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Research article

Asymptotic behavior of ordered random variables in mixture of two Gaussian sequences with random index

H. M. Barakat[1] **and M. H. Dwes**[2,*]

[1] Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt.
[2] Department of Mathematics, Faculty of Science, Alexandria University, Alexandria, Egypt

* **Correspondence:** Email: marwa.dwes@alexu.edu.eg.

Abstract: When the random sample size is assumed to converge weakly and to be independent of the basic variables, the asymptotic distributions of extreme, intermediate, and central order statistics, as well as record values, for a mixture of two stationary Gaussian sequences under an equi-correlated setup are derived. Furthermore, sufficient conditions for convergence are derived in each case. An interesting fact is revealed that in several cases, the limit distributions of the aforementioned statistics are the same when the sample size is random and non-random. e.g., when one mixture component has a correlation that converges to a non-zero value.

Keywords: mixture distribution; Gaussian sequences; random index; order statistics; record values
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1. Introduction

In the literature, mixture distributions appear frequently and naturally when a statistical population contains two or more subpopulations. They are also occasionally used to represent non-normal distributions. A mixture distribution, which is a convex combination of two or more probability density functions (PDFs), is a powerful and flexible tool for modelling complex data as it combines the properties of the individual PDFs, see [1–6]. Besides this major use, Doğru and Arslan [7] and Titterington et al. [8] showed that mixture models are frequently used in a variety of applications. Given two distribution functions (DFs) $F_{X_1}(x) = P(X_1 \leq x)$ and $F_{X_2}(x) = P(X_2 \leq x)$, and weights p and $q = 1 - p$, such that $p, q \geq 0$, the mixing model of F_{X_1} and F_{X_2} can be defined by its DF as $F_X(x) = pF_{X_1}(x) + qF_{X_2}(x)$. The random variable (RV) X in the mixing model is defined (cf. [6, 9]) as

$$X = B_p X_1 + B_q X_2, \tag{1.1}$$

where $B_p = 1 - B_q$ is a Bernoulli distributed RV with parameter $p = 1 - q$, and the RVs X_1 and X_2 are independent of B_p (and B_q). The mixing model can now be expressed in terms of RVs rather than DFs. Furthermore, the dependence structure between the two RVs X_1 and X_2 has no bearing on the mixing model. Recently, the representation (1.1) was employed by Barakat et al. [6, 10] to obtain the quantile function of the mixing model. Moreover, this depiction was a key component in investigating the limit distributions of extreme, intermediate, and central order statistics (OSs), as well as record values, of the mixture of two stationary Gaussian sequences (SGSs) under an equi-correlated setup by Barakat and Dwes [11]. The limit DFs of order RVs under the equi-correlated setting have been researched by many academics, [12–16] are a few examples of publications on this topic and its importance. Our goal in this work is to extend the results of Barakat and Dwes [11] when the sample size is assumed to be an RV, which is independent of the basic variables.

Random sample sizes come up naturally in topics like sequential analysis, branching processes, damage models, and rarefaction of point processes. Random minima and maxima also appear in the study of floods, droughts, and breaking strength difficulties. One of the most important reasons for random sizes to appear in statistical experiments is that some observations may be lost in many biological and agricultural situations for a variety of causes, making it impossible to have a set sample size. Also, the sample size can sometimes be determined by the occurrence of certain random events making the sample size random. In reality, there are two scenarios in every application. The statistician does not influence the relationship between the sample size and the underlying RVs in the first scenario since the random sample size is generated by the problem itself. Among the authors who worked on this scenario are [17–21]. On the other hand, if the random sample size is introduced as a model extension (primarily for statistical inference), it may normally be assumed to be independent of the underlying model. In this study, we adopt the second scenario, where we assume the random size is a positive integer-valued RV ν_n, which is independent of the basic variables, and the DF $P(\nu_n \leq x) = A_n(x)$ converges weakly to a non-degenerate limit DF. Among the authors who worked on this scenario are [17–19, 22, 23].

In the rest of this introductory section, we give a concrete formulation of the main problem of the paper and display some auxiliary results. Let's say there are two SGSs, $\{X_{1,i}\}$ and $\{X_{2,i}\}$, $i = 1, 2, ..., n$. Furthermore, let $\{X_{j,i}\}$, $j = 1, 2$, have zero mean, unit variance and constant correlation coefficient $r_{j,n} = E(X_{j,i}X_{j,k}) \geq 0$, $i \neq k$, written $X_{j,i} \sim Gas(0, 1, r_{j,n})$. The sequence $\{X_{j,i}\}$, $j = 1, 2$, can be represented by $X_{j,i} = \sqrt{r_{j,n}}Y_{j,0} + \sqrt{1 - r_{j,n}}Y_{j,i}$, $i = 1, 2, ..., n$, where $Y_{j,0}, Y_{j,1}, ..., Y_{j,n}$ are i.i.d standard normal variables (cf. [21]). Moreover, if we assume that the two SGSs $\{X_{1,i}\}$ and $\{X_{2,i}\}$ are independent without sacrificing generality, then by (1.1), the mixture of these sequences are

$$X_i = B_p X_{1,i} + B_q X_{2,i}, \quad i = 1, 2, ..., n, \tag{1.2}$$

where each of the sequences $\{X_{1,i}\}$ and $\{X_{2,i}\}$ is independent of B_p (and B_q). On the other hand, (1.2) can be exemplified by $X_i = Z_{0,n} + Z_{i,n}$, $i = 1, 2, ..., n$, where

$$Z_{i,n} := \begin{cases} B_p \sqrt{r_{1,n}}Y_{1,0} + B_q \sqrt{r_{2,n}}Y_{2,0}, & \text{if } i = 0, \\ B_p \sqrt{1 - r_{1,n}}Y_{1,i} + B_q \sqrt{1 - r_{2,n}}Y_{2,i}, & \text{if } i > 0, \end{cases} \tag{1.3}$$

$P(Z_{0,n} \leq z) = p\Phi\left(\frac{z}{\sqrt{r_{1,n}}}\right) + q\Phi\left(\frac{z}{\sqrt{r_{2,n}}}\right)$, $\Phi(.)$ is the standard normal DF, and $Z_{1,n}, Z_{2,n}, ..., Z_{n,n}$ are i.i.d RVs with common DF

$$\mathcal{F}(z) := P(Z_{i,n} \le z) = p\Phi\left(\frac{z}{\sqrt{1-r_{1,n}}}\right) + q\Phi\left(\frac{z}{\sqrt{1-r_{2,n}}}\right). \tag{1.4}$$

Thus, for any $1 \le s \le n$, the sth OS based on the sequence $\{X_i\}$ can be written as

$$X_{s:n} = Z_{0,n} + Z_{s:n}, \tag{1.5}$$

where $Z_{s:n}$ is the sth OS based on the sequence $\{Z_{i,n}\},\ i = 1, 2, ..., n$.

The OSs $X_{s:n}$ and $X_{s'_n:n} := X_{n-s+1:n}$ are called the sth lower and upper extremes, respectively, if the rank $s \ge 1$ was fixed with respect to n. Using the well-known connection $\max(x_1, x_2, ..., x_n) = -\min(-x_1, -x_2, ..., -x_n)$, any result for the lower OSs may be deduced from the upper OSs, and vice versa. There are two categories of OSs based on their rank nature, plus extreme OSs. If $\max(s_n, n - s_n + 1) \to \infty$, as $n \to \infty$, a sequence $X_{s_n:n}$ is termed a sequence of OSs with variable rank. As a result, two specific variable ranks are of particular interest: (1) $\frac{s_n}{n} \to 0$ (or $\frac{s_n}{n} \to 1$), as $n \to \infty$, which we shall call the lower (or the upper) intermediate rank case, and (2) $\frac{s_n}{n} \to \lambda$ $(0 < \lambda < 1)$, as $n \to \infty$, which will be referred to as the case of central ranks. The λth sample quantile is a familiar example of central OSs, where $s_n = [\lambda n], 0 < \lambda < 1$, and $[x]$ denotes the largest integer not exceeding x.

In a series of RVs, successive maxima, or values that rigorously exceed all previous values, are recorded. Let $\{X_n,\ n \ge 1\}$ be i.i.d RVs with a common DF $F_X(x)$. Then, X_j is an (upper) record value if $X_j > X_i,\ \forall i < j$, and as a result X_1 is a record value. The record time sequence $\{T_n,\ n \ge 1\}$ is the sequence of times at which records occur. Thus, $T_1 = 1$, $T_n = \min\{j : X_j > X_{T_{n-1}},\ n > 1\}$. Consequently, the record value sequence $\{R_n\}$ is given by $R_n = X_{T_n}$ (cf. Arnold et al. [24]). The record value ${}_XR_n$ based on the sequence $\{X_i\}$, represented by (1.2), can be represented as

$${}_XR_n = Z_{0,n} + {}_ZR_n, \tag{1.6}$$

where ${}_ZR_n$ is the record value based on the sequence $\{Z_{i,n}\}$, given by (1.3).

In the sequel, the following result will be frequently needed:

Lemma 1.1 (cf. [11]). *Let $\eta_1, \eta_2, ..., \eta_n$ be i.i.d RVs with a mixture DF*

$$F_\eta(x) = \sum_{j=1}^{k} p_j F_j(\alpha_{j,n} x),\ \ \sum_{j=1}^{k} p_j = 1,\ \alpha_{j,n} > 0,$$

where $\{F_j(.)\}$ is a sequence of non-degenerate DFs. Furthermore, let $G_n(x) = a_n x + b_n$, $a_n > 0$, be a suitable linear transformation. Then,

1) *the limit distribution of the extreme OS $\eta_{n-s+1:n}$ of the sequence $\{\eta_i\}$ is given by*

$$F_{\eta_{n-s+1:n}}(G_n(x)) := P(\eta_{n-s+1:n} \le G_n(x)) \xrightarrow[n]{w} \left[\prod_{j=1}^{k} \Psi_j^{p_j}(x)\right] \sum_{l=0}^{s-1} \frac{\left(-\sum_{j=1}^{k} p_j \log \Psi_j(x)\right)^l}{l!},$$

if $F_j(\alpha_{j,n} G_n(x))$ belongs to the max-domain of attraction of the non-degenerate max-type $\Psi_j(x)$, written $F_j(\alpha_{j,n} G_n(x)) \in D(\Psi_j(x)),\ j = 1, 2, ..., k$,

2) *the limit distribution of central OS* $\eta_{n-s_n+1:n}$ *(where* $\sqrt{n}(\frac{s_n}{n}-\lambda)\underset{n}{\longrightarrow}0$*) of the sequence* $\{\eta_i\}$ *is given by*

$$F_{\eta_{n-s_n+1:n}}(G_n(x)) := P(\eta_{n-s_n+1:n}\leq G_n(x)) \xrightarrow[n]{w} \Phi\left(\sum_{j=1}^{k} p_j u_j(x;\lambda)\right),$$

if $F_j(\alpha_{j,n}G_n(x))$ *belongs to the central-domain of attraction of the non-degenerate type* $\Phi(u_j(x;\lambda))$*, written* $F_j(\alpha_{j,n}G_n(x))\in D_\lambda(\Phi(u_j(x;\lambda)))$*,* $j=1,2,...,k$*,*

3) *the limit distribution of intermediate OS* $\eta_{n-s_n+1:n}$ *of the sequence* $\{\eta_i\}$ *(where* $\frac{s_n}{n}\underset{n}{\longrightarrow}0$*) is given by*

$$F_{\eta_{n-s_n+1:n}}(G_n(x)) := P(\eta_{n-s_n+1:n}\leq G_n(x)) \xrightarrow[n]{w} \Phi\left(\sum_{j=1}^{k} p_j \nu_j(x)\right),$$

if $F_j(\alpha_{j,n}G_n(x))$*belongs to the intermediate-domain of attraction of the non-degenerate type* $\Phi(\nu_j(x))$*, written* $F_j(\alpha_{j,n}G_n(x))\in D_{in}(\Phi(\nu_j(x)))$*,* $j=1,2,...,k$*.*

In the first part of the lemma, there are only three conceivable max-types, according to the Extremal Type Theorem (cf. [21, 25]), namely max-Weibull, Fréchet, and Gumbel types. Moreover, in the second part of the lemma, according to the result of [26], there are only four possible limit types for $\Phi(u_i(x;\lambda))$*. Finally, in the third part of the lemma, according to the result of [27], there are only three possible limit types for* $\Phi(\nu_i(x))$*.*

In the second section of this paper, we study the asymptotic distribution of upper extreme OSs based on the mixture of two SGSs given by (1.2), when the random sample size is assumed to converge weakly and to be independent of the basic variables. In the third section, we obtain the parallel results for the central OSs. In the fourth section, the asymptotic behavior of the intermediate OSs of the mixture of two SGSs is studied under the previous assumptions. In the last section, the asymptotic behavior of the record in the mixture of two SGSs, given in (1.6), is studied under the previous assumptions.

Everywhere in what follows, the symbols $\underset{n}{\longrightarrow}$, $\xrightarrow[n]{w}$, and $\xrightarrow[n]{p}$ symbolize convergence, converge weakly, and converge in probability, as $n\to\infty$, respectively. Moreover, $(*)$ denotes the convolution operation.

2. Asymptotic behavior of extreme OSs in a mixture of two SGSs with a random index

According to the relation $\Phi(a_nx+b_n)\in D(\Psi_3(x))$ (cf. [25]), where

$$\left.\begin{aligned} a_n &= (2\log n)^{-\frac{1}{2}},\\ b_n &= \tfrac{1}{a_n}-\tfrac{a_n}{2}(\log\log n+\log 4\pi), \end{aligned}\right\} \tag{2.1}$$

the Gumbel type $\Psi_3(x)=\exp(-\exp(-x))$ is the only important type in our study. The asymptotic distribution of extreme OS $X_{\nu_n-s+1:\nu_n}$ regarding the sequence (1.2) (and consequently the sequence (1.5)) is determined by the following theorem when the sample size ν_n is assumed to converge weakly and to be independent of the basic variables.

Theorem 2.1. *Let a_n, b_n be defined as in (2.1) and ν_n be a sequence of integer-valued RVs independent of $\{X_i\}$ such that $A_n(nx) \xrightarrow[n]{w} A(x)$, where $P(\nu_n \leq x) = A_n(x)$, $A(+0) = 0$, and $A(x)$ is a non-degenerate DF. Furthermore, let $r_{j,n} \log n \xrightarrow[n]{} \tau_j \geq 0$, $j = 1, 2$. Then,*

$$F_{X_{\nu_n-s+1:\nu_n}}(a_n x + b_n) := P(X_{\nu_n-s+1:\nu_n} \leq a_n x + b_n) \xrightarrow[n]{w} \int_0^\infty \Psi(x, \tau_1, \tau_2, z) dA(z),$$

where

$$\Psi(x, \tau_1, \tau_2, z) = H\left[pu\left(x + \tau_1 - \log z\right) + qu\left(x + \tau_2 - \log z\right)\right] * \left[p\Omega\left(x, \tau_1\right) + q\Omega\left(x, \tau_2\right)\right],$$

$u(x) = e^{-x}$, $H(x) = e^{-x} \sum_{l=0}^{s-1} \frac{x^l}{l!}$,

$$\Omega(x, \tau) := \begin{cases} \Phi\left(\frac{x}{\sqrt{2\tau}}\right), & \text{if } \tau > 0, \\ I_{(0,\infty)}(x), & \text{if } \tau = 0, \end{cases}$$

and $I_A(x)$ is the indicator function of the set A.

Additionally, let $\max(r_{1,n} \log n, r_{2,n} \log n) \xrightarrow[n]{} \infty$. *Then,*

1) $F_{X_{\nu_n-s+1:\nu_n}}(\sqrt{r_{1,n}} x + b_n) \xrightarrow[n]{w} p\Phi(x) + q\Phi(\tau x)$, *if* $r_{j,n} \log n \xrightarrow[n]{} \infty$, $j = 1, 2$, $\sqrt{\frac{r_{1,n}}{r_{2,n}}} \xrightarrow[n]{} \tau > 0$, *and* $r_{1,n}$ *is a slowly varying function (SVF) of n (cf. [28]), i.e.,* $\dfrac{r_{1,n\theta}}{r_{1,n}} \xrightarrow[n]{} 1$, $\forall \theta > 0$.
2) $F_{X_{\nu_n-s+1:\nu_n}}(\sqrt{r_{2,n}} x + b_n) \xrightarrow[n]{w} pI_{(0,\infty)}(x) + q\Phi(x)$, *if* $r_{j,n} \log n \xrightarrow[n]{} \infty$, $j = 1, 2$, $\frac{r_{1,n}}{r_{2,n}} \xrightarrow[n]{} 0$, *and* $r_{2,n}$ *is an SVF of n, or if* $r_{1,n} \log n \xrightarrow[n]{} \tau_1 \geq 0$, $r_{2,n} \log n \xrightarrow[n]{} \infty$, *and* $r_{2,n}$ *is an SVF of n.*
3) $F_{X_{\nu_n-s+1:\nu_n}}(\sqrt{r_{1,n}} x + b_n) \xrightarrow[n]{w} p\Phi(x) + qI_{(0,\infty)}(x)$, *if* $r_{j,n} \log n \xrightarrow[n]{} \infty$, $j = 1, 2$, $\frac{r_{2,n}}{r_{1,n}} \xrightarrow[n]{} 0$, *and* $r_{1,n}$ *is an SVF of n, or if* $r_{1,n} \log n \xrightarrow[n]{} \infty$, $r_{2,n} \log n \xrightarrow[n]{} \tau_2 \geq 0$, *and* $r_{1,n}$ *is an SVF of n.*

Proof. Let $P_{nm} = P(\nu_n = m)$. Thus, from the law of total probability, we obtain

$$F_{X_{\nu_n-s+1:\nu_n}}(a_n x + b_n) = \sum_{m=s}^{\infty} F_{X_{m-s+1:m}}(a_n x + b_n) P_{nm}. \tag{2.2}$$

Assume that $m = [nz]$, then the sum in (2.2) can be represented by the Riemann-Stieltjes integral as

$$F_{X_{\nu_n-s+1:\nu_n}}(a_n x + b_n) = \int_0^\infty F_{X_{nz-s+1:nz}}(a_n x + b_n) dA_n(nz). \tag{2.3}$$

Under the condition $r_{j,n} \log n \xrightarrow[n]{} \tau_j \geq 0$, $j = 1, 2$, and by (1.5), the DF of $X_{nz-s+1:nz}$ is expressed by

$$F_{X_{nz-s+1:nz}}(a_n x + b_n) = P\left(X_{nz-s+1:nz} \leq a_n x + b_n\right) = P\left(U_{nz} + V_{nz} \leq x\right), \tag{2.4}$$

where $U_{nz} = \frac{Z_{0,nz}}{a_n}$ and $V_{nz} = \frac{Z_{nz-s+1:nz} - b_n}{a_n}$ are independent. From (1.3), $U_{nz} = B_p \frac{\sqrt{r_{1,nz}}}{a_n} Y_{1,0} + B_q \frac{\sqrt{r_{2,nz}}}{a_n} Y_{2,0}$, but $\frac{\sqrt{r_{j,nz}}}{a_n} = \sqrt{2 r_{j,nz} \log n} \xrightarrow[n]{} \sqrt{2\tau_j}$, then

$$U_{nz} \xrightarrow[n]{p} \begin{cases} 0, & \text{if } \tau_1 = \tau_2 = 0, \\ B_q \sqrt{2\tau_2} Y_{2,0}, & \text{if } \tau_1 = 0, \tau_2 > 0, \\ B_p \sqrt{2\tau_1} Y_{1,0}, & \text{if } \tau_1 > 0, \tau_2 = 0, \\ B_p \sqrt{2\tau_1} Y_{1,0} + B_q \sqrt{2\tau_2} Y_{2,0}, & \text{if } \tau_1, \tau_2 > 0. \end{cases}$$

Thus, by using the law of total probability and some simple algebra, we get

$$P\left(U_{nz}\leq x\right)\xrightarrow[n]{w}\begin{cases}pI_{(0,\infty)}\left(x\right)+q\Phi\left(\frac{x}{\sqrt{2\tau_2}}\right), & \text{if } \tau_1=0,\tau_2>0,\\ p\Phi\left(\frac{x}{\sqrt{2\tau_1}}\right)+qI_{(0,\infty)}\left(x\right), & \text{if } \tau_1>0,\tau_2=0,\\ p\Phi\left(\frac{x}{\sqrt{2\tau_1}}\right)+q\Phi\left(\frac{x}{\sqrt{2\tau_2}}\right), & \text{if } \tau_1,\tau_2>0.\end{cases}$$

Consequently,

$$\left.\begin{array}{ll}U_{nz}\xrightarrow[n]{p}0, & \text{if } \tau_1=\tau_2=0,\\ P\left(U_{nz}\leq x\right)\xrightarrow[n]{w}p\Omega\left(x,\tau_1\right)+q\Omega\left(x,\tau_2\right), & \text{if } \max(\tau_1,\tau_2)>0.\end{array}\right\} \tag{2.5}$$

In addition, we have

$$P(V_{nz}\leq x)=P(Z_{nz-s+1:nz}\leq a_nx+b_n), \tag{2.6}$$

where $Z_{nz-s+1:nz}$ is the sth upper extreme OS based on the sequence $\{Z_{i,n}\}$ given by (1.3), and $Z_{1,nz},Z_{2,nz},...Z_{nz,nz}$ are i.i.d RVs with the common DF $\mathcal{F}(.)$ given by (1.4). Hence,

$$\mathcal{F}(a_nx+b_n)=p\Phi\left(\frac{a_nx+b_n}{\sqrt{1-r_{1,nz}}}\right)+q\Phi\left(\frac{a_nx+b_n}{\sqrt{1-r_{2,nz}}}\right). \tag{2.7}$$

The limit DF of $Z_{nz-s+1:nz}$ can be found from Lemma 1.1 by determining the domain of attraction for the DF $\Phi\left(\frac{a_nx+b_n}{\sqrt{1-r_{j,nz}}}\right):=\Phi\left(a^*_{j,nz}x+b^*_{j,nz}\right)$. We can do that using the Khinchin's type theorem and Extreme Value Theorem. First, from the assumption $r_{j,n}\log n\xrightarrow[n]{}\tau_j$ (i.e., $r_{j,nz}\log nz\xrightarrow[n]{}\tau_j$), thus $r_{j,n}\xrightarrow[n]{}0$ (i.e., $r_{j,nz}\xrightarrow[n]{}0$), we get $\frac{a^*_{j,nz}}{a_{nz}}=\frac{1}{\sqrt{1-r_{j,nz}}}\frac{\sqrt{\log nz}}{\sqrt{\log n}}\xrightarrow[n]{}1$. By using the relations $(1-r_{j,nz})^{-\frac{1}{2}}=1+\frac{1}{2}r_{j,nz}(1+o(1))$, $a^{-1}_{nz}=\sqrt{2\log n}+\frac{\log z}{\sqrt{2\log n}}(1+o(1))$, $\log\log nz=\log\log n+\log(1+\frac{\log z}{\log n})$ and $\frac{b_{nz}}{a_{nz}}=2\log nz-\frac{1}{2}(\log\log nz+\log 4\pi)$ and taking into consideration that $\frac{\log\ \log\ n}{\log n}\xrightarrow[n]{}0$ (by using the L'Hopital's rule), we get

$$\begin{aligned}\frac{b^*_{j,nz}-b_{nz}}{a_{nz}}&=\frac{a^{-1}_{nz}\,b_n}{\sqrt{1-r_{j,nz}}}-\frac{b_{nz}}{a_{nz}}=\left[1+\frac{1}{2}r_{j,nz}(1+o(1))\right]\left[\sqrt{2\log n}+\frac{\log z}{\sqrt{2\log n}}(1+o(1))\right]\\ &\times\left[\sqrt{2\log n}-\frac{1}{2\sqrt{2\log n}}(\log\log n+\log 4\pi)\right]-2\log nz+\frac{1}{2}(\log\log nz+\log 4\pi)\\ &=2\log n-\frac{1}{2}(\log\log n+\log 4\pi)+\log z(1+o(1))-\frac{\log z}{4\log n}(1+o(1))(\log\log n+\log 4\pi)\\ &+\left[\log n+\log nz+o(1)\log z-\frac{1}{2}(\log\log n+\log 4\pi)-\frac{\log z}{4\log n}(1+o(1))(\log\log n+\log 4\pi)\right]\\ &\times\frac{1}{2}r_{j,nz}(1+o(1))-2\log n-2\log z+\frac{1}{2}\log\log n+\frac{1}{2}\log(1+\frac{\log z}{\log n})+\frac{1}{2}\log 4\pi\xrightarrow[n]{}-\log z+\tau_j.\end{aligned}$$

Consequently, the Khinchin's type theorem and Extreme Value Theorem yield

$$\Phi^{nz}\left(\frac{a_n x + b_n}{\sqrt{1-r_{j,nz}}}\right) \xrightarrow[n]{w} \Psi_3(x+\tau_j - \log z). \tag{2.8}$$

Therefore, from (2.6)–(2.8), and Lemma 1.1, we get

$$\begin{aligned} P(V_{nz} \le x) \quad &\xrightarrow[n]{w} \quad \Psi_3^p(x+\tau_1-\log z)\Psi_3^q(x+\tau_2-\log z) \\ &\quad \times \sum_{l=0}^{s-1} \frac{[-p\log\Psi_3(x+\tau_1-\log z) - q\log\Psi_3(x+\tau_2-\log z)]^l}{l!} \\ &= \quad \exp\{-[pu(x+\tau_1-\log z) + qu(x+\tau_2-\log z)]\} \\ &\quad \times \sum_{l=0}^{s-1} \frac{[pu(x+\tau_1-\log z) + qu(x+\tau_2-\log z)]^l}{l!} \\ &= \quad H[pu(x+\tau_1-\log z) + qu(x+\tau_2-\log z)], \end{aligned} \tag{2.9}$$

for any finite interval of length z. Therefore, from (2.4), (2.5), (2.9) and Lemma 2.2.1 in [21], we get $F_{X_{nz-s+1:nz}}(a_n x + b_n) \xrightarrow[n]{w} \Psi(x,\tau_1,\tau_2,z)$ uniformly with respect to x over any finite interval of z (the convergence is uniform because of the continuity of the limit in x), where Ψ is defined in the theorem. Now, let c be a continuity point of $A(x)$ such that $1-A(c) < \varepsilon$. By using (2.3) and the triangle inequality, we get

$$\begin{aligned} &\left|F_{X_{\nu_n-s+1:\nu_n}}(a_n x+b_n) - \int_0^\infty \Psi(x,\tau_1,\tau_2,z)dA(z)\right| \\ &= \left|\int_0^\infty F_{X_{nz-s+1:nz}}(a_n x+b_n)dA_n(nz) \; - \; \int_0^\infty \Psi(x,\tau_1,\tau_2,z)dA(z)\right| \\ &= \left|\int_0^c F_{X_{nz-s+1:nz}}(a_n x+b_n)dA_n(nz) \; - \; \int_0^c \Psi(x,\tau_1,\tau_2,z)dA(z)\right. \\ &+ \left.\int_c^\infty F_{X_{nz-s+1:nz}}(a_n x+b_n)dA_n(nz) \; - \; \int_c^\infty \Psi(x,\tau_1,\tau_2,z)dA(z)\right| \\ &\le \left|\int_0^c F_{X_{nz-s+1:nz}}(a_n x+b_n)dA_n(nz) \; - \; \int_0^c \Psi(x,\tau_1,\tau_2,z)dA(z)\right| \\ &+ \left|\int_c^\infty F_{X_{nz-s+1:nz}}(a_n x+b_n)dA_n(nz)\right| - \left|\int_c^\infty \Psi(x,\tau_1,\tau_2,z)dA(z)\right|. \end{aligned} \tag{2.10}$$

The second term of the right-hand side in (2.10) can be estimated by

$$\begin{aligned} \left|\int_c^\infty F_{X_{nz-s+1:nz}}(a_n x+b_n)dA_n(nz)\right| &\le |1-A_n(nc)| \\ &\le |1-A(c)-[A_n(nc)-A(c)]| \le 2\varepsilon. \end{aligned} \tag{2.11}$$

The third term of the right-hand side in (2.10) can be estimated by

$$\left|\int_c^\infty \Psi(x,\tau_1,\tau_2,z)dA(z)\right| \le \; |1-A(c)| \le \varepsilon. \tag{2.12}$$

Moreover, using the triangle inequality, the first term of the right-hand side in (2.10) can be estimated by

$$\begin{aligned}
&\left|\int_0^c F_{X_{nz-s+1:nz}}(a_nx+b_n)dA_n(nz)-\int_0^c \Psi(x,\tau_1,\tau_2,z)\,dA(z)\right|\\
&\leq\left|\int_0^c F_{X_{nz-s+1:nz}}(a_nx+b_n)dA_n(nz)-\int_0^c \Psi(x,\tau_1,\tau_2,z)dA_n(nz)\right| \qquad (2.13)\\
&\quad+\left|\int_0^c \Psi(x,\tau_1,\tau_2,z)dA_n(nz)-\int_0^c \Psi(x,\tau_1,\tau_2,z)dA(z)\right|.
\end{aligned}$$

Additionally, the first term of the right-hand side in (2.13) can be estimated by

$$\begin{aligned}
&\left|\int_0^c F_{X_{nz-s+1:nz}}(a_nx+b_n)dA_n(nz)-\int_0^c \Psi(x,\tau_1,\tau_2,z)dA_n(nz)\right|\\
&=\int_0^c \left|F_{X_{nz-s+1:nz}}(a_nx+b_n)-\Psi(x,\tau_1,\tau_2,z)\right|dA_n(nz) \qquad (2.14)\\
&\leq\int_0^c \varepsilon\,dA_n(nz)=\varepsilon\,(A_n(nc)-A_n(0))\leq\varepsilon,
\end{aligned}$$

since $F_{X_{nz-s+1:nz}}(a_nx+b_n)\xrightarrow[n]{w}\Psi(x,\tau_1,\tau_2,z)$ uniformly over the finite interval $[0,c]$. Moreover, the second term of the right-hand side in (2.13) can be estimated by constructing Riemann sums. Specifically, assume that n_0 is a fixed number and that $0=c_0<c_1<...<c_{n_0}=c$ are continuity points of $A(x)$. Moreover, n_0 and c_i are chosen such that

$$\left|\int_0^c \Psi(x,\tau_1,\tau_2,z)dA_n(nz)-\sum_{i=1}^{n_0}\Psi(x,\tau_1,\tau_2,c_i)\left[A_n(nc_i)-A_n(nc_{i-1})\right]\right|<\varepsilon,$$

and

$$\left|\int_0^c \Psi(x,\tau_1,\tau_2,z)dA(z)-\sum_{i=1}^{n_0}\Psi(x,\tau_1,\tau_2,c_i)\left[A(c_i)-A(c_{i-1})\right]\right|<\varepsilon.$$

Thus, once again, by the triangle inequality,

$$\begin{aligned}
&\left|\int_0^c \Psi(x,\tau_1,\tau_2,z)dA_n(nz)-\int_0^c \Psi(x,\tau_1,\tau_2,z)dA(z)\right|\\
&\leq\left|\int_0^c \Psi(x,\tau_1,\tau_2,z)dA_n(nz)-\sum_{i=1}^{n_0}\Psi(x,\tau_1,\tau_2,c_i)\left[A_n(nc_i)-A_n(nc_{i-1})\right]\right|\\
&\quad+\left|\int_0^c \Psi(x,\tau_1,\tau_2,z)dA(z)-\sum_{i=1}^{n_0}\Psi(x,\tau_1,\tau_2,c_i)\left[A(c_i)-A(c_{i-1})\right]\right|\\
&\quad+\left|\sum_{i=1}^{n_0}\Psi(x,\tau_1,\tau_2,c_i)\left\{\left[A_n(nc_i)-A(c_i)\right]-\left[A_n(nc_{i-1})-A(c_{i-1})\right]\right\}\right|<3\varepsilon.
\end{aligned}$$

Combining this fact with (2.14), the left-hand side term of (2.13) becomes smaller than 4ε for large n. Also, combining this fact with (2.11) and (2.12), the left-hand side term of (2.10) becomes smaller than 7ε for large n. The proof for the first part of the theorem is completed.

Turn now to the conditions $r_{j,n}\log n \underset{n}{\longrightarrow} \infty$, $j=1,2$, $\sqrt{\frac{r_{1,n}}{r_{2,n}}} \underset{n}{\longrightarrow} \tau > 0$, and $r_{1,n}$ is an SVF of n. From (1.5), we get

$$F_{X_{nz-s+1:nz}}(\sqrt{r_{1,n}}x+b_n) = P\left(X_{nz-s+1:nz} \leq \sqrt{r_{1,n}}x+b_n\right) = P\left(U_{nz}+V_{nz} \leq x\right), \tag{2.15}$$

where $U_{nz} = \frac{Z_{0,nz}}{\sqrt{r_{1,n}}}$ and $V_{nz} = \frac{Z_{nz-s+1:nz}-b_n}{\sqrt{r_{1,n}}}$ are independent. From (1.3), $U_{nz} = B_p\sqrt{\frac{r_{1,nz}}{r_{1,n}}}\ Y_{1,0} + B_q\sqrt{\frac{r_{2,nz}}{r_{1,n}}}Y_{2,0} \xrightarrow[n]{p} B_pY_{1,0}+B_q\frac{1}{\tau}Y_{2,0}$ since $\sqrt{\frac{r_{2,nz}}{r_{1,n}}} = \sqrt{\frac{r_{2,nz}}{r_{1,nz}}}\sqrt{\frac{r_{1,nz}}{r_{1,n}}} \xrightarrow[n]{p} \frac{1}{\tau}$ (from our conditions). Therefore,

$$P\left(U_{nz} \leq x\right) \xrightarrow[n]{w} p\Phi(x)+q\Phi(\tau x). \tag{2.16}$$

While, $|V_{nz}| \leq \left|\frac{Z_{nz-s+1:nz}-b_{nz}}{\sqrt{r_{1,n}}}\right| + |L_n|$, where $L_n = \frac{b_{nz}-b_n}{\sqrt{r_{1,n}}}$. Then, for every $\varepsilon>0$, we get

$$\begin{aligned} P\left(|V_{nz}| \geq \varepsilon\right) &\leq P\left(\left|\frac{Z_{nz-s+1:nz}-b_{nz}}{a_{nz}}\right| \frac{a_{nz}}{\sqrt{r_{1,n}}} + |L_n| \geq \varepsilon\right) \\ &= P\left(\left|\frac{Z_{nz-s+1:nz}-b_{nz}}{a_{nz}}\right| \geq \frac{\sqrt{r_{1,n}}}{a_{nz}}(\varepsilon-|L_n|)\right) \underset{n}{\longrightarrow} 0, \end{aligned} \tag{2.17}$$

since $\frac{\sqrt{r_{1,n}}}{a_{nz}} = \frac{\sqrt{r_{1,nz}}}{a_{nz}}\sqrt{\frac{r_{1,n}}{r_{1,nz}}} = \sqrt{2r_{1,nz}\log nz}\sqrt{\frac{r_{1,n}}{r_{1,nz}}} \underset{n}{\rightarrow} \infty$ and $L_n \underset{n}{\longrightarrow} 0$ (as we will show). Using the relations $a_{nz}^{-1} = \sqrt{2\log n}+\frac{\log z}{\sqrt{2\log n}}(1+o(1))$, $a_{nz} = \frac{1}{\sqrt{2\log n}}[1-\frac{\log z}{2\log n}(1+o(1))]$ and $\log\log nz = \log\log n+\log(1+\frac{\log z}{\log n})$, we obtain

$$L_n = \frac{1}{\sqrt{r_{1,n}}}\left\{\frac{1}{a_{nz}}-\frac{1}{a_n}-\frac{1}{2}\left[a_{nz}(\log\log nz+\log 4\pi)-a_n(\log\log n+\log 4\pi)\right]\right\},$$

but

$$\begin{aligned} \frac{1}{\sqrt{r_{1,n}}}(\frac{1}{a_{nz}}-\frac{1}{a_n}) &= \frac{1}{\sqrt{r_{1,n}}}\left[\sqrt{2\log n}+\frac{\log z}{\sqrt{2\log n}}(1+o(1))-\sqrt{2\log n}\right] \\ &= \frac{\log z}{\sqrt{2r_{1,n}\log n}}(1+o(1)) \underset{n}{\longrightarrow} 0, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2\sqrt{r_{1,n}}}\left[a_{nz}(\log\log nz+\log 4\pi)-a_n(\log\log n+\log 4\pi)\right] \\ &= \frac{1}{2\sqrt{r_{1,n}}}\left[\frac{1}{\sqrt{2\log n}}\left(1-\frac{\log z}{2\log n}(1+o(1))\right)(\log\log n+\log(1+\frac{\log z}{\log n})+\log 4\pi)\right. \\ &\qquad \left. -\frac{1}{\sqrt{2\log n}}(\log\log n+\log 4\pi)\right] = \\ &\frac{1}{2\sqrt{2r_{1,n}\log n}}\left[\log(1+\frac{\log z}{\log n})-\frac{\log z}{2\log n}(1+o(1))(\log\log n+\log(1+\frac{\log z}{\log n})+\log 4\pi)\right] \\ &\underset{n}{\longrightarrow} 0. \end{aligned}$$

Finally, from (2.15)–(2.17), and Lemma 2.2.1 in [21], we get

$$F_{X_{nz-s+1:nz}}(\sqrt{r_{1,n}}x + b_n) \xrightarrow[n]{w} p\Phi(x) + q\Phi(\tau x).$$

Thus, the remainder proof of this case is precisely the same as that of the first case.

Consider the conditions $r_{j,n} \log n \underset{n}{\longrightarrow} \infty$, $j = 1, 2$, $\frac{r_{1,n}}{r_{2,n}} \underset{n}{\longrightarrow} 0$, and $r_{2,n}$ is an SVF of n, or if $r_{1,n} \log n \underset{n}{\longrightarrow} \tau_1 \geq 0$, $r_{2,n} \log n \underset{n}{\longrightarrow} \infty$, and $r_{2,n}$ is an SVF of n. Using the same technique, we get

$$F_{X_{nz-s+1:nz}}(\sqrt{r_{2,n}}x + b_n) = P(U_{nz} + V_{nz} \leq x) \xrightarrow[n]{w} pI_{(0,\infty)}(x) + q\Phi(x),$$

where $U_{nz} = \frac{Z_{0,nz}}{\sqrt{r_{2,n}}} = B_p \sqrt{\frac{r_{1,nz}}{r_{2,n}}} Y_{1,0} + B_q \sqrt{\frac{r_{2,nz}}{r_{2,n}}} Y_{2,0} \xrightarrow[n]{p} B_q Y_{2,0}$ and $P(|V_{nz}| \geq \varepsilon) \underset{n}{\longrightarrow} 0$ since $|V_{nz}| = \left|\frac{Z_{nz-s+1:nz} - b_n}{\sqrt{r_{2,n}}}\right| \leq \left|\frac{Z_{nz-s+1:nz} - b_{nz}}{\sqrt{r_{2,n}}}\right| + |L_n|$, and $L_n = \frac{b_{nz} - b_n}{\sqrt{r_{2,n}}} \underset{n}{\rightarrow} 0$. The remainder proof of this case is precisely the same as that of the first case. Finally, it is easy to see that the proof of the last case is similar as that of the second case. The theorem is now fully proved. □

Example 2.1. *When $0 \leq \tau_1, \tau_2 < \infty$, it is natural to look for the limitations on ν_n, under which we get the relation $\lim_{n\to\infty} F_{X_{\nu_n - s+1:\nu_n}}(a_n x + b_n) = \lim_{n\to\infty} F_{X_{n-s+1:n}}(a_n x + b_n)$. In view of Theorem 2.1, the last equation is satisfied, if and only if, the DF $A(z)$ is degenerate at one, which means the asymptotically almost randomlessness of ν_n. This situation practically happens if we have the RV ν_n following shifted Poisson DF, with probability mass function (PMF) $P(\nu_n = x) = \frac{e^{-\lambda_n}\lambda_n^{x-\rho}}{(x-\rho)!}$, $x = \rho, \rho+1, ...,$ for some integer $\rho > 1$, where $\frac{\lambda_n}{n} \underset{n}{\longrightarrow} 1$. Another practical and important case is when we assume that the random sample size follows the shifted geometric RV ν_n, with PMF $P(\nu_n = x) = p_n(1 - p_n)^{x-\rho}, x = \rho + 1, \rho + 2, ...,$ where $np_n \underset{n}{\longrightarrow} 1$. Clearly, the characteristic function $\psi(t) = E(e^{\frac{i\nu_n t}{n}}) = \frac{p_n e^{\frac{i\rho t}{n}}}{1-(1-p_n)e^{\frac{it}{n}}} \underset{n}{\longrightarrow} \frac{1}{1-it}$. Therefore, $P(\nu_n \leq nz) \xrightarrow[n]{w} 1 - e^{-z}, z > 0$. As an important result of this case when $r_{j,n} \log n \underset{n}{\longrightarrow} \tau_j = 0$, $j = 1, 2$, since $u(x - \log z) = zu(x)$ and $\Psi(x, 0, 0, z) = H(u(x - \log z))$, we get*

$$F_{X_{\nu_n - s+1:\nu_n}}(a_n x + b_n) \xrightarrow[n]{w} \sum_{l=0}^{s-1} \frac{u^l(x)}{l!} \int_0^\infty z^l e^{-z(1-u(x))} dz = \sum_{l=0}^{s-1} \frac{e^{-lx}}{(1 - e^{-x})^{l+1}}.$$

3. Asymptotic behavior of central OSs in a mixture of two SGSs with a random index

There are only four conceivable limit types for $\Phi(u_i(x; \lambda))$ in Lemma 1.1, as shown in [26]. But the only used type in our study is the normal type because of the following lemma.

Lemma 3.1 (cf. [11])**.** *Let $\eta_1, \eta_2, ..., \eta_n$ be i.i.d RVs with a common DF F_η and a PDF f_η. Then, $F_{\eta_{n-s_n+1:n}}(c_n x + x_\lambda) \xrightarrow[n]{w} \Phi(x)$ as $\sqrt{n}(\frac{s_n}{n} - \lambda) \underset{n}{\longrightarrow} 0$, where $F_\eta(x_\lambda) = 1 - \lambda$, $f_\eta(x_\lambda) > 0$, $0 < \lambda < 1$, and $c_n = \frac{\sqrt{\lambda(1-\lambda)}}{\sqrt{n} f_\eta(x_\lambda)}$.*

Lemma 3.1 was previously presented in [26] for $\eta_{s_n:n}$ with the same limit, but the normalizing constants are $c'_n = \frac{\sqrt{\lambda'(1-\lambda')}}{\sqrt{n} f_\eta(x_{\lambda'})}$ and $x_{\lambda'}$, where $F_\eta(x_{\lambda'}) = \lambda' = 1 - \lambda$. Consequently, $c'_n = c_n$ and $x_{\lambda'} = x_\lambda$.

When the random sample size is considered to converge weakly and to be independent of the basic variables, the asymptotic distribution of the central OS $X_{\nu_n - s_{\nu_n}+1:\nu_n}$ regarding the sequence (1.5) is derived by the following theorem.

Theorem 3.1. *Let $\Phi(x_\lambda) = 1-\lambda$, $c_n = \frac{\sqrt{\lambda(1-\lambda)}}{\varphi(x_\lambda)\sqrt{n}}$ and $\{\nu_n\}$ be a sequence of integer-valued RVs independent of $\{X_i\}$ such that $A_n(nx) \xrightarrow[n]{w} A(x)$, where $P(\nu_n \le x) = A_n(x)$, $A(x)$ is a non-degenerate DF, and $\varphi(x)$ is the PDF of a standard normal variable. When $\sqrt{n}(\frac{s_n}{n} - \lambda) \xrightarrow[n]{} 0$, the asymptotic distribution of the central OS $X_{s'(\nu_n):\nu_n} := X_{\nu_n - s_{\nu_n}+1:\nu_n}$ is given by:*

1) $F_{X_{s'(\nu_n):\nu_n}}(c_n x + x_\lambda) \xrightarrow[n]{w} \int_0^\infty \Psi^*(x,\tau_1,\tau_2,z)dA(z)$, *if* $nr_{j,n} \xrightarrow[n]{} \tau_j \ge 0$, *for* $j = 1,2$, *where* $\Psi^*(x,\tau_1,\tau_2,z) = p\Phi\left(\left(1+\frac{\tau_1\varphi^2(x_\lambda)}{\lambda(1-\lambda)}\right)^{-\frac{1}{2}}\sqrt{z}x\right) + q\Phi\left(\left(1+\frac{\tau_2\varphi^2(x_\lambda)}{\lambda(1-\lambda)}\right)^{-\frac{1}{2}}\sqrt{z}x\right)$.
2) $F_{X_{s'(\nu_n):\nu_n}}\left(\sqrt{r_{1,n}}x + x_\lambda\right) \xrightarrow[n]{w} p\Phi(x) + q\Phi(\tau x)$, *if* $nr_{j,n} \xrightarrow[n]{} \infty$, $j = 1,2$, $\sqrt{\frac{r_{1,n}}{r_{2,n}}} \xrightarrow[n]{} \tau > 0$ *and* $r_{1,n}$ *is an SVF of* n.
3) $F_{X_{s'(\nu_n):\nu_n}}\left(\sqrt{r_{2,n}}x + x_\lambda\right) \xrightarrow[n]{w} pI_{(0,\infty)}(x) + q\Phi(x)$, *if* $nr_{j,n} \xrightarrow[n]{} \infty$, $j = 1,2$, $\frac{r_{1,n}}{r_{2,n}} \xrightarrow[n]{} 0$, *and* $r_{2,n}$ *is an SVF of* n, *or if* $nr_{1,n} \xrightarrow[n]{} \tau_1 \ge 0$, $nr_{2,n} \xrightarrow[n]{} \infty$, *and* $r_{2,n}$ *is an SVF of* n.
4) $F_{X_{s'(\nu_n):\nu_n}}\left(\sqrt{r_{1,n}}x + x_\lambda\right) \xrightarrow[n]{w} p\Phi(x) + qI_{(0,\infty)}(x)$, *if* $nr_{j,n} \xrightarrow[n]{} \infty$, $j = 1,2$, $\frac{r_{2,n}}{r_{1,n}} \xrightarrow[n]{} 0$, *and* $r_{1,n}$ *is an SVF of* n, *or if* $nr_{1,n} \xrightarrow[n]{} \infty$, $nr_{2,n} \xrightarrow[n]{} \tau_2 \ge 0$, *and* $r_{1,n}$ *is an SVF of* n.

Proof. By starting the proof as we have done in Theorem 2.1, we get the corresponding equation to (2.3) as

$$F_{X_{s'(\nu_n):\nu_n}}(c_n x + x_\lambda) = \int_0^\infty F_{X_{s'(nz):nz}}(c_n x + x_\lambda)dA_n(nz). \tag{3.1}$$

The condition, $nr_{j,n} \xrightarrow[n]{} \tau_j \ge 0$, $j = 1,2$, implies that the DF of $X_{s'(nz):nz}$ is expressed as

$$F_{X_{s'(nz):nz}}(c_n x + x_\lambda) = P\left(X_{s'(nz):nz} \le c_n x + x_\lambda\right) = P\left(U_{nz} + V_{nz} \le x\right), \tag{3.2}$$

where $U_{nz} = \frac{Z_{0,nz}}{c_n}$ and $V_{nz} = \frac{Z_{s'(nz):nz} - x_\lambda}{c_n}$ are independent. According to (1.3), $U_{nz} = B_p\frac{\sqrt{r_{1,nz}}}{c_n}Y_{1,0} + B_q\frac{\sqrt{r_{2,nz}}}{c_n}Y_{2,0}$, but $\frac{\sqrt{r_{j,nz}}}{c_n} = \varphi(x_\lambda)\sqrt{\frac{nr_{j,nz}}{\lambda(1-\lambda)}} \xrightarrow[n]{} \varphi(x_\lambda)\sqrt{\frac{\tau_j}{\lambda(1-\lambda)z}}$, thus

$$U_{nz} \xrightarrow[n]{p} B_p\sqrt{\frac{\tau_1\varphi^2(x_\lambda)}{\lambda(1-\lambda)z}}Y_{1,0} + B_q\sqrt{\frac{\tau_2\varphi^2(x_\lambda)}{\lambda(1-\lambda)z}}Y_{2,0}. \tag{3.3}$$

Additionally, we have

$$P(V_{nz} \le x) = P(Z_{s'(nz):nz} \le c_n x + x_\lambda) = F_{Z_{s'(nz):nz}}(c_n x + x_\lambda), \tag{3.4}$$

where $Z_{s'(nz):nz}$ is a central OS based on the sequence $\{Z_{i,n}\}$, given by (1.3), and $Z_{1,nz}, Z_{2,nz}, ..., Z_{nz,nz}$ are i.i.d RVs with the common DF $\mathcal{F}(.)$ given by (1.4). Hence,

$$\mathcal{F}(c_n x + x_\lambda) = p\Phi\left(\frac{c_n x + x_\lambda}{\sqrt{1 - r_{1,nz}}}\right) + q\Phi\left(\frac{c_n x + x_\lambda}{\sqrt{1 - r_{2,nz}}}\right). \tag{3.5}$$

Using the Khinchin's type theorem and Lemma 3.1, we will show that

$$\Phi\left(\frac{c_n}{\sqrt{1 - r_{j,nz}}}x + \frac{x_\lambda}{\sqrt{1 - r_{j,nz}}}\right) := \Phi\left(c^*_{j,nz}x + b^*_{j,nz}\right) \in D_\lambda(\Phi(\sqrt{z}x)),\ j = 1,2, \tag{3.6}$$

because $\frac{c^*_{j,nz}}{c_{nz}} = \frac{\sqrt{z}}{\sqrt{1-r_{j,nz}}} \underset{n}{\longrightarrow} \sqrt{z}$ (from $nr_{j,n} \underset{n}{\longrightarrow} \tau_j$) and $\frac{b^*_{j,nz}-x_\lambda}{c_{nz}} = \frac{x_\lambda}{c_{nz}}\left(\frac{1}{\sqrt{1-r_{j,nz}}} - 1\right) \underset{n}{\longrightarrow} 0$, which can be proved using $(1 - r_{j,nz})^{-\frac{1}{2}} = 1 + \frac{1}{2}r_{j,nz}(1 + o(1))$ and $r_{j,nz}\sqrt{nz} \underset{n}{\longrightarrow} 0$. Therefore, from (3.4)–(3.6) and Lemma 1.1, we get

$$P(V_{nz} \leq x) \overset{w}{\underset{n}{\longrightarrow}} \Phi\left(\sqrt{z}x\right), \tag{3.7}$$

over any finite interval of length z. Thus, from (3.2), a combination of (3.3) and (3.7) yields $F_{X_{s'(nz):nz}}(c_n x + x_\lambda) \overset{w}{\underset{n}{\longrightarrow}} \Psi^*(x, \tau_1, \tau_2, z)$ uniformly with respect to x for any finite interval of z (the convergence is uniform because of the continuity of the limit in x). Using this convergence and (3.1), the remainder proof of this case is precisely the same as that of the first case of Theorem 2.1.

Turn now to the conditions $nr_{j,n} \underset{n}{\longrightarrow} \infty$, $j = 1, 2$, $\sqrt{\frac{r_{1,n}}{r_{2,n}}} \underset{n}{\longrightarrow} \tau > 0$, and $r_{1,n}$ is an SVF of n. From (1.5), we get

$$F_{X_{s'(nz):nz}}\left(\sqrt{r_{1,n}}x + x_\lambda\right) = P\left(\frac{Z_{0,nz}}{\sqrt{r_{1,n}}} + \frac{Z_{s'(nz):nz} - x_\lambda}{\sqrt{r_{1,n}}} \leq x\right). \tag{3.8}$$

From (1.3), $\frac{Z_{0,nz}}{\sqrt{r_{1,n}}} = B_p\sqrt{\frac{r_{1,nz}}{r_{1,n}}}\,Y_{1,0} + B_q\sqrt{\frac{r_{2,nz}}{r_{1,n}}}Y_{2,0} \overset{p}{\underset{n}{\longrightarrow}} B_pY_{1,0} + B_q\,\frac{1}{\tau}Y_{2,0}$ since $\sqrt{\frac{r_{2,nz}}{r_{1,n}}} = \sqrt{\frac{r_{2,nz}}{r_{1,nz}}}\sqrt{\frac{r_{1,nz}}{r_{1,n}}} \overset{p}{\underset{n}{\longrightarrow}} \frac{1}{\tau}$ (from our conditions). Thus,

$$P\left(\frac{Z_{0,nz}}{\sqrt{r_{1,n}}} \leq x\right) \overset{w}{\underset{n}{\longrightarrow}} p\Phi(x) + q\Phi(\tau x). \tag{3.9}$$

In addition, for every $\varepsilon > 0$, we get

$$p\left(\frac{\left|Z_{s'(nz):nz} - x_\lambda\right|}{\sqrt{r_{1,n}}} > \varepsilon\right) = p\left(\frac{\left|Z_{s'(nz):nz} - x_\lambda\right|}{c_{nz}} > \frac{\sqrt{r_{1,n}}}{c_{nz}}\varepsilon\right) \underset{n}{\longrightarrow} 0, \tag{3.10}$$

since $\frac{\sqrt{r_{1,n}}}{c_{nz}} = \varphi(x_\lambda)\sqrt{\frac{nzr_{1,n}}{\lambda(1-\lambda)}} \underset{n}{\longrightarrow} \infty$. Finally, from (3.8) and Lemma 2.2.1 in [21], a combination of (3.9) and (3.10) yields $F_{X_{s'(nz):nz}}\left(\sqrt{r_{1,n}}x + x_\lambda\right) \overset{w}{\underset{n}{\longrightarrow}} p\Phi(x) + q\Phi(\tau x)$. Using this convergence and (3.1), the remainder proof of this case is precisely the same as that of the first case of Theorem 2.1.

Consider the conditions $nr_{j,n} \underset{n}{\longrightarrow} \infty$, $j = 1, 2$, $\frac{r_{1,n}}{r_{2,n}} \underset{n}{\longrightarrow} 0$, and $r_{2,n}$ is an SVF of n, or $nr_{1,n} \underset{n}{\longrightarrow} \tau_1 \geq 0$, $nr_{2,n} \underset{n}{\longrightarrow} \infty$, and $r_{2,n}$ is an SVF of n. In the same manner, we get $F_{X_{s'(nz):nz}}(\sqrt{r_{2,n}}x + x_\lambda) = P\left(\frac{Z_{0,nz}}{\sqrt{r_{2,n}}} + \frac{Z_{s'(nz):nz}-x_\lambda}{\sqrt{r_{2,n}}} \leq x\right) \overset{w}{\underset{n}{\longrightarrow}} pI_{(0,\infty)}(x) + q\Phi(x)$ since $\frac{Z_{0,nz}}{\sqrt{r_{2,n}}} = B_p\sqrt{\frac{r_{1,nz}}{r_{2,n}}}Y_{1,0} + B_q\sqrt{\frac{r_{2,nz}}{r_{2,n}}}Y_{2,0} \overset{p}{\underset{n}{\longrightarrow}} B_qY_{2,0}$ and $P\left(\frac{\left|Z_{s'(nz):nz}-x_\lambda\right|}{\sqrt{r_{2,n}}} \geq \varepsilon\right) \underset{n}{\longrightarrow} 0$ because $\frac{\sqrt{r_{2,n}}}{c_n} = \frac{\varphi(x_\lambda)\sqrt{nr_{2,n}}}{\sqrt{\lambda(1-\lambda)}} \underset{n}{\longrightarrow} \infty$. The remainder proof of this case is precisely the same as that of Theorem 2.1. Finally, it is easy to see that the proof of the fourth case is similar as that of the third case. □

Example 3.1. *Let $0 \leq \tau_1, \tau_2 < \infty$. Furthermore, let the random sample size follow the shifted geometric RV ν_n, with PMF $P(\nu_n = x) = p_n(1 - p_n)^{x-\rho}, x = \rho + 1, \rho + 2, ...$, where $np_n \underset{n}{\longrightarrow} 1$. In view of Example 2.1, we get $A_n(nz) \overset{w}{\underset{n}{\longrightarrow}} A(x)$, where $A(x)$ is the standard negative exponential distribution. Now, an application of Theorem 3.1 yields*

$$\Lambda(x) := \int_0^\infty \Psi^*(x, \tau_1, \tau_2, z)dA(z) = \int_0^\infty (p\Phi(\sigma_1 x\sqrt{z}) + q\Phi(\sigma_2 x\sqrt{z}))e^{-z}dz,$$

where $\sigma_i = (1 + \frac{\tau_i \varphi^2(x_\lambda)}{\lambda(1-\lambda)})^{-\frac{1}{2}},\ i = 1, 2$. *Therefore, if* $x > 0$, *we get after some algebra*

$$\Lambda(x) = \frac{p}{2}\left(1 + \frac{\sigma_1 x}{2 + \sigma_1^2 x^2}\right) + \frac{q}{2}\left(1 + \frac{\sigma_2 x}{2 + \sigma_2^2 x^2}\right).$$

Similarly, if $x < 0$ *after some calculations, we get* $\Lambda(x) = \frac{p}{2}\left(1 + \frac{\sigma_1 x}{2+\sigma_1^2 x^2}\right) + \frac{q}{2}\left(1 + \frac{\sigma_2 x}{2+\sigma_2^2 x^2}\right)$. *Therefore, for all* x, *we have* $\Lambda(x) = \frac{p}{2}\left(1 + \frac{\sigma_1 x}{2+\sigma_1^2 x^2}\right) + \frac{q}{2}\left(1 + \frac{\sigma_2 x}{2+\sigma_2^2 x^2}\right)$. *This DF has no moments even if the order is less than one.*

4. Asymptotic behavior of intermediate OSs in a mixture of two SGSs with a random index

There are only three conceivable limit types for $\Phi(v_i(x))$ in Lemma 1.1, as shown in [27]. But the only used type in our study is the normal type because of the following lemma.

Lemma 4.1 (cf. [11])**.** *Let* $\eta_1, \eta_2, ..., \eta_n$ *be i.i.d RVs from the standard normal distribution* $\Phi(x)$, *then the asymptotic distribution of any upper intermediate OS* $\eta_{n-s_n+1:n}$ *is given by* $\Phi_{\eta_{n-s_n+1:n}}(a_n x + b_n) \xrightarrow[n]{w} \Phi(x)$, *where* $a_n = \frac{\sqrt{s_n}}{n\varphi(b_n)} \sim \frac{1}{b_n\sqrt{s_n}}$, $\Phi(b_n) = 1 - \frac{s_n}{n}$, *and* $b_n \sim \sqrt{2\log\frac{n}{s_n}}$, *as* $n \longrightarrow \infty$.

Lemma 4.1 was previously presented, but for a lower intermediate OS $\eta_{s_n:n}$ in [29].

The next theorem gives the asymptotic distribution of the upper intermediate OS $X_{\nu_n - s_{\nu_n}+1:\nu_n}$ of the sequence (1.5) under the Chibisov rank sequence $s_n \sim ln^{\alpha}$, $0 < \alpha < 1$ (see [3, 27]), where the sample size ν_n is supposed to converge weakly and to be independent of the basic variables.

Theorem 4.1. *let* $s_n \sim ln^{\alpha}$, $0 < \alpha < 1$ $a_n = \frac{\sqrt{s_n}}{n\varphi(b_n)} \sim \frac{1}{b_n\sqrt{s_n}}$, $\Phi(b_n) = 1 - \frac{s_n}{n}$ *and* $b_n \sim \sqrt{2\log\frac{n}{s_n}}$, *as* $n \longrightarrow \infty$. *Furthermore, let* $\{\nu_n\}$ *be a sequence of integer-valued RVs independent of* $\{X_i\}$ *such that* $A_n(nx) \xrightarrow[n]{w} A(x)$, *where* $A(x)$ *is a non-degenerate DF. Then, for any upper intermediate OS* $X_{s'(\nu_n):\nu_n} := X_{\nu_n - s_{\nu_n}+1:\nu_n}$ *we have*

1) $F_{X_{s'(\nu_n):\nu_n}}(a_n x + b_{n_\nu}) \xrightarrow[n]{w} \int_0^\infty \Psi^\dagger(x, \tau_1, \tau_2, z)dA(z)$, *if* $s_n r_{j,n}\log n \xrightarrow[n]{} \tau_j \geq 0$, *for* $j = 1, 2$, *where* $\Psi^\dagger(x, \tau_1, \tau_2, z) = p\Phi\left(\frac{z^{\alpha/2}x}{\sqrt{1+2\tau_1(1-\alpha)}}\right) + q\Phi\left(\frac{z^{\alpha/2}x}{\sqrt{1+2\tau_2(1-\alpha)}}\right)$.
2) $F_{X_{s'(\nu_n):\nu_n}}\left(\sqrt{r_{1,n}}x + b_{\nu_n}\right) \xrightarrow[n]{w} p\Phi(x) + q\Phi(\tau x)$, *if* $s_n r_{j,n}\log n \xrightarrow[n]{} \infty$, $j = 1, 2$, $\sqrt{\frac{r_{1,n}}{r_{2,n}}} \xrightarrow[n]{} \tau > 0$, *and* $r_{1,n}$ *is an SVF of* n.
3) $F_{X_{s'(\nu_n):\nu_n}}\left(\sqrt{r_{2,n}}x + b_{\nu_n}\right) \xrightarrow[n]{w} pI_{(0,\infty)}(x) + q\Phi(x)$, *if* $s_n r_{j,n}\log n \xrightarrow[n]{} \infty$, $j = 1, 2$, $\frac{r_{1,n}}{r_{2,n}} \xrightarrow[n]{} 0$, *and* $r_{2,n}$ *is an SVF of* n, *or* $s_n r_{1,n}\log n \xrightarrow[n]{} \tau_1 \geq 0$, $s_n r_{2,n}\log n \xrightarrow[n]{} \infty$, *and* $r_{2,n}$ *is an SVF of* n.
4) $F_{X_{s'(\nu_n):\nu_n}}\left(\sqrt{r_{1,n}}x + b_{\nu_n}\right) \xrightarrow[n]{w} p\Phi(x) + qI_{(0,\infty)}(x)$, *if* $s_n r_{j,n}\log n \xrightarrow[n]{} \infty$, $j = 1, 2$, $\frac{r_{2,n}}{r_{1,n}} \xrightarrow[n]{} 0$, *and* $r_{1,n}$ *is an SVF of* n, *or* $s_n r_{1,n}\log n \xrightarrow[n]{} \infty$, $s_n r_{2,n}\log n \xrightarrow[n]{} \tau_2 \geq 0$, *and* $r_{1,n}$ *is an SVF of* n.

Proof. By starting the proof as we have done in Theorem 2.1, we get the corresponding equation to (2.3) as

$$F_{X_{s'(\nu_n):\nu_n}}(a_n x + b_{n_\nu}) = \int_0^\infty F_{X_{s'(nz):nz}}(a_n x + b_{nz})dA_n(nz). \tag{4.1}$$

First, under the condition $s_n r_{j,n}\log n \underset{n}{\longrightarrow} \tau_j \geq 0$, for $j = 1,2$, and from (1.5), the DF of $X_{s'(nz):nz}$ is expressed as

$$F_{X_{s'(nz):nz}}(a_n x + b_{nz}) = P\left(X_{s'(nz):nz} \leq a_n x + b_{nz}\right) = P\left(U_{nz} + V_{nz} \leq x\right), \tag{4.2}$$

where $U_{nz} = \frac{Z_{0,nz}}{a_n}$ and $V_{nz} = \frac{Z_{s'(nz):nz} - b_{nz}}{a_n}$ are independent. From (1.3), we get $U_{nz} = B_p \frac{\sqrt{r_{1,nz}}}{a_n} Y_{1,0} + B_q \frac{\sqrt{r_{2,nz}}}{a_n} Y_{2,0}$, but $\frac{\sqrt{r_{j,nz}}}{a_n} \sim \sqrt{2 r_{j,nz} s_n \log n\,(1 - \frac{\log s_n}{\log n})} = \sqrt{2 s_{nz} r_{j,nz}\log(nz)}\,\sqrt{\frac{s_n}{s_{nz}}\frac{\log n}{\log(nz)}\,(1 - \frac{\log s_n}{\log n})} \underset{n}{\longrightarrow} \sqrt{2\tau_j z^{-\alpha}(1-\alpha)}$ since $\log\frac{n}{s_n} = \log n\,(1 - \frac{\log s_n}{\log n})$, $\frac{s_{nz}}{s_n} \sim z^{\alpha}$, $\frac{\log s_n}{\log n} \underset{n}{\longrightarrow} \alpha$ and $\frac{\log(nz)}{\log n} = 1 + \frac{\log z}{\log n} \underset{n}{\longrightarrow} 1$. Thus,

$$U_{nz} \overset{p}{\underset{n}{\longrightarrow}} B_p\sqrt{2\tau_1(1-\alpha)}\, z^{-\frac{\alpha}{2}} Y_{1,0} + B_q\sqrt{2\tau_2(1-\alpha)}\, z^{-\frac{\alpha}{2}} Y_{2,0}. \tag{4.3}$$

In addition,

$$P(V_{nz} \leq x) = P(Z_{s'(nz):nz} \leq a_n x + b_{nz}) = F_{Z_{s'(nz):nz}}(a_n x + b_{nz}), \tag{4.4}$$

where $Z_{s'(nz):nz}$ is an intermediate OS based on the sequence $\{Z_{i,n}\}$, given by (1.3), and $Z_{1,nz}, Z_{2,nz}, ...Z_{nz,nz}$ are i.i.d RVs with the common DF (1.4). Therefore,

$$\mathcal{F}(a_n x + b_{nz}) = p\Phi\left(\frac{a_n x + b_{nz}}{\sqrt{1 - r_{1,nz}}}\right) + q\Phi\left(\frac{a_n x + b_{nz}}{\sqrt{1 - r_{2,nz}}}\right). \tag{4.5}$$

Using Khinchin's type theorem and Lemma 4.1, we will show that

$$\Phi\left(\frac{a_n}{\sqrt{1 - r_{j,nz}}}x + \frac{b_{nz}}{\sqrt{1 - r_{j,nz}}}\right) := \Phi\left(a^*_{j,nz}x + b^*_{j,nz}\right) \in D_{in}(\Phi(z^{\alpha/2}x)),\ j = 1,2. \tag{4.6}$$

Now, we need the limit of $\frac{a^*_{j,nz}}{a_{nz}}$ and $\frac{b^*_{j,nz} - b_{nz}}{a_{nz}}$. First, we get $\frac{a^*_{j,nz}}{a_{nz}} = \frac{a_n}{a_{nz}}\frac{1}{\sqrt{1 - r_{j,nz}}}$, but $r_{j,nz} \underset{n}{\longrightarrow} 0$ (from $s_n r_{j,n}\log n \underset{n}{\longrightarrow} \tau_j$) and $\frac{a_n}{a_{nz}} \sim \sqrt{\frac{s_{nz}}{s_n}}\sqrt{\frac{\log\frac{nz}{s_{nz}}}{\log\frac{n}{s_n}}} \underset{n}{\longrightarrow} z^{\alpha/2}$ since $\frac{s_{nz}}{s_n} \sim z^{\alpha}$, $\log\frac{nz}{s_{nz}} = \log(nz)\,(1 - \frac{\log s_{nz}}{\log(nz)})$, $\log\frac{n}{s_n} = \log n\,(1 - \frac{\log s_n}{\log n})$, $\frac{\log s_{nz}}{\log(nz)} \underset{n}{\longrightarrow} \alpha$, $\frac{\log s_n}{\log n} \underset{n}{\longrightarrow} \alpha$ and $\frac{\log(nz)}{\log n} = 1 + \frac{\log z}{\log n} \underset{n}{\longrightarrow} 1$. Thus $\frac{a^*_{j,nz}}{a_{nz}} \underset{n}{\longrightarrow} z^{\alpha/2}$. Second, we get $\frac{b^*_{j,nz} - b_{nz}}{a_{nz}} = \frac{b_{nz}}{a_{nz}}\left(\frac{1}{\sqrt{1 - r_{j,nz}}} - 1\right)$, but $(1 - r_{j,nz})^{-\frac{1}{2}} = 1 + \frac{1}{2}r_{j,nz}(1 + o(1))$ and $\frac{b_{nz}}{a_{nz}} \sim 2\sqrt{s_{nz}}\log\frac{nz}{s_{nz}}$. Thus $\frac{b^*_{j,nz} - b_{nz}}{a_{nz}} \sim r_{j,nz}\sqrt{s_{nz}}\log(nz)\,(1 - \frac{\log s_{nz}}{\log(nz)})(1 + o(1)) \underset{n}{\longrightarrow} 0$ (from $s_n r_{j,n}\log n \underset{n}{\longrightarrow} \tau_j$). Consequently, from (4.4)–(4.6) and Lemma 1.1, we get

$$P(V_{nz} \leq x) \overset{w}{\underset{n}{\longrightarrow}} \Phi\left(z^{\alpha/2}x\right), \tag{4.7}$$

over any finite interval of length z. Thus, from (4.2), a combination of (4.3) and (4.7) yields $F_{X_{s'(nz):nz}}(a_n x + b_n) \overset{w}{\underset{n}{\longrightarrow}} \Psi^{\dagger}(x, \tau_1, \tau_2, z)$ uniformly with respect to x for any finite interval of z (the convergence is uniform because of the continuity of the limit in x). Using this convergence and (4.1), the remainder proof of this case is precisely the same as that of the first case of Theorem 2.1.

Turn now to the conditions $s_n r_{j,n}\log n \underset{n}{\longrightarrow} \infty$, $j = 1,2$, $\sqrt{\frac{r_{1,n}}{r_{2,n}}} \underset{n}{\longrightarrow} \tau > 0$ and $r_{1,n}$ is an SVF of n. From (1.5), we get

$$F_{X_{s'(nz):nz}}\left(\sqrt{r_{1,n}}x + b_{nz}\right) = P\left(\frac{Z_{0,nz}}{\sqrt{r_{1,n}}} + \frac{Z_{s'(nz):nz} - b_{nz}}{\sqrt{r_{1,n}}} \leq x\right). \tag{4.8}$$

From (1.3), $\frac{Z_{0,nz}}{\sqrt{r_{1,n}}} = B_p \sqrt{\frac{r_{1,nz}}{r_{1,n}}}\, Y_{1,0} + B_q \sqrt{\frac{r_{2,nz}}{r_{1,n}}} Y_{2,0} \xrightarrow[n]{p} B_p Y_{1,0} + B_q\, \frac{1}{\tau} Y_{2,0}$. Therefore,

$$P\left(\frac{Z_{0,nz}}{\sqrt{r_{1,n}}} \leq x\right) \xrightarrow[n]{w} p\Phi(x) + q\Phi(\tau x). \tag{4.9}$$

Additionally, for every $\varepsilon > 0$, we obtain

$$p\left(\frac{\left|Z_{s'(nz):nz} - b_{nz}\right|}{\sqrt{r_{1,n}}} > \varepsilon\right) = p\left(\frac{\left|Z_{s'(nz):nz} - b_{nz}\right|}{a_{nz}} > \frac{\sqrt{r_{1,n}}}{a_{nz}}\varepsilon\right) \xrightarrow[n]{} 0, \tag{4.10}$$

since $\frac{\sqrt{r_{1,n}}}{a_{nz}} \sim \sqrt{2 s_{nz} r_{1,nz} \log(nz)\left(1 - \frac{\log s_{nz}}{\log(nz)}\right)} \sqrt{\frac{r_{1,n}}{r_{1,nz}}} \xrightarrow[n]{} \infty$. Finally, from (4.8) and Lemma 2.2.1 in [21], a combination of (4.9) and (4.10) yields $F_{X_{s'(nz):nz}}\left(\sqrt{r_{1,n}}x + b_{nz}\right) \xrightarrow[n]{w} p\Phi(x) + q\Phi(\tau x)$. Using this convergence and Eq (4.1), the remainder proof of this case is precisely the same as that of the first case of Theorem 2.1.

In the same manner, under the conditions $s_n r_{j,n} \log n \xrightarrow[n]{} \infty$, $j = 1, 2$, $\frac{r_{1,n}}{r_{2,n}} \xrightarrow[n]{} 0$, and $r_{2,n}$ is an SVF of n, or $s_n r_{1,n} \log n \xrightarrow[n]{} \tau_1 \geq 0$, $s_n r_{2,n} \log n \xrightarrow[n]{} \infty$, and $r_{2,n}$ is an SVF of n, we get $F_{X_{s'(nz):nz}}\left(\sqrt{r_{2,n}}x + b_{nz}\right) = P\left(\frac{Z_{0,nz}}{\sqrt{r_{2,n}}} + \frac{Z_{s'(nz):nz} - b_{nz}}{\sqrt{r_{2,n}}} \leq x\right) \xrightarrow[n]{w} pI_{(0,\infty)}(x) + q\Phi(x)$ since $\frac{Z_{0,nz}}{\sqrt{r_{2,n}}} = B_p \sqrt{\frac{r_{1,nz}}{r_{2,n}}} Y_{1,0} + B_q \sqrt{\frac{r_{2,nz}}{r_{2,n}}} Y_{2,0} \xrightarrow[n]{p} B_q Y_{2,0}$ and $P\left(\frac{\left|Z_{s'(nz):nz} - b_{nz}\right|}{\sqrt{r_{2,n}}} \geq \varepsilon\right) \xrightarrow[n]{} 0$, because $\frac{\sqrt{r_{2,n}}}{a_{nz}} \xrightarrow[n]{} \infty$. The remainder proof of this case is precisely the same as that of Theorem 2.1.

Again, in the same manner, under the conditions $s_n r_{j,n} \log n \xrightarrow[n]{} \infty$, $j = 1, 2$, $\frac{r_{2,n}}{r_{1,n}} \xrightarrow[n]{} 0$, and $r_{1,n}$ is an SVF of n, or $s_n r_{1,n} \log n \xrightarrow[n]{} \infty$, $s_n r_{2,n} \log n \xrightarrow[n]{} \tau_2 \geq 0$, and $r_{1,n}$ is an SVF of n, we get $F_{X_{s'(nz):nz}}\left(\sqrt{r_{1,n}}x + b_{nz}\right) \xrightarrow[n]{w} p\Phi(x) + qI_{(0,\infty)}(x)$ since $\frac{Z_{0,nz}}{\sqrt{r_{1,n}}} \xrightarrow[n]{p} B_p Y_{1,0}$ and $p\left(\frac{\left|Z_{s'(nz):nz} - b_{nz}\right|}{\sqrt{r_{1,n}}} \geq \varepsilon\right) \xrightarrow[n]{} 0$, $\forall \varepsilon > 0$. The remainder proof of this case is precisely the same as that of Theorem 2.1. The theorem is now fully proved. □

5. Asymptotic behavior of record values in a mixture of two SGSs with a random index

Chandler [30] wrote the seminal paper on the statistical treatment of record values. He studied the stochastic behaviour of random record values generated by i.i.d observations in a continuous DF F. The cumulative hazard function $H_F(x) = -\log(\overline{F}(x))$ and its inverse $\Psi(u) = H_F^{-1}(u) = F^{-1}(1 - \exp(-u))$ determine the basic features of the record values. For example, the DF of the upper record value R_n may be expressed in terms of $H_F(x)$ as $P(R_n \leq x) = \Gamma_n(H_F(x))$ (cf. [24]), where $\Gamma_n(x) = \frac{1}{\Gamma(n)} \int\limits_0^x t^{n-1} e^{-t} dt$ is the incomplete gamma ratio function. Resnick [31] uncovered the class of conceivable limit laws for the upper record R_n. He connected these limit laws to the max-limit laws via the Duality theorem. For further fascinating work on the relationship between OSs and record values, see [32]. Because the upper record based on the standard normal distribution belongs to the domain of attraction of the normal type, written $D_R(\Phi)$, (cf. [24]), the normal type is the only important type in our investigation. Namely,

$$P\left(R_n \leq (\Phi^{-1}(1 - e^{-(n+\sqrt{n})}) - \Phi^{-1}(1 - e^{-n}))x + \Phi^{-1}(1 - e^{-n})\right) \xrightarrow[n]{w} \Phi(x),$$

where $\Phi^{-1}(x)$ is the usual inverse function of $\Phi(x)$. Furthermore, using the mean value theorem, an analogous simplified form of this limiting result is $P(R_n \leq a_n x + b_n) \xrightarrow[n]{w} \Phi(x)$, i.e., $\Phi(a_n x + b_n) \in D_R(\Phi(x))$, where $a_n = \frac{1}{\sqrt{2}}$ and $b_n = \Phi^{-1}(1 - e^{-n})$ (cf. Example 2.3.4 of [24]). We will need the following lemma due to [11].

Lemma 5.1. *Suppose* ${}_{Z}R_n$ *is the upper record value corresponding to the DF* $\mathcal{F}(z)$, *represented in the Eq (1.4). Then,* $P\left({}_{Z}R_n \leq \sqrt{1 - r_n}(a_n x + b_n)\right) \xrightarrow[n]{w} \Phi(x)$, *where* $r_n = \min(r_{1,n}, r_{2,n})$, $1 \neq \max(r_{1,n}, r_{2,n}) \nrightarrow 1$, *as* $n \to \infty$, $a_n = \frac{1}{\sqrt{2}}$ *and* $b_n = \Phi^{-1}(1 - e^{-n})$.

The following theorem gives the limit distribution of record values regarding the sequence (1.6) when the sample size is assumed to converge weakly and to be independent of the basic variables.

Theorem 5.1. *Let* ${}_{X}R_{\nu_n}$ *be the upper record value based on* $X_1, X_2, ...X_{\nu_n}$, *given by (1.2), where the random sample size* ν_n *is a sequence of integer-valued RVs independent of* $\{X_i\}$ *such that* $A_n(nx) \xrightarrow[n]{w} A(x)$, *and* $A(x)$ *is a non-degenerate DF. Furthermore, let* $r_n = \min(r_{1,n}, r_{2,nz})$, $1 \neq \max(r_{1,n}, r_{2,nz}) \nrightarrow 1$, *as* $n \to \infty$, $a_n = \frac{1}{\sqrt{2}}$ *and* $b_n = \Phi^{-1}(1 - e^{-n})$. *Then, the asymptotic distribution of the record* ${}_{X}R_{\nu_n}$ *is given by:*

1) $P({}_{X}R_{\nu_n} \leq a_n x + b_{\nu_n}) \xrightarrow[n]{w} \Phi(x + \tau)$, *where* $\tau = \begin{cases} \tau_1, & \text{if } r_n = r_{1,n}, \\ \tau_2, & \text{if } r_n = r_{2,nz}, \end{cases}$ *if* $r_{j,n}\sqrt{n} \xrightarrow[n]{} \tau_j \geq 0$, *for* $j = 1, 2$,
2) $P({}_{X}R_{\nu_n} \leq a_n x + b_{\nu_n}) \xrightarrow[n]{w} p\Phi(x + \tau) + q\Phi(\frac{x+\tau}{\sqrt{2r+1}})$, *if* $r_{1,n}\sqrt{n} \xrightarrow[n]{} \tau \geq 0$, $r_{2,n}\sqrt{n} \xrightarrow[n]{} \infty$, *and* $r_{2,n} \xrightarrow[n]{} r > 0$,
3) $P({}_{X}R_{\nu_n} \leq a_n x + b_{\nu_n}) \xrightarrow[n]{w} p\Phi(\frac{x+\tau}{\sqrt{2r+1}}) + q\Phi(x + \tau)$, *if* $r_{1,n}\sqrt{n} \xrightarrow[n]{} \infty$, $r_{2,n}\sqrt{n} \xrightarrow[n]{} \tau \geq 0$, *and* $r_{1,n} \xrightarrow[n]{} r > 0$,
4) $P({}_{X}R_{\nu_n} \leq a_n x + \sqrt{1 - r_{\nu_n}} b_{\nu_n}) \xrightarrow[n]{w} p\Phi(\frac{x}{\sqrt{1+2r_1-r}}) + q\Phi(\frac{x}{\sqrt{1+2r_2-r}})$, *where* $r = \begin{cases} r_1, & \text{if } r_n = r_{1,n}, \\ r_2, & \text{if } r_n = r_{2,nz}, \end{cases}$ *if* $r_{j,n}\sqrt{n} \xrightarrow[n]{} \infty$ *and* $r_{j,n} \xrightarrow[n]{} r_j$, $j = 1, 2$.

Proof. By starting the proof as we have done in Theorem 2.1, we obtain the corresponding equation to (2.3) as

$$P({}_{X}R_{\nu_n} \leq a_n x + b_{\nu_n}) = \int_0^\infty P({}_{X}R_{nz} \leq a_n x + b_{nz})\, dA_n(nz). \tag{5.1}$$

Under the condition $r_{j,n}\sqrt{n} \xrightarrow[n]{} \tau_j \geq 0$, for $j = 1, 2$, by using (1.6), the DF of the record value ${}_{X}R_{nz}$ is expressed as

$$P({}_{X}R_{nz} \leq a_n x + b_{nz}) = P\left(U_{nz} + V_{nz} \leq x\right), \tag{5.2}$$

where $U_{nz} = \frac{Z_{0,nz}}{a_n}$ and $V_{nz} = \frac{{}_{Z}R_{nz} - b_{nz}}{a_n}$ are independent. Clearly, $U_{nz} = B_p \frac{\sqrt{r_{1,nz}}}{a_n} Y_{1,0} + B_q \frac{\sqrt{r_{2,nz}}}{a_n} Y_{2,0} \xrightarrow[n]{p} 0$, from the condition $r_{j,n}\sqrt{n} \xrightarrow[n]{} \tau_j \geq 0$. On the other hand, assuming $A_n = a_n\sqrt{1 - r_n}$ and $B_n = b_n\sqrt{1 - r_n}$, we get $\frac{a_n}{A_{nz}} = \frac{1}{\sqrt{1 - r_{nz}}} \xrightarrow[n]{} 1$ (from $r_{j,n}\sqrt{n} \xrightarrow[n]{} \tau_j$, $j = 1, 2$) and $\frac{b_{nz} - B_{nz}}{A_{nz}} = \frac{b_{nz}}{a_{nz}}\left[\frac{1}{\sqrt{1 - r_{nz}}} - 1\right] \xrightarrow[n]{} \tau$, that can be proved using $(1 - r_{nz})^{-\frac{1}{2}} = 1 + \frac{1}{2} r_{nz}(1 + \circ(1))$ and $b_{nz} \sim \sqrt{2n}$ (cf. [14]). Therefore, from Lemma 5.1 and the Khinchin's type theorem, we get

$$P(V_{nz} \leq x) = P({}_{Z}R_{nz} \leq a_n x + b_{nz}) \xrightarrow[n]{w} \Phi(x + \tau). \tag{5.3}$$

Now, from the Eqs (5.2) and (5.3), and Lemma 2.2.1 in [21] plus $U_{nz} \xrightarrow[n]{p} 0$, we get $P({}_X R_{nz} \le a_n x + b_{nz}) \xrightarrow[n]{w} \Phi(x + \tau)$. The remainder proof of this case is precisely the same as that of the first case of Theorem 2.1 by using the relations (5.1) and the last relation.

Now, turning to the second case of the theorem, i.e., under the conditions $r_{1,n}\sqrt{n} \xrightarrow[n]{} \tau \ge 0$, $r_{2,n}\sqrt{n} \xrightarrow[n]{} \infty$, and $r_{2,n} \xrightarrow[n]{} r > 0$. We note that $\min(r_{1,n}, r_{2,nz}) = r_{1,n}$, for large n. Moreover, the Eqs (5.2) and (5.3) still hold with the same previous sequences U_{nz} and V_{nz}, but $U_{nz} \xrightarrow[n]{p} B_q\sqrt{2r}Y_{2,0}$. Then, we get $P({}_X R_{nz} \le a_n x + b_{nz}) \xrightarrow[n]{w} p\Phi(x+\tau) + q\Phi(\frac{x+\tau}{\sqrt{2r+1}})$. The remainder proof of this case is precisely the same as that of the first case of Theorem 2.1 by using the relations (5.1) and the last relation. For the third case, clearly we have $\min(r_{1,n}, r_{2,nz}) = r_{2,n}$, for large n. Moreover, the rest of the proof of this case is similar to the proof of the second case. For brevity, the proof is omitted.

Finally, for proving the fourth case of the theorem, i.e., we adopt the conditions $r_{j,n}\sqrt{n} \xrightarrow[n]{} \infty$, and $r_{j,n} \xrightarrow[n]{} r_j$, $j = 1, 2$, use the representation (2.1), the distribution of the record value ${}_X R_{nz}$ can be written as

$$P({}_X R_{nz} \le a_n x + \sqrt{1 - r_{nz}}b_{nz}) = P\left(U_{nz} + V_{nz} \le x\right),$$

where $U_{nz} = \frac{Z_{0,nz}}{a_n}$ and $V_{nz} = \frac{{}_Z R_{nz} - \sqrt{1-r_{nz}}b_{nz}}{a_n}$ are independent. Clearly, $U_{nz} \xrightarrow[n]{p} B_p\sqrt{2r_1}Y_{1,0} + B_q\sqrt{2r_2}Y_{2,0}$. On the other hand, again by assuming $A_n = a_n\sqrt{1 - r_n}$ and $B_n = b_n\sqrt{1 - r_n}$, we get $\frac{a_n}{A_{nz}} = \frac{1}{\sqrt{1-r_{nz}}} \xrightarrow[n]{} \frac{1}{\sqrt{1-r}}$ and $\frac{\sqrt{1-r_{nz}}b_{nz} - B_{nz}}{A_{nz}} = 0$. Therefore, from Lemma 5.1 and the Khinchin's type theorem, we get

$$P(V_{nz} \le x) = P({}_Z R_{nz} \le a_n x + \sqrt{1 - r_{nz}}b_{nz}) \xrightarrow[n]{w} \Phi(\frac{x}{\sqrt{1-r}}).$$

The rest of the proof is self-evident. The theorem is now fully proved. □

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Conflict of interest

All authors declare no conflict of interest in this paper.

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