) = \mathbb{J}_0(\kappa), \quad \kappa \in \mathbb{R}.$$

A model for a variety of physical phenomena, including shock waves and three-phase flow in porous media, uses (1.1). Traffic flow, explosions, aeroplanes breaking the sound barrier, glacier waves, and other phenomena all use shock waves. Nonlinear hyperbolic PDEs are used to formulate them [32–34]. The Adomian decomposition method (ADM) [35–37] and the homotopy perturbation method (HPM) [38–40] have both been used to study the shock wave and wave equations. Since solutions are simple to find, the classical shock wave and wave equations provide excellent predictive equations. We introduce a fractional-order generalization of shock wave (1.1) in this article, which has the following form:

$$D^{\rho}_{\vartheta}\mathbb{J}(\kappa,\vartheta) + f(\mathbb{J}(\kappa,\vartheta))_{\kappa} = 0, \quad \kappa \in \mathbb{R},\ \vartheta > 0, \tag{1.2}$$

where a parameter named ρ designates the order of the time-derivative. In the Caputo sense, the derivative is regarded. A parameter specifying the order of the fractional-order shock wave equation is included in the general response expression and can be changed to generate different responses. The fractional-order shock wave equation is reduced to the standard shock wave equation when $\rho = 1$.

This study solves the fractional-order shock wave and wave equations analytically and numerically using the new homotopy perturbation transform technique and the Yang transform decomposition method. This method's ability to combine two effective approaches for generating precise and approximative analytical solutions for nonlinear equations is an advantage. It is important to note that the suggested methodology can perform better overall since it can reduce the amount of computational work required compared to the other techniques while still keeping the high accuracy of the numerical result.

The rest of the paper is organized as follows: In Section 2, we introduce some fundamental preliminaries. In Section 3, we presented the general methodology of HPYTM to obtain the solution of general arbitrary order PDE, while in Section 4, we presented the general method of YDTM to get the solution of general arbitrary order PDE. In Section 5, the suggested techniques are implemented for solving nonlinear fractional-order shock wave equations, and in Section 6, results and discussion are given. Similarly, in the end, a brief conclusion is given in Section 7.

2. Preliminaries

In this section, we discuss a number of important concepts, terminology and ideas relevant to the fractional derivative operators used in the current study.

Definition 2.1. *The fractional Caputo derivative is stated as*

$$D^{\rho}_{\mathbb{J}}\mathbb{J}(\kappa,\vartheta) = \frac{1}{\Gamma(k-\rho)}\int_0^{\vartheta}(\vartheta-\rho)^{k-\rho-1}\mathbb{J}^{(k)}(\kappa,\rho)d\rho, \quad k-1 < \rho \le k, \quad k \in \mathbb{N}. \tag{2.1}$$

Definition 2.2. *The Yang transform (YT) of a function* $\mathbb{J}(\vartheta)$ *is stated as*

$$Y(\mathbb{J}(\vartheta)) = M(u) = \int_0^{\infty} e^{\frac{-\vartheta}{u}} \mathbb{J}(\vartheta) d\vartheta, \quad \vartheta > 0,\ u \in (-\vartheta_1, \vartheta_2), \tag{2.2}$$

having Yang inverse transform as

$$Y^{-1}\{M(u)\} = \mathbb{J}(\vartheta). \tag{2.3}$$

Definition 2.3. *The Yang transform of nth derivatives is stated as*

$$Y(\mathbb{J}^n(\vartheta)) = \frac{M(u)}{u^n} - \sum_{k=0}^{n-1} \frac{\mathbb{J}^k(0)}{u^{n-k-1}}, \tag{2.4}$$

for all $n = 1, 2, 3, \ldots$.

Definition 2.4. *The Yang transform of fractional-order derivative is stated as*

$$Y(\mathbb{J}^{\rho}(\vartheta)) = \frac{M(u)}{u^{\rho}} - \sum_{k=0}^{n-1} \frac{\mathbb{J}^k(0)}{u^{\rho-(k+1)}}, \quad 0 < \rho \leq n. \tag{2.5}$$

3. General methodology of HPYTM

This section is concerned with fundamental solution process of HPYTM to get the solution of general arbitrary order partial differential equation

$$D_{\vartheta}^{\rho} \mathbb{J}(\kappa, \vartheta) = \mathcal{P}_1[\kappa]\mathbb{J}(\kappa, \vartheta) + \mathcal{Q}_1[\kappa]\mathbb{J}(\kappa, \vartheta), \quad 0 < \rho \leq 2, \tag{3.1}$$

subjected to the initial conditions

$$\mathbb{J}(\kappa, 0) = \xi(\kappa), \quad \frac{\partial}{\partial \vartheta}\mathbb{J}(\kappa, 0) = \zeta(\kappa),$$

where $D_{\vartheta}^{\rho} = \frac{\partial^{\rho}}{\partial \vartheta^{\rho}}$ is the fractional Caputo derivative, $\mathcal{P}_1[\kappa]$ and $\mathcal{Q}_1[\kappa]$ are linear and nonlinear operators.

By employing YT, we have

$$Y\left[D_{\vartheta}^{\rho} \mathbb{J}(\kappa, \vartheta)\right] = Y[\mathcal{P}_1[\kappa]\mathbb{J}(\kappa, \vartheta) + \mathcal{Q}_1[\kappa]\mathbb{J}(\kappa, \vartheta)], \tag{3.2}$$

$$\frac{1}{u^{\rho}}\left\{M(u) - u\mathbb{J}(0) - u^2\mathbb{J}'(0)\right\} = Y[\mathcal{P}_1[\kappa]\mathbb{J}(\kappa, \vartheta) + \mathcal{Q}_1[\kappa]\mathbb{J}(\kappa, \vartheta)]. \tag{3.3}$$

On simplification, we get

$$M(\mathbb{J}) = u\mathbb{J}(0) + u^2\mathbb{J}'(0) + u^{\rho} Y[\mathcal{P}_1[\kappa]\mathbb{J}(\kappa, \vartheta) + \mathcal{Q}_1[\kappa]\mathbb{J}(\kappa, \vartheta)]. \tag{3.4}$$

Operating inverse YT both sides, we have

$$\mathbb{J}(\kappa, \vartheta) = \mathbb{J}(0) + \mathbb{J}'(0) + Y^{-1}[u^{\rho} Y[\mathcal{P}_1[\kappa]\mathbb{J}(\kappa, \vartheta) + \mathcal{Q}_1[\kappa]\mathbb{J}(\kappa, \vartheta)]]. \tag{3.5}$$

Now, by using the HPM

$$\mathbb{J}(\kappa, \vartheta) = \sum_{k=0}^{\infty} \epsilon^k \mathbb{J}_k(\kappa, \vartheta), \tag{3.6}$$

with perturbation parameter $\epsilon \in [0,1]$.

The nonlinear terms are decomposed as

$$\mathcal{Q}_1[\kappa]\mathbb{J}(\kappa,\vartheta) = \sum_{k=0}^{\infty} \epsilon^k H_n(\mathbb{J}), \tag{3.7}$$

and He's polynomials $H_k(\mathbb{J})$ are given as

$$H_n(\mathbb{J}_0, \mathbb{J}_1, \ldots, \mathbb{J}_n) = \frac{1}{\Gamma(n+1)} D_\epsilon^k \left[\mathcal{Q}_1 \left(\sum_{k=0}^{\infty} \epsilon^i \mathbb{J}_i \right) \right]_{\epsilon=0}, \tag{3.8}$$

where $D_\epsilon^k = \frac{\partial^k}{\partial \epsilon^k}$.

Using (3.7) and (3.8) in (3.5), we have

$$\sum_{k=0}^{\infty} \epsilon^k \mathbb{J}_k(\kappa,\vartheta) = \mathbb{J}(0) + \mathbb{J}^{'}(0) + \epsilon \left(Y^{-1} \left\{ u^\rho Y \left(\mathcal{P}_1 \sum_{k=0}^{\infty} \epsilon^k \mathbb{J}_k(\kappa,\vartheta) + \sum_{k=0}^{\infty} \epsilon^k H_k(\mathbb{J}) \right) \right\} \right). \tag{3.9}$$

Thus, by equating the coefficient of ϵ, we get

$$\begin{aligned}
\epsilon^0 &: \mathbb{J}_0(\kappa,\vartheta) = \mathbb{J}(0) + \mathbb{J}^{'}(0), \\
\epsilon^1 &: \mathbb{J}_1(\kappa,\vartheta) = Y^{-1}\left[u^\rho Y(\mathcal{P}_1[\kappa]\mathbb{J}_0(\kappa,\vartheta) + H_0(\mathbb{J}))\right], \\
\epsilon^2 &: \mathbb{J}_2(\kappa,\vartheta) = Y^{-1}\left[u^\rho Y(\mathcal{P}_1[\kappa]\mathbb{J}_1(\kappa,\vartheta) + H_1(\mathbb{J}))\right], \\
&\vdots \\
\epsilon^k &: \mathbb{J}_k(\kappa,\vartheta) = Y^{-1}\left[u^\rho Y(\mathcal{P}_1[\kappa]\mathbb{J}_{k-1}(\kappa,\vartheta) + H_{k-1}(\mathbb{J}))\right],
\end{aligned} \tag{3.10}$$

for $k > 0$, $k \in \mathbb{N}$.

Finally, we use truncated series to approximate the analytical solution $\mathbb{J}_k(\kappa,\vartheta)$,

$$\mathbb{J}(\kappa,\vartheta) = \lim_{M\to\infty} \sum_{k=1}^{M} \mathbb{J}_k(\kappa,\vartheta). \tag{3.11}$$

The solutions to the aforementioned series usually converge quickly.

4. General methodology of YTDM

This section is concerned with the fundamental solution process of YTDM to get the solution of general arbitrary order partial differential equation

$$D_\vartheta^\rho \mathbb{J}(\kappa,\vartheta) = \mathcal{P}_1(\kappa,\vartheta) + \mathcal{Q}_1(\kappa,\vartheta), \quad 0 < \rho \le 1, \tag{4.1}$$

subjected to the initial conditions

$$\mathbb{J}(\kappa,0) = \xi(\kappa), \quad \frac{\partial}{\partial\vartheta}\mathbb{J}(\kappa,0) = \zeta(\kappa),$$

where $D_{\vartheta}^{\rho} = \frac{\partial^{\rho}}{\partial\vartheta^{\rho}}$ is the fractional Caputo derivative, $\mathcal{P}_1$ and $\mathcal{Q}_1$ are linear and non-linear operators, respectively.

By employing YT, we have

$$Y\left[D_{\vartheta}^{\rho}\mathbb{J}(\kappa,\vartheta)\right] = Y\left[\mathcal{P}_1(\kappa,\vartheta) + \mathcal{Q}_1(\kappa,\vartheta)\right], \tag{4.2}$$

using the differentiation property of the YT, we have

$$\frac{1}{u^{\rho}}\left\{M(u) - u\mathbb{J}(0) - u^2\mathbb{J}^{'}(0)\right\} = Y\left[\mathcal{P}_1(\kappa,\vartheta) + \mathcal{Q}_1(\kappa,\vartheta)\right]. \tag{4.3}$$

On simplification, we get

$$M(\mathbb{J}) = u\mathbb{J}(0) + u^2\mathbb{J}^{'}(0) + u^{\rho}Y\left[\mathcal{P}_1(\kappa,\vartheta) + \mathcal{Q}_1(\kappa,\vartheta)\right]. \tag{4.4}$$

Operating inverse YT both sides, we have

$$\mathbb{J}(\kappa,\vartheta) = \mathbb{J}(0) + \mathbb{J}^{'}(0) + Y^{-1}[u^{\rho}Y[\mathcal{P}_1(\kappa,\vartheta) + \mathcal{Q}_1(\kappa,\vartheta)]. \tag{4.5}$$

Thus, by means of YTDM the infinite series solution of $\mathbb{J}(\kappa,\vartheta)$,

$$\mathbb{J}(\kappa,\vartheta) = \sum_{m=0}^{\infty}\mathbb{J}_m(\kappa,\vartheta). \tag{4.6}$$

The decomposition of of nonlinear terms $\mathcal{Q}_1$ in terms of the Adomian polynomials are given as

$$\mathcal{Q}_1(\kappa,\vartheta) = \sum_{m=0}^{\infty}\mathcal{A}_m, \tag{4.7}$$

where

$$\mathcal{A}_m = \frac{1}{m!}\left[\frac{\partial^m}{\partial\ell^m}\left\{\mathcal{Q}_1\left(\sum_{k=0}^{\infty}\ell^k\kappa_k, \sum_{k=0}^{\infty}\ell^k\vartheta_k\right)\right\}\right]_{\ell=0}. \tag{4.8}$$

By putting (4.6) and (4.8) into (4.5), we obtain

$$\sum_{m=0}^{\infty}\mathbb{J}_m(\kappa,\vartheta) = \mathbb{J}(0) + \mathbb{J}^{'}(0) + Y^{-1}u^{\rho}\left[Y\left(\mathcal{P}_1\left(\sum_{m=0}^{\infty}\kappa_m, \sum_{m=0}^{\infty}\vartheta_m\right) + \sum_{m=0}^{\infty}\mathcal{A}_m\right)\right]. \tag{4.9}$$

We illustrate the below terms:

$$\mathbb{J}_0(\kappa,\vartheta) = \mathbb{J}(0) + \vartheta\mathbb{J}^{'}(0), \tag{4.10}$$
$$\mathbb{J}_1(\kappa,\vartheta) = Y^{-1}\left[u^{\rho}Y^{+}\left\{\mathcal{P}_1(\kappa_0,\vartheta_0) + \mathcal{A}_0\right\}\right]. \tag{4.11}$$

Hence, in general for $m \geq 1$, we have

$$\mathbb{J}_{m+1}(\kappa,\vartheta) = Y^{-1}\left[u^{\rho}Y^{+}\{\mathcal{P}_1(\kappa_m,\vartheta_m) + \mathcal{A}_m\}\right].$$

5. Applications

This section is concerned with the implementation of YTDM and HPYTM to solve fractional-order nonlinear shock wave equations.

Example 5.1. Consider the nonlinear fractional-order shock wave equation

$$\frac{\partial^{\rho}\mathbb{J}}{\partial\vartheta^{\rho}}+\left(\frac{1}{c_0}-\frac{\Upsilon+1}{2}\frac{\mathbb{J}}{c_0^2}\right)\frac{\partial\mathbb{J}}{\partial\kappa}=0,\quad 0<\rho\leq 1, \tag{5.1}$$

with the initial condition

$$\mathbb{J}(\kappa,0)=\exp\left(-\frac{\kappa^2}{2}\right),$$

where c_0 and Υ are constants with Υ representing the specific heat. In the present study, we choose $c_0 = 2$ and $\Upsilon = 1.5$, which represent the airflow.

By employing YT, we have

$$Y\left(\frac{\partial^{\rho}\mathbb{J}}{\partial\vartheta^{\rho}}\right)=-Y\left[\left(\frac{1}{c_0}-\frac{\Upsilon+1}{2}\frac{\mathbb{J}}{c_0^2}\right)\frac{\partial\mathbb{J}}{\partial\kappa}\right]. \tag{5.2}$$

Now, by YT differentiation property, we get

$$\frac{1}{u^{\rho}}\{M(u)-u\mathbb{J}(0)\}=-Y\left[\left(\frac{1}{c_0}-\frac{\Upsilon+1}{2}\frac{\mathbb{J}}{c_0^2}\right)\frac{\partial\mathbb{J}}{\partial\kappa}\right] \tag{5.3}$$

and

$$M(u)=u\mathbb{J}(0)-u^{\rho}Y\left[\left(\frac{1}{c_0}-\frac{\Upsilon+1}{2}\frac{\mathbb{J}}{c_0^2}\right)\frac{\partial\mathbb{J}}{\partial\kappa}\right]. \tag{5.4}$$

Operating inverse YT both sides, we have

$$\begin{aligned}\mathbb{J}(\kappa,\vartheta)&=\mathbb{J}(0)-Y^{-1}\left\{u^{\rho}\left[Y\left(\left(\frac{1}{c_0}-\frac{\Upsilon+1}{2}\frac{\mathbb{J}}{c_0^2}\right)\frac{\partial\mathbb{J}}{\partial\kappa}\right)\right]\right\},\\ \mathbb{J}(\kappa,\vartheta)&=\exp\left(-\frac{\kappa^2}{2}\right)-Y^{-1}\left\{u^{\rho}\left[Y\left(\left(\frac{1}{c_0}-\frac{\Upsilon+1}{2}\frac{\mathbb{J}}{c_0^2}\right)\frac{\partial\mathbb{J}}{\partial\kappa}\right)\right]\right\}.\end{aligned} \tag{5.5}$$

Thus, by using the HPM, we obtain

$$\sum_{k=0}^{\infty}\epsilon^k\mathbb{J}_k(\kappa,\vartheta)=\gamma+\epsilon\left(Y^{-1}\left\{u^{\rho}Y\left[\frac{1}{c_0}\frac{\partial\mathbb{J}}{\partial\kappa}\left(\sum_{k=0}^{\infty}\epsilon^k\mathbb{J}_k(\kappa,\vartheta)\right)-\frac{\Upsilon+1}{2}\frac{1}{c_0^2}\left(\sum_{k=0}^{\infty}\epsilon^kH_k(\mathbb{J})\right)\right]\right\}\right), \tag{5.6}$$

where non-linear terms are represented by He's polynomial $H_k(\mathbb{J})$ and is stated as

$$\sum_{k=0}^{\infty}\epsilon^kH_k(\mathbb{J})=\mathbb{J}\frac{\partial\mathbb{J}}{\partial\kappa}. \tag{5.7}$$

Some components of He's polynomials are calculated as

$$H_0(\mathbb{J})=\mathbb{J}_0\frac{\partial\mathbb{J}_0}{\partial\kappa},\quad H_1(\mathbb{J})=\mathbb{J}_0\frac{\partial\mathbb{J}_1}{\partial\kappa}+\mathbb{J}_1\frac{\partial\mathbb{J}_0}{\partial\kappa}.$$

By comparing the coefficient of ϵ, we obtain

$$
\begin{aligned}
\epsilon^0 &: \mathbb{J}_0(\kappa,\vartheta) = \exp\left(-\frac{\kappa^2}{2}\right),\\
\epsilon^1 &: \mathbb{J}_1(\kappa,\vartheta) = Y^{-1}\left\{u^\rho Y\left(\frac{1}{c_0}\frac{\partial \mathbb{J}_0}{\partial\kappa} - \frac{\Upsilon+1}{2}\frac{1}{c_0^2}H_0(\mathbb{J})\right)\right\} = -\left[\frac{1}{c_0} - \frac{\Upsilon+1}{2c_0^2}\exp\left(-\frac{\kappa^2}{2}\right)\right]\kappa\exp\left(-\frac{\kappa^2}{2}\right)\frac{\vartheta^\rho}{\Gamma(\rho+1)},\\
\epsilon^2 &: \mathbb{J}_2(\kappa,\vartheta) = Y^{-1}\left\{u^\rho Y\left[(\frac{1}{c_0}\frac{\partial \mathbb{J}_1}{\partial\kappa} - \frac{\Upsilon+1}{2}\frac{1}{c_0^2}H_1(\mathbb{J})\right)\right\} = \exp\left(-\frac{\kappa^2}{2}\right)\left[-\frac{1}{c_0^2} + \frac{\kappa^2}{c_0^2} - \frac{\Upsilon+1}{c_0^3}\exp\left(-\frac{\kappa^2}{2}\right)\right.\\
&\left. - \frac{2(\Upsilon+1)}{c_0^3}\kappa^2\exp\left(-\frac{\kappa^2}{2}\right) - \frac{(\Upsilon+1)^2}{4c_0^4}\exp\left(-\kappa^2\right) + \frac{3(\Upsilon+1)^2}{4c_0^4}\kappa^2\exp\left(-\kappa^2\right)\right]\frac{\vartheta^{2\rho}}{\Gamma(2\rho+1)},\\
&\vdots
\end{aligned}
$$

The remaining elements of the HPYTM solution can be achieved by continuing in the same way. Thus,

$$
\begin{aligned}
\mathbb{J}(\kappa,\vartheta) = {}& \exp\left(-\frac{\kappa^2}{2}\right) - \left[\frac{1}{c_0} - \frac{\Upsilon+1}{2c_0^2}\exp\left(-\frac{\kappa^2}{2}\right)\right]\kappa\exp\left(-\frac{\kappa^2}{2}\right)\frac{\vartheta^\rho}{\Gamma(\rho+1)} + \exp\left(-\frac{\kappa^2}{2}\right)\\
&\times\left[-\frac{1}{c_0^2} + \frac{\kappa^2}{c_0^2} - \frac{\Upsilon+1}{c_0^3}\exp\left(-\frac{\kappa^2}{2}\right) - \frac{2(\Upsilon+1)}{c_0^3}\kappa^2\exp\left(-\frac{\kappa^2}{2}\right) - \frac{(\Upsilon+1)^2}{4c_0^4}\exp\left(-\kappa^2\right)\right.\\
&\left.+\frac{3(\Upsilon+1)^2}{4c_0^4}\kappa^2\exp\left(-\kappa^2\right)\right]\frac{\vartheta^{2\rho}}{\Gamma(2\rho+1)} + \cdots.
\end{aligned}
$$

By employing YT, we have

$$
Y\left(\frac{\partial^\rho \mathbb{J}}{\partial\vartheta^\rho}\right) = -Y\left(\left(\frac{1}{c_0} - \frac{\Upsilon+1}{2}\frac{\mathbb{J}}{c_0^2}\right)\frac{\partial \mathbb{J}}{\partial\kappa}\right). \tag{5.8}
$$

Now, by YT differentiation property, we get

$$
\frac{1}{u^\rho}\left\{M(u) - u\mathbb{J}(0)\right\} = -Y\left(\left(\frac{1}{c_0} - \frac{\Upsilon+1}{2}\frac{\mathbb{J}}{c_0^2}\right)\frac{\partial \mathbb{J}}{\partial\kappa}\right), \tag{5.9}
$$

$$
M(u) = u\mathbb{J}(0) + u^\rho Y\left(\left(\frac{1}{c_0} - \frac{\Upsilon+1}{2}\frac{\mathbb{J}}{c_0^2}\right)\frac{\partial \mathbb{J}}{\partial\kappa}\right). \tag{5.10}
$$

Operating inverse YT both sides, we have

$$
\begin{aligned}
\mathbb{J}(\kappa,\vartheta) &= \mathbb{J}(0) - Y^{-1}\left\{u^\rho\left[Y\left(\left(\frac{1}{c_0} - \frac{\Upsilon+1}{2}\frac{\mathbb{J}}{c_0^2}\right)\frac{\partial \mathbb{J}}{\partial\kappa}\right)\right]\right\},\\
\mathbb{J}(\kappa,\vartheta) &= \exp\left(-\frac{\kappa^2}{2}\right) - Y^{-1}\left\{u^\rho\left[Y\left(\left(\frac{1}{c_0} - \frac{\Upsilon+1}{2}\frac{\mathbb{J}}{c_0^2}\right)\frac{\partial \mathbb{J}}{\partial\kappa}\right)\right]\right\}.
\end{aligned} \tag{5.11}
$$

Thus, by means of YTDM the infinite series solution of $\mathbb{J}(\kappa,\vartheta)$,

$$
\mathbb{J}(\kappa,\vartheta) = \sum_{m=0}^{\infty}\mathbb{J}_m(\kappa,\vartheta), \tag{5.12}
$$

and by means of Adomian polynomial, the nonlinear terms are derived as $\mathbb{J}\frac{\partial \mathbb{J}}{\partial \kappa} = \sum_{m=0}^{\infty} \mathcal{A}_m$, we have

$$\begin{aligned}\sum_{m=0}^{\infty} \mathbb{J}_m(\kappa, \vartheta) &= \mathbb{J}(\kappa, 0) - Y^{-1}\left\{u^{\rho}\left[Y\left(\left(\frac{1}{c_0}\frac{\partial \mathbb{J}}{\partial \kappa} - \frac{\Upsilon+1}{2c_0^2}\sum_{m=0}^{\infty}\mathcal{A}_m\right)\right)\right]\right\},\\ \sum_{m=0}^{\infty} \mathbb{J}_m(\kappa, \vartheta) &= \exp\left(-\frac{\kappa^2}{2}\right) - Y^{-1}\left\{u^{\rho}\left[Y\left(\left(\frac{1}{c_0}\frac{\partial \mathbb{J}}{\partial \kappa} - \frac{\Upsilon+1}{2c_0^2}\sum_{m=0}^{\infty}\mathcal{A}_m\right)\right)\right]\right\}.\end{aligned} \tag{5.13}$$

In this way, the nonlinear terms are decomposed as

$$\mathcal{A}_0 = \mathbb{J}_0\frac{\partial \mathbb{J}_0}{\partial \kappa}, \quad \mathcal{A}_1 = \mathbb{J}_0\frac{\partial \mathbb{J}_1}{\partial \kappa} + \mathbb{J}_1\frac{\partial \mathbb{J}_0}{\partial \kappa}.$$

On comparing both sides, we get

$$\mathbb{J}_0(\kappa, \vartheta) = \exp\left(-\frac{\kappa^2}{2}\right).$$

On $m = 0$, we have

$$\mathbb{J}_1(\kappa, \vartheta) = -\left[\frac{1}{c_0} - \frac{\Upsilon+1}{2c_0^2}\exp\left(-\frac{\kappa^2}{2}\right)\right]\kappa\exp\left(-\frac{\kappa^2}{2}\right)\frac{\vartheta^{\rho}}{\Gamma(\rho+1)}.$$

On $m = 1$, we have

$$\begin{aligned}\mathbb{J}_2(\kappa, \vartheta) = \exp\left(-\frac{\kappa^2}{2}\right)&\left[-\frac{1}{c_0^2} + \frac{\kappa^2}{c_0^2} - \frac{\Upsilon+1}{c_0^3}\exp\left(-\frac{\kappa^2}{2}\right) - \frac{2(\Upsilon+1)}{c_0^3}\kappa^2\exp\left(-\frac{\kappa^2}{2}\right)\right.\\ &\left.-\frac{(\Upsilon+1)^2}{4c_0^4}\exp\left(-\kappa^2\right) + \frac{3(\Upsilon+1)^2}{4c_0^4}\kappa^2\exp\left(-\kappa^2\right)\right]\frac{\vartheta^{2\rho}}{\Gamma(2\rho+1)}.\end{aligned}$$

Thus, the remaining elements ρ_m of YTDM for $(m \geq 2)$ are easy to get. Hence, we write the series form solution as

$$\begin{aligned}\mathbb{J}(\kappa, \vartheta) &= \sum_{m=0}^{\infty}\mathbb{J}_m(\kappa, \vartheta) = \mathbb{J}_0(\kappa, \vartheta) + \mathbb{J}_1(\kappa, \vartheta) + \mathbb{J}_2(\kappa, \vartheta) + \cdots,\\ \mathbb{J}(\kappa, \vartheta) &= \exp\left(-\frac{\kappa^2}{2}\right) - \left[\frac{1}{c_0} - \frac{\Upsilon+1}{2c_0^2}\exp\left(-\frac{\kappa^2}{2}\right)\right]\kappa\exp\left(-\frac{\kappa^2}{2}\right)\frac{\vartheta^{\rho}}{\Gamma(\rho+1)}\\ &\quad+\exp\left(-\frac{\kappa^2}{2}\right)\left[-\frac{1}{c_0^2} + \frac{\kappa^2}{c_0^2} - \frac{\Upsilon+1}{c_0^3}\exp\left(-\frac{\kappa^2}{2}\right) - \frac{2(\Upsilon+1)}{c_0^3}\kappa^2\exp\left(-\frac{\kappa^2}{2}\right) - \frac{(\Upsilon+1)^2}{4c_0^4}\exp\left(-\kappa^2\right)\right.\\ &\quad\left.+\frac{3(\Upsilon+1)^2}{4c_0^4}\kappa^2\exp\left(-\kappa^2\right)\right]\frac{\vartheta^{2\rho}}{\Gamma(2\rho+1)} + \cdots.\end{aligned}$$

Example 5.2. Consider the nonlinear fractional-order shock wave equation

$$\frac{\partial^{\rho}\mathbb{J}}{\partial \vartheta^{\rho}} + \mathbb{J}\frac{\partial \mathbb{J}}{\partial \kappa} - \frac{\partial}{\partial \vartheta}\frac{\partial \mathbb{J}^2}{\partial \kappa^2} = 0, \quad 0 < \rho \leq 1, \tag{5.14}$$

with initial source

$$\mathbb{J}(\kappa, 0) = 3\,\mathrm{sech}^2\left(\frac{\kappa - 15}{2}\right).$$

By employing YT, we have

$$Y\left(\frac{\partial^{\rho}\mathbb{J}}{\partial\vartheta^{\rho}}\right)=Y\left(\frac{\partial}{\partial\vartheta}\frac{\partial\mathbb{J}^{2}}{\partial\kappa^{2}}-\mathbb{J}\frac{\partial\mathbb{J}}{\partial\kappa}\right). \tag{5.15}$$

Now, by YT differentiation property, we get

$$\frac{1}{u^{\rho}}\{M(u)-u\mathbb{J}(0)\}=Y\left(\frac{\partial}{\partial\vartheta}\frac{\partial\mathbb{J}^{2}}{\partial\kappa^{2}}-\mathbb{J}\frac{\partial\mathbb{J}}{\partial\kappa}\right) \tag{5.16}$$

and

$$M(u)=u\mathbb{J}(0)+u^{\rho}Y\left(\frac{\partial}{\partial\vartheta}\frac{\partial\mathbb{J}^{2}}{\partial\kappa^{2}}-\mathbb{J}\frac{\partial\mathbb{J}}{\partial\kappa}\right). \tag{5.17}$$

Operating inverse YT both sides, we have

$$\begin{aligned}\mathbb{J}(\kappa,\vartheta)&=\mathbb{J}(0)+Y^{-1}\left\{u^{\rho}\left[Y\left(\frac{\partial}{\partial\vartheta}\frac{\partial\mathbb{J}^{2}}{\partial\kappa^{2}}-\mathbb{J}\frac{\partial\mathbb{J}}{\partial\kappa}\right)\right]\right\},\\ \mathbb{J}(\kappa,\vartheta)&=3\operatorname{sech}^{2}\left(\frac{\kappa-15}{2}\right)+Y^{-1}\left\{u^{\rho}\left[Y\left(\frac{\partial}{\partial\vartheta}\frac{\partial\mathbb{J}^{2}}{\partial\kappa^{2}}-\mathbb{J}\frac{\partial\mathbb{J}}{\partial\kappa}\right)\right]\right\}.\end{aligned} \tag{5.18}$$

Now, by using the HPM, we obtain

$$\sum_{k=0}^{\infty}\epsilon^{k}\mathbb{J}_{k}(\kappa,\vartheta)=3\operatorname{sech}^{2}\left(\frac{\kappa-15}{2}\right)+\epsilon\left(Y^{-1}\left\{u^{\rho}Y\left[\left(\sum_{k=0}^{\infty}\epsilon^{k}\mathbb{J}_{k}(\kappa,\vartheta)\right)_{\kappa\kappa\vartheta}-\left(\sum_{k=0}^{\infty}\epsilon^{k}H_{k}(\mathbb{J})\right)\right]\right\}\right), \tag{5.19}$$

where non-linear terms are presented by He's polynomial $H_k(\mathbb{J})$ and is stated as

$$\sum_{k=0}^{\infty}\epsilon^{k}H_{k}(\mathbb{J})=\mathbb{J}\frac{\partial\mathbb{J}}{\partial\kappa}. \tag{5.20}$$

Some terms of He's polynomials are calculated as

$$H_{0}(\mathbb{J})=\mathbb{J}_{0}\frac{\partial\mathbb{J}_{0}}{\partial\kappa},\quad H_{1}(\mathbb{J})=\mathbb{J}_{0}\frac{\partial\mathbb{J}_{1}}{\partial\kappa}+\mathbb{J}_{1}\frac{\partial\mathbb{J}_{0}}{\partial\kappa}.$$

By comparing the coefficient of ϵ, we obtain

$$\begin{aligned}\epsilon^{0}:\mathbb{J}_{0}(\kappa,\vartheta)&=3\operatorname{sech}^{2}\left(\frac{\kappa-15}{2}\right),\\ \epsilon^{1}:\mathbb{J}_{1}(\kappa,\vartheta)&=Y^{-1}\left\{u^{\rho}Y\left(\frac{\partial}{\partial\vartheta}\frac{\partial\mathbb{J}_{0}^{2}}{\partial\kappa^{2}}-H_{0}(\mathbb{J})\right)\right\}\\ &=9\operatorname{sech}^{4}\left(\frac{\kappa-15}{2}\right)\tanh\left(\frac{\kappa-15}{2}\right)\frac{\vartheta^{\rho}}{\Gamma(\rho+1)},\\ \epsilon^{2}:\mathbb{J}_{2}(\kappa,\vartheta)&=Y^{-1}\left\{u^{\rho}Y\left(\frac{\partial}{\partial\vartheta}\frac{\partial\mathbb{J}_{1}^{2}}{\partial\kappa^{2}}-H_{1}(\mathbb{J})\right)\right\}\\ &=\left[\frac{189}{2}\operatorname{sech}^{6}\left(\frac{\kappa-15}{2}\right)\tanh^{2}\left(\frac{\kappa-15}{2}\right)-\frac{27}{2}\operatorname{sech}^{6}\left(\frac{\kappa-15}{2}\right)\right]\frac{\vartheta^{2\rho}}{\Gamma(2\rho+1)}\\ &+\left[\frac{135}{2}\operatorname{sech}^{4}\left(\frac{\kappa-15}{2}\right)\tanh^{3}\left(\frac{\kappa-15}{2}\right)-\frac{63}{2}\operatorname{sech}^{4}\left(\frac{\kappa-15}{2}\tanh\left(\frac{\kappa-15}{2}\right)\right)\right]\frac{\Gamma(\rho+1)\vartheta^{2\rho-1}}{\Gamma(2\rho)},\\ &\vdots\end{aligned}$$

The remaining elements of the HPYTM solution can be achieved by continuing in the same way. Thus,

$$\begin{aligned}\mathbb{J}(\kappa,\vartheta) &= 3\,\mathrm{sech}^2\left(\frac{\kappa-15}{2}\right)+9\,\mathrm{sech}^4\left(\frac{\kappa-15}{2}\right)\tanh\left(\frac{\kappa-15}{2}\right)\frac{\vartheta^{\rho}}{\Gamma(\rho+1)}\\ &+\left[\frac{189}{2}\,\mathrm{sech}^6\left(\frac{\kappa-15}{2}\right)\tanh^2\left(\frac{\kappa-15}{2}\right)-\frac{27}{2}\,\mathrm{sech}^6\left(\frac{\kappa-15}{2}\right)\right]\frac{\vartheta^{2\rho}}{\Gamma(2\rho+1)}\\ &+\left[\frac{135}{2}\,\mathrm{sech}^4\left(\frac{\kappa-15}{2}\right)\tanh^3\left(\frac{\kappa-15}{2}\right)-\frac{63}{2}\,\mathrm{sech}^4\left(\frac{\kappa-15}{2}\tanh\left(\frac{\kappa-15}{2}\right)\right)\right]\frac{\Gamma(\rho+1)\vartheta^{2\rho-1}}{\Gamma(2\rho)}\\ &+\cdots.\end{aligned}$$

By employing YT, we have

$$Y\left(\frac{\partial^{\rho}\mathbb{J}}{\partial\vartheta^{\rho}}\right)=Y\left(\frac{\partial}{\partial\vartheta}\frac{\partial\mathbb{J}^2}{\partial\kappa^2}-\mathbb{J}\frac{\partial\mathbb{J}}{\partial\kappa}\right). \tag{5.21}$$

Now, by YT differentiation property, we have

$$\frac{1}{u^{\rho}}\{M(u)-u\mathbb{J}(0)\}=Y\left(\frac{\partial}{\partial\vartheta}\frac{\partial\mathbb{J}^2}{\partial\kappa^2}-\mathbb{J}\frac{\partial\mathbb{J}}{\partial\kappa}\right) \tag{5.22}$$

and

$$M(u)=u\mathbb{J}(0)+u^{\rho}Y\left(\frac{\partial}{\partial\vartheta}\frac{\partial\mathbb{J}^2}{\partial\kappa^2}-\mathbb{J}\frac{\partial\mathbb{J}}{\partial\kappa}\right). \tag{5.23}$$

Operating inverse YT both sides, we have

$$\begin{aligned}\mathbb{J}(\kappa,\vartheta)&=\mathbb{J}(0)+Y^{-1}\left\{u^{\rho}\left[Y\left(\frac{\partial}{\partial\vartheta}\frac{\partial\mathbb{J}^2}{\partial\kappa^2}-\mathbb{J}\frac{\partial\mathbb{J}}{\partial\kappa}\right)\right]\right\},\\ \mathbb{J}(\kappa,\vartheta)&=\frac{1}{(1+\exp(\kappa))^2}+Y^{-1}\left\{u^{\rho}\left[Y\left(\frac{\partial}{\partial\vartheta}\frac{\partial\mathbb{J}^2}{\partial\kappa^2}-\mathbb{J}\frac{\partial\mathbb{J}}{\partial\kappa}\right)\right]\right\}.\end{aligned} \tag{5.24}$$

Thus, by means of YTDM the infinite series solution of $\mathbb{J}(\kappa,\vartheta)$,

$$\mathbb{J}(\kappa,\vartheta)=\sum_{m=0}^{\infty}\mathbb{J}_m(\kappa,\vartheta), \tag{5.25}$$

and by means of Adomian polynomial, the nonlinear terms are derived as $\mathbb{J}\frac{\partial\mathbb{J}}{\partial\kappa}=\sum_{m=0}^{\infty}\mathcal{A}_m$, we have

$$\begin{aligned}\sum_{m=0}^{\infty}\mathbb{J}_m(\kappa,\vartheta)&=\mathbb{J}(\kappa,0)+Y^{-1}\left\{u^{\rho}\left[Y\left(\frac{\partial}{\partial\vartheta}\frac{\partial\mathbb{J}^2}{\partial\kappa^2}-\sum_{m=0}^{\infty}\mathcal{A}_m\right)\right]\right\},\\ \sum_{m=0}^{\infty}\mathbb{J}_m(\kappa,\vartheta)&=\frac{1}{(1+\exp(\kappa))^2}+Y^{-1}\left\{u^{\rho}\left[Y\left(\frac{\partial}{\partial\vartheta}\frac{\partial\mathbb{J}^2}{\partial\kappa^2}-\sum_{m=0}^{\infty}\mathcal{A}_m\right)\right]\right\}.\end{aligned} \tag{5.26}$$

In this way, the nonlinear terms are decomposed as

$$\mathcal{A}_0=\mathbb{J}_0\frac{\partial\mathbb{J}_0}{\partial\kappa},\quad \mathcal{A}_1=\mathbb{J}_0\frac{\partial\mathbb{J}_1}{\partial\kappa}+\mathbb{J}_1\frac{\partial\mathbb{J}_0}{\partial\kappa}.$$

On comparing both sides, we get

$$\mathbb{J}_0(\kappa,\vartheta) = 3\,\mathrm{sech}^2\left(\frac{\kappa-15}{2}\right).$$

On $m = 0$, we have

$$\mathbb{J}_1(\kappa,\vartheta) = 9\,\mathrm{sech}^4\left(\frac{\kappa-15}{2}\right)\tanh\left(\frac{\kappa-15}{2}\right)\frac{\vartheta^{\rho}}{\Gamma(\rho+1)}.$$

On $m = 1$, we have

$$\begin{aligned}\mathbb{J}_2(\kappa,\vartheta) = &\left[\frac{189}{2}\,\mathrm{sech}^6\left(\frac{\kappa-15}{2}\right)\tanh^2\left(\frac{\kappa-15}{2}\right) - \frac{27}{2}\,\mathrm{sech}^6\left(\frac{\kappa-15}{2}\right)\right]\frac{\vartheta^{2\rho}}{\Gamma(2\rho+1)}\\ &+\left[\frac{135}{2}\,\mathrm{sech}^4\left(\frac{\kappa-15}{2}\right)\tanh^3\left(\frac{\kappa-15}{2}\right) - \frac{63}{2}\,\mathrm{sech}^4\left(\frac{\kappa-15}{2}\,\tanh\left(\frac{\kappa-15}{2}\right)\right)\right]\frac{\Gamma(\rho+1)\vartheta^{2\rho-1}}{\Gamma(2\rho)}.\end{aligned}$$

Thus, the remaining elements ρ_m of YTDM for ($m \geq 3$) are easy to get. Hence, we write the series form solution as

$$\begin{aligned}\mathbb{J}(\kappa,\vartheta) &= \sum_{m=0}^{\infty}\mathbb{J}_m(\kappa,\vartheta) = \mathbb{J}_0(\kappa,\vartheta) + \mathbb{J}_1(\kappa,\vartheta) + \cdots,\\ \mathbb{J}(\kappa,\vartheta) &= 3\,\mathrm{sech}^2\left(\frac{\kappa-15}{2}\right) + 9\,\mathrm{sech}^4\left(\frac{\kappa-15}{2}\right)\tanh\left(\frac{\kappa-15}{2}\right)\frac{\vartheta^{\rho}}{\Gamma(\rho+1)}\\ &\quad+\left[\frac{189}{2}\,\mathrm{sech}^6\left(\frac{\kappa-15}{2}\right)\tanh^2\left(\frac{\kappa-15}{2}\right) - \frac{27}{2}\,\mathrm{sech}^6\left(\frac{\kappa-15}{2}\right)\right]\frac{\vartheta^{2\rho}}{\Gamma(2\rho+1)}\\ &\quad+\left[\frac{135}{2}\,\mathrm{sech}^4\left(\frac{\kappa-15}{2}\right)\tanh^3\left(\frac{\kappa-15}{2}\right) - \frac{63}{2}\,\mathrm{sech}^4\left(\frac{\kappa-15}{2}\,\tanh\left(\frac{\kappa-15}{2}\right)\right)\right]\frac{\Gamma(\rho+1)\vartheta^{2\rho-1}}{\Gamma(2\rho)}\\ &\quad+\cdots.\end{aligned}$$

At $\rho = 1$, we get in closed form the exact solution as

$$\mathbb{J}(\kappa,\vartheta) = 3\,\mathrm{sech}^2\left(\frac{\kappa-15-\vartheta}{2}\right). \tag{5.27}$$

6. Results and discussion

In this section, we present the physical explanations for the presented problems. We note that our approaches can handle time fractional-order shock wave equations perfectly. It can be shown that $\mathbb{J}(\kappa,\vartheta)$ reduces with the growth of κ and ϑ for $\rho = 0.25, 0.50, 0.75$ and 1 in Figure 1, which shows the surface solutions of $\mathbb{J}(\kappa,\vartheta)$ for various time fractional equations in Brownian. The analytical solution derived using the suggested approaches is shown in Figure 2, together with the exact solution for various values of κ and ϑ. Figure 2 also shows the surface solutions of $\mathbb{J}(\kappa,\vartheta)$. Figure 3 shows the absolute error behavior of Example 5.2. Similarly, in Table 1, we show the behavior of the accurate and proposed method solutions at various fractional orders of ρ having different values of κ and ϑ. The figures and table show that the solution of the proposed method is in good contact with the exact solution.

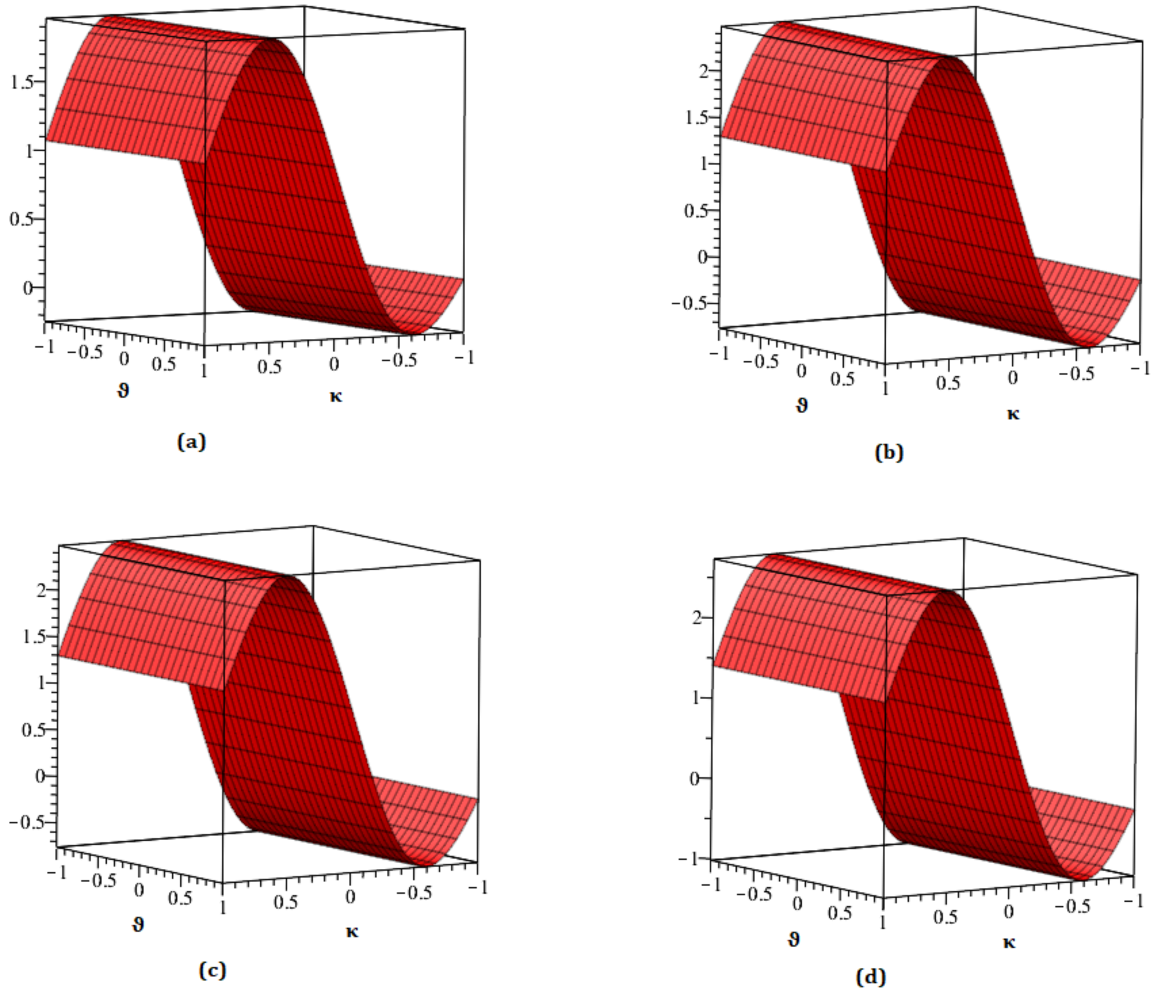

Figure 1. The $\mathbb{J}(\kappa, \vartheta)$ surface solutions with regard to κ and ϑ for various ρ values within the domain $-1 \leq \kappa, \vartheta \leq 1$. (a) Surface solution of the function $\mathbb{J}(\kappa, \vartheta)$ at the value of $\rho = 0.25$. (b) Surface solution of the function $\mathbb{J}(\kappa, \vartheta)$ at the value of $\rho = 0.50$. (c) Surface solution of the function $\mathbb{J}(\kappa, \vartheta)$ at the value of $\rho = 0.75$. (d) Surface solution of the function $\mathbb{J}(\kappa, \vartheta)$ at the value of $\rho = 1$.

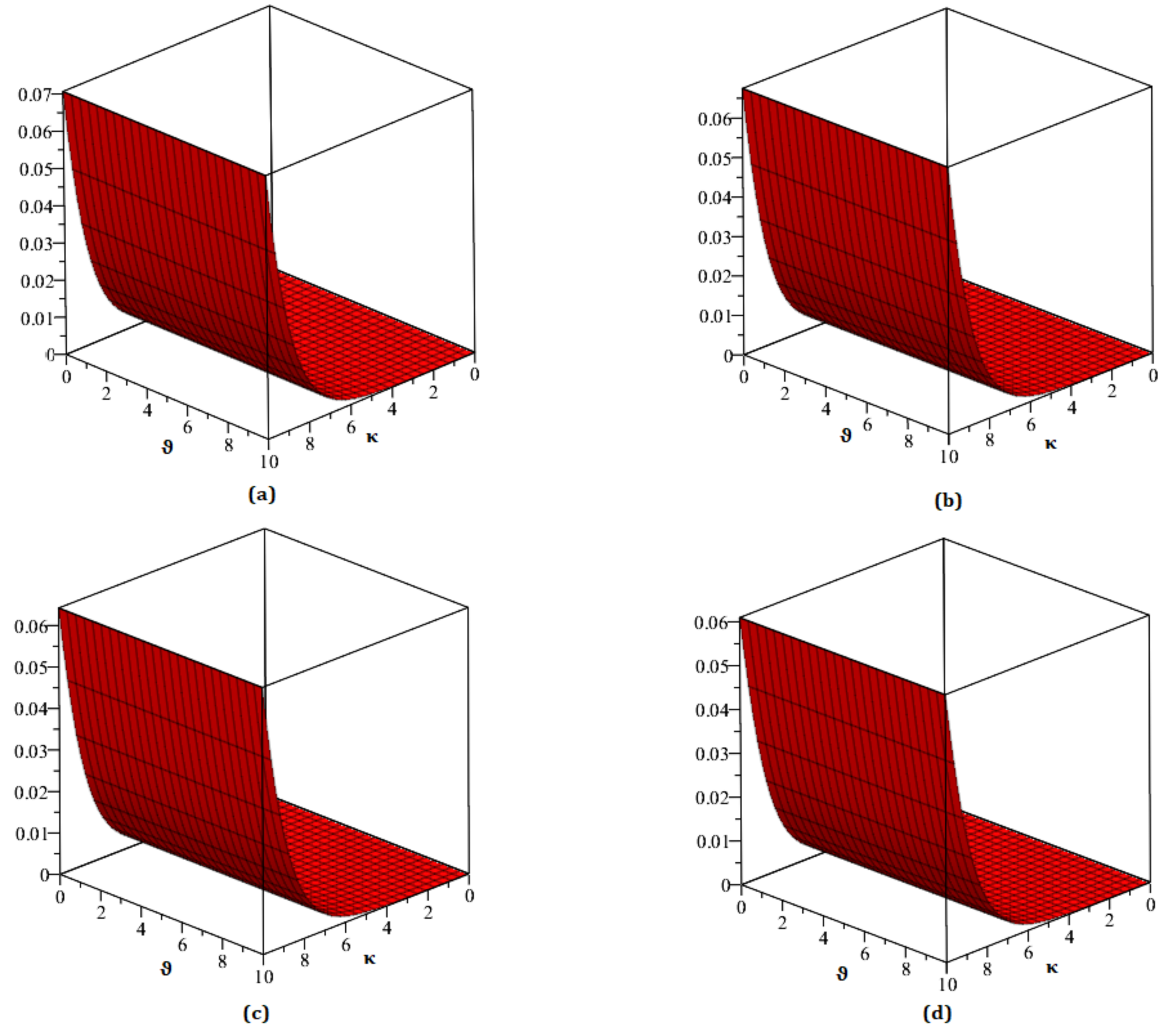

Figure 2. The $\mathbb{J}(\kappa, \vartheta)$ surface solutions with regard to κ and ϑ for various ρ values within the domain $0 \leq \kappa, \vartheta \leq 10$. (a) Surface solution of the function $\mathbb{J}(\kappa, \vartheta)$ at the value of $\rho = 0.25$. (b) Surface solution of the function $\mathbb{J}(\kappa, \vartheta)$ at the value of $\rho = 0.50$. (c) Surface solution of the function $\mathbb{J}(\kappa, \vartheta)$ at the value of $\rho = 0.75$. (d) Surface solution of the function $\mathbb{J}(\kappa, \vartheta)$ at the value of $\rho = 1$.

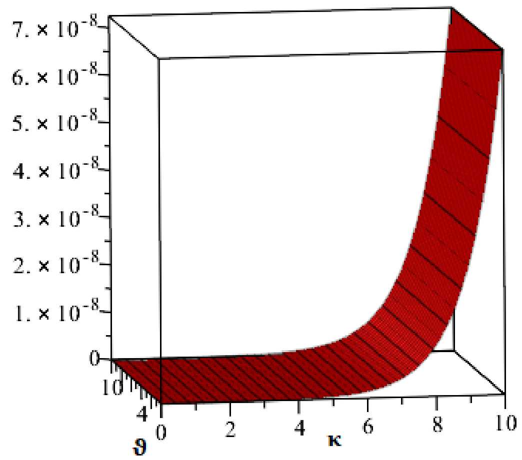

Figure 3. The absolute error graph for $\mathbb{J}(\kappa, \vartheta)$ of Example 5.2.

Table 1. Example 5.1: Exact, YTDM and HPYTM solution for $\mathbb{J}(\kappa, \vartheta)$ at different fractional-order of ρ, κ and ϑ.

ϑ	κ	$\rho = 0.6$	$\rho = 0.8$	$\rho = 1(approx)$	$\rho = 1(approx)$	$\rho = 1(exact)$
	0.2	0.000004490	0.000004487	0.000004483	0.000004483	0.000004479
	0.4	0.000005481	0.000005479	0.000005476	0.000005476	0.000005470
0.01	0.6	0.000006693	0.000006691	0.000006688	0.000006688	0.000006681
	0.8	0.000008171	0.000008171	0.000008169	0.000008169	0.000008161
	1	0.000009982	0.000009980	0.000009978	0.000009978	0.000009968
	0.2	0.000004490	0.000004487	0.000004483	0.000004483	0.000004474
	0.4	0.000005482	0.000005478	0.000005476	0.000005476	0.000005465
0.02	0.6	0.000006693	0.000006690	0.000006688	0.000006688	0.000006675
	0.8	0.000008177	0.000008172	0.000008169	0.000008169	0.000008153
	1	0.000009987	0.000009982	0.000009978	0.000009978	0.000009958
	0.2	0.000004489	0.000004485	0.000004483	0.000004483	0.000004470
	0.4	0.000005489	0.00000583	0.000005476	0.000005476	0.000005459
0.03	0.6	0.000006693	0.000006690	0.000006688	0.000006688	0.000006668
	0.8	0.000008179	0.000008174	0.000008169	0.000008169	0.000008145
	1	0.000009989	0.000009982	0.000009978	0.000009978	0.000009948
	0.2	0.000004492	0.000004487	0.000004483	0.000004483	0.000004465
	0.4	0.000005487	0.000005481	0.000005476	0.000005476	0.000005454
0.04	0.6	0.000006699	0.000006691	0.000006688	0.000006688	0.000006661
	0.8	0.000008183	0.000008177	0.000008169	0.000008169	0.000008136
	1	0.000009999	0.000009983	0.000009978	0.000009978	0.000009938
	0.2	0.000004497	0.000004489	0.000004483	0.000004483	0.000004461
	0.4	0.000005490	0.000005479	0.000005476	0.000005476	0.000005448
0.05	0.6	0.000006697	0.000006691	0.000006688	0.000006688	0.000006655
	0.8	0.000008191	0.000008185	0.000008169	0.000008169	0.000008128
	1	0.000009997	0.000009988	0.000009978	0.000009978	0.000009928

7. Conclusions

The approximate and analytical solutions of the fractional-order nonlinear shock wave and wave equations are successfully obtained using the HPYTM and YTDM. The proposed methods are completed in two steps. First, we implement the Yang transform to simplify the proposed problems. After simplification, Adomian decomposition and homotopy perturbation methods are used to get the close form solution to the issues presented. The significance of fractional derivatives and the method for handling the recurrence relation are both shown in this study. The HPYTM and YTDM are used directly to generate the series solutions. Compared to previous methods examined in the literature, the current schemes exhibit more efficiency and require less computation. With the aid of MAPLE Software, all iterations were calculated. According to the solution graphs, this method can be applied to a wide range of nonlinear fractional differential equations in science and engineering. In subsequent research, these strategies could be expanded to address diverse nonlinear obstacle issues.

Funding

This work was supported by Princess Nourah bint Abdulrahman University Researchers Supporting Project (No. PNURSP2022R183).

Acknowledgements

This work was supported by Princess Nourah bint Abdulrahman University Researchers Supporting Project (No. PNURSP2022R183).

Conflict of interest

The authors declare that they have no competing interests.

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