AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(10): 19292–19305.
DOI:10.3934/math.20221059
Received: 11 July 2022
Revised: 14 August 2022
Accepted: 23 August 2022
Published: 31 August 2022

Research article

Existence of infinitely many normalized radial solutions for a class of quasilinear Schrödinger-Poisson equations in $\mathbb{R}^3$

Jinfu Yang, Wenmin Li, Wei Guo and Jiafeng Zhang*

School of Data Science and Information Engineering, Guizhou Minzu University, Guiyang 550025, China

* **Correspondence:** Email: jiafengzhang@163.com.

Abstract: In this paper, we study the existence of infinitely many normalized radial solutions for the following quasilinear Schrödinger-Poisson equations:

$$-\Delta u - \lambda u + (|x|^{-1} * |u|^2)u - \Delta(u^2)u - |u|^{p-2}u = 0,\ x \in \mathbb{R}^3,$$

where $p \in (\frac{10}{3}, 6)$, $\lambda \in \mathbb{R}$. Firstly, the quasilinear equations are transformed into semilinear equations by making a appropriate change of variables, whose associated variational functionals are well defined in $H_r^1(\mathbb{R}^3)$. Secondly, by constructing auxiliary functional and combining pohožaev identity, we prove that under constraints, the energy functionals related to the equation have bounded Palais-Smale sequences on each level set. Finally, it is obtained that there are infinitely many normalized radial solutions for this kind of quasilinear Schrödinger-Poisson equations.

Keywords: mountain pass geometry; pohožaev identity; normalized radial solutions
Mathematics Subject Classification: 35A15, 35B38, 49J35

1. Introduction and main results

In this paper, the following quasilinear Schrödinger-Poisson equations varying with time t are considered

$$i\partial_t\varphi + \Delta\varphi - (|x|^{-1} * |\varphi|^2)\varphi + k[\Delta\rho(|\varphi|^2)]\rho'(|\varphi|^2)\varphi + |\varphi|^{p-2}\varphi = 0,\ (t,x) \in \mathbb{R}\times\mathbb{R}^3, \qquad \text{E(i)}$$

where $\varphi = \varphi(x,t)$ is the wave function, k is a positive constant, ρ is a real function. Here we focus on the case $k = 1$, $\rho(s) = s$. This class of Schrödinger type equations with a repulsive nonlocal Coulombic potential is obtained by approximation of the Hartree-Fock equation which has been used to describe a quantum mechanical system of many particles, more physical meanings can be found in references [1–4] and its references.

We are interested in whether the equations $E(i)$ have solutions in the following form

$$\varphi = \varphi(x,t) = e^{-i\lambda t}u(x),$$

where $\lambda \in \mathbb{R}$ is the frequency of occurrence of $u(x)$. After standing wave transformation $\varphi = \varphi(x,t) = e^{-i\lambda t}u(x)$, the following steady-state equation is obtained

$$-\Delta u - \lambda u + (|x|^{-1} * |u|^2)u - \Delta(u^2)u - |u|^{p-2}u = 0, \quad x \in \mathbb{R}^3. \tag{1.1}$$

Therefore, if $e^{-i\lambda t}u(x)$ is the standing wave solution of equation $E(i)$, if and only if $u(x)$ is the solution of Eq (1.1). At this time, there are two research methods for finding the solution of Eq (1.1), one of them is to treat λ as a fixed parameter and the another is to treat λ as a Lagrange multiplier. When λ is regarded as a fixed parameter, the solution of Eq (1.1) can be obtained by finding the critical point of the following functional

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}(1+2u^2)|\nabla u|^2dx - \frac{\lambda}{2}\int_{\mathbb{R}^3}u^2dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}dydx - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p dx. \tag{1.2}$$

When λ is a Lagrange multiplier, its value range is unknown, and the study of this situation will be more interesting. Therefore, in this paper, we regard λ as a Lagrange multiplier. After giving the mass $\int_{\mathbb{R}^3}u^2dx = c$ in advance, we study the solution of Eq (1.1) satisfying $\|u\|_2^2 = c$. So, the solution of Eq (1.1) satisfying this condition can be obtained by the critical point of the following functional $K(u)$ under constraint S_c

$$K(u) = \frac{1}{2}\int_{\mathbb{R}^3}(1+2u^2)|\nabla u|^2dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}dydx - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p dx, \tag{1.3}$$

where

$$S_c = \{u \in H^1(\mathbb{R}^3) : \|u\|_2^2 = c\}.$$

In this case, the parameter λ cannot be fixed but instead appears as a Lagrange multiplier, if $u \in S_c$ is a minimizer of problem

$$\delta(c) := \inf_{u \in S_c} K(u),$$

then there exists $\lambda \in \mathbb{R}$ such that $K'(u) = \lambda u$, namely, (u, λ) is the solution of Eq (1.1) and satisfies $\|u\|_2^2 = c$. Since in literature there are no results available for the quasilinear Schrödinger-Poisson equation studied in this paper, we can only provide references for similar problems.

In [5–9], many authors studied Schrödinger-Poisson equations similar to the following

$$i\psi_t + \Delta\psi + V(x,t)\psi + f(\psi) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3,$$

where $f \in C(\mathbb{R}, \mathbb{R})$, $V(x,t)$ is the potential function, the unknown function $\psi = \psi(x,t) : \mathbb{R}^3 \times [0,T] \to \mathbb{C}$ is the wave function. The existence and nonexistence of normalized solutions of the general Schrödinger-Poisson equation were established, depending strongly on the value of $p \in (2,6)$ and of the parameter $c > 0$. That is, it is precisely proved that when $p \in (2,3)$ and $c > 0$ are sufficiently small, the energy functional corresponding to Schrödinger-Poisson equation has a global minimum solution on S_c. When $p \in (3, \frac{10}{3})$, there exists a $c_0 > 0$ such that a solution exists if and only if $c \geq c_0$.

When $p \in (\frac{10}{3}, 6)$, since the energy functional corresponding to Schrödinger-Poisson equations have no lower bound on S_c, it is impossible to find the global minimum solution on S_c. But in [10], the authors proved that for any $C > 0$ is small enough, the following Schrödinger-Poisso equations

$$-\Delta u - \lambda u + (|x|^{-1} * |u|^2)u - |u|^{p-2}u = 0, \quad x \in \mathbb{R}^3$$

has an energy minimum solution on S_c. However, in [11], considering that $H_r^1(\mathbb{R}^3)$ is compact embedded in $L^q(\mathbb{R}^3)$ $(q \in (2, 6))$, the author proved that the above equation has infinitely many normalized radial solutions when $p \in (\frac{10}{3}, 6)$ and $c > 0$ is sufficiently small.

In [12–14], different authors studied quasilinear Schrödinger-Poisson equations similar to the following

$$-\Delta u + V(x)u - \Delta(u^2)u = f(u), \quad x \in \mathbb{R}^N,$$

where $f \in C(\mathbb{R}, \mathbb{R})$, $V(x)$ is the potential function. Firstly, the quasilinear equation is transformed into a semilinear equation by using a change of variables, by conditionally limiting f, the existence results of positive solutions, ground state solutions and bound state solutions of the above equations were established by various analysis methods. For example, in [12], the authors studied the following quasilinear Schrödinger-Poisson equations

$$-\Delta u + V(x)u - \Delta(u^2)u = h(u), \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $V \in C(\mathbb{R}^N, \mathbb{R})$, $h \in C(\mathbb{R}^+, \mathbb{R})$, is Hölder continuous and satisfy

(V_0) there exists $V_0 > 0$ such that $V(x) \geq V_0 > 0$.

(V_1) $\lim_{|x|\to\infty} V(x) = V(\infty)$ and $V(x) \leq V(\infty)$.

(h_0) $\lim_{s\to 0} \frac{h(s)}{s} = 0$.

(h_1) for any $s \in \mathbb{R}$, $C > 0$, there exist $p < \frac{3N+2}{N-2}$($when\ N = 1, 2, p < \infty$) such that $|h(s)| \leq C(1 + |s|^p)$.

If one of the following conditions hold, then Eq (1.4) has a positive nontrivial solution:

(h_2) There exists $\mu > 4$, such that, for any $s > 0$, $0 < \mu H(s) \leq h(s)s$ hold, where $H(s) = \int_0^s h(t)dt$.

(h_3) For any $s > 0$, $0 < 4H(s) \leq h(s)s$ hold and when $N \geq 4$, $p < \frac{3N+4}{N}$ $(N = 3, p \leq 5)$, where $H(s) = \int_0^s h(t)dt$.

Therefore, we are curious that does the quasilinear Schrödinger-Poisson equation like (1.1) has similar results as those in the above literature under some conditions. Compared with [5–9], we study the existence of infinitely many normalized radial solutions of Schrödinger-Poisson equation with quasilinear term. To our knowledge, there are very few results in this direction in the existing literature. Moreover, compared with references [12–14], in this paper, we establish the existence results of infinitely many normalized radial solutions for this kind of equation, this can be regarded as the supplement and generalization of quasilinear Schrödinger-Poisson equations in this research direction of gauge solution.

Our result is as follows.

Theorem 1.1. *Assume that $p \in (\frac{10}{3}, 6)$. There exists $c_0 > 0$ sufficient small such that for any $c \in (0, c_0]$, (1.1) admits an unbounded sequence of distinct pairs of radial solutions $(\pm u_n, \lambda_n) \in S_c \times \mathbb{R}^-$ with $\|u_n\|_2^2 = c$ and $\lambda_n < 0$ for each $n \in \mathbb{N}$, and such that*

$$-\Delta u_n - \lambda_n u_n + (|x|^{-1} * |u_n|^2)u_n - \Delta(u_n^2)u_n - |u_n|^{p-2}u_n = 0.$$

Remark 1.1. *Firstly, when $p \in (\frac{10}{3}, 6)$, the energy functionals $K(u)$ corresponding to Eq (1.1) have no lower bounds on S_c, which will result in the absence of global minimum solution. Secondly, it can be easily checked that the functionasl $K(u)$, restricted to S_c, do not satisfy the Palais-Smale condition.*

Remark 1.2. *Firstly, due to the existence of quasilinear term, the energy functionals corresponding to Eq (1.1) are not well defined in $H^1_r(\mathbb{R}^3)$), hence, the usual variational method can not be used directly. In order to overcome this difficulty, we have two methods: one is to re-establish an appropriate variational framework so that the energy functional corresponding to Eq (1.1) have a good definition, the another is to convert Eq (1.1) into semi-linear equations through a change of variables, and then we can use a general variational method to study it. In this paper, we select second method. Secondly, because of the existence of the nonlocal term $|x|^{-1} * |u|^2$, this will make the proof more complex.*

Definition 1.1. *For given $c > 0$, we say that $I(u)$ possesses a mountain pass geometry on S_c if there exists $\rho_c > 0$ such that*

$$\gamma(c) = \inf_{g\in\Gamma_c} \max_{\tau\in[0,1]} I(g(\tau)) > \max_{g\in\Gamma_c} \max\{I(g(0)), I(g(1))\},$$

where $I : H^1(\mathbb{R}^3) \to \mathbb{R}$, $\Gamma_c = \{g \in C([0, 1], S_c) : \|\nabla g(0)\|_2^2 \le \rho_c, I(g(1)) < 0\}, S_c = \{u \in H^1(\mathbb{R}^3) : \|u\|_2^2 = c\}$.

2. Preliminaries and proof of Theorem 1.1

In the following, we will introduce some notations.
(1) $H^1(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$.
(2) $H^1_r(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|)\}$.
(3) $\|u\| = \left(\int_{\mathbb{R}^3} |\nabla u|^2 + u^2 dx\right)^{\frac{1}{2}}, \forall u \in H^1_r(\mathbb{R}^3)$.
(4) $\|u\|_s = \left(\int_{\mathbb{R}^3} |u|^s dx\right)^{\frac{1}{s}}, \forall s \in [1, +\infty)$.
(5) $\langle u, v\rangle_{H^1_r} = \int_{\mathbb{R}^3} \nabla u \nabla v + uv dx$.
(6) c, c_i, C, C_i denote various positive constants.

From the above description, we know that the energy functional $J_\lambda : H^1(\mathbb{R}^3) \to \mathbb{R}$ associated with problem (1.1) by $J_\lambda(u)$, where $J_\lambda(u)$ is given in (1.2).

Then such solutions of Eq (1.1) satisfying condition $\|u\|_2^2 = c$ and $u(x) = u(|x|)$ can be obtained by looking for critical points of the functionals K limited to $S'_c = \{u \in H^1_r(\mathbb{R}^3) : \|u\|_2^2 = c\}$, where $K(u)$ is given in (1.3), $H^1_r(\mathbb{R}^3)$ is the space composed of the radial function of $H^1(\mathbb{R}^3)$, and the embedding $H^1_r(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ is compact for $q \in (2, 6)$.

In order to prove our results, we first do a appropriate change of variables, $\forall u, v \in H^1_r(\mathbb{R}^3)$, $v := f^{-1}(u)$, f satisfies the following conditions:
(f_1) f is uniquely defined, smooth and invertible;
(f_2) $f'(t) = \frac{1}{\sqrt{1+2f^2(t)}}$, $t \in [0, +\infty)$;
(f_3) $f(0) = 0$ and $f(t) = -f(-t)$, $t \in (-\infty, 0]$.
In addition, we summarize some properties of f which have been proved in [12–14].
(f_4) $|f(t)| \le |t|$, $\forall t \in \mathbb{R}$;

(f_5) there exists a positive constant C such that

$$|f(t)| \geq \begin{cases} C|t|, & |t| \leq 1, \\ C|t|^{\frac{1}{2}}, & |t| \geq 1. \end{cases}$$

By making a change of variables, we get the following functional

$$I_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx - \frac{\lambda}{2}\int_{\mathbb{R}^3} f^2(v) dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|f(v(x))|^2|f(v(y))|^2}{|x-y|} dydx - \frac{1}{p}\int_{\mathbb{R}^3} |f(v)|^p dx.$$

It can be seen that functional I_λ is well defined in $H_r^1(\mathbb{R}^3)$ and the corresponding equation is

$$-\Delta v - \lambda f(v) f'(v) + (|x|^{-1} * |f(v)|^2) f(v) f'(v) - |f(v)|^p f(v) f'(v) = 0. \tag{2.1}$$

Then such solutions of Eq (1.1) satisfying condition $\|u\|_2^2 = c$ and $u(x) = u(|x|)$ can be obtained by looking for critical points of the following functional F under the corresponding constraints

$$F(v) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|f(v(x))|^2|f(v(y))|^2}{|x-y|} dydx - \frac{1}{p}\int_{\mathbb{R}^3} |f(v)|^p dx.$$

The corresponding constraint condition is

$$S_r(c) = \{v \in H_r^1(\mathbb{R}^3) : \|f(v)\|_2^2 = c,\ c > 0\}.$$

Then, if $u \in S_c'$ is the solution of Eq (1.1), if and only if $v \in S_r(c)$ is the solution of Eq (2.1). Moreover we define, for short, the following quantities

$$A(v) := \int_{\mathbb{R}^3} |\nabla v|^2 dx, \quad B(v) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|f(v(x))|^2|f(v(y))|^2}{|x-y|} dydx,$$
$$C(v) := \int_{\mathbb{R}^3} |f(v)|^p dx, \quad D(v) := \int_{\mathbb{R}^3} f^2(v) dx.$$

We first establish some preliminary results. Let $\{V_n\} \subset H_r^1(\mathbb{R}^3)$ be a strictly increasing sequence of finite-dimensional linear subspaces in $H_r^1(\mathbb{R}^3)$, such that $\bigcup_n V_n$ is dense in $H_r^1(\mathbb{R}^3)$. We denote by $V_n^\perp$ the orthogonal space of V_n in $H_r^1(\mathbb{R}^3)$. Then we have

Lemma 2.1. *(See [5, Lemma 2.1]) Assume that* $p \in (2, 6)$. *Then there holds*

$$\mu_n := \inf_{v \in V_{n-1}^\perp} \frac{\int_{\mathbb{R}^3} |\nabla v|^2 + |v|^2 dx}{(\int_{\mathbb{R}^3} |v|^p dx)^{\frac{2}{p}}} = \inf_{v \in V_{n-1}^\perp} \frac{\|v\|^2}{\|v\|_p^2} \to \infty, \quad n \to \infty.$$

Now for $c > 0$ fixed and for each $n \in \mathbb{N}$, we define

$$\rho_n := L^{-\frac{2}{p-2}} \mu_n^{\frac{p}{p-2}}, \quad L = \max_{x > 0} \frac{(x^2 + c)^{\frac{p}{2}}}{x^p + c^{\frac{p}{2}}},$$

and

$$B_n := \{v \in V_{n-1}^\perp \cap S_r(c) : \|\nabla f(v)\|_2^2 = \rho_n\}. \tag{2.2}$$

We also define

$$b_n := \inf_{v \in B_n} F(v). \tag{2.3}$$

Lemma 2.2. *For every $p \in (2,6)$, we have $b_n \to \infty$ as $n \to \infty$. Particularly, there exists $n_0 \in \mathbb{N}$ such that $b_n \geq 1$ for all $n \geq n_0$, $n \in \mathbb{N}$.*

Proof. For any $v \in B_n$, we have that

$$\begin{aligned} F(v) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|f(v(x))|^2 |f(v(y))|^2}{|x-y|} dydx - \frac{1}{p}\int_{\mathbb{R}^3} |f(v)|^p dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} |f(v)|^p dx \\ &\geq \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{p\mu_n^{\frac{p}{2}}}(\|\nabla f(v)\|_2^2 + c)^{\frac{p}{2}} \\ &\geq \frac{1}{2}\|\nabla v\|_2^2 - \frac{L}{p\mu_n^{\frac{p}{2}}}(\|\nabla f(v)\|_2^p + c^{\frac{p}{2}}) = \frac{p-2}{2p}\rho_n - \frac{L}{p\mu_n^{\frac{p}{2}}} c^{\frac{p}{2}}. \end{aligned}$$

From this estimation and Lemma 2.1, it follows since $p > 2$, that $b_n \to +\infty$ as $n \to \infty$. Now, considering the sequence $V_n \in H_r^1(\mathbb{R}^3)$ only from an $n_0 \in \mathbb{N}$ such that $b_n \geq 1$ for any $n \geq n_0$ it concludes the proof of the lemma. □

Next we start to set up our min-max scheme. First we introduce the map $\kappa : H := H_r^1(\mathbb{R}^3) \times \mathbb{R} \to H_r^1(\mathbb{R}^3)$ by

$$\kappa(v,\theta)(x) = e^{\frac{3\theta}{2}} v(e^\theta x),\ v \in H_r^1(\mathbb{R}^3),\ \theta \in \mathbb{R},\ x \in \mathbb{R}^3, \tag{2.4}$$

where H is a Banach space equipped with the product norm $\|(v,\theta)\|_H = \left(\|v\|^2 + |\theta|^2\right)^{\frac{1}{2}}$. Observe that for any given $v \in S_r(c)$, we have $\kappa(v,\theta) \to H_r^1(\mathbb{R}^3)$ for all $\theta \in \mathbb{R}$. Also from [8, Lemma 2.1], we know that

$$\begin{cases} A(\kappa(v,\theta)) \to 0, & F(\kappa(v,\theta)) \to 0, \quad \theta \to -\infty, \\ A(\kappa(v,\theta)) \to +\infty, & F(\kappa(v,\theta)) \to -\infty, \quad \theta \to +\infty. \end{cases} \tag{2.5}$$

Thus, using the fact that V_n is finite dimensional, we deduce that for each $n \in \mathbb{N}$, there exists an $\theta_n > 0$, such that

$$\overline{g}_n : [0,1] \times (S_r(c) \cap V_n) \to S_r(c), \quad \overline{g}_n(t,v) = \kappa(v,(2t-1)\theta_n), \tag{2.6}$$

satisfies

$$\begin{cases} A(\overline{g}_n(0,v)) < \rho_n, & A(\overline{g}_n(1,v)) > \rho_n, \\ F(\overline{g}_n(0,v)) < b_n, & F(\overline{g}_n(1,v)) < b_n. \end{cases} \tag{2.7}$$

Now we define

$$\begin{aligned} \Gamma_n := \Big\{ & g : [0,1] \times (S_r(c) \cap V_n) \to S_r(c) | g \text{ is continuous, odd in v and such that} \\ & \forall v : g(0,v) = \overline{g}_n(0,v),\ \ g(1,v) = \overline{g}_n(1,v) \Big\}. \end{aligned}$$

Clearly $\overline{g}_n \in \Gamma_n$. Let

$$f(v^t(x)) = t^{\frac{3}{2}} f[v(tx)],\ \forall t > 0,\ v \in H_r^1(\mathbb{R}^3),$$

where $v^t(x) = t^{\frac{3}{2}} v(tx)$, then $\|f(v^t)\|_2^2 = \|f(v)\|_2^2$, and so, for any $v \in S_r(c),\ t > 0$, we have $v^t \in S_r(c)$, and take into account

$$F(v^t) = \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{t}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|f(v(x))|^2|f(v(y))|^2}{|x-y|} dydx - \frac{t^{\frac{3p-6}{p}}}{p}\int_{\mathbb{R}^3} |f(v)|^p dx,$$

let

$$Q(v) := \frac{\partial F(v^t)}{\partial t}\Big|_{t=1} = \|\nabla v\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|f(v(x))|^2|f(v(y)|^2}{|x-y|} dydx$$
$$- \frac{3(p-2)}{2p}\int_{\mathbb{R}^3} |f(v)|^p dx = A(v) + \frac{1}{4}B(v) - \frac{3(p-2)}{2p}C(v).$$

Now we give the key intersection result, due to [15].

Lemma 2.3. *For each $n \in \mathbb{N}$,*

$$\gamma_n(c) := \inf_{g\in\Gamma_n} \max_{t\in[0,1], v\in S_r(c)\cap V_n} F(t, v) \geq b_n. \tag{2.8}$$

Proof. The point is to show that for each $g \in \Gamma_n$ there exists a pair $(t, v) \in [0, 1] \times (S_r(c) \cap V_n)$, such that $g(t, v) \in B_n$ with B_n defined in (2.2). but this proof is completely similar to the proof of Lemma 2.3 in [15], and this proof is omitted here. □

According to Lemma 2.3 and (2.7), we derive that, F satisfies mountain pass geometry, that is, for any $g \in \Gamma_n$, we have

$$\gamma_n(c) \geq b_n > \max\{\max_{v\in S_r(c)\cap V_n} F(g(0, v)), \max_{v\in S_r(c)\cap V_n} F(g(1, v))\}.$$

Next, we shall prove that the sequence $\{\gamma_n(c)\}$ is indeed a sequence of critical values for F restricted to $S_r(c)$. To this purpose, we first show that there exists a bounded Palais-Smale sequence at each level $\{\gamma_n(c)\}$. From now on, we fix an arbitrary $n \in \mathbb{N}$, then the following lemma can be obtained.

Lemma 2.4. *For any fixed $c > 0$, there exists a sequence $\{v_k\} \subset S_r(c)$ satisfying*

$$\begin{cases} F(v_k) \to \gamma_n(c), \\ F'|_{S_r(c)(v_k)} \to 0, \quad k \to \infty, \\ Q(v_k) \to 0, \end{cases} \tag{2.9}$$

where

$$Q(v) = \|\nabla v\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|f(v(x))|^2|f(v(y)|^2}{|x-y|} dydx$$
$$- \frac{3(p-2)}{2p}\int_{\mathbb{R}^3} |f(v)|^p dx = A(v) + \frac{1}{4}B(v) - \frac{3(p-2)}{2p}C(v).$$

In particular $\{v_k\} \subset S_r(c)$ is bounded.

To find such a Palais-Smale sequence, we consider the following auxiliary functional

$$\widetilde{F} : S_r(c) \times \mathbb{R} \to \mathbb{R},\ (v, \theta) \mapsto F(\kappa(v, \theta)), \tag{2.10}$$

where $\kappa(v, \theta)$ is given in (2.4), it is checked easily that $Q(v) = \frac{\partial F(\kappa(v,\theta))}{\partial \theta}|_{\theta=0}$, set

$$\widetilde{\Gamma}_n := \left\{\widetilde{g} : [0, 1] \times (S_r(c) \cap V_n) \to S_r(c) \times \mathbb{R}\ |\widetilde{g} \text{ is continuous, odd in } v, \text{ and such that } k \circ \widetilde{g} \in \Gamma_n\right\}.$$

Clearly, for any $g \in \Gamma_n$, $\widetilde{g} := (g, 0) \in \widetilde{\Gamma}_n$.

Define

$$\widetilde{\gamma}_n(c) := \inf_{\widetilde{g}\in\widetilde{\Gamma}_n} \max_{t\in[0,1],v\in S_r(c)\cap V_n} \widetilde{F}(\widetilde{g}(t, v)).$$

According to [10, 16, 17], it is checked easily that F and $\widetilde{F}$ have the same mountain pass geometry and $\widetilde{\gamma}_n(c) = \gamma_n(c)$.

Following [18], we recall that for any $c > 0$, $S_r(c)$ is a submanifold of $H^1_r(\mathbb{R}^3)$ with codimension 1 and the tangent space at $S_r(c)$ is defined as

$$T_{u_0} = \left\{v \in H^1_r(\mathbb{R}^3) : \int_{\mathbb{R}^3} u_0 v dx = 0\right\}.$$

The norm of the derivative of the C^1 restriction functional $F|_{S_r(c)}$ is defined by

$$\|F|'_{S_r(c)}(u_0)\| = \sup_{v\in T_{u_0},\|v\|=1} \langle F'(u_0), v\rangle.$$

Similarly, the tangent space at $(u_0, \theta_0) \in S_r(c) \times \mathbb{R}$ is given as

$$\tilde{T}_{u_0,\theta_0} = \left\{(z_1, z_2) \in H : \int_{\mathbb{R}^3} u_0 z_1 dx = 0\right\}.$$

The norm of the derivative of the C^1 restriction functional $\tilde{F}|_{S_r(c)\times\mathbb{R}}$ is defined by

$$\|\tilde{F}|'_{S_r(c)\times\mathbb{R}}(u_0, \theta_0)\| = \sup_{(z_1,z_2)\in\tilde{T}_{(u_0,\theta_0)},\|(z_1,z_2)\|_H=1} \langle\tilde{F}|'_{S_r(c)\times\mathbb{R}}(u_0, \theta_0), (z_1, z_2)\rangle.$$

As in [19, Lemma 2.3], we have the following lemma, which was established by using Ekeland's variational principle.

Lemma 2.5. *For any $\varepsilon > 0$, if $\widetilde{g}_0 \in \widetilde{\Gamma}_n$ satisfies*

$$\max_{v\in S_r(c)\cap V_n} \widetilde{F}(\widetilde{g}_0(t, v)) \le \widetilde{\gamma}_n(c) + \varepsilon.$$

Then there exists a pair of $(v_0, \theta_0) \in S_r(c) \times \mathbb{R}$ such that:

(i) $\tilde{F}(v_0, \theta_0) \in [\tilde{\gamma}_n(c) - \varepsilon, \tilde{\gamma}_n(c) + \varepsilon]$;

(ii) $\min_{t\in[0,1],v\in S_r(c)\cap V_n} \|(v_0, \theta_0) - \tilde{g}_k(t, v)\|_H \le \sqrt{\varepsilon}$;

(iii) $\|\tilde{F}'|_{S_r(c)\times\mathbb{R}}(v_0, \theta_0)\| \le 2\sqrt{\varepsilon}$, i.e.,

$$|\langle\tilde{F}'|_{S_r(c)\times\mathbb{R}}(v_0, \theta_0), z\rangle_{H^*\times H}| \le 2\sqrt{\varepsilon}\|z\|_H,\ \forall z \in \tilde{T}_{(v_0,\theta_0)}.$$

Next, we will give the proof of Lemma 2.4.

Proof. From the definition of $\gamma_n(c)$, we know that for each $k \in \mathbb{N}$, there exists an $g_k \in \Gamma_n$ such that

$$\max_{t\in[0,1],v\in S_r(c)\cap V_n} F(g_k(t,v)) \le \gamma_n(c) + \frac{1}{k}.$$

Since $\widetilde{\gamma}_n(c) = \gamma_n(c)$, $\widetilde{g}_k = (g_k, 0) \in \widetilde{\Gamma}_n$ satisfy

$$\max_{t\in[0,1],v\in S_r(c)\cap V_n} \widetilde{F}(\widetilde{g}_k(t,v)) \le \widetilde{\gamma}_n(c) + \frac{1}{k}.$$

Thus applying Lemma 2.5, we obtain a sequence $\{v_k, \theta_k\} \subset S_r(c) \times \mathbb{R}$ such that:
(i) $\tilde{F}(v_k, \theta_k) \in [\gamma_n(c) - \frac{1}{k}, \gamma_n(c) + \frac{1}{k}]$;
(ii) $\min_{t\in[0,1],v\in S_r(c)\cap V_n} \|(v_k, \theta_k) - (g_k(t,v), 0)\|_H \le \frac{1}{\sqrt{k}}$;
(iii) $\|\tilde{F}'|_{S_r(c)\times\mathbb{R}}(v_k, \theta_k)\| \le 2\sqrt{k}$, that is

$$|\langle \tilde{F}'|_{S_r(c)\times\mathbb{R}}(v_k, \theta_k), z\rangle_{H^*\times H}| \le 2\sqrt{k}\|z\|_H, \quad \forall z \in \tilde{T}_{(v_k,\theta_k)}.$$

For each $k \in \mathbb{N}$, let $u_k = \kappa(v_k, \theta_k)$. We shall prove that $u_k \in S_r(c)$ satisfies (2.9). Indeed, firstly, from (i) we have that $F(u_k) \to \gamma_n(c) (k \to \infty)$, since $F(u_k) = F(\kappa(v_k, \theta_k)) = \widetilde{F}(v_k, \theta_k)$. Secondly, note that

$$Q(u_k) = A(u_k) + \frac{1}{4}B(u_k) - \frac{3(p-2)}{2p}C(u_k) = \langle \widetilde{F}'(v_k, \theta_k), (0, 1)\rangle_{H^*\times H},$$

and $(0, 1) \in \widetilde{T}_{(v_k,\theta_k)}$. Thus ($iii$) yields $Q(u_k) \to 0 (k \to \infty)$. Finally, to verify that $F'|_{S_r(c)(u_k)} \to 0 (k \to \infty)$, it suffices to prove for $k \in \mathbb{N}$ sufficiently large, that

$$\langle F'(u_k), \omega\rangle_{(H_r^1)^*\times H_r^1} \le \frac{4}{\sqrt{k}}\|\omega\|, \quad \forall \omega \in T_{u_k}. \tag{2.11}$$

To this end, we note that, for $\omega \in T_{u_k}$, setting $\widetilde{\omega} = \kappa(\omega, -\theta_k)$, $v_k' = f(v_k)$, one has

$$\begin{aligned} &\langle F'(u_k), \omega\rangle_{(H_r^1)^*\times H_r^1} \\ &= \int_{\mathbb{R}^3} \nabla u_k \nabla \omega dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|f(u_k(x))|^2 f(u_k(y))\omega(y)}{|x-y|} dydx \\ &- \int_{\mathbb{R}^3} |f(u_k)|^{p-2} f(u_k)\omega dx \\ &= e^{2\theta_k}\int_{\mathbb{R}^3} \nabla v_k \nabla\widetilde{\omega} dx + e^{\theta_k}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v_k'(x))|^2 v_k'(y)\widetilde{\omega}(y)}{|x-y|} dydx \\ &- e^{\frac{3(p-2)\theta_k}{2}}\int_{\mathbb{R}^3} |v_k'|^{p-2} v_k'\widetilde{\omega} dx \\ &= e^{2\theta_k}\int_{\mathbb{R}^3} \nabla v_k \nabla\widetilde{\omega} dx + e^{\theta_k}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|f(v_k(x))|^2 f(v_k(y))\widetilde{\omega}(y)}{|x-y|} dydx \\ &- e^{\frac{3(p-2)\theta_k}{2}}\int_{\mathbb{R}^3} |f(v_k)|^{p-2} f(v_k)\widetilde{\omega} dx \\ &= \langle \widetilde{F}'(v_k, \theta_k), (\widetilde{\omega}, 0)\rangle_{H^*\times H}. \end{aligned} \tag{2.12}$$

If $(\widetilde{\omega}, 0) \in \widetilde{T}_{(v_k,\theta_k)}$ and $\|(\widetilde{\omega}, 0)\|_H^2 \le 4\|\omega\|^2$ and $k \in \mathbb{N}$ is sufficiently large, then (*iii*) implies (2.11). To verify these conditions, observe that $(\widetilde{\omega}, 0) \in \widetilde{T}_{(v_k,\theta_k)} \Leftrightarrow \omega \in T_{u_k}$. Also from (*ii*) it follows that

$$|\theta_k| = |\theta_k - 0| \le \min_{t\in[0,1], v\in S_r(c)\cap V_n} \|(u_k, \theta_k) - (g_k(t, v), 0)\|_H \le \frac{1}{\sqrt{k}},$$

by which we deduce that

$$\|(\widetilde{\omega}, 0)\|_H^2 = \|\widetilde{\omega}\|^2 = \int_{\mathbb{R}^3} |\omega(x)|^2 dx + e^{-2\theta_k} \int_{\mathbb{R}^3} |\nabla\omega(x)|^2 dx \le 2\|\omega\|^2$$

holds for $k \in \mathbb{N}$ large enough. At this point, (2.11) has been verified. To end the proof of the lemma it remains to show that $\{u_k\} \in S_r(c)$ is bounded. But one notes that for any $v \in H^1(\mathbb{R}^3)$, there holds that

$$F(v) - \frac{2}{3(p-2)} Q(v) = \frac{3p-10}{6(p-2)} A(v) + \frac{3p-2}{12(p-2)} B(v). \tag{2.13}$$

Thus we have

$$\gamma_n(c) + o_k(1) = F(u_k) - \frac{2}{3(p-2)} Q(u_k) = \frac{3p-10}{6(p-2)} A(u_k) + \frac{3p-2}{12(p-2)} B(u_k). \tag{2.14}$$

Since $p \in (\frac{10}{3}, 6)$ it follows immediately from (2.14) that $\{u_k\} \in S_r(c)$ is bounded in $H^1(\mathbb{R}^3)$. □

Lemma 2.6. *Assume that (f_1)–(f_4) hold and v is a weak solution of (2.1). Then $Q(v) = 0$. Furthermore, there exists a constant $c_0 > 0$ independing on $\lambda \in \mathbb{R}$ such that if $\|f(v)\|_2^2 \le c_0$ sufficiently small, then $\lambda < 0$.*

Proof. Let v be a weak solution of (2.1), the following Pohožaev-type identity holds

$$\frac{1}{2}\|\nabla v\|_2^2 + \frac{5}{4} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|f(v(x))|^2 |f(v(y)|^2}{|x-y|} dydx - \frac{3}{p} \int_{\mathbb{R}^3} |f(v)|^p dx = \frac{3\lambda}{2} \|f(v)\|_2^2. \tag{2.15}$$

By multiplying (1.1) by u and integrating. After a change of variables, we obtain the following identity

$$\|\nabla v\|_2^2 + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|f(v(x))|^2 |f(v(y)|^2}{|x-y|} dydx - \int_{\mathbb{R}^3} |f(v)|^p dx = \lambda \|f(v)\|_2^2. \tag{2.16}$$

By multiplying (2.16) by $\frac{3}{2}$ and minus (2.15), we obtain

$$\|\nabla v\|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|f(v(x))|^2 |f(v(y)|^2}{|x-y|} dydx - \frac{3(p-2)}{2p} \int_{\mathbb{R}^3} |f(v)|^p dx = 0. \tag{2.17}$$

From the above equation $Q(v) = 0$. By multiplying (2.16) by $\frac{3}{p}$ and minus (2.15), we obtain

$$\frac{p-6}{2p-6} A(v) + \frac{5p-12}{2(3p-6)} B(v) = \lambda D(v). \tag{2.18}$$

On the one hand, by (f_4), Hardy-Littlewood-Sobolev inequality and Gagliardo-Nirenberg inequality, we have

$$B(v) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|f(v(x))|^2 |f(v(y)|^2}{|x-y|} dydx \le C\|f(v)\|_{\frac{12}{5}}^4 \le C\|v\|_{\frac{12}{5}}^4 \le \widetilde{C}\|\nabla v\|_2^3 \|v\|_2. \tag{2.19}$$

On the other hand, by (f_4), (2.17) and Gagliardo-Nirenberg inequality, we derive that, there exists a $C(p) > 0$ such that

$$\begin{aligned}\|\nabla v\|_2^2 - C(p)\|\nabla v\|_2^{\frac{3(p-2)}{2}}\|v\|_2^{\frac{6-p}{2}} &\leq \|\nabla v\|_2^2 - \frac{3(p-2)}{2p}\|v|_p^p \\ &\leq \|\nabla v\|_2^2 - \frac{3(p-2)}{2p}\|f(v)\|_p^p \\ &= -\frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|f(v(x))|^2|f(v(y)|^2}{|x-y|}dydx \leq 0,\end{aligned}$$

implies

$$\|\nabla v\|_2^{\frac{10-3p}{2}} \leq C(p)\|v\|_2^{\frac{6-p}{2}}. \tag{2.20}$$

Due to $p \in (\frac{10}{3}, 6)$, then $\frac{10-3p}{2} < 0$. Note that (2.20) tells us that, for any solution v of (2.1) with small L^2-norm, $\|\nabla v\|_2$ must be large. By the property of f, it is easy to see that f is monotonically increasing on $(-\infty, +\infty)$. By this time, we just taking $\|f(v)\|_2^2 = c$ is small enough so as to $|v| < 1$, then by (f_5), (2.18) and (2.19), we derive that, there exist $C > 0$ and $\widetilde{C}(p) > 0$ such that

$$\lambda C^2\|v\|_2^2 \leq \lambda\|f(v)\|_2^2 \leq \frac{p-6}{2p-6}\|\nabla v\|_2^2 + \widetilde{C}(p)\|\nabla v\|_2^3\|v\|_2. \tag{2.21}$$

It follws from (2.20) that when $\|\nabla u\|_2$ is sufficiently small, the left-hand side of (2.21) is negative, that is, $\lambda C^2\|v\|_2^2 < 0(\|\nabla u\|_2 \to 0)$. Therefore, there exists a constant $c_0 > 0$ independing on $\lambda \in \mathbb{R}$, such that a solution v of (2.1), if satisfies $\|f(v)\|_2^2 \leq c_0$ sufficiently small, then $\lambda < 0$. □

Similar to [8, 16, 17], we have the following proposition.

Proposition 2.1. *Let $\{v_k\} \subset S_r(c)$ be the Palais-Smale sequence obtained in Lemma 2.4. Then there exist $\lambda_n \in \mathbb{R}$ and $v_n \in H_r^1(\mathbb{R}^3)$, such that, up to a subsequence,*

(i) $v_k \rightharpoonup v_n$, *in* $H_r^1(\mathbb{R}^3)$,

(ii) $-\Delta v_k - \lambda_n f(v_k)f'(v_k) + (|x|^{-1} * |f(v_k)|^2)f(v_k)f'(v_k) - |f(v_k)|^{p-2}f(v_k)f'(v_k) \to 0$, *in* $H_r^{-1}(\mathbb{R}^3)$,

(iii) $-\Delta v_n - \lambda_n f(v_n)f'(v_n) + (|x|^{-1} * |f(v_n)|^2)f(v_n)f'(v_n) - |f(v_n)|^{p-2}f(v_n)f'(v_n) = 0$, *in* $H_r^{-1}(\mathbb{R}^3)$.

Moreover, if $\lambda_n < 0$, then we have

$$v_k \to v_n \ (k \to \infty), \quad \text{in } H_r^1(\mathbb{R}^3).$$

In particular, $\|f(v_n)\|_2^2 = c$, $F(v_n) = \gamma_n(c)$ and $F'(v_n) = \lambda_n v_n$ in $H_r^{-1}(\mathbb{R}^3)$.

Proof. Since $\{v_k\} \subset S_r(c)$ is bounded, up to a subsequence, there exists a $v_n \in H_r^1(\mathbb{R}^3)$, such that

$$\begin{cases} v_k \rightharpoonup v_n & \text{in } H^1(\mathbb{R}^3); \\ v_k \to v_n & \text{in } L^p(\mathbb{R}^3)(\forall p \in (2,6)); \\ v_k \to v_n \quad a.e. & \text{in } \mathbb{R}^3. \end{cases}$$

Now let's prove that $v_n \neq 0$. Suppose $v_n = 0$, then by the strong convergence of $v_k \to v_n$ in $L^p(\mathbb{R}^3)$, it follows that $C(v_k) \to 0$. Taking into account that $Q(v_k) \to 0$, it then implies that $A(v_k) \to 0$ and $B(v_k) \to 0$. Thus $F(v_k) \to 0$ and this contradicts the fact that $\gamma_n(c) \geq b_n \geq 1$. Thus (i) holds.

The proofs of (*ii*) and (*iii*) can be found in [10, Proposition 4.1]. Now using (*ii*), (*iii*) and the convergence $C(v_k) \to C(v_n)$, it follows that

$$A(v_k) - \lambda_n D(v_k) + B(v_k) \to A(v_n) - \lambda_n D(v_n) + B(v_n)(k \to \infty).$$

If $\lambda_n < 0$, then we conclude from the weak convergence of $v_k \rightharpoonup v_n$ in $H_r^1(\mathbb{R}^3)$, that

$$A(v_k) \to A(v_n), \quad -\lambda_n D(v_k) \to \lambda_n D(v_n), \;\; B(v_k) \to B(v_n)(k \to \infty).$$

Thus $v_k \to v_n(k \to \infty)$ in $H_r^1(\mathbb{R}^3)$, and in particular, $\|f(v_n)\|_2^2 = c$, $F(v_n) = \gamma_n(c)$ and $F'(v_n) = \lambda_n v_n$ in $H_r^{-1}(\mathbb{R}^3)$. □

Next, we will give the proof of Theorem 1.1.

Proof. We recall that in Lemma 2.6, it has been proved that if $(v, \lambda) \in S_r(c) \times \mathbb{R}$ solves (2.1), then it only need $\lambda < 0$ provided $c > 0$ is sufficiently small. Thus by Lemma 2.4 and Proposition 2.1, when $c > 0$ is small enough, for each $n \in \mathbb{N}$, we obtain a couple solution $(v_n, \lambda_n) \in H_r^1(\mathbb{R}^3) \times \mathbb{R}^-$ solving (2.1) with $\|f(v_n)\|_2^2 = c$ and $F(v_n) = \gamma_n(c)$. Note from Lemmas 2.2 and 2.3, we have $\gamma_n(c) \to \infty(n \to \infty)$ and then we deduce that the sequence of solutions $\{(v_n, \lambda_n)\}$ is unbounded. In conclusion, it is proved that there are infinitely many normalized radial solutions to Eq (2.1), that is, there are infinitely many normalized radial solutions for Eq (1.1) and such that

$$-\Delta u_n - \lambda_n u_n + (|x|^{-1} * |u_n|^2)u_n - \Delta(u_n^2)u_n - |u_n|^{p-2}u_n = 0.$$

At this point, the proof of the theorem is completed. As the paper is about an elliptic PDE with a non-local term in the equation, for readers interested in non-local problems, there should also be a reference to recent articels about elliptic PDEs with non-local boundary conditions, e.g., [20]. □

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Grant Number 11661021 and 11861021); Science and Technology Foundation of Guizhou Province (Grant Number KY[2017]1084).

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. C. Bardos, F. Golse, A. Gottlieb, N. Mauser, Mean field dynamics of fermions and the time-dependent Hartree-Fock equation, *J. Math. Pure. Appl.*, **82** (2003), 665–683. https://doi.org/10.1016/S0021-7824(03)00023-0
2. P. Lions, Solutions of Hartree-Fock equations for Coulomb systems, *Commun. Math. Phys.*, **109** (1987), 33–97. https://doi.org/10.1007/BF01205672

3. E. Lieb, B. Simon, The Thomas-Fermi theory of atoms, molecules, and solids, *Adv. Math.*, **23** (1977), 22–116. https://doi.org/10.1016/0001-8708(77)90108-6
4. N. Mauser, The Schrödinger-Poisson-X^{α} equation, *Appl. Math. Lett.*, **14** (2001), 759–763. https://doi.org/10.1016/S0893-9659(01)80038-0
5. J. Bellazzini, G. Siciliano, Stable standing waves for a class of nonlinear Schrödinger-Poisson equations, *Z. Angew. Math. Phys.*, **62** (2011), 267–280. https://doi.org/10.1007/s00033-010-0092-1
6. J. Bellazzini, G. Siciliano, Scaling properties of functionals and existence of constrained minimizers, *J. Funct. Anal.*, **261** (2011), 2486–2507. https://doi.org/10.1016/j.jfa.2011.06.014
7. I. Catto, J. Dolbeault, O. Sańchez, J. Soler, Existence of steady states for the Maxwell-Schrödinger-Poisson system: exploring the applicability of the concentration-compactness principle, *Math. Mod. Meth. Appl. S.*, **23** (2013), 1915–1938. https://doi.org/10.1142/S0218202513500541
8. L. Jeanjean, T. Luo, Sharp non-existence results of prescribed L^2-norm solutions for some class of Schrödinger-Poisson and quasi-linear equations, *Z. Angew. Math. Phys.*, **64** (2013), 937–954. https://doi.org/10.1007/s00033-012-0272-2
9. O. Sańchez, J. Soler, Long-time dynamics of the Schrödinger-Poisson-Slater system, *J. Stat. Phys.*, **114** (2004), 179–204. https://doi.org/10.1023/B:JOSS.0000003109.97208.53
10. J. Bellazzini, L. Jeanjean, T. Luo, Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations, *P. Lond. Math. Soc.*, **107** (2013), 303–339. https://doi.org/10.1112/plms/pds072
11. T. Luo, Multiplicity of normalized solutions for a class of nonlinear Schrödinger-Poisson-Slater equations, *J. Math. Anal. Appl.*, **416** (2014), 195–204. https://doi.org/10.1016/j.jmaa.2014.02.038
12. M. Colin, L. Jeanjean, Solutions for a quasilinear Schrödinger equations: a dual approach, *Nonlinear Anal.-Theor.*, **56** (2004), 213–226. https://doi.org/10.1016/j.na.2003.09.008
13. J. Liu, Y. Wang, Z. Wang, Soliton solutions for quasilinear Schrödinger equations, II, *J. Differ. Equations*, **187** (2003), 473–493. https://doi.org/10.1016/S0022-0396(02)00064-5
14. Y. Xue, C. Tang, Existence of a bound state solution for quasilinear Schrödinger equations, *Adv. Nonlinear Anal.*, **8** (2019), 323–338. https://doi.org/10.1515/anona-2016-0244
15. T. Bartsch, S. de Valeriola, Normalized solutions of nonlinear Schrödinger equations, *Arch. Math.*, **100** (2013), 75–83. https://doi.org/10.1007/s00013-012-0468-x
16. S. Chen, X. Tang, S. Yuan, Normalized solutions for Schrödinger-Poisson equations with general nonlinearities, *J. Math. Anal. Appl.*, **481** (2019), 123447. https://doi.org/10.1016/j.jmaa.2019.123447
17. W. Xie, H. Chen, H. Shi, Existence and multiplicity of normalized solutions for a class of Schrödinger-Poisson equations with general nonlinearities, *Math. Method. Appl. Sci.*, **43** (2020), 3602–3616. https://doi.org/10.1002/mma.6140
18. M. Willem, *Minimax theorems, progress in nonlinear differential equations and their applications*, Boston: Birkhauser, 1996. http://dx.doi.org/10.1007/978-1-4612-4146-1
19. L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal.-Theor.*, **28** (1997), 1633–1659. https://doi.org/10.1016/S0362-546X(96)00021-1

20. P. Agarwal, J. Merker, G. Schuldt, Singular integral Neumann boundary conditions for semilinear elliptic PDEs, *Axioms*, **10** (2021), 74. https://doi.org/10.3390/axioms10020074

AIMS Press