Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(10): 19221–19236.
DOI: 10.3934/math.20221055
Received: 08 June 2022
Revised: 16 August 2022
Accepted: 19 August 2022
Published: 30 August 2022

Research article

On solvability of BVP for a coupled Hadamard fractional systems involving fractional derivative impulses

Hui Huang[1,*], **Kaihong Zhao**[2,*] **and Xiuduo Liu**[3]

[1] School of Science, Shaoyang University, Shaoyang 422000, Hunan, China
[2] Department of Mathematics, School of Electronics & Information Engineering, Taizhou University, Taizhou 318000, Zhejiang, China
[3] Puyang Petrochemical Vocational and Technical College, Puyang 457001, Henan, China

* **Correspondence:** Email: 1224754144@qq.com, zhaokaihongs@126.com.

Abstract: Hadamard fractional calculus is one of the most important fractional calculus theories. Compared with a single Hadamard fractional order equation, Hadamard fractional differential equations have a more complex structure and a wide range of applications. It is difficult and challenging to study the dynamic behavior of Hadamard fractional differential equations. This manuscript mainly deals with the boundary value problem (BVP) of a nonlinear coupled Hadamard fractional system involving fractional derivative impulses. By applying nonlinear alternative of Leray-Schauder, we find some new conditions for the existence of solutions to this nonlinear coupled Hadamard fractional system. Our findings reveal that the impulsive function and its impulsive point have a great influence on the existence of the solution. As an application, we discuss an interesting example to verify the correctness and validity of our results.

Keywords: coupled Hadamard fractional system; boundary value problem; solvability; fractional derivative impulse; nonlinear alternative of Leray-Schauder
Mathematics Subject Classification: 34B10, 34B15, 34B37

1. Introduction

In 1892, Hadamard [1] proposed a new fractional integral and derivative with logarithmic function as integral kernel. This kind of fractional calculus is called Hadamard type fractional calculus, which is as famous as Riemann-Liouville and Caputo fractional calculus. Compared with Riemann-Liouville and Caputo fractional derivatives, Hadamard fractional derivative is significantly different in that the kernel function is invariant to expansion and contraction. In fact, for a given order $\alpha > 0$, the kernel function of Hadamard fractional derivative is $K_H(t, s) = (\log \frac{t}{s})^{\alpha-1}$, while the kernel function of Riemann-Liouville and Caputo fractional derivative is $K_R(t,s) = (t-s)^{\alpha-1}$. For any expansion coefficient $k > 0$, it is easy to see that $K_H(kt,ks) = K_H(t,s)$ and $K_R(kt,ks) \neq K_R(t,s)$. Scholars have made various generalizations on fractional derivatives and integrals in theory, and these generalizations are still continuing up to now. At the same time, the application of fractional calculus has been widely studied. The theoretical and applied research results of fractional calculus have been published in the form of monographs (see [2–9]). The fractional boundary value problem is one of the important contents in the study of fractional differential system. Many practical application problems such as blood flow, chemical engineering, thermo elasticity, underground water flow, population dynamics, and so forth ultimately come down to the study of boundary value problem of fractional differential equation. Therefore, many kinds of fractional boundary value problem and their dynamic properties have been focused and deeply studied. The types of fractional boundary value problem mainly include nonlocal (multi-point) BVP [10–15], integral BVP [13, 14, 16–19], impulse BVP [15, 18–22] and delay BVP [10, 12, 19], etc. The published papers on fractional differential system mainly study the existence [10–18, 22–27], stability [24, 28–36] and multiplicity [19, 21, 37, 38] of system solutions. Certainly, there have some previous works dealing with the Hadamard fractional system (see [39–47]). However, there are relatively few papers on the coupled system with Hadamard fractional derivative impulses. Compared with a single Hadamard fractional differential equation, Hadamard fractional differential equations have a more complex structure and a wide range of applications. In the process of applying fixed point theory to study the existence of solutions of Hadamard fractional differential equations, it is more difficult to construct the existence region of solutions and prove the compactness of operators than single equation.

In a recent paper [41], the authors considered the following impulsive integral BVP for a class of single Hadamard fractional equation.

$$\begin{cases} {}^H D_{t_k}^{\alpha} y(t) = \sum_{i=1}^{m} f_i(t, y(t), {}^H D_{t_k}^{\alpha} y(t), {}^H D_{t_k}^{\beta_i} y(t)), \ t \in (t_k, t_{k+1}] \subset J, \quad 0 \le k \le n, \\ {}^H J_{t_k}^{1-\alpha} y(t_k^+) - {}^H J_{t_{k-1}}^{1-\alpha} y(t_k^-) = I_k(y(t_k)), \ 1 \le k \le n, \\ {}^H J_a^{1-\alpha} y(a) = \lambda \cdot {}^H J_{t_n}^{1-\alpha} y(T), \end{cases}$$

where $J = [a, T]$, $0 < a < T$, $0 < \beta_i < \alpha < 1$ $(i = 1, 2, \ldots, m)$ and $\lambda \in R$ are some constants. ${}^H D_{t_k}^{*}$ stands the left-sided Hadamard fractional derivatives of order $*$. ${}^H J_{t_k}^{1-\alpha}$ is the left-sided Hadamard fractional integrals of order $1-\alpha$. $f_i \in C(J \times R^3, R)$, $I_k \in C(R, R)$. The impulsive point sequence $\{t_k\}_{k=1}^n$ satisfies $a = t_0 < t_1 < t_2 < t_3 < \ldots < t_n < t_{n+1} = T$. ${}^H J_{t_k}^{1-\alpha} y(t_k^+)$ and ${}^H J_{t_{k-1}}^{1-\alpha} y(t_k^-)$ represent the right and left limits at $t = t_k$ and satisfy ${}^H J_{t_{k-1}}^{1-\alpha} y(t_k^-) = {}^H J_{t_{k-1}}^{1-\alpha} y(t_k)$, respectively. Applying the contraction mapping principle, they investigated the existence, uniqueness and stability of solution. This system gives a great inspiration to the structure of the equations studied in this paper.

Motivated by the above mentioned, this manuscript mainly focuses on the following BVP of nonlinear Hadamard fractional differential coupling system with fractional impulses.

$$\begin{cases} {}^H D_{1^+}^{\alpha} x(t) = f(t, x(t), {}^H D_{1^+}^{\alpha_1} x(t), {}^H D_{1^+}^{\beta_1} y(t)), \ t \in J, \ t \neq t_k, \\ {}^H D_{1^+}^{\beta} y(t) = g(t, y(t), {}^H D_{1^+}^{\alpha_1} x(t), {}^H D_{1^+}^{\beta_1} y(t)), \ t \in J, \ t \neq t_k, \\ {}^H D_{1^+}^{\gamma_1} x(t_k^+) - {}^H D_{1^+}^{\gamma_1} x(t_k^-) = I_{1k}(x(t_k)), \ k = 1, \ldots, n, \\ {}^H D_{1^+}^{\gamma_2} y(t_k^+) - {}^H D_{1^+}^{\gamma_2} y(t_k^-) = I_{2k}(y(t_k)), \ k = 1, \ldots, n, \\ x(1) = x(e), \ y(1) = y(e), \end{cases} \tag{1.1}$$

where $J = [1, e]$, $1 < \alpha, \beta < 2$, $0 < \alpha_1, \beta_1, \gamma_1, \gamma_2 < 1$, ${}^H D^*_{1^+}$ is the Hadamard fractional derivative of order $*$, $f, g \in C(J \times R^3, R)$, $I_{1k}, I_{2k} \in C(R, R)$, and the pulse sequence $\{t_k\}$ satisfies $1 = t_0 < t_1 < \ldots < t_n < t_{n+1} = e$, ${}^H D^{\gamma_1}_{t_k^+} x(t_k^+)$, ${}^H D^{\gamma_2}_{t_k^+} y(t_k^+)$ and ${}^H D^{\gamma_1}_{t_{k-1}^+} x(t_k^-)$, ${}^H D^{\gamma_2}_{t_{k-1}^+} y(t_k^-)$ denote the right and left limits at $t = t_k$ such that ${}^H D^{\gamma_1}_{t_{k-1}^+} x(t_k^-) = {}^H D^{\gamma_1}_{t_{k-1}^+} x(t_k)$, ${}^H D^{\gamma_2}_{t_{k-1}^+} y(t_k^-) = {}^H D^{\gamma_1}_{t_{k-1}^+} y(t_k)$, $k = 1, 2, \ldots, n$.

In this manuscript, our contributions mainly include two aspects. On the one hand, our system structure is relatively complex. For example, system (1.1) is a coupled system and involves six Hadamard fractional derivatives. However, most of the previous work is single equation and involves fewer fractional derivatives. On the other hand, we try to study the existence of solutions of Hadamard fractional equations by using fixed point theory. In fact, we prove the existence of solution of system (1.1) by applying nonlinear alternative of Leray-Schauder. The most important steps in the proof are the construction of the existence region of the solution of system (1.1) and the compactness verification of the defined nonlinear operator.

The rest of this paper is organized as follows. In Section 2, we recall some definitions and lemmas of the Hadamard fractional calculus. In Section 3, we obtain the solvability of system (1.1). In Section 4, an example is given to demonstrate the application of our main results. Finally, Section 5 is a brief summary.

2. Preliminaries

Let $C(J, R)$ be a Banach space of continuous functions from J to R with the norm $\|\omega\|_C = \sup_{t \in J} |\omega(t)|$. A function set $PC(J, R)$ is defined by

$$\begin{aligned} PC(J,R) = \{\omega(t) \in C(J,R) :\ & {}^H D^{\alpha_1}_{1^+}\omega(t),\ {}^H D^{\beta_1}_{1^+}\omega(t) \in C(J,R),\ {}^H D^{\gamma_1}_{1^+}\omega(t_k^+),\ {}^H D^{\gamma_1}_{1^+}\omega(t_k^-), \\ & {}^H D^{\gamma_2}_{1^+}\omega(t_k^+) \text{ and } {}^H D^{\gamma_2}_{1^+}\omega(t_k^-) \text{ all exist, and satisfy } {}^H D^{\gamma_1}_{1^+}\omega(t_k^-) = {}^H D^{\gamma_1}_{1^+}\omega(t_k), \\ & {}^H D^{\gamma_2}_{1^+}\omega(t_k^-) = {}^H D^{\gamma_2}_{1^+}\omega(t_k), 0 < \alpha_1, \beta_1, \gamma_1, \gamma_2 < 1, 1 \le k \le n\}. \end{aligned}$$

Obviously, $PC(J, R)$ is a real Banach space equipped with the norm

$$\|\omega\|_{PC} = \max\left\{\|\omega\|_C,\ \|{}^H D^{\alpha_1}_{1^+}\omega\|_C,\ \|{}^H D^{\beta_1}_{1^+}\omega\|_C\right\}.$$

Let $X = PC(J, R) \times PC(J, R)$. It is easy to verify that X is a Banach space with the norm $\|(u, v)\| = \max\{\|u\|_{PC},\ \|v\|_{PC}\}$, for all $(u, v) \in X$.

Definition 2.1. *[5] For $a > 0$, the left-sided Hadamard fractional integral of order $\alpha > 0$ for a function $h : [a, \infty) \to R$ is defined by*

$${}^H J^{\alpha}_{a^+} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds,$$

provided the integral exists, where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2. *[5] For $a > 0$, if $h \in C^n[a, \infty)$ and $\alpha > 0$, the left-sided Hadamard fractional derivative of order α is defined by*

$${}^H D^{\alpha}_{a^+} h(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \frac{h(s)}{s} ds,$$

where $n - 1 < \alpha \le n$, $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1. *[5] For $a > 0$, assume that $h \in C^n(a,T) \cap L^1(a,T)$ with the left-sided Hadamard fractional derivative of order $\alpha > 0$, then*

$$ {}^HJ_{a^+}^{\alpha}({}^HD_{a^+}^{\alpha}h(t)) = h(t) + c_1\left(\log\frac{t}{a}\right)^{\alpha-1} + c_2\left(\log\frac{t}{a}\right)^{\alpha-2} + \ldots + c_n\left(\log\frac{t}{a}\right)^{\alpha-n}, $$

where $c_i \in R$, $i = 1,...,n-1,n$ and $n = [\alpha]+1$.

Lemma 2.2. *[5] Assume that $\alpha,\beta > 0$, then the following properties are true:*

$$ {}^HD_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{x}{a}\right)^{\beta-\alpha-1}, $$

$$ {}^HJ_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{x}{a}\right)^{\beta+\alpha-1}, $$

$$ {}^HD_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\alpha-j}(x) = 0, \ \ j = 1,2,\ldots,[\alpha]+1, $$

$$ {}^HD_{a^+}^{\alpha}({}^HJ_{a^+}^{\beta}y(t)) = {}^HJ_{a^+}^{\beta-\alpha}y(t), \ \ {}^HJ_{a^+}^{\beta}({}^HJ_{a^+}^{\alpha}y(t)) = {}^HJ_{a^+}^{\alpha+\beta}y(t). $$

Lemma 2.3. *(Nonlinear alternative of Leray-Schauder [48]) Let X be a Banach space, C be a nonempty convex subset of X, Ω be an open subset of C with $\theta \in \Omega$. Suppose that $T : \overline{\Omega} \to C$ is a completely continuous mapping. Then, either*

(i) *the mapping T has a fixed point in $\overline{\Omega}$, or*
(ii) *there exists a $u \in \partial\Omega$ and $\lambda \in (0,1)$ with $u = \lambda Tu$.*

Lemma 2.4. *Let $1 < \alpha < 2$, $0 < \gamma_1 < 1$, $I_{1k} \in C(R, R)$ $(k = 1,2,\ldots,n)$ and $z \in C(J,R)$. Then a function $x \in PC(J,R)$ is a solution of the boundary value problem*

$$ \begin{cases} {}^HD_{1^+}^{\alpha}x(t) = z(t), \ \ t \in J, \ \ t \neq t_k, \\ {}^HD_{1^+}^{\gamma_1}x(t_k^+) - {}^HD_{1^+}^{\gamma_1}x(t_k^-) = I_{1k}(x(t_k)), \ \ 1 \le k \le n, \\ x(1) = x(e), \end{cases} \tag{2.1} $$

iff $x \in PC(J,R)$ is a solution of the following integral equation:

$$ x(t) = \begin{cases} {}^HJ_{1^+}^{\alpha}z(t) + [\mathcal{B}_n - \mathcal{A}_n - {}^HJ_{1^+}^{\alpha}x(e)](\log t)^{\alpha-1}, \ \ t \in [1,t_1], \\ {}^HJ_{1^+}^{\alpha}z(t) + [\mathcal{B}_n - (\mathcal{A}_n - \mathcal{A}_k) - {}^HJ_{1^+}^{\alpha}x(e)](\log t)^{\alpha-1} - \mathcal{B}_k(\log t)^{\alpha-2}, \ \ t \in (t_k,t_{k+1}], \end{cases} \tag{2.2} $$

where $k = 1,2,\ldots,n$,

$$ \mathcal{A}_k = \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{k}(\log t_j)^{1+\gamma_1-\alpha}I_{1j}(x(t_j)), \ \ \ \mathcal{B}_k = \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}(\log t_k)^{2+\gamma_1-\alpha}I_{1k}(x(t_k)). $$

Proof. If $x \in PC(J,R)$ is a solution of (2.1), then, when $t \in [1,t_1]$, from Lemma 2.1, we have

$$ x(t) = {}^HJ_{1^+}^{\alpha}z(t) + c_{10}(\log t)^{\alpha-1} + c_{11}(\log t)^{\alpha-2}. \tag{2.3} $$

By the right continuity of $x(t)$ at $t = 1$, we get $c_{11} = 0$. It follows from (2.3) and Lemma 2.2 that

$$ {}^{H}D^{\gamma_1}_{1^+}x(t) = {}^{H}D^{\gamma_1}_{t_0^+}[{}^{H}J^{\alpha}_{1^+}z(t) + c_{10}(\log t)^{\alpha-1}] = {}^{H}J^{\alpha-\gamma_1}_{1^+}z(t) + c_{10}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}(\log t)^{\alpha-\gamma_1-1}, \tag{2.4} $$

and

$$ {}^{H}D^{\gamma_1}_{1^+}x(t_1^-) = {}^{H}D^{\gamma_1}_{1^+}x(t_1) = {}^{H}J^{\alpha-\gamma_1}_{1^+}z(t_1) + c_{10}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}(\log t_1)^{\alpha-\gamma_1-1}. \tag{2.5} $$

When $t \in (t_1, t_2]$, we similarly have

$$ x(t) = {}^{H}J^{\alpha}_{1^+}z(t) + c_{20}(\log t)^{\alpha-1} + c_{21}(\log t)^{\alpha-2}, \tag{2.6} $$

$$ {}^{H}D^{\gamma_1}_{1^+}x(t) = {}^{H}J^{\alpha-\gamma_1}_{1^+}z(t) + c_{20}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}(\log t)^{\alpha-\gamma_1-1} + c_{21}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma_1-1)}(\log t)^{\alpha-\gamma_1-2}, \tag{2.7} $$

and

$$ {}^{H}D^{\gamma_1}_{1^+}x(t_1^+) = {}^{H}J^{\alpha-\gamma_1}_{1^+}z(t_1) + c_{20}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}(\log t_1)^{\alpha-\gamma_1-1} + c_{21}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\gamma_1-1)}(\log t_1)^{\alpha-\gamma_1-2}. \tag{2.8} $$

By (2.5), (2.8) and ${}^{H}D^{\gamma_1}_{1^+}x(t_1^+) - {}^{H}D^{\gamma_1}_{1^+}x(t_1^-) = I_{11}(x(t_1))$, we obtain

$$ c_{20} - c_{10} = \frac{\Gamma(\alpha-\gamma_1)}{\Gamma(\alpha)}(\log t_1)^{1+\gamma_1-\alpha}I_{11}(x(t_1)) - c_{21}\frac{\alpha-\gamma_1-1}{\alpha-1}(\log t_1)^{-1}. \tag{2.9} $$

From the continuity of $x(t)$ at $t = t_1$, (2.3) and (2.6), we get

$$ c_{21} = -(c_{20} - c_{10})\log t_1. \tag{2.10} $$

Equations (2.9) and (2.10) lead to

$$ \begin{cases} c_{20} - c_{10} = \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}(\log t_1)^{1+\gamma_1-\alpha}I_{11}(x(t_1)), \\ c_{21} = -\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}(\log t_1)^{2+\gamma_1-\alpha}I_{11}(x(t_1)). \end{cases} \tag{2.11} $$

When $t \in (t_k, t_{k+1}]$, $k = 2, 3, \ldots, n$, repeating the above calculation, we get

$$ x(t) = {}^{H}J^{\alpha}_{1^+}z(t) + c_{k+1,0}(\log t)^{\alpha-1} + c_{k+1,1}(\log t)^{\alpha-2}, \tag{2.12} $$

and

$$ \begin{cases} c_{k+1,0} - c_{k0} = \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}(\log t_k)^{1+\gamma_1-\alpha}I_{1k}(x(t_k)), \\ c_{k+1,1} = -\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}(\log t_k)^{2+\gamma_1-\alpha}I_{1k}(x(t_k)). \end{cases} \tag{2.13} $$

It follows from (2.11) and (2.13) that

$$ \begin{cases} c_{k+1,0} = c_{10} + \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum\limits_{j=1}^{k}(\log t_j)^{1+\gamma_1-\alpha}I_{1j}(x(t_j)), \\ c_{k+1,1} = -\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}(\log t_k)^{2+\gamma_1-\alpha}I_{1k}(x(t_k)). \end{cases} \tag{2.14} $$

Thus,

$$\begin{aligned} x(e) &= {}^H J_{1^+}^{\alpha} z(e) + c_{n+1,0}(\log e)^{\alpha-1} + c_{n+1,1}(\log e)^{\alpha-2} \\ &= {}^H J_{1^+}^{\alpha} z(e) + c_{10} + \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}(\log t_j)^{1+\gamma_1-\alpha} I_{1j}(x(t_j)) - \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}(\log t_n)^{2+\gamma_1-\alpha} I_{1n}(x(t_n)) \\ &= x(1) = 0, \end{aligned}$$

which implies that

$$c_{10} = \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}(\log t_n)^{2+\gamma_1-\alpha} I_{1n}(x(t_n)) - {}^H J_{1^+}^{\alpha} z(e) - \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}(\log t_j)^{1+\gamma_1-\alpha} I_{1j}(x(t_j)). \tag{2.15}$$

Substituting (2.15) and (2.14) into (2.3) and (2.12), one can easily obtain (2.2), that is, $x \in PC(J, R)$ is also a solution of (2.2). Since the above derivation is completely reversible, so vice versa. The proof is completed. □

Similarly, we have the following assertion.

Lemma 2.5. *Let* $1 < \beta < 2$, $0 < \gamma_2 < 1$, $I_{2k} \in C(R, R)$ $(k = 1, 2, \ldots, n)$ *and* $w \in C(J, R)$. *Then a function* $y \in PC(J, R)$ *is a solution of the boundary value problem*

$$\begin{cases} {}^H D_{1^+}^{\beta} y(t) = w(t), \ t \in J, \ t \neq t_k, \\ {}^H D_{1^+}^{\gamma_2} y(t_k^+) - {}^H D_{1^+}^{\gamma_2} y(t_k^-) = I_{2k}(y(t_k)), \ 1 \le k \le n, \\ y(1) = y(e), \end{cases} \tag{2.16}$$

iff $y \in PC(J, R)$ *is a solution of the following integral equation:*

$$y(t) = \begin{cases} {}^H J_{1^+}^{\beta} w(t) + [\mathcal{D}_n - C_n - {}^H J_{1^+}^{\beta} y(e)](\log t)^{\beta-1}, \ t \in [1, t_1], \\ {}^H J_{1^+}^{\beta} w(t) + [\mathcal{D}_k - (C_n - C_k) - {}^H J_{1^+}^{\beta} y(e)](\log t)^{\beta-1} - \mathcal{D}_k(\log t)^{\beta-2}, \ t \in (t_k, t_{k+1}], \end{cases} \tag{2.17}$$

where $k = 1, 2, \ldots, n$,

$$C_k = \frac{\Gamma(\beta-\gamma_2)}{\gamma_2\Gamma(\beta-1)}\sum_{j=1}^{k}(\log t_j)^{1+\gamma_2-\beta} I_{2j}(y(t_j)), \quad \mathcal{D}_k = \frac{\Gamma(\beta-\gamma_2)}{\gamma_2\Gamma(\beta-1)}(\log t_k)^{2+\gamma_2-\beta} I_{2k}(y(t_k)).$$

3. Main results

In this section, we shall investigate the solvability of BVP (1.1) by employing the nonlinear alternative of Leray-Schauder.

Theorem 3.1. *If the following conditions* (H_1)–(H_4) *hold, then the boundary value problem* (1.1) *has at least a pair of solution.*

(H_1) *The functions* $f, g \in C(J \times R^3, R)$, *and* $I_{1k}, I_{2k} \in C(R, R)$, $k = 1, 2, \ldots, n$;

(H_2) *For all* $u, v, w \in R$, $t \in J$, *there exist* $a_i, b_i, d_i, p_i \in C(J, R_+)$ *with* $a_i^* = \sup_{t\in J} a_i(t)$, $b_i^* = \sup_{t\in J} b_i(t)$, $d_i^* = \sup_{t\in J} d_i(t)$ *and* $p_i^* = \sup_{t\in J} p_i(t)$ $(i = 1, 2)$, *such that*

$$|f(t, u, v, w)| \le a_1(t) + b_1(t)|u| + d_1(t)|v| + p_1(t)|w|,$$
$$|g(t, u, v, w)| \le a_2(t) + b_2(t)|u| + d_2(t)|v| + p_2(t)|w|;$$

(H_3) *For any* $u \in R$, *there exist some constants* $M_k, N_k > 0$, $k = 1, 2, \ldots, n$, *such that*

$$|I_{1k}(u)| \le M_k|u|, \ \ |I_{2k}(u)| \le N_k|u|;$$

(H_4) $0 < \lambda_1, \lambda_2 < 1$, *where*

$$\lambda_1 = \frac{b_1^* + d_1^* + p_1^* + 1}{\Gamma(\alpha + 1)} + \frac{\Gamma(\alpha - \gamma_1)}{\gamma_1\Gamma(\alpha - 1)}\Big[M_n + \sum_{j=1}^{n} M_j(\log t_j)^{1+\gamma_1-\alpha} + (\log t_1)^{\alpha-2}\sum_{j=1}^{n} M_j\Big],$$

$$\lambda_2 = \frac{b_2^* + d_2^* + p_2^* + 1}{\Gamma(\beta + 1)} + \frac{\Gamma(\beta - \gamma_2)}{\gamma_2\Gamma(\beta - 1)}\Big[N_n + \sum_{j=1}^{n} N_j(\log t_j)^{1+\gamma_2-\beta} + (\log t_1)^{\beta-2}\sum_{j=1}^{n} N_j\Big].$$

Proof. Let $\Omega = \{(x, y) \in X : \|(x, y)\| < r\}$, where $r \ge \max\left\{\frac{a_1^*}{(1-\lambda_1)\Gamma(\alpha+1)}, \frac{a_2^*}{(1-\lambda_2)\Gamma(\beta+1)}\right\}$. Then $\overline{\Omega} = \{(x, y) \in X : \|(x, y)\| \le r\}$, $\partial\Omega = \{(x, y) \in X : \|(x, y)\| = r\}$. According to Lemmas 2.4 and 2.5, define an operator $T : \overline{\Omega} \to X$ as follows:

$$T(x, y)(t) = (T_1(x, y)(t),\ T_2(x, y)(t))^T,\ \forall\ (x, y) \in X,\ t \in J, \tag{3.1}$$

where

$$T_1(x, y)(t) = \begin{cases} {}^HJ^{\alpha}_{1^+} f(t, x(t), {}^HD^{\alpha_1}_{1^+}x(t), {}^HD^{\beta_1}_{1^+}y(t)) + [\mathcal{B}_n - \mathcal{A}_n - {}^H J^{\alpha}_{1^+}x(e)](\log t)^{\alpha-1}, & t \in [1, t_1], \\ {}^HJ^{\alpha}_{1^+} f(t, x(t), {}^HD^{\alpha_1}_{1^+}x(t), {}^HD^{\beta_1}_{1^+}y(t)) + [\mathcal{B}_n - (\mathcal{A}_n - \mathcal{A}_k) \\ \quad - {}^HJ^{\alpha}_{1^+}x(e)](\log t)^{\alpha-1} - \mathcal{B}_k(\log t)^{\alpha-2}, & t \in (t_k, t_{k+1}], \end{cases} \tag{3.2}$$

and

$$T_2(x, y)(t) = \begin{cases} {}^HJ^{\beta}_{1^+} g(t, y(t), {}^HD^{\alpha_1}_{1^+}x(t), {}^HD^{\beta_1}_{1^+}y(t)) + [\mathcal{D}_n - C_n - {}^H J^{\beta}_{1^+}y(e)](\log t)^{\beta-1}, & t \in [1, t_1], \\ {}^HJ^{\beta}_{1^+} g(t, y(t), {}^HD^{\alpha_1}_{1^+}x(t), {}^HD^{\beta_1}_{1^+}y(t)) + [\mathcal{D}_n - (C_n - C_k) \\ \quad - {}^HJ^{\beta}_{1^+}y(e)](\log t)^{\beta-1} - \mathcal{D}_k(\log t)^{\beta-2}, & t \in (t_k, t_{k+1}], \end{cases} \tag{3.3}$$

where $\mathcal{A}_k$, $\mathcal{B}_k$, C_k and $\mathcal{D}_k$ are defined as Lemmas 2.4 and 2.5. Thus, the existence of solution of system (1.1) is equivalent to the existence of fixed point of an operator T defined by (3.1)–(3.3). Now we shall apply Lemma 2.3 to prove that T exists a fixed point $(x^*(t), y^*(t)) \in \overline{\Omega}$. Firstly, we need to verify that $T : \overline{\Omega} \to X$ is completely continuous. In fact, for all $(x, y) \in \overline{\Omega}$, $t \in J = [1, e]$, when $t \in [1, t_1]$, we derive from conditions (H_1)–(H_4) that

$$
\begin{aligned}
&|T_1(x,y)(t)|\\
\le&{}^HJ^{\alpha}_{1^+}|f(t,x(t),\,{}^HD^{\alpha_1}_{1^+}x(t),\,{}^HD^{\beta_1}_{1+}y(t))|+[|\mathcal{B}_n|+|\mathcal{A}_n|+\,{}^HJ^{\alpha}_{1^+}|x(e)|](\log t_1)^{\alpha-1}\\
\le&{}^HJ^{\alpha}_{1^+}[a_1(t)+b_1(t)|x(t)|+d_1(t)|{}^HD^{\alpha_1}_{1^+}x(t)|+p_1(t)|{}^HD^{\beta_1}_{1+}y(t)|]\\
&+\Big[\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}(\log t_n)^{2+\gamma_1-\alpha}M_n|x(t_n)|+\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}(\log t_j)^{1+\gamma_1-\alpha}M_j|x(t_j)|+\,{}^HJ^{\alpha}_{1^+}|x(e)|\Big](\log t_1)^{\alpha-1}\\
\le&{}^HJ^{\alpha}_{1^+}[a_1^*+b_1^*\|x\|_{PC}+d_1^*\|x\|_{PC}+p_1^*\|y\|_{PC}]+\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}M_n\|x\|_{PC}\\
&+\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}(\log t_j)^{1+\gamma_1-\alpha}M_j\|x\|_{PC}+\,{}^HJ^{\alpha}_{1^+}\|x\|_{PC}\\
\le&{}^HJ^{\alpha}_{1^+}[a_1^*+b_1^*\|x\|_{PC}+d_1^*\|x\|_{PC}+p_1^*\|y\|_{PC}]+\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}M_j\|x\|_{PC}\\
&+\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}(\log t_j)^{1+\gamma_1-\alpha}M_j\|x\|_{PC}+\,{}^HJ^{\alpha}_{1^+}\|x\|_{PC}\\
\le&[a_1^*+b_1^*\|x\|_{PC}+d_1^*\|x\|_{PC}+p_1^*\|y\|_{PC}]\cdot\frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{ds}{s}+\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}M_j\|x\|_{PC}\\
&+\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}(\log t_j)^{1+\gamma_1-\alpha}M_j\|x\|_{PC}+\|x\|_{PC}\cdot\frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{ds}{s}\\
=&[a_1^*+b_1^*\|x\|_{PC}+d_1^*\|x\|_{PC}+p_1^*\|y\|_{PC}]\cdot\frac{1}{\Gamma(\alpha+1)}(\log t)^{\alpha}+\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}M_n\|x\|_{PC}\\
&+\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}(\log t_j)^{1+\gamma_1-\alpha}M_j\|x\|_{PC}+\|x\|_{PC}\cdot\frac{1}{\Gamma(\alpha+1)}(\log t)^{\alpha}\\
\le&\frac{a_1^*}{\Gamma(\alpha+1)}+\Big\{\frac{b_1^*+d_1^*+p_1^*+1}{\Gamma(\alpha+1)}+\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\Big[M_n+\sum_{j=1}^{n}M_j(\log t_j)^{1+\gamma_1-\alpha}\Big]\Big\}\|(x,y)\|\\
<&\frac{a_1^*}{\Gamma(\alpha+1)}+\lambda_1\|(x,y)\|\le r.
\end{aligned}\tag{3.4}
$$

When $t\in(t_k,t_{k+1}]$, $k=1,2,\ldots,n$, we have

$$
\begin{aligned}
|T_1(x,y)(t)|\le&{}^HJ^{\alpha}_{1^+}|f(t,x(t),\,{}^HD^{\alpha_1}_{1^+}x(t),\,{}^HD^{\beta_1}_{1+}y(t))|\\
&+[|\mathcal{B}_n|+|\mathcal{A}_n-\mathcal{A}_k|+\,{}^HJ^{\alpha}_{1^+}|x(e)|](\log t)^{\alpha-1}+|\mathcal{B}_k|(\log t)^{\alpha-2}\\
\le&{}^HJ^{\alpha}_{1^+}[a_1(t)+b_1(t)|x(t)|+d_1(t)|{}^HD^{\alpha_1}_{1^+}x(t)|+p_1(t)|{}^HD^{\beta_1}_{1+}y(t)|]\\
&+\Big[\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}(\log t_n)^{2+\gamma_1-\alpha}M_n|x(t_n)|+\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=k+1}^{n}(\log t_j)^{1+\gamma_1-\alpha}M_j|x(t_j)|\\
&+\,{}^HJ^{\alpha}_{1^+}|x(e)|\Big](\log t_{k+1})^{\alpha-1}+\frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}(\log t_k)^{2+\gamma_1-\alpha}M_k|x(t_k)|(\log t_k)^{\alpha-2}
\end{aligned}
$$

$$
\begin{aligned}
&\le {}^H J^{\alpha}_{1^+}[a_1^* + b_1^*\|x\|_{PC} + d_1^*\|x\|_{PC} + p_1^*\|y\|_{PC}] + \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}M_n\|x\|_{PC} + {}^H J^{\alpha}_{1^+}\|x\|_{PC}\\
&\quad + \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}(\log t_j)^{1+\gamma_1-\alpha}M_j\|x\|_{PC} + \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}M_k|x(t_k)|(\log t_1)^{\alpha-2}\\
&\le {}^H J^{\alpha}_{1^+}[a_1^* + b_1^*\|x\|_{PC} + d_1^*\|x\|_{PC} + p_1^*\|y\|_{PC}] + \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}M_n\|x\|_{PC} + {}^H J^{\alpha}_{1^+}\|x\|_{PC}\\
&\quad + \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}(\log t_j)^{1+\gamma_1-\alpha}M_j\|x\|_{PC} + \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}M_j\|x\|_{PC}(\log t_1)^{\alpha-2}\\
&\le [a_1^* + b_1^*\|x\|_{PC} + d_1^*\|x\|_{PC} + p_1^*\|y\|_{PC}]\cdot\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\\
&\quad + \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}M_n\|x\|_{PC} + \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}(\log t_j)^{1+\gamma_1-\alpha}M_j\|x\|_{PC}\\
&\quad + \|x\|_{PC}\cdot\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} + \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}M_j\|x\|_{PC}(\log t_1)^{\alpha-2}\\
&= [a_1^* + b_1^*\|x\|_{PC} + d_1^*\|x\|_{PC} + p_1^*\|y\|_{PC}]\cdot\frac{1}{\Gamma(\alpha+1)}(\log t)^{\alpha}\\
&\quad + \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}M_n\|x\|_{PC} + \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}(\log t_j)^{1+\gamma_1-\alpha}M_j\|x\|_{PC}\\
&\quad + \|x\|_{PC}\cdot\frac{1}{\Gamma(\alpha+1)}(\log t)^{\alpha} + \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\sum_{j=1}^{n}M_j\|x\|_{PC}(\log t_1)^{\alpha-2}\\
&\le \frac{a_1^*}{\Gamma(\alpha+1)} + \left\{\frac{b_1^* + d_1^* + p_1^* + 1}{\Gamma(\alpha+1)} + \frac{\Gamma(\alpha-\gamma_1)}{\gamma_1\Gamma(\alpha-1)}\left[M_n + \sum_{j=1}^{n}M_j(\log t_j)^{1+\gamma_1-\alpha}\right.\right.\\
&\quad \left.\left. + (\log t_1)^{\alpha-2}\sum_{j=1}^{n}M_j\right]\right\}\|(x,y)\| = \frac{a_1^*}{\Gamma(\alpha+1)} + \lambda_1\|(x,y)\| \le r. \qquad (3.5)
\end{aligned}
$$

Similar to (3.4) and (3.5), we also have

$$
\begin{aligned}
|T_2(x,y)(t)| &\le \frac{a_2^*}{\Gamma(\beta+1)} + \left\{\frac{b_2^* + d_2^* + p_2^* + 1}{\Gamma(\beta+1)} + \frac{\Gamma(\beta-\gamma_2)}{\gamma_2\Gamma(\beta-1)}\left[\sum_{j=1}^{n}N_j(\log t_j)^{1+\gamma_2-\beta} + N_n\right]\right\}\|(x,y)\|\\
&< \frac{a_2^*}{\Gamma(\beta+1)} + \lambda_2\|(x,y)\| \le r,\ \ t\in[1,t_1], \qquad (3.6)
\end{aligned}
$$

and

$$
\begin{aligned}
&|T_2(x,y)(t)|\\
\le& \frac{a_2^*}{\Gamma(\beta+1)} + \left\{\frac{b_2^* + d_2^* + p_2^* + 1}{\Gamma(\beta+1)} + \frac{\Gamma(\beta-\gamma_2)}{\gamma_2\Gamma(\beta-1)}\left[\sum_{j=1}^{n}N_j(\log t_j)^{1+\gamma_2-\beta} + N_n + (\log t_1)^{\beta-2}\sum_{j=1}^{n}N_j\right]\right\}\|(x,y)\|\\
=& \frac{a_2^*}{\Gamma(\beta+1)} + \lambda_2\|(x,y)\| \le r,\ \ t\in(t_k,t_{k+1}]. \qquad (3.7)
\end{aligned}
$$

From (3.4)–(3.7), one knows that T is uniformly bounded and $T(\overline{\Omega}) \subset \overline{\Omega}$.

Next, we show that the operator T is equicontinuous. Indeed, let $\tau_2, \tau_1 \in J = [1, e]$ with $\tau_1 < \tau_2$ and $(x, y) \in \overline{\Omega}$, then when $\tau_1, \tau_2 \in [1, t_1]$, we have

$$\begin{aligned}
&|T_1(x, y)(\tau_2) - T_1(x, y)(\tau_1)| \\
\leq& |{}^H J^{\alpha}_{1^+} f(\tau_2, x(\tau_2), {}^H D^{\alpha_1}_{1^+} x(\tau_2), {}^H D^{\beta_1}_{1+} y(\tau_2)) - {}^H J^{\alpha}_{1^+} f(\tau_1, x(\tau_1), {}^H D^{\alpha_1}_{1^+} x(\tau_1), {}^H D^{\beta_1}_{1+} y(\tau_1))| \\
&+ [|\mathcal{B}_n| + |\mathcal{A}_n| + {}^H J^{\alpha}_{1^+} |x(e)|] \Big| (\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1} \Big| \\
=& \Big| \frac{1}{\Gamma(\alpha)} \int_1^{\tau_2} \Big(\log \frac{\tau_2}{s} \Big)^{\alpha-1} f(s, x(s), {}^H D^{\alpha_1}_{1^+} x(s), {}^H D^{\beta_1}_{1+} y(s)) \frac{ds}{s} \\
&\frac{1}{\Gamma(\alpha)} \int_1^{\tau_1} \Big(\log \frac{\tau_1}{s} \Big)^{\alpha-1} f(s, x(s), {}^H D^{\alpha_1}_{1^+} x(s), {}^H D^{\beta_1}_{1+} y(s)) \frac{ds}{s} \Big| \\
&+ [|\mathcal{B}_n| + |\mathcal{A}_n| + {}^H J^{\alpha}_{1^+} |x(e)|] \Big| (\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1} \Big| \\
=& \Big| \frac{1}{\Gamma(\alpha)} \int_1^{\tau_1} \Big[\Big(\log \frac{\tau_2}{s} \Big)^{\alpha-1} - \Big(\log \frac{\tau_1}{s} \Big)^{\alpha-1} \Big] f(s, x(s), {}^H D^{\alpha_1}_{1^+} x(s), {}^H D^{\beta_1}_{1+} y(s)) \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \Big(\log \frac{\tau_2}{s} \Big)^{\alpha-1} f(s, x(s), {}^H D^{\alpha_1}_{1^+} x(s), {}^H D^{\beta_1}_{1+} y(s)) \frac{ds}{s} \Big| \\
&+ [|\mathcal{B}_n| + |\mathcal{A}_n| + {}^H J^{\alpha}_{1^+} |x(e)|] \Big| (\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1} \Big| \\
\leq& \frac{1}{\Gamma(\alpha)} \int_1^{\tau_1} \Big[\Big(\log \frac{\tau_2}{s} \Big)^{\alpha-1} - \Big(\log \frac{\tau_1}{s} \Big)^{\alpha-1} \Big] |f(s, x(s), {}^H D^{\alpha_1}_{1^+} x(s), {}^H D^{\beta_1}_{1+} y(s))| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \Big(\log \frac{\tau_2}{s} \Big)^{\alpha-1} |f(s, x(s), {}^H D^{\alpha_1}_{1^+} x(s), {}^H D^{\beta_1}_{1+} y(s))| \frac{ds}{s} \\
&+ [|\mathcal{B}_n| + |\mathcal{A}_n| + {}^H J^{\alpha}_{1^+} |x(e)|] \Big| (\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1} \Big| \\
\leq& [a_1^* + b_1^* \|x\|_{PC} + d_1^* \|x\|_{PC} + p_1^* \|y\|_{PC}] \cdot \frac{1}{\Gamma(\alpha)} \int_1^{\tau_1} \Big[\Big(\log \frac{\tau_2}{s} \Big)^{\alpha-1} - \Big(\log \frac{\tau_1}{s} \Big)^{\alpha-1} \Big] \frac{ds}{s} \\
&+ [a_1^* + b_1^* \|x\|_{PC} + d_1^* \|x\|_{PC} + p_1^* \|y\|_{PC}] \cdot \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \Big(\log \frac{\tau_2}{s} \Big)^{\alpha-1} \frac{ds}{s} \\
&+ [|\mathcal{B}_n| + |\mathcal{A}_n| + {}^H J^{\alpha}_{1^+} |x(e)|] \Big| (\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1} \Big| \\
=& \frac{1}{\Gamma(\alpha+1)} [a_1^* + b_1^* \|x\|_{PC} + d_1^* \|x\|_{PC} + p_1^* \|y\|_{PC}] \Big[(\log \tau_2)^{\alpha} - \Big(\log \frac{\tau_2}{\tau_1} \Big)^{\alpha} - (\log \tau_1)^{\alpha} \Big] \\
&+ \frac{1}{\Gamma(\alpha+1)} [a_1^* + b_1^* \|x\|_{PC} + d_1^* \|x\|_{PC} + p_1^* \|y\|_{PC}] \Big(\log \frac{\tau_2}{\tau_1} \Big)^{\alpha} \\
&+ [|\mathcal{B}_n| + |\mathcal{A}_n| + {}^H J^{\alpha}_{1^+} |x(e)|] \Big| (\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1} \Big| \to 0, \text{ as } \tau_1 \to \tau_2.
\end{aligned} \tag{3.8}$$

When $\tau_1, \tau_2 \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, n$, similar to (3.8), we get

$$\begin{aligned}
&|T_1(x, y)(\tau_2) - T_1(x, y)(\tau_1)| \\
\leq& |{}^H J^{\alpha}_{1^+} f(\tau_2, x(\tau_2), {}^H D^{\alpha_1}_{1^+} x(\tau_2), {}^H D^{\beta_1}_{1+} y(\tau_2)) - {}^H J^{\alpha}_{1^+} f(\tau_1, x(\tau_1), {}^H D^{\alpha_1}_{1^+} x(\tau_1), {}^H D^{\beta_1}_{1+} y(\tau_1))| \\
&+ [|\mathcal{B}_n| + |\mathcal{A}_n - \mathcal{A}_k| + {}^H J^{\alpha}_{1^+} |x(e)|] \Big| (\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1} \Big| + |\mathcal{B}_k| \Big| (\log \tau_2)^{\alpha-2} - (\log \tau_1)^{\alpha-2} \Big| \\
\leq& \frac{1}{\Gamma(\alpha+1)} [a_1^* + b_1^* \|x\|_{PC} + d_1^* \|x\|_{PC} + p_1^* \|y\|_{PC}] \Big[(\log \tau_2)^{\alpha} - \Big(\log \frac{\tau_2}{\tau_1} \Big)^{\alpha} - (\log \tau_1)^{\alpha} \Big]
\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha+1)}[a_1^*+b_1^*\|x\|_{PC}+d_1^*\|x\|_{PC}+p_1^*\|y\|_{PC}]\Big(\log\frac{\tau_2}{\tau_1}\Big)^{\alpha}$$
$$+[|\mathcal{B}_n|+|\mathcal{A}_n-\mathcal{A}_k|+{}^HJ_{1^+}^{\alpha}|x(e)|]\Big|(\log\tau_2)^{\alpha-1}-(\log\tau_1)^{\alpha-1}\Big|$$
$$+|\mathcal{B}_k|\Big|(\log\tau_2)^{\alpha-2}-(\log\tau_1)^{\alpha-2}\Big|\to 0,\ \text{as}\ \tau_1\to\tau_2. \tag{3.9}$$

Similar to (3.8) and (3.10), we obtain

$$|T_2(x,y)(\tau_2)-T_2(x,y)(\tau_1)|$$
$$\le\frac{1}{\Gamma(\beta+1)}[a_2^*+b_2^*\|x\|_{PC}+d_2^*\|x\|_{PC}+p_2^*\|y\|_{PC}]\Big[(\log\tau_2)^{\beta}-\Big(\log\frac{\tau_2}{\tau_1}\Big)^{\beta}-(\log\tau_1)^{\beta}\Big]$$
$$+\frac{1}{\Gamma(\beta+1)}[a_2^*+b_2^*\|x\|_{PC}+d_2^*\|x\|_{PC}+p_2^*\|y\|_{PC}]\Big(\log\frac{\tau_2}{\tau_1}\Big)^{\beta}+[|\mathcal{D}_n|+|C_n|+{}^HJ_{1^+}^{\beta}|y(e)|]$$
$$\times\Big|(\log\tau_2)^{\beta-1}-(\log\tau_1)^{\beta-1}\Big|\to 0,\ \text{as}\ \tau_1\to\tau_2,\ \tau_1,\tau_2\in[1,t_1], \tag{3.10}$$

and

$$|T_2(x,y)(\tau_2)-T_2(x,y)(\tau_1)|$$
$$\le\frac{1}{\Gamma(\beta+1)}[a_2^*+b_2^*\|x\|_{PC}+d_2^*\|x\|_{PC}+p_2^*\|y\|_{PC}]\Big[(\log\tau_2)^{\beta}-\Big(\log\frac{\tau_2}{\tau_1}\Big)^{\beta}-(\log\tau_1)^{\beta}\Big]$$
$$+\frac{1}{\Gamma(\beta+1)}[a_2^*+b_2^*\|x\|_{PC}+d_2^*\|x\|_{PC}+p_2^*\|y\|_{PC}]\Big(\log\frac{\tau_2}{\tau_1}\Big)^{\beta}$$
$$+[|\mathcal{D}_n|+|C_n-C_k|+{}^HJ_{1^+}^{\beta}|y(e)|]\Big|(\log\tau_2)^{\beta-1}-(\log\tau_1)^{\beta-1}\Big|$$
$$+|\mathcal{D}_k|\Big|(\log\tau_2)^{\beta-2}-(\log\tau_1)^{\beta-2}\Big|\to 0,\ \text{as}\ \tau_1\to\tau_2,\ \tau_1,\tau_2\in(t_k,t_{k+1}]. \tag{3.11}$$

From (3.8)–(3.11), we conclude that, for all $\epsilon>0$ and $(x,y)\in X$, there exists $\nu=\nu(\epsilon)>0$ with $\nu\le\max\{t_{k+1}-t_k : k=0,1,2,\dots,n\}$ such that $\|T(x,y)\|<\epsilon$ provided that $|\tau_2-\tau_1|<\nu$ for all $\tau_2,\tau_2\in[1,e]$. That is, the operator T is equicontinuous. Hence, by the Arzela-Ascoli theorem, we know that $T:\overline{\Omega}\to\overline{\Omega}$ is completely continuous.

Finally, We prove that the condition (ii) of Lemma 2.3 is not true. In fact, for all $(\overline{x},\overline{y})\in\partial\Omega$, $0<\lambda<1$ and $t\in[1,e]$, analogous to (3.5)–(3.7), we have

$$|\lambda T_1(\overline{x},\overline{y})(t)|\le\lambda\Big(\frac{a_1^*}{\Gamma(\alpha+1)}+\lambda_1\|(\overline{x},\overline{y})\|\Big)\le\lambda r<r, \tag{3.12}$$

and

$$|\lambda T_2(\overline{x},\overline{y})(t)|\le\lambda\Big(\frac{a_2^*}{\Gamma(\beta+1)}+\lambda_2\|(\overline{x},\overline{y})\|\Big)\le\lambda r<r. \tag{3.13}$$

Equations (3.12) and (3.13) implies that $\|\lambda T(\overline{x},\overline{y})\|<\|(\overline{x},\overline{y})\|=r$, that is, $(\overline{x},\overline{y})\ne\lambda T(\overline{x},\overline{y})$, for all $(\overline{x},\overline{y})\in\partial\Omega$. According to Lemma 2.5, we know that the system (1.1) has at least a pair of solution $(x^*,y^*)\in\overline{\Omega}$. The proof is completed. □

4. Illustrative example

Consider the following boundary value problem for nonlinear Hadamard fractional differential coupling system with fractional order impulses.

$$\begin{cases} {}^HD^{\alpha}_{1^+}x(t) = f(t, x(t), {}^HD^{\alpha_1}_{1^+}x(t), {}^HD^{\beta_1}_{1^+}y(t)), \ t \in [1, e], \ t \neq t_k, \\ {}^HD^{\beta}_{1^+}y(t) = g(t, y(t), {}^HD^{\alpha_1}_{1^+}x(t), {}^HD^{\beta_1}_{1^+}y(t)), \ t \in [1, e], \ t \neq t_k, \\ {}^HD^{\gamma_1}_{1^+}x(t_k^+) - {}^HD^{\gamma_1}_{1^+}x(t_k^-) = I_{1k}(x(t_k)), \ k = 1, \dots, n, \\ {}^HD^{\gamma_2}_{1^+}y(t_k^+) - {}^HD^{\gamma_2}_{1^+}y(t_k^-) = I_{2k}(y(t_k)), \ k = 1, \dots, n, \\ x(1) = x(e), \ y(1) = y(e), \end{cases} \tag{4.1}$$

where $\alpha = \frac{3}{2}, \beta = \frac{5}{4}, \alpha_1 = \frac{1}{2}, \beta_1 = \frac{3}{4}, \gamma_1 = \frac{1}{8}, \gamma_2 = \frac{7}{8}, n = 2, t_1 = \frac{5}{4}, t_2 = 2, f(t, u, v, w) = \frac{2+u+v+w}{100+100e^{t+5}}$, $g(t, u, v) = \frac{t+\arctan(u+v+w)}{100e^{t+5}}$, $I_{11}(u) = I_{22}(u) = \frac{u}{200}$, $I_{12}(u) = I_{21}(u) = \frac{\sqrt[3]{u}}{200}$. Obviously, $f, g \in C(J \times R^3, R)$, $I_{11}, I_{12}, I_{21}, I_{22} \in C(R, R)$. By a simple calculation, we obtain

$$a_1(t) = \frac{2}{100 + 100e^{t+5}}, \ b_1(t) = d_1(t) = p_1(t) = \frac{1}{100 + 100e^{t+5}}, \ a_1^* = \frac{1}{50 + 50e^6},$$

$$b_1^* = d_1^* = p_1^* = \frac{1}{100 + 100e^6}, \ a_2(t) = \frac{t}{100e^{t+5}}, \ b_2(t) = d_2(t) = p_2(t) = \frac{1}{100e^{t+5}},$$

$$a_2^* = b_2^* = d_2^* = p_2^* = \frac{1}{100e^6}, \ M_1 = M_2 = \frac{1}{200}, \ N_1 = N_2 = \frac{1}{600}.$$

Thus, we get

$$\lambda_1 = \frac{b_1^* + d_1^* + p_1^* + 1}{\Gamma(\alpha + 1)} + \frac{\Gamma(\alpha - \gamma_1)}{\gamma_1\Gamma(\alpha - 1)}\Big[M_n + \sum_{j=1}^{n} M_j(\log t_j)^{1+\gamma_1-\alpha} + (\log t_1)^{\alpha-2}\sum_{j=1}^{n} M_j\Big] \approx 0.9155 < 1,$$

$$\lambda_2 = \frac{b_2^* + d_2^* + p_2^* + 1}{\Gamma(\beta + 1)} + \frac{\Gamma(\beta - \gamma_2)}{\gamma_2\Gamma(\beta - 1)}\Big[N_n + \sum_{j=1}^{n} N_j(\log t_j)^{1+\gamma_2-\beta} + (\log t_1)^{\beta-2}\sum_{j=1}^{n} N_j\Big] \approx 0.8931 < 1,$$

So all the conditions (H_1)–(H_4) are true. According to Theorem 3.1, the boundary value problem (4.1) exists at least a pair of solution.

5. Conclusions

Hadamard fractional calculus, like Riemann Liouville and Caputo fractional calculus, is an important generalization and extension of the classical integral order calculus theory. Based on the need of theoretical development and wide application, Hadamard fractional differential equation has been paid much attention and studied by many scholars. However, the previous findings are few for Hadamard fractional coupled equations. Therefore, this paper deals with a Hadamard fractional coupled system involving fractional derivative impulses. We obtain some new sufficient criteria for the existence of solutions by use of the Leray-Schauder alternative theorem. The condition (H_4) shows that the value of the impulse point t_k directly affects the existence of the solution of the system (1.1). At the same time, our methods and techniques can be used for reference to similar problems. Since system (1.1) only involves the impulsive effect and two-point boundary value conditions, we can further explore the dynamic behavior of Hadamard fractional coupled system under the influence of delay effect, nonlocal boundary value condition and integral boundary value condition in the future.

Conflict of interest

All authors declare that they have no competing interests.

References

1. J. Hadamard, Essai sur l'étude des fonctions données par leur développment de Taylor, *J. Math. Pures Appl.*, **8** (1892), 101–186. Available from: `https://eudml.org/doc/233965`.
2. I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, San Diego: Academic Press, 1999. Available from: `http://lib.ugent.be/catalog/rug01:002178612`.
3. K. Diethelm, *The analysis of fractional differential equations*, Berlin, Heidelberg: Springer, 2010. https://doi.org/10.1007/978-3-642-14574-2
4. B. Ahmad, A. Alsaedi, S. K. Ntouyas, J. Tariboon, *Hadamard-type fractional differential equations, inclusions and inequalities*, Cham: Springer, 2017. https://doi.org/10.1007/978-3-319-52141-1
5. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.
6. D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo, *Fractional calculus: Models and numerical methods*, Singapore: World Scientific, 2012. https://doi.org/10.1142/8180
7. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: John Wiley & Sons, Inc., 1993.
8. K. B. Oldham, J. Spanier, *The fractional calculus*, New York: Academic Press, 1974.
9. Y. Zhou, *Basic theory of fractional differential equations*, Singapore: World Scientific, 2014. https://doi.org/10.1142/9069
10. K. H. Zhao, K. Wang, Existence of solutions for the delayed nonlinear fractional functional differential equations with three-point integral boundary value conditions, *Adv. Differ. Equ.*, **2016** (2016), 1–18. https://doi.org/10.1186/s13662-016-1012-2
11. Y. H. Zhang, Z. B. Bai, T. T. Feng, Existence results for a coupled system of nonlinear fractional three-point boundary value problems at resonance, *Comput. Math. Appl.*, **61** (2011), 1032–1047. https://doi.org/10.1016/j.camwa.2010.12.053
12. K. H. Zhao, P. Gong, Positive solutions of m-point multi-term fractional integral BVP involving time-delay for fractional differential equations, *Bound. Value Probl.*, **2015** (2015), 1–19. https://doi.org/10.1186/s13661-014-0280-6

13. B. Ahmad, S. K. Ntouyas, A. Alsaedi, On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions, *Chaos Soliton Fract.*, **83** (2016), 234–241. https://doi.org/10.1016/j.chaos.2015.12.014

14. K. H. Zhao, P. Gong, Positive solutions of nonlocal integral BVPs for the nonlinear coupled system involving high-order fractional differential, *Math. Slovaca*, **67** (2017), 447–466. https://doi.org/10.1515/ms-2016-0281

15. K. H. Zhao, H. Huang, Existence results of nonlocal boundary value problem for a nonlinear fractional differential coupled system involving fractional order impulses, *Adv. Differ. Equ.*, **2019** (2019), 1–13. https://doi.org/10.1186/s13662-019-1982-y

16. B. Ahmad, J. J. Nieto, Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions, *Bound. Value Probl.*, **2009** (2009), 1–11. https://doi.org/10.1155/2009/708576

17. K. H. Zhao, P. Gong, Positive solutions of Riemann-Stieltjes integral boundary problems for the nonlinear coupling system involving fractional-order differential, *Adv. Differ. Equ.*, **2014** (2014), 1–18. https://doi.org/10.1186/1687-1847-2014-254

18. B. Ahmad, S. Sivasundaram, Existence of solutions for impulsive integral boundary value problems of fractional order, *Nonlinear Anal. Hybrid Syst.*, **4** (2010), 134–141. https://doi.org/10.1016/j.nahs.2009.09.002

19. K. H. Zhao, Multiple positive solutions of integral BVPs for high-order nonlinear fractional differential equations with impulses and distributed delays, *Dyn. Syst.*, **30** (2015), 208–223. https://doi.org/10.1080/14689367.2014.995595

20. J. R. Wang, Y. Zhou, M. Fečkan, On recent developments in the theory of boundary value problems for impulsive fractional differential equations, *Comput. Math. Appl.*, **64** (2012), 3008–3020. https://doi.org/10.1016/j.camwa.2011.12.064

21. Y. Tian, W. G. Ge, Variational methods to Sturm-Liouville boundary value problem for impulsive differential equations, *Nonlinear Anal.*, **72** (2010), 277–287. https://doi.org/10.1016/j.na.2009.06.051

22. K. H. Zhao, P. Gong, Positive solutions for impulsive fractional differential equations with generalizedperiodic boundary value conditions, *Adv. Differ. Equ.*, **2014** (2014), 1–19. https://doi.org/10.1186/1687-1847-2014-255

23. T. W. Zhang, L. L. Xiong, Periodic motion for impulsive fractional functional differential equations with piecewise Caputo derivative, *Appl. Math. Lett.*, **101** (2020), 106072. https://doi.org/10.1016/j.aml.2019.106072

24. M. A. Almalahi, M. S. Abdo, S. K. Panchal, Existence and Ulam-Hyers stability results of a coupled system of ψ-Hilfer sequential fractional differential equations, *Results Appl. Math.*, **10** (2021), 100142. https://doi.org/10.1016/j.rinam.2021.100142

25. M. A. Almalahi, O. Bazighifan, S. K. Panchal, S. S. Askar, G. I. Oros, Analytical study of two nonlinear coupled hybrid systems involving generalized Hilfer fractional operators, *Fractal Fract.*, **5** (2021), 1–21. https://doi.org/10.3390/fractalfract5040178

26. M. A. Almalahi, S. K. Panchal, Some properties of implicit impulsive coupled system via φ-Hilfer fractional operator, *Bound. Value Probl.*, **2021** (2021), 1–22. https://doi.org/10.1186/s13661-021-01543-4

27. P. Bedi, A. Kumar, T. Abdeljawad, A. Khan, Existence of mild solutions for impulsive neutral Hilfer fractional evolution equations, *Adv. Differ. Equ.*, **2020** (2020), 1–16. https://doi.org/10.1186/s13662-020-02615-y

28. H. Khan, C. Tunç, A. Khan, Stability results and existence theorems for nonlinear delay-fractional differential equations with φ_P^*-operator, *J. Appl. Anal. Comput.*, **10** (2020), 584–597. https://doi.org/10.11948/20180322

29. H. Khan, J. F. Gomez-Aguilar, T. Abdeljawad, A. Khan, Existence results and stability criteria for ABC-fuzzy-Volterra integro-differential equation, *Fractals*, **28** (2020), 2040048. https://doi.org/10.1142/S0218348X20400484

30. P. Bedi, A. Kumar, T. Abdeljawad, A. Khan, Study of Hilfer fractional evolution equations by the properties of controllability and stability, *Alex. Eng. J.*, **60** (2021), 3741–3749. https://doi.org/10.1016/j.aej.2021.02.014

31. K. H. Zhao, Stability of a nonlinear ML-nonsingular kernel fractional Langevin system with distributed lags and integral control, *Axioms*, **11** (2022), 1–14. https://doi.org/10.3390/axioms11070350

32. K. H. Zhao, Local exponential stability of four almost-periodic positive solutions for a classic Ayala-Gilpin competitive ecosystem provided with varying-lags and control terms, *Int. J. Control*, 2022. https://doi.org/10.1080/00207179.2022.2078425

33. T. W. Zhang, J. W. Zhou, Y. Z. Liao, Exponentially stable periodic oscillation and Mittag-Leffler stabilization for fractional-order impulsive control neural networks with piecewise Caputo derivatives, *IEEE Trans. Cybernet.*, **52** (2002), 9670–9683. https://doi.org/10.1109/TCYB.2021.3054946

34. T. W. Zhang, Y. K. Li, Exponential Euler scheme of multi-delay Caputo-Fabrizio fractional-order differential equations, *Appl. Math. Lett.*, **124** (2022), 107709. https://doi.org/10.1016/j.aml.2021.107709

35. T. W. Zhang, Y. K. Li, Global exponential stability of discrete-time almost automorphic Caputo-Fabrizio BAM fuzzy neural networks via exponential Euler technique, *Knowl-Based Syst.*, **246** (2022), 108675. https://doi.org/10.1016/j.knosys.2022.108675

36. Z. H. Li, W. Zhang, C. D. Huang, J. W. Zhou, Bifurcation for a fractional-order Lotka-Volterra predator-Cprey model with delay feedback control, *AIMS Math.*, **6** (2021), 675–687. https://doi.org/10.3934/math.2021040

37. J. W. Zhou, B. X. Zhou, L. P. Tian, Y. N. Wang, Variational approach for the variable-order fractional magnetic Schrödinger equation with variable growth and steep potential in $\mathbb{R}^{N*}$, *Adv. Math. Phys.*, **2020** (2020), 1–15. https://doi.org/10.1155/2020/1320635

38. J. W. Zhou, B. X. Zhou, Y. N. Wang, Multiplicity results for variable-order nonlinear fractional magnetic Schrödinger equation with variable growth, *J. Funct. Space.*, **2020** (2020), 1–15. https://doi.org/10.1155/2020/7817843

39. A. A. Kilbas, Hadamard-type fractional calculus, *J. Korean Math. Soc.*, **38** (2001), 1191–1204.
40. P. L. Butzer, A. A. Kilbas, J. J. Trujillo, Compositions of Hadamard-type fractional integration operators and the semigroup property, *J. Math. Anal. Appl.*, **269** (2002), 387–400. https://doi.org/10.1016/S0022-247X(02)00049-5
41. K. H. Zhao, S. Ma, Ulam-Hyers-Rassias stability for a class of nonlinear implicit Hadamard fractional integral boundary value problem with impulses, *AIMS Math.*, **7** (2022), 3169–3185. https://doi.org/10.3934/math.2022175
42. B. Ahmad, S. K. Ntouyas, On Hadamard fractional integro-differential boundary value problems, *J. Appl. Math. Comput.*, **47** (2015), 119–131. https://doi.org/10.1007/s12190-014-0765-6
43. S. Aljoudi, B. Ahmad, J. J. Nieto, A. Alsaedi, A coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions, *Chaos Soliton Fract.*, **91** (2016), 39–46. https://doi.org/10.1016/j.chaos.2016.05.005
44. Y. Y. Gambo, F. Jarad, D. Baleanu, T. Abdeljawad, On Caputo modification of the Hadamard fractional derivatives, *Adv. Differ. Equ.*, **2014** (2014), 1–12. https://doi.org/10.1186/1687-1847-2014-10
45. M. Benchohra, S. Bouriah, J. R. Graef, Boundary value problems for nonlinear implicit Caputo-Hadamard-type fractional differential equations with impulses, *Mediterr. J. Math.*, **14** (2017), 1–21. https://doi.org/10.1007/s00009-017-1012-9
46. G. Rahman, T. Abdeljawad, F. Jarad, A. Khan, K. S. Nisar, Certain inequalities via generalized proportional Hadamard fractional integral operators, *Adv. Differ. Equ.*, **2019** (2019), 1–10. https://doi.org/10.1186/s13662-019-2381-0
47. Y. Adjabi, F. Jarad, D. Baleanu, T. Abdeljawad, On Cauchy problems with Caputo Hadamard fractional derivatives, *J. Comput. Anal. Appl.*, **21** (2016), 661–681.
48. E. Zeidler, *Nonlinear functional analysis and applications I: Fixed point theorems*, New York: Springer, 1986.

AIMS Press