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Research article

When does a double-layer potential equal to a single-layer one?

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Abstract: Let D be a bounded domain in $\mathbb{R}^3$ with a closed, smooth, connected boundary S, N be the outer unit normal to S, $k > 0$ be a constant, $u_{N^\pm}$ are the limiting values of the normal derivative of u on S from D, respectively $D' := \mathbb{R}^3 \setminus \bar{D}$; $g(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$, $w := w(x,\mu) := \int_S g_N(x,s)\mu(s)ds$ be the double-layer potential, $u := u(x,\sigma) := \int_S g(x,s)\sigma(s)ds$ be the single-layer potential.
In this paper it is proved that for every w there is a unique u, such that $w = u$ in D and vice versa. This result is new, although the potential theory has more than 150 years of history.
Necessary and sufficient conditions are given for the existence of u and the relation $w = u$ in D', given w in D', and for the existence of w and the relation $w = u$ in D', given u in D'.

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1. Introduction

Let D be a bounded domain in $\mathbb{R}^3$ with a closed, smooth, connected boundary S, $N = N_s$ be the outer unit normal to S at the point $s \in S$, $k > 0$ be a constant, $g(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$, $w := w(x,\mu) := \int_S g_N(x,s)\mu(s)ds$ be the double-layer potential, $w^\pm$ are the limiting values of w on S from D, respectively, from D', $u := u(x,\sigma) := \int_S g(x,s)\sigma(s)ds$ be the single-layer potential, $u_{N^\pm}$ are the limiting values of the normal derivative of u on S from D, respectively $D' := \mathbb{R}^3 \setminus \bar{D}$, $\bar{D}$ is the closure of D, $\bar{\sigma}$, denotes the complex conjugate of σ, $H^0 := L^2(S)$, $H^1 := W_2^1(S)$ is the Sobolev space, $\mu \in H^0$. We write iff for if and only if and use the known formulas for the limiting values of the potentials on S, see [3], pp. 148, 153:

$$w^\pm = 0.5(A' \mp I)\mu; \quad {u_N}^\pm = 0.5(A \pm I)\sigma; \quad w_N^+ = w_N^-; \quad Q(\sigma) := \int_S g(t,s)\sigma(s)ds, \tag{1.1}$$

where $A'\mu = \int_S g_{N_t}(t,s)\mu(s)ds$, $A\sigma = \int_S g_{N_s}(t,s)\sigma(s)ds$. In this paper it is proved that for every w in D there is a unique u, such that $w = u$ in D, and vice versa.

Necessary and sufficient conditions are given for $w = u$ in D' and for $u = w$ in D'.

In [4] the problem for the Laplace's equations was studied in Lipschitz domains. In this case the integral equations, based on potential of double layer, are uniquely solvable for any right-hand side. This is not so, in general, for the corresponding equations for the Helmholtz operator. Our method of proof and the results are new. We assume that the boundary S is smooth and connected. This is done for brevity and simplicity: we do not want to make the presentation more difficult for the reader than it is necessary. Our results and proofs are valid for Lipschitz boundaries. Theorem 1 can be used for looking for the solution of the Dirichlet problem in the form of the single-layer potential.

Let us state these results:

Theorem 1. *For every w, defined in D, there is a unique u such that w = u in D, and vice versa.*

For every w, defined in D′, there is a unique u such that w = u in D′ iff

$$\int_S w^- p ds = 0 \quad \forall p \in N(Q). \tag{1.2}$$

For every u, defined in D′, there is a unique w such that w = u in D′ iff

$$\int_S u r ds = 0 \quad \forall r \in N(A + I). \tag{1.3}$$

This result is new, although the potential theory has more than 150 years of history.

In Section 2 proofs are given. In Section 3 it is proved that $Q : H^0 \to H^1$ is a Fredholm operator.

2. Proofs

a) Assume that $w = w(x, \mu)$ is given in D. Let us prove that $u = u(x, \sigma)$ exists such that $u = w$ in D, and u is uniquely defined by w. First, let us prove the last claim. Suppose $u_1 = w$ and $u_2 = w$ in D. Let $u_1 - u_2 := u$, $u = \int_S g(x, s)\sigma ds$ in D, $\sigma = \sigma_1 - \sigma_2$. Then $u|_S = 0$, $(\nabla^2 + k^2)u = 0$ in D' and u satisfies the radiation condition, so $u = 0$ in D'. Thus, $u = 0$ in $D \cup D'$, $u_1 = u_2$, and the claim is proved.

Let us now prove the existence of u such that $w = u$ in D. One has

$$w^+ = 0.5(A' - I)\mu = u|_S = Q(\sigma) \tag{2.1}$$

This is an equation for σ while w^+ is given.

Note that $Q = Q_0 + Q_1$, where $Q_0\sigma = \int_S g_0(t, s)\sigma(s)ds$, $g_0(t, s) := \frac{1}{4\pi|t-s|}$. The operator Q_0 is an isomorphism of H^0 onto H^1, see Lemma 1 in Section 3. Therefore, the operator Q_0^{-1} is well defined and maps H^1 onto H^0.

Consequently, equation (2.1) is equivalent to

$$w^+ = (I + Q_1Q_0^{-1})Q_0(\sigma) = (I + Q_1Q_0^{-1})\eta, \quad \eta := Q_0\sigma, \quad \sigma = Q_0^{-1}\eta. \tag{2.2}$$

The operator $Q_1Q_0^{-1}$ is compact in H^0, see Section 3. Therefore a necessary and sufficient condition for the solvability of equation (2.2), and the equivalent equation (2.1), is:

$$\int_S w^+\bar{\eta} ds = 0 \quad \forall \eta \in N((I + Q_1Q_0^{-1})^\star), \tag{2.3}$$

where $N(B)$ is the null space of the operator B and $B^\star$ is the adjoint operator to B in H^0, $B = I + Q_1 Q_0^{-1}$, B is of Fredholm type in H^0. The kernel $g(t,s)$ of Q, the function $g(t,s) = \frac{e^{ik|t-s|}}{4\pi|t-s|}$, is symmetric: $g(t,s) = g(s,t)$. Therefore, the kernel of $Q^\star$ is $\bar{g}(t,s)$. Clearly, $(I + Q_1 Q_0^{-1})^\star \eta = 0$ iff $(I + Q_0^{-1} \bar{Q}_1)\eta = 0$, or, taking the complex conjugate,

$$(I + Q_0^{-1} Q_1)\bar{\eta} = 0, \tag{2.4}$$

where we have used the real-valuedness of the kernel of Q_0. Applying the operator Q_0 to the last equation, one gets an equivalent equation

$$(Q_0 + Q_1)\bar{\eta} = 0, \tag{2.5}$$

since Q_0 is an isomorphism. Let $u = u(x, \bar{\eta})$. Then $u|_S = 0$ according to equation (2.5). Since $(\nabla^2 + k^2)u = 0$ in D' and u satisfies the radiation condition, it follows that $u = 0$ in D' and $\bar{\eta} = u_N^+ - u_N^- = u_N^+$. Therefore, using the Green's formula, one obtains:

$$\int_S w^+ \bar{\eta} ds = \int_S w^+ u_N^+ ds = \int_S w_N^+ u ds = 0,$$

and, since $u = 0$ on S, it follows that condition (2.3) is always satisfied. Thus, the necessary and sufficient condition (2.3) for the solvability of equation (2.1) is always satisfied. Therefore, $u(x,\eta) = w(x,\mu)$ in D, $Q_0^{-1}\eta = \sigma$.

b) Asume now that u is given in D and let us prove the existence of a unique w such that $u = w$ in D. First, we prove that w is uniquely determined by u if $u = w$ in D.

Indeed, assume that there are two w_j, $j = 1, 2$, such that $u = w_j$ in D. Then $w := w_1 - w_2 = 0$ in D. Therefore $w_{N^+} = 0$ on S. It is known (see [3], p. 154) that $w_{N^+} = w_{N^-}$, so $w_{N^-} = 0$. Therefore, $(\nabla^2 + k^2)w = 0$ in D', $w_{N^-} = 0$ on S, and w satisfies the radiation condition at infinity. This implies $w = 0$ in D', so $w = 0$ in $D \cup D'$. Therefore $\mu = w^- - w^+ = 0$, so $w_1 = w_2$ if $u = w_j$, $j = 1, 2$, in D. We have proved that w is uniquely determined by u if $u = w$ in D.

Let us now prove the existence of the solution μ to equation (2.1) and the relation $u = w$ in D.

The operator $A' - I$ is Fredholm in H^0, so a necessary and sufficient condition for the equation (2.1) to be solvable is:

$$\int_S Q(\sigma)\bar{h} ds = 0 \quad \forall h \in N((A' - I)^\star). \tag{2.6}$$

One has $(A' - I)^\star h = (\bar{A} - I)h = 0$ iff $(A - I)\bar{h} = 0$.

If $(A - I)\bar{h} = 0$, then $u_N^-(s, \bar{h}) = 0$, so $u(s, \bar{h}) = 0$ in D'. Note that $u(s, \bar{h}) = Q(\bar{h})$. Therefore, $Q(\bar{h}) = 0$ in D'. Since Q is a symmetric operator in H^0, one has:

$$\int_S Q(\sigma)\bar{h} ds = \int_S \sigma Q(\bar{h}) ds = 0.$$

Consequently, condition (2.6) is always satisfied, the solution μ to equation (2.1) exists and $u = w$ in D.

c) Assume that w is given in D'. Let us prove that u exists such that $u = w$ in D' iff

$$\int_S w^- \bar{p} ds = 0 \quad \forall p \in N(Q^\star). \tag{2.7}$$

Consider the equation for σ:

$$w^- = Q(\sigma). \tag{2.8}$$

The operator $Q : H^0 \to H^1$ is Fredholm-type. Thus, a necessary and sufficient condition for the solvability of the above equation is equation (2.7). If σ solves (2.8), then $u(x,\sigma) = w$ in D' because the value of u on S determines uniquely u in D'.

d) Assume now that u is given in D'. Let us prove that w exists such that $u = w$ in D' iff

$$\int_S u\bar{p}ds = 0 \quad \forall p \in N(\bar{A} + I). \tag{2.9}$$

Note that $p \in N(A + I)$ iff $\bar{p} \in N(\bar{A} + I)$. The equation for μ, given u in D', is:

$$0.5(A' + I)\mu = u. \tag{2.10}$$

Since the operator $A' + I$ is Fredholm in H^0, a necessary and sufficient condition for the solvability of (2.10) for μ is:

$$\int_S u\bar{p}ds = 0 \quad \forall p \in N((A' + I)^\star). \tag{2.11}$$

If $p \in N((A' + I)^\star)$, then $p \in N(\bar{A} + I)$, so condition (2.9) is the same as (2.11). As in section c), the relation $u = w$ in D' is a consequence of the fact that $u = w^-$ on S.

Theorem 1 is proved. □

Remark 1. Our proofs remain valid if $k = 0$, that is, for the potentials corresponding to the Laplace equation, rather than the Helmholtz equation.

3. Auxiliary lemma

Recall that H^1 is the Sobolev space on S, $H^0 = L^2(S)$.

Lemma 1. *The operator $Q = Q_0 + Q_1 : H^0 \to H^1$ is of Fredholm-type, where Q_0 is the operator with the kernel $\frac{1}{4\pi|t-s|}$ and Q_1 has the kernel $\frac{e^{ik|t-s|}-1}{4\pi|s-t|}$. The operator Q_0 is an isomorhism of H^0 onto H^1, which has a continuous inverse. The operator $Q_1Q_0^{-1}$ is compact in H^0.*

Proof. Let us check that $Q_0 : H^0 \to H^1$ is an isomorphism. The Fourier transform of the kernel $\frac{1}{4\pi|x-y|}$ is positive: $\int_{\mathbb{R}^3} \frac{e^{i\xi\cdot x}}{4\pi|x|}dx = \frac{1}{|\xi|^2}$. So, $Q_0 : H^0 \to H^1$ is injective. The kernel of the operator Q_1 is smooth enough for $Q_1 : H^0 \to H^1$ to be compact. Let us check that $Q_0 : H^0 \to H^1$ is surjective. Let $f \in H^1$ and $Q_0\sigma = f$. Then $u := u(x,\sigma) = \int_S g(x,s)\sigma ds$ solves the problem: $\nabla^2 u = 0$ in D, $u|_S = f$. By the known elliptic estimates (see, e.g., [1]) one has $||u||_{H^{3/2}(D)} \leq ||u||_{H^1(S)}$. Therefore, $\nabla u \in H^{1/2}(D)$ and, by the trace theorem, $u|_S \in H^0(S)$. This proves surjectivity of $Q_0 : H^0 \to H^1$. Thus, Q_0 is an isomorphism of H^0 onto H^1 which has a continuous inverse. The sum of an isomorphism Q_0 and a compact operator Q_1 is a Fredholm operator, see, e.g., [2]. The operator $Q_1Q_0^{-1}$ is compact in H^0 because the kernel of Q_1 is sufficiently smooth. Although the operator $Q_1Q_0^{-1}$ is defined on a dense subset H^1 of H^0, but since this operator is bounded in H^0 its closure is a bounded operator in H^0. Since the kernel of Q_1 is $O(|s-t|)$, the kernel of $Q_1Q_0^{-1}$ is a continuous function of $|s-t|$ and the surface S is a compact set. Therefore, the operator $Q_1Q_0^{-1}$ is compact in H^0.

Lemma 1 is proved. □

4. Conclusions

It is proved that every double layer potential w in a bounded domain is equal to a single layer potential u in a bounded domain D with a smooth closed connected boundary. Necessary and sufficient conditions are given for $w = u$ in the exterior domain D'.

Conflict of interest

The authors declare that there are no conflicts of interest.

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