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Research article

On boundedness of fractional integral operators via several kinds of convex functions

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Abstract: For generalizations of concepts of different fields fractional derivative operators as well as fractional integral operators are useful notions. Our aim in this paper is to discuss boundedness of the integral operators which contain Mittag-Leffler function in their kernels. The results are obtained for strongly $(\alpha, h-m)$-convex functions which hold for different kinds of convex functions at the same time. They also give improvements/refinements of many already published results.

Keywords: strongly $(\alpha, h-m)$-convex function; generalized fractional integral operators; unified Mittag-Leffler function; bounds
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1. Introduction

Fractional integral operators have great significance in extensive fields of science and engineering. They are widely used to construct and solve fractional order models and fractional dynamical systems. In recent decades fractional integral operators are frequently used to study different types of integral inequalities including well known inequalities of Hadamard [1–5], Ostrowski [6–9], Grüss [10, 11], Opial [12], Chebsheve [13, 14] and Minkowski [15, 16].

We have motivated by ongoing research in integral inequalities, and interested to establish inequalities for fractional integral operators defined in [17]. To attain the desired results, we have used a generalized class of functions called strongly $(\alpha, h-m)$-convex functions. The findings of this paper simultaneously give generalizations as well as refinements of many recently published inequalities.

The unified Mittag-Leffler function is defined as follows:

Definition 1.1. *[17] For* $\underline{a} = (a_1, a_2, ..., a_n)$, $\underline{b} = (b_1, b_2, ..., b_n)$, $\underline{c} = (c_1, c_2, ..., c_n)$, *where* a_i, b_i,

$c_i \in \mathbb{C}$; $i = 1, 2, 3, ..., n$ such that $\Re(a_i), \Re(b_i), \Re(c_i) > 0$, $\forall$ i. Also let α, β, γ, δ, μ, ν, λ, ρ, $\theta, t \in \mathbb{C}$, $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta), \Re(\theta)\} > 0$ and $k \in (0,1) \cup \mathbb{N}$ with $k + \Re(\rho) < \Re(\delta + \nu + \alpha)$, $Im(\rho) = Im(\delta + \nu + \alpha)$, then Mittag-Leffler function is defined by

$$M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(z; \underline{a}, \underline{b}, \underline{c}, \varrho\right) = \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{n} \beta_{\varrho}(b_i, a_i)(\lambda)_{\rho l}(\theta)_{kl} z^l}{\prod_{i=1}^{n} \beta(c_i, a_i)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l + \beta)}, \tag{1.1}$$

where $\Gamma(\mu)$ is the gamma function, $\Gamma(\mu) = \int_0^{\infty} e^{-z} z^{\mu-1} dz$, $(\theta)_{kl}$ is the Pochhammer symbol, $(\theta)_{kl} = \frac{\Gamma(\theta+lk)}{\Gamma(\theta)}$ and β_{ϱ} is the extension of beta function and it is defined as follows:

$$\beta_{\varrho}(\vartheta, y) = \int_0^1 \vartheta^{\zeta-1}(1-\vartheta)^{y-1} e^{-\left(\frac{\varrho}{\vartheta(1-\vartheta)}\right)} d\vartheta. \tag{1.2}$$

Along with the convergence conditions of the unified Mittag-Leffler function given in Definition 1.1, the unified fractional operators are defined as follows:

Definition 1.2. *[17] Let $\Phi \in L_1[\xi_1, \xi_2]$. Then $\forall\ \zeta \in [\xi_1, \xi_2]$, the fractional integral operator containing the unified Mittag-Leffler function $M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(z; \underline{a}, \underline{b}, \underline{c}, \varrho\right)$ satisfying all the convergence conditions is defined as follows:*

$$\left(\Upsilon^{\omega,\lambda,\rho,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi\right)(\zeta; \underline{a}, \underline{b}, \underline{c}, \varrho) = \int_{\xi_1}^{\zeta} (\zeta-\vartheta)^{\alpha-1} M^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega(\zeta-\vartheta)^{\mu}; \underline{a}, \underline{b}, \underline{c}, \varrho\right)\Phi(\vartheta)d\vartheta, \tag{1.3}$$

$$\left(\Upsilon^{\omega,\lambda,\rho,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi\right)(\zeta; \underline{a}, \underline{b}, \underline{c}, \varrho) = \int_{\zeta}^{\xi_2} (\vartheta-\zeta)^{\alpha-1} M^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega(\zeta-\vartheta)^{\mu}; \underline{a}, \underline{b}, \underline{c}, \varrho\right)\Phi(\vartheta)d\vartheta. \tag{1.4}$$

By setting $a_i = l$, $\varrho = 0$ and $\Re(\varrho) > 0$ in (1.3) and (1.4), we get the fractional integral operator associated with generalized Q function as (see [18]):

$$\left({}_{Q}\Upsilon^{\omega,\lambda,\rho,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi\right)(\zeta; \underline{a}, \underline{b}) = \int_{\xi_1}^{\zeta} (\zeta-\vartheta)^{\alpha-1} Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega(\zeta-\vartheta)^{\mu}; \underline{a}, \underline{b}\right)\Phi(\vartheta)d\vartheta, \tag{1.5}$$

$$\left({}_{Q}\Upsilon^{\omega,\lambda,\rho,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi\right)(\zeta; \underline{a}, \underline{b}) = \int_{\zeta}^{\xi_2} (\vartheta-\zeta)^{\alpha-1} Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega(\vartheta-\zeta)^{\mu}; \underline{a}, \underline{b}\right)\Phi(\vartheta)d\vartheta, \tag{1.6}$$

where

$$Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(z; \underline{a}, \underline{b}\right) = \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{n} \beta(b_i, l)(\lambda)_{\rho l}(\theta)_{kl} z^l}{\prod_{i=1}^{n} \beta(a_i, l)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l + \beta)}$$

is a generalized Q function defined in [19].

The more generalized and extended version of integral operator given in Definition 1.2 is defined as follows:

Definition 1.3. *[20] Let $\Delta \in L_1[\xi_1, \xi_2]$, $0 < \xi_1, \xi_2 < \infty$ be a positive function and let $\Psi : [\xi_1, \xi_2] \to \mathbb{R}$ be a differentiable and strictly increasing function. Also let $\frac{\Delta}{\zeta}$ be an increasing function on $[\xi_1, \infty)$.*

Then for $\zeta \in [\xi_1, \xi_2]$ the unified integral operator in its generalized form satisfying all the convergence conditions is defined by:

$$({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi)(\zeta;\varrho) = \int_u^\zeta \Lambda_\zeta^\vartheta(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)\Phi(\vartheta)d(\Psi(\vartheta)), \tag{1.7}$$

$$({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi)(\zeta;\varrho) = \int_\zeta^v \Lambda_\vartheta^\zeta(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)\Phi(\vartheta)d(\Psi(\vartheta)), \tag{1.8}$$

where

$$\Lambda_\zeta^\vartheta(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta) = \frac{\Delta(\Psi(\zeta)-\Psi(\vartheta))}{\Psi(\zeta)-\Psi(\vartheta)} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\Psi(\zeta)-\Psi(\vartheta))^\mu; \underline{a}, \underline{b}, \underline{c}, \varrho). \tag{1.9}$$

Definition 1.4. *[18] By setting $a_i = l$, $\varrho = 0$ and $\Re(\varrho) > 0$ in (1.7) and (1.8), we get the fractional integral operator associated with generalized Q function as follows:*

$$\left({}^{\Psi}_{Q}\Upsilon^{\Delta,\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi\right)(\zeta;\underline{a},\underline{b}) = \int_{\xi_1}^\zeta \Lambda_\zeta^y(Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)\Phi(\vartheta)d(\Psi(\vartheta)), \tag{1.10}$$

$$\left({}^{\Psi}_{Q}\Upsilon^{\Delta,\omega,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi\right)(\zeta;\underline{a},\underline{b}) = \int_\zeta^b \Lambda_\zeta^y(Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)\Phi(\vartheta)d(\Psi(\vartheta)), \tag{1.11}$$

where $\Lambda_\zeta^\vartheta(Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta) = \dfrac{\Delta(\Psi(\zeta)-\Psi(\vartheta))}{\Psi(\zeta)-\Psi(\vartheta)} Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\Psi(\zeta)-\Psi(\vartheta))^\mu, \underline{a}, \underline{b}, \varrho)$.

One can note that if Ψ and $\frac{\Delta}{\zeta}$ are increasing functions, then for $u < \vartheta < v, u, v \in [\xi_1, \xi_2]$, the kernel $\Lambda_\zeta^\vartheta(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)$ satisfies the following inequality:

$$\Lambda_\vartheta^u(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta) \le \Lambda_v^u(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta). \tag{1.12}$$

For more details, one can see [21]. In the whole paper we will use

$$I(\xi_1, \xi_2, \Psi) = \frac{1}{\xi_2 - \xi_1}\int_{\xi_1}^{\xi_2} \Psi(\vartheta)d\vartheta. \tag{1.13}$$

Convex functions are extensively utilized in mathematics, physics, mathematical statistics and economics. The geometrical visualization of a convex function can be seen in the Hadamard inequality. Several new classes of functions have been defined to prove generalizations and refinements of many known mathematical inequalities. The definition of strongly $(\alpha, h-m)$-convex function is defined as follows:

Definition 1.5. *[22] Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. A function $\Phi : [0, \xi_2] \to \mathbb{R}$ is called strongly $(\alpha, h-m)$-convex function with modulus $C \ge 0$, if f is non-negative and for all $\zeta, y \in [0, \xi_2]$, $\vartheta \in (0, 1)$ and $m \in (0, 1]$, one have the inequality*

$$\Phi(\zeta\vartheta + m(1-\vartheta)y) \le h(\vartheta^\alpha)\Phi(\zeta) + mh(1-\vartheta^\alpha)f(y) - mCh(\vartheta^\alpha)h(1-\vartheta^\alpha)|y-\zeta|^2. \tag{1.14}$$

Remark 1. *The above definition produces several types of convex functions like $(h-m)$-convex, (s,m)-convex, strongly (α,m)-convex, (α,m)-convex functions etc.*

The upcoming section consists of bounds of fractional integral operators given in (1.7) and (1.8). They are constructed by using definition of strongly $(\alpha, h-m)$-convex function. We have established a Hadamard type inequality, by using a lemma for strongly $(\alpha, h-m)$-convex functions. The results of this paper are connected with various inequalities that have been published in recent decades. Finally, by using strongly $(\alpha, h-m)$-convexity of the function $|\Phi'|$ further bounds are given.

2. Main results

Theorem 2.1. *Let $\Phi \in L_1[\xi_1,\xi_2]$ be an integrable strongly $(\alpha, h-m)$-convex function. Also let $\frac{\Delta}{\zeta}$ be an increasing function on $[\xi_1,\xi_2]$ and, let $\Psi : [\xi_1,\xi_2] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function, also let $\frac{\Delta}{\zeta}$ be an increasing function on $[\xi_1,\xi_2]$. Then $\forall\ \zeta \in [\xi_1,\xi_2]$, we have the following inequality containing the unified Mittag-Leffler function $M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(z; \underline{a}, \underline{b}, \underline{c}, \varrho\right)$ satisfying all the convergence conditions:*

$$\begin{aligned}
&\left({}_{\Psi}^{\Delta}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}\,\Phi\right)(\zeta;\varrho) + \left({}_{\Psi}^{\Delta}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}\,\Phi\right)(\zeta;\varrho) \leq \Lambda^{\xi_1}_{\zeta}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\,\Psi;\Delta)(\zeta-\xi_1) \\
&\times\left(\Phi(\xi_1)X^{\xi_1}_{\zeta}(r^{\alpha},h,\Psi') + m\Phi\left(\frac{\zeta}{m}\right)X^{\xi_1}_{\zeta}(r^{\alpha},h,\Psi') - \frac{C(\zeta-\xi_1 m)^2 h(1)(\Psi(\zeta)-\Psi(\xi_1))}{m(\zeta-\xi_1)}\right) \\
&+ \Lambda^{\zeta}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\,\Psi;\Delta)\left(\Phi(\xi_2)X^{\xi_1}_{\zeta}(r^{\alpha},h,\Psi') + m\Phi\left(\frac{\zeta}{m}\right)X^{\xi_1}_{\zeta}(1-r^{\alpha},h,\Psi')\right. \\
&\left.- \frac{C(\xi_2 m-\zeta)^2 h(1)(\Psi(\xi_2)-\Psi(x))}{m(\xi_2-\zeta)}\right),
\end{aligned} \tag{2.1}$$

where $X^{\xi_1}_{\zeta}(r^{\alpha},h,\Psi') = \int_{\xi_1}^{\zeta} h\left(r^{\alpha}\right)\Psi'(\zeta - r(\zeta-\xi_1))dr$, $X^{\xi_1}_{\zeta}(1-r^{\alpha},h,\Psi') = \int_{\xi_1}^{\zeta} h\left(1-r^{\alpha}\right)\Psi'(\zeta - r(\zeta-\xi_1))dr$.

Proof. Using (1.12), we can write the following inequalities:

$$\Lambda^{\vartheta}_{\zeta}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\,\Psi;\Delta)\Psi'(\vartheta) \leq \Lambda^{\xi_1}_{\zeta}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\,\Psi;\Delta)\Psi'(\vartheta), \tag{2.2}$$

$$\Lambda^{\zeta}_{\vartheta}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\,\Psi;\Delta)\Psi'(\vartheta) \leq \Lambda^{\zeta}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\,\Psi;\Delta)\Psi;\Delta)\Psi'(\vartheta). \tag{2.3}$$

Using strongly $(\alpha, h-m)$-convexity of Φ, we have

$$\Phi(\vartheta) \leq h\left(\frac{\zeta-\vartheta}{\zeta-\xi_1}\right)^{\alpha}\Phi(\xi_1) + mh\left(1-\left(\frac{\zeta-\vartheta}{\zeta-\xi_1}\right)^{\alpha}\right)\Phi\left(\frac{\zeta}{m}\right) - \frac{C(\zeta-\xi_1 m)^2}{m}h\left(\frac{\zeta-\vartheta}{\zeta-\xi_1}\right)^{\alpha}h\left(1-\left(\frac{\zeta-\vartheta}{\zeta-\xi_1}\right)^{\alpha}\right), \tag{2.4}$$

$$\Phi(\vartheta) \leq h\left(\frac{\vartheta-\zeta}{\xi_2-\zeta}\right)^{\alpha}\Phi(\xi_1) + mh\left(1-\left(\frac{\vartheta-\zeta}{\xi_2-\zeta}\right)^{\alpha}\right)\Phi\left(\frac{\zeta}{m}\right) - \frac{C(\zeta-\xi_1 m)^2}{m}h\left(\frac{\vartheta-\zeta}{\xi_2-\zeta}\right)^{\alpha}h\left(1-\left(\frac{\vartheta-\zeta}{\xi_2-\zeta}\right)^{\alpha}\right). \tag{2.5}$$

From (2.2) and (2.4), one can obtain the following inequality

$$\int_{\xi_1}^{\zeta}\Lambda^{\vartheta}_{\zeta}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\,\Psi;\Delta)\Phi(\vartheta)d(\Psi(\vartheta)) \leq \Lambda^{\xi_1}_{\zeta}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\,\Psi;\Delta)\left(\Phi(\xi_1)\int_{\xi_1}^{\zeta} h\left(\frac{\zeta-\vartheta}{\zeta-\xi_1}\right)^{\alpha} d(\Psi(\vartheta))\right.$$

$$+\, m\Phi\Big(\frac{\zeta}{m}\Big)\int_{\xi_1}^{\zeta} h\left(1-\left(\frac{\zeta-\vartheta}{\zeta-\xi_1}\right)^{\alpha}\right) d(\Psi(\vartheta)) - \frac{C(\zeta-\xi_1 m)^2}{m}\int_{\xi_1}^{\zeta} h\Big(\frac{\zeta-\vartheta}{\zeta-\xi_1}\Big)^{\alpha} h\Big(1-\Big(\frac{\zeta-\vartheta}{\zeta-\xi_1}\Big)^{\alpha}\Big) d(\Psi(\vartheta))\Big).$$

Using Definition 1.3 and setting $r=\frac{\zeta-\vartheta}{\zeta-\xi_1}$, we obtain

$$\Big({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}\,\Phi\Big)(\zeta;\varrho) \le \Lambda^{\xi_1}_{\zeta}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\Psi;\Delta)(\zeta-\xi_1)\Big(\Phi(\xi_1)\int_0^1 h\,(r^{\alpha})\,\Psi(\zeta-r(\zeta-\xi_1))dr \tag{2.6}$$
$$+\, m\Phi\Big(\frac{\zeta}{m}\Big)\int_0^1 h\,(1-r^{\alpha})\,\Psi'(\zeta-r(\zeta-\xi_1)))dr - \frac{C(\zeta-\xi_1 m)^2}{m}\int_0^1 h(r^{\alpha})h(1-r^{\alpha})\Psi'(\zeta-r(\zeta-\xi_1))dr\Big).$$

The above inequality will become

$$\Big({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}\,\Phi\Big)(\zeta;\varrho) \le \Lambda^{\xi_1}_{\zeta}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\Psi;\Delta)\Big(\Phi(\xi_1)X^{\xi_1}_{\zeta}(r^{\alpha},h,\Psi') \tag{2.7}$$
$$+\, m\Phi\Big(\frac{\zeta}{m}\Big)X^{\xi_1}_{\zeta}(r^{\alpha},h,\Psi') - \frac{C(\zeta-\xi_1 m)^2 h(1)(\Psi(\zeta)-\Psi(\xi_1))}{m(\zeta-\xi_1)}.$$

On the other hand, using the same technique that we did for (2.2) and (2.4), the following inequality from (2.3) and (2.5) can be obtained:

$$\Big({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}\,\Phi\Big)(\zeta;\varrho) \le \Lambda^{\zeta}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\,\Phi;\Delta)(\xi_2-\zeta)\Big(\Phi(\xi_1)\int_0^1 h\,(r^{\alpha})\,\Psi'(\zeta-r(\xi_2-\zeta))dr \tag{2.8}$$
$$+ m\Phi\Big(\frac{\zeta}{m}\Big)\int_0^1 h\,(1-r^{\alpha})\,\Psi'(\zeta-r(\xi_2-\zeta)))dr - \frac{C(m\xi_2-\zeta)^2}{m}\int_0^1 h(r^{\alpha})h(1-r^{\alpha})\Psi'(\zeta-r(\xi_1-\zeta))dr\Big).$$

The above inequality can takes the following form

$$\Big({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}\,\Phi\Big)(\zeta;\varrho) \le \Lambda^{\zeta}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\,\Phi;\Delta)(\xi_2-\zeta)\Big(\Phi(\xi_2)X^{\xi_2}_{\zeta}(r^{\alpha},h,\Psi') \tag{2.9}$$
$$+\, m\Phi\Big(\frac{\zeta}{m}\Big)X^{\xi_2}_{\zeta}(1-r^{\alpha},h,\Psi') - \frac{C(\zeta-\xi_1 m)^2 h(1)(\Psi(\xi_2)-\Psi(\zeta))}{m(\xi_2-\zeta)}\Big).$$

By adding (2.7) and (2.9), (2.1) can be obtained.

□

Remark 2. *(i) If* $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$ *in (2.1), [22, Theorem 1] is obtained.*
(ii) If $h(\zeta)=\zeta$ *in the result of (i), then [23, Theorem 7] is obtained.*
(iii) If $C=0$, $\Delta(\vartheta)=\vartheta^{\beta}$, $\Psi(\zeta)=\zeta$, $h(\zeta)=\zeta^{s}$ *and* (α,m) *=(1,1) in the result of (i), then [24, Theorem 2.1] is obtained.*
(iv) If $\Delta(\vartheta)=\vartheta^{\beta}$, $h(\zeta)=\Psi(\zeta)=\zeta$ *in the result of (i), then [25, Theorem 4] can be obtained.*
(v) If $\Delta(\vartheta)=\vartheta^{\beta}$, $\Psi(\zeta)=\zeta$ *and* $C=0$ *in the result of (i), then [26, Theorem 1] can be obtained.*
(vi) If $C=0$ *and* $h(\zeta)=\zeta$ *in (2.1), then [4, Theorem 1] is obtained.*
(vii) If $C=0$ *in the result of (i), [27, Theorem 1] is obtained.*

Corollary 1. *If* $h(\zeta)=\zeta$*, then the following inequality holds for strongly* (α,m)*-convex function:*

$$\Big({}^{\Upsilon}_{\Psi}\Omega^{\omega,\lambda,\rho,\theta,k,\eta}_{\sigma^+,\alpha,\beta,\gamma,\mu,\nu}\,\Phi\Big)(\vartheta;\varrho) + \Big({}^{\Upsilon}_{\Psi}\Omega^{\omega,\lambda,\rho,\theta,k,\eta}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}\,\Phi\Big)(\vartheta;\varrho) \le \Lambda^{\xi_1}_{\vartheta}(M^{\lambda,\rho,\theta,k,\eta}_{\alpha,\beta,\gamma,\mu,\nu}\,\Psi;\Upsilon)\left(\left(m\Phi\left(\frac{\vartheta}{m}\right)\Psi(\vartheta)-\Phi(\xi_1)\Psi(\xi_1)\right)\right.$$

$$-\frac{\Gamma(\alpha+1)}{(\vartheta-\xi_1)^\alpha}\left(m\Phi\left(\frac{\vartheta}{m}\right)-\Phi(\xi_1)\right){}^{\alpha}I_{\xi_1^+}\Psi(\vartheta)\Bigg)-\frac{C(\zeta-m\xi_1)}{m(\zeta-\xi_1)^\alpha}\left(\Gamma(\alpha+1)\,{}^{\alpha}I_{\xi_1^+}\Psi(\vartheta)-\frac{\Gamma(2\alpha+1)^{2\alpha}I_{\xi_1^+}\Psi(\vartheta)}{(\zeta-\xi_1)^\alpha}\right)$$
$$+\Lambda_{\xi_2}^{\vartheta}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,\eta}\,\Psi;\Upsilon)\left(\left(\Phi(\xi_2)\Psi(\xi_2)-m\Phi\left(\frac{\vartheta}{m}\right)\Psi(\vartheta)\right)-\frac{\Gamma(\alpha+1)}{(\xi_2-\vartheta)^\alpha}\left(\Phi(\xi_2)-m\Phi\left(\frac{\vartheta}{m}\right)\right){}^{\alpha}I_{\xi_2^-}\Psi(\vartheta)\right)$$
$$-\frac{C(m\xi_2-\zeta)}{m(\xi_2-\zeta)^\alpha}\left(\frac{\Gamma(2\alpha+1)^{2\alpha}I_{\xi_2^-}\Psi(\vartheta)}{(\xi_2-\zeta)^\alpha}-\Gamma(\alpha+1)\,{}^{\alpha}I_{\xi_2^-}\Psi(\vartheta)\right).$$

Corollary 2. *If $(\alpha,m)=(1,1)$ and $h(\zeta)=\zeta$, then the following inequality holds for strongly convex function:*

$$\left({}_{\Psi}^{\Upsilon}\Omega_{\sigma^+,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,\eta}\,\Phi\right)(\vartheta;\varrho)+\left({}_{\Psi}^{\Upsilon}\Omega_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,\eta}\,\Phi\right)(\vartheta;\varrho)\le\Lambda_{\vartheta}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,\eta}\,\Psi;\Upsilon)\left((\Psi(\vartheta)-\Psi(\xi_1))(\Phi(\vartheta)+\Phi(\xi_1))\right.$$
$$-C(\zeta-\xi_1)(2I(\xi_1,\zeta,I_d\Psi)-(\xi_1+\zeta)I(\xi_1,\zeta,\Psi)+\Lambda_{\xi_2}^{\vartheta}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,\eta}\,\Psi;\Upsilon)\left((\Psi(\xi_2)-\Psi(\vartheta))(\Phi(\xi_2)+\Phi(\vartheta))\right.$$
$$-C(\xi_2-\zeta)(2I(\zeta,\xi_2,I_d\Psi)-(\xi_2+\zeta)I(\zeta,\xi_2,\Psi)),$$

where I_d is the identity function.

For positive values of C all the results obtained in the aforementioned Remarks/Corollaries get their refinements. The following lemma is required to establish the next result.

Lemma 2.1. *[22] Let $\Phi:[\xi_1,\xi_2]\longrightarrow\mathbb{R}$, be a strongly $(\alpha,h-m)$-convex function with modulus $C\ge 0$, $m\in(0,1]$, $0\le\xi_1\le m\xi_2$. If $\Phi(\zeta)=\Phi(\frac{\xi_1+m\xi_2-\zeta}{m})$, then the following inequality holds:*

$$\Phi\left(\frac{\xi_1+m\xi_2}{2}\right)\le\left(h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\right)\Phi(\zeta)-\frac{C}{m}h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)(\xi_1-\zeta+m\xi_2-m\zeta)^2. \tag{2.10}$$

The following result provides upper and lower bounds of sum of operators (1.7) and (1.8) in the form of a Hadamard type inequality.

Theorem 2.2. *Under the assumptions of Theorem 2.1 in addition, if $\Phi(\zeta)=\Phi\left(\frac{\xi_1+m\xi_2-\zeta}{m}\right)$, then we have*

$$\frac{1}{h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)}\left(\Phi\left(\frac{\xi_1+\xi_2}{2}\right)\left(\left({}_{\Psi}^{\varDelta}\Upsilon_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n}1\right)(\xi_1;\varrho)+\left({}_{\Psi}^{\varDelta}\Upsilon_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n}1\right)(\xi_2;\varrho)\right)\right. \tag{2.11}$$
$$+\frac{C}{m}h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(\left({}_{\Psi}^{\varDelta}\Upsilon_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n}(\xi_1-\zeta+m\xi_2-m\zeta)^2\right)(\xi_1;\varrho)\right.$$
$$\left.\left.+\left({}_{\Psi}^{\varDelta}\Upsilon_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n}(\xi_1-\zeta+m\xi_2-m\zeta)^2\right)(\xi_1;\varrho)\right)\right)\le\left({}_{\Psi}^{\varDelta}\Upsilon_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n}\,\Phi\right)(\xi_1;\varrho)$$
$$+\left({}_{\Psi}^{\varDelta}\Upsilon_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n}\,\Phi\right)(\xi_2;\varrho)\le(\xi_2-\xi_1)\left(\Lambda_{\xi_2}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\,\Psi;\varDelta)+\Lambda_{\xi_2}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\,\Psi;\varDelta)\right)$$
$$\times\left(\Phi(\xi_2)X_{\xi_2}^{\xi_1}(r^\alpha,h;\Psi')+m\Phi\left(\frac{\xi_1}{m}\right)X_{\xi_2}^{\xi_1}(1-r^\alpha,h;\Psi')-\frac{C(\xi_1-m\xi_2)^2h(1)(\Psi(\xi_2)-\Psi(\xi_1))}{m(\xi_2-\xi_1)}\right).$$

Proof. Using (1.12), we can write the following inequalities

$$\Lambda_{\zeta}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\,\Psi;\varDelta)\Psi'(\zeta)\le\Lambda_{\xi_2}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\,\Psi;\varDelta)\Psi'(\zeta), \tag{2.12}$$

$$\Lambda_{\xi_2}^{\zeta}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\ \Psi;\Delta)\Psi'(\zeta)\leq\Lambda_{\xi_2}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\ \Psi;\Delta)\Psi'(\zeta).\tag{2.13}$$

Using strongly $(\alpha, h-m)$-convexity of Φ for $\zeta\in[\xi_1,\xi_2]$, we have

$$\Phi(\zeta)\leq h\left(\frac{\zeta-\xi_1}{\xi_2-\xi_1}\right)^{\alpha}\Phi(\xi_2)+mh\left(1-\left(\frac{\zeta-\xi_1}{\xi_2-\xi_1}\right)^{\alpha}\right)\Phi\left(\frac{\xi_1}{m}\right)-\frac{C(\xi_1-m\xi_2)^2}{m}h\left(\frac{\zeta-\xi_1}{\xi_2-\xi_1}\right)^{\alpha}h\left(1-\left(\frac{\zeta-\xi_1}{\xi_2-\xi_1}\right)^{\alpha}\right).\tag{2.14}$$

Multiplying (2.12) and (2.14) and integrating the resulting inequality over $[\xi_1,\xi_2]$, one can obtain

$$\begin{aligned}&\int_{\xi_1}^{\xi_2}\Lambda_{\zeta}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\ \Psi;\Delta)\Phi(\zeta)d(\Psi(\zeta))\leq\Lambda_{\xi_2}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\ \Psi;\Delta)\Bigg(m\Phi\left(\frac{\xi_1}{m}\right)\\&\times\int_{\xi_1}^{\xi_2}h\left(1-\left(\frac{\zeta-\xi_1}{\xi_2-\xi_1}\right)^{\alpha}\right)d(\Psi(\zeta))+\Phi(\xi_2)\int_{\xi_1}^{\xi_2}h\left(\frac{\zeta-\xi_1}{\xi_2-\xi_1}\right)^{\alpha}d(\Psi(\zeta))\\&-\frac{C(\xi_1-m\xi_2)^2}{m}\int_{\xi_1}^{\xi_2}h\left(\frac{\zeta-\xi_1}{\xi_2-\xi_1}\right)^{\alpha}h\left(1-\left(\frac{\zeta-\xi_1}{\xi_2-\xi_1}\right)^{\alpha}\right)d(\Psi(\zeta))\Bigg).\end{aligned}$$

By using Definition 1.3 and setting $r=\frac{\zeta-\xi_1}{\xi_2-\xi_1}$, the following inequality is obtained:

$$\begin{aligned}&\left({}_{\Psi}^{\Delta}\Upsilon_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n}\ \Phi\right)(\xi_1;\varrho)\leq\Lambda_{\xi_2}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\ \Psi;\Delta)(\xi_2-\xi_1)\Bigg(\bigg(\Phi(\xi_2)X_{\xi_2}^{\xi_1}(r^{\alpha},h;\Psi')\\&+m\Phi\left(\frac{\xi_1}{m}\right)X_{\xi_2}^{\xi_1}(1-r^{\alpha},h;\Psi')-\frac{C(\xi_1-m\xi_2)^2h(1)(\Psi(\xi_2)-\Psi(\xi_1))}{m(\xi_2-\xi_1)}\bigg).\end{aligned}\tag{2.15}$$

Using the same technique that we did for (2.12) and (2.14), the following inequality can be observed from (2.14) and (2.13)

$$\begin{aligned}&\left({}_{\Psi}^{\Delta}\Upsilon_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n}\ \Phi\right)(b;\varrho)\leq\Lambda_{\xi_2}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\ \Psi;\Delta)(\xi_2-\xi_1)\bigg(\Phi(\xi_2)X_{\xi_2}^{\xi_1}(r^{\alpha},h;\Psi')\\&+m\Phi\left(\frac{\xi_1}{m}\right)X_{\xi_2}^{\xi_1}(1-r^{\alpha},h;\Psi')-\frac{C(\xi_1-m\xi_2)^2h(1)(\Psi(\xi_2)-\Psi(\xi_1))}{m(\xi_2-\xi_1)}\bigg).\end{aligned}\tag{2.16}$$

By adding (2.15) and (2.16), following inequality can be obtained:

$$\begin{aligned}&\left({}_{\Psi}^{\Delta}\Upsilon_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n}\Phi\right)(\xi_1;\varrho)+\left({}_{\Psi}^{\Delta}\Upsilon_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}^{\omega,\lambda,\rho,\theta,k,n}\ \Phi\right)(\xi_2;\varrho)\leq(\xi_2-\xi_1)\\&\times\left(\Lambda_{\xi_2}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\ \Psi;\Delta)+\Lambda_{\xi_2}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\ \Psi;\Delta)\right)\bigg(\Phi(\xi_2)X_{\xi_2}^{\xi_1}(r^{\alpha},h;\Psi')\\&+m\Phi\left(\frac{\xi_1}{m}\right)X_{\xi_2}^{\xi_1}(1-r^{\alpha},h;\Psi')-\frac{C(\xi_1-m\xi_2)^2h(1)(\Psi(\xi_2)-\Psi(\xi_1))}{m(\xi_2-\xi_1)}\bigg).\end{aligned}\tag{2.17}$$

Multiplying both sides of (2.10) by $\Lambda_{\zeta}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\ \Psi;\Delta)d(\Psi(\zeta))$, and integrating over $[\xi_1,\xi_2]$ we have

$$\begin{aligned}&\Phi\left(\frac{\xi_1+m\xi_2}{2}\right)\int_{\xi_1}^{\xi_2}\Lambda_{\zeta}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\ \Psi;\Delta)d(\Psi(\zeta))\leq h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_{\xi_1}^{\xi_2}\Lambda_{\zeta}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\ \Psi;\Delta)\Phi(\zeta)d(\Psi(\zeta))\\&-\frac{C}{m}h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_{\xi_1}^{\xi_2}\Lambda_{\zeta}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\ \Psi;\Delta)(\xi_1-\zeta+m\xi_2-m\zeta)^2d(\Psi(\zeta)).\end{aligned}$$

From Definition 1.3, the following inequality is obtained:

$$\frac{1}{h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)}\left(\Phi\left(\frac{\xi_1+\xi_2}{2}\right)\left({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}1\right)(\xi_1;\varrho)+\frac{C}{m}h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)\right. \tag{2.18}$$
$$\left.\times\left({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}(\xi_1-\zeta+m\xi_2-m\zeta)^2\right)(\xi_1;\varrho)\right)\leq\left({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}\Phi\right)(\xi_1;\varrho).$$

Similarly multiplying both sides of (2.10) by $\Lambda^{\zeta}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\ \Psi;\Delta)\Psi'(\zeta)$, and integrating over $[\xi_1,\xi_2]$ we have

$$\frac{1}{h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)}\left(\Phi\left(\frac{\xi_1+\xi_2}{2}\right)\left({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}1\right)(\xi_2;\varrho)+\frac{C}{m}h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)\right. \tag{2.19}$$
$$\left.\times\left({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}(\xi_1-\zeta+m\xi_2-m\zeta)^2\right)(\xi_2;\varrho)\right)\leq\left({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}\Phi\right)(\xi_1;\varrho).$$

From (2.17)–(2.19), inequality (2.11) can be achieved. □

Remark 3. *(i) If $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, in (2.11), then [22, Theorem 23] is obtained.*
(ii) If $h(\zeta)=\zeta$ in the result of (i), then [23, Theorem 11] is obtained.
(iii) If $\omega=\varrho=C=0$, (α,m) =(1,1), $\Phi(\vartheta)=\Gamma(\beta)\vartheta^{\beta+1}$ and $h(\zeta)=\Psi(\zeta)=\zeta$ in the result of (i), then [28, Theorem 3] is obtained.
(iv) If $\Delta(\vartheta)=\vartheta^{\beta+1}$ and $h(\vartheta)=\Psi(\vartheta)=\vartheta$ in the result of (i), then [25, Theorem 6] can be obtained.
(v) If $\Delta(\vartheta)=\vartheta^{\beta+1}$, $\Psi(\vartheta)=\vartheta$ and $C=0$ in the result of (i), then [26, Theorem 4] can be obtained.
(vi) If $C=0$ and $h(\zeta)=\zeta$ in (2.11), then [4, Theorem 2] is obtained.

Corollary 3. *If $h(\zeta)=\zeta$ in (2.11), then the following inequality holds for strongly (α,m)-convex function:*

$$\frac{2^\alpha}{(1+m(2^\alpha-1))}\left(\Phi\left(\frac{\xi_1+m\xi_2}{2}\right)\left(\left({}^{\Upsilon}_{\Psi}\Omega^{\omega,\lambda,\rho,\theta,k,\eta}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}1\right)(\xi_1;\varrho)+\left({}^{\Upsilon}_{\Psi}\Omega^{\omega,\lambda,\rho,\theta,k,\eta}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}1\right)(\xi_2;\varrho)\right)\right.$$
$$\left.+\frac{C(2^\alpha-1)}{2^{2\alpha}m}\left(\left({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}(\xi_1-\zeta+m\xi_2-m\zeta)^2\right)(\xi_1;\varrho)+\left({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}(\xi_1-\zeta+m\xi_2-m\zeta)^2\right)(\xi_1;\varrho)\right)\right)$$
$$\leq\left({}^{\Upsilon}_{\Psi}\Omega^{\omega,\lambda,\rho,\theta,k,\eta}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}\Phi\right)(a;\varrho)+\left({}^{\Upsilon}_{\Psi}\Omega^{\omega,\lambda,\rho,\theta,k,\eta}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}\Phi\right)(b;\varrho)\leq 2\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,\eta}_{\alpha,\beta,\gamma,\mu,\nu}\ \Psi;\Upsilon)\left(\left(\Phi(\xi_2)\Psi(\xi_2)-m\Phi\left(\frac{\xi_1}{m}\right)\Psi(\xi_1)\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{(\xi_2-\xi_1)^\alpha}\left(\Phi(\xi_2)-m\Phi\left(\frac{\xi_1}{m}\right)\right)^\alpha I_{\xi_2^-}\Psi(\vartheta)\right)\right)-\frac{C(m\xi_2-\xi_1)}{m(\xi_2-\xi_1)^\alpha}\left(\frac{\Gamma(2\alpha+1)I^{2\alpha}_{\xi_2^-}\Psi(\xi_1)}{(\xi_2-\xi)^\alpha}-\Gamma(\alpha+1)I^{\alpha}_{\xi_2^-}\Psi(\xi_1)\right).$$

Corollary 4. *If $(\alpha,m)=(1,1)$ and $h(\zeta)=\zeta$ in (2.11), then the following inequality holds for strongly convex function:*

$$\left(\Phi\left(\frac{\xi_1+\xi_2}{2}\right)\left(\left({}^{\Upsilon}_{\Psi}\Omega^{\omega,\lambda,\rho,\theta,k,\eta}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}1\right)(\xi_1;\varrho)+\left({}^{\Upsilon}_{\Psi}\Omega^{\omega,\lambda,\rho,\theta,k,\eta}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}1\right)(\xi_2;\varrho)\right)\right.$$
$$\left.+\frac{C}{4}\left(\left({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}(\xi_1-\zeta+\xi_2-\zeta)^2\right)(\xi_1;\varrho)+\left({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}(\xi_1-\zeta+\xi_2-\zeta)^2\right)(\xi_1;\varrho)\right)\right)$$
$$\leq\left({}^{\Upsilon}_{\Psi}\Omega^{\omega,\lambda,\rho,\theta,k,\eta}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}\Phi\right)(a;\varrho)+\left({}^{\Upsilon}_{\Psi}\Omega^{\omega,\lambda,\rho,\theta,k,\eta}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}\Phi\right)(b;\varrho)\leq 2\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,\eta}_{\alpha,\beta,\gamma,\mu,\nu}\ \Psi;\Upsilon)$$
$$\times((\Psi(\xi_2)-\Psi(\xi_1))(\Phi(\xi_2)+\Phi(\xi_1))-(\xi_2-\xi_1)\lambda 2I(\xi_1,\xi_2,I_d\Psi)-(\xi_1+\xi_2)I(\xi_1,\xi_2,g))$$

For positive values of λ all the results obtained in the aforementioned Remarks/Corollaries got their refinements.

Theorem 2.3. *Let $\Phi : [\xi_1, \xi_2] \longrightarrow \mathbb{R}$ be a differentiable function. If $|\Phi'|$ is (s, m)-convex and let $\Psi : [\xi_1, \xi_2] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function, also let $\frac{\Delta}{\zeta}$ be an increasing function on $[\xi_1, \xi_2]$. Then $\forall\ \zeta \in [\xi_1, \xi_2]$, we have the following inequality containing the unified Mittag-Leffler function $M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(z; \underline{a}, \underline{b}, \underline{c}, \varrho\right)$ satisfying all the convergence conditions:*

$$\begin{aligned}
&\left|\left({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi * \Psi\right)(\zeta; \varrho) + \left({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi * \Psi\right)(\zeta; \varrho)\right| \leq \Lambda^{\xi_1}_{\zeta}(M^{\omega,\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\ \Psi; \Delta)(\zeta - \xi_1) \qquad (2.20)\\
&\times\left(\left(m\left|\Phi'\left(\frac{\zeta}{m}\right)\right| X^{\xi_1}_{\zeta}(1 - r^{\alpha}, h, \Psi') - |\Phi'(\xi_1)| X^{\xi_1}_{\zeta}(r^{\alpha}, h, \Psi')\right) - \frac{C(\zeta - \xi_1 m)^2 h(1)(\Psi(\zeta) - \Psi(\xi_1))}{m(\zeta - \xi_1)}\right)\\
&+ \Lambda^{\zeta}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \Psi; \Delta)(\xi_2 - \zeta)\left(\left(m\left|\Phi'\left(\frac{\zeta}{m}\right)\right| X^{\xi_2}_{\zeta}(1 - r^{\alpha}, h, \Psi') - |\Phi'(\xi_2)| X^{\xi_2}_{\zeta}(r^{\alpha}, h, \Psi')\right)\right.\\
&\left. - \frac{C(\xi_2 m - \zeta)^2 h(1)(\Psi(\xi_2) - \Psi(\zeta))}{m(\xi_2 - \zeta)}\right),
\end{aligned}$$

where

$$\left({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi * \Psi\right)(\zeta; \varrho) := \int_{\xi_1}^{\zeta} \Lambda^{\vartheta}_{\zeta}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \Psi; \Delta)\Phi'(\vartheta) d(\Psi(\vartheta)),$$

$$\left({}^{\Delta}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi * \Psi\right)(\zeta; \varrho) := \int_{\zeta}^{\xi_2} \Lambda^{\zeta}_{t}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\ \Psi; \Delta)\Phi'(\vartheta) d(\Psi(\vartheta)).$$

Proof. Let $\zeta \in [\xi_1, \xi_2]$ and $\vartheta \in [\xi_1, \zeta]$. Then using strongly $(\alpha, h - m)$-convexity of $|\Phi'|$ we have

$$|\Phi'(\vartheta)| \leq h\left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha} |\Phi'(\xi_1)| + mh\left(1 - \left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha}\right)\left|\Phi'\left(\frac{\zeta}{m}\right)\right| - \frac{C(\vartheta - \xi_1 m)}{m} h\left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha} h\left(1 - \left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha}\right). \qquad (2.21)$$

The inequality (2.21) can be written as follows:

$$\begin{aligned}
&-\left(h\left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha} |\Phi'(\xi_1)| + mh\left(1 - \left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha}\right)\left|\Phi'\left(\frac{\zeta}{m}\right)\right| - \frac{C(\vartheta - \xi_1 m)}{m} h\left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha} h\left(1 - \left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha}\right)\right) \qquad (2.22)\\
&\leq \Phi'(\vartheta) \leq h\left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha} |\Phi'(\xi_1)| + mh\left(1 - \left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha}\right)\left|\Phi'\left(\frac{\zeta}{m}\right)\right| - \frac{C(\vartheta - \xi_1 m)}{m} h\left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha} h\left(1 - \left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha}\right).
\end{aligned}$$

Let we consider the second inequality of (2.22)

$$\Phi'(\vartheta) \leq h\left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha} |\Phi'(\xi_1)| + mh\left(1 - \left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha}\right)\left|\Phi'\left(\frac{\zeta}{m}\right)\right| - \frac{C(\vartheta - \xi_1 m)}{m} h\left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha} h\left(1 - \left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha}\right). \qquad (2.23)$$

Multiplying (2.2) and (2.23) and integrating over $[\xi_1, x]$, we can obtain:

$$\int_{\xi_1}^{\zeta} \Lambda^{\vartheta}_{\zeta}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\ \Psi; \Delta) d(\Psi(\vartheta)) \leq \Lambda^{\xi_1}_{\zeta}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\ \Psi; \Delta)\left(|\Phi(\xi_1)| \int_{\xi_1}^{\zeta} h\left(\frac{\zeta - \vartheta}{\zeta - \xi_1}\right)^{\alpha} d(\Psi(\vartheta))\right.$$

$$+ m\left|\Phi\left(\frac{\zeta}{m}\right)\right|\int_{\xi_1}^{\zeta} h\left(1-\left(\frac{\zeta-\vartheta}{\zeta-\xi_1}\right)^{\alpha}\right)d(\Psi(\vartheta)) - \frac{C(\vartheta-\xi_1 m)}{m}\int_{\xi_1}^{\zeta} h\left(\frac{\zeta-\vartheta}{\zeta-\xi_1}\right)^{\alpha} h\left(1-\left(\frac{\zeta-\vartheta}{\zeta-\xi_1}\right)^{\alpha}\right)d(\Psi(\vartheta))\Big).$$

This gives

$$\begin{aligned}&\left({}_{\Psi}^{\Delta}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}\,\Phi*\Psi\right)(\zeta;\varrho) \le \Lambda_{\zeta}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\,\Psi;\Delta)(\zeta-\xi_1) \\ &\times\left(\left(m\left|\Phi'\left(\frac{\zeta}{m}\right)\right|X_{\zeta}^{\xi_1}(1-r^{\alpha},h,\Psi') - |\Phi'(\xi_1)|X_{\zeta}^{\xi_1}(r^{\alpha},h,\Psi')\right) - \frac{C(\zeta-\xi_1 m)^2 h(1)(\Psi(\zeta)-\Psi(\xi_1))}{m(\zeta-\xi_1)}\right).\end{aligned} \tag{2.24}$$

Considering the left hand side from the inequality (2.22) and adopt the same pattern as did for the right hand side inequality, then

$$\begin{aligned}&\left({}_{\Psi}^{\Delta}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}(\Phi*\Psi)\right)(\zeta;\varrho) \ge -\Lambda_{\zeta}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\,\Psi;\Delta)(\zeta-\xi_1) \\ &\times\left(\left(m\left|\Phi'\left(\frac{\zeta}{m}\right)\right|X_{\zeta}^{\xi_1}(1-r^{\alpha},h,\Psi') - |\Phi'(\xi_1)|X_{\zeta}^{\xi_1}(r^{\alpha},h,\Psi')\right) - \frac{C(\zeta-\xi_1 m)^2 h(1)(\Psi(\zeta)-\Psi(\xi_1))}{m(\zeta-\xi_1)}\right).\end{aligned} \tag{2.25}$$

From (2.24) and (2.25), following inequality is observed:

$$\begin{aligned}&\left|\left({}_{\Psi}^{\Delta}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}(\Phi*\Psi)\right)(\zeta;\varrho)\right| \le \Lambda_{\zeta}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\,\Psi;\Delta)(\zeta-\xi_1) \\ &\times\left(\left(m\left|\Phi'\left(\frac{\zeta}{m}\right)\right|X_{\zeta}^{\xi_1}(1-r^{\alpha},h,\Psi') - |\Phi'(\xi_1)|X_{\zeta}^{\xi_1}(r^{\alpha},h,\Psi')\right) - \frac{C(\zeta-\xi_1 m)^2 h(1)(\Psi(\zeta)-\Psi(\xi_1))}{m(\zeta-\xi_1)}\right).\end{aligned} \tag{2.26}$$

Now using strongly $(\alpha, h-m)$-convexity of $|\Phi'|$ on $(\zeta, \xi_2]$ for $\zeta \in [\xi_1, \xi_2]$ we have

$$|\Phi'(\vartheta)| \le h\left(\frac{\vartheta-\zeta}{\xi_2-\zeta}\right)^{\alpha}|\Phi'(\xi_2)| + mh\left(1-\left(\frac{\vartheta-\zeta}{\xi_2-\zeta}\right)^{\alpha}\right)\left|\Phi'\left(\frac{\zeta}{m}\right)\right| - \frac{C(\zeta-\xi_1 m)^2}{m}h\left(\frac{\vartheta-\zeta}{\xi_2-\zeta}\right)^{\alpha}h\left(1-\left(\frac{\vartheta-\zeta}{\xi_2-\zeta}\right)^{\alpha}\right). \tag{2.27}$$

On the same procedure as we did for (2.2) and (2.21), one can obtain following inequality from (2.3) and (2.27):

$$\begin{aligned}&\left|\left({}_{\Psi}^{\Delta}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}(\Phi*\Psi)\right)(\zeta;\varrho)\right| \le \Lambda_{\xi_2}^{\zeta}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\Psi;\Delta)(\xi_2-\zeta) \\ &\times\left(\left(m\left|\Phi'\left(\frac{\zeta}{m}\right)\right|X_{\zeta}^{\xi_2}(1-r^{\alpha},h,\Psi') - |\Phi'(\xi_2)|X_{\zeta}^{\xi_2}(r^{\alpha},h,\Psi')\right) - \frac{C(\xi_2 m-\zeta)^2 h(1)(\Psi(\xi_2)-\Psi(\zeta))}{m(\xi_2-\zeta)}\right).\end{aligned} \tag{2.28}$$

By adding (2.26) and (2.28), inequality (2.20) can be achieved. □

Remark 4. *(i) If* $n = 1$, $b_1 = \lambda + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, $\rho = \nu = 0$, *in (2.20), [22, Theorem 26].*
(ii) If $C = 0$ *in the result of (i), [27, Theorem 4] is obtained.*
(iii) If $h(\zeta) = \zeta$ *in the result of (i), then [23, Theorem 14] is obtained.*
(iv) If $\Delta(\vartheta) = \vartheta^{\beta}$ *and* $\Psi(\zeta) = h(\zeta) = \zeta$ *in the result of (i), then [25, Theorem 5] is obtained.*
(v) If $\Delta(\vartheta) = \vartheta^{\beta+1}$, $\Psi(\zeta) = \zeta$ *and* $C = 0$ *in the result of (i), then [26, Theorem 2] is obtained.*
(vi) If $C = 0$ *and* $h(\zeta) = \zeta$ *in (2.20), then [4, Theorem 3] is obtained.*

Corollary 5. *If $h(\zeta) = \zeta$ in (2.20), then the following inequality holds for strongly (α, m)-convex function:*

$$\left|\left({}^{\Upsilon}_{\Psi}\Omega^{\omega,\lambda,\rho,\theta,k,\eta}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi * \Psi\right)(\zeta;\varrho) + \left({}^{\Upsilon}_{\Psi}\Omega^{\omega,\lambda,\rho,\theta,k,\eta}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi * \Psi\right)(\zeta;\varrho)\right|$$
$$\leq \Lambda_{\zeta}^{\xi_1}(M^{\omega,\lambda,\rho,\theta,k,\eta}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\ \Psi;\Upsilon)\left(\left(m\left|\Phi'\left(\frac{\zeta}{m}\right)\right|\Psi(\zeta) - |\Phi'(\xi_1)|\Psi(\xi_1)\right) - \frac{\Gamma(\alpha+1)}{(\zeta-\xi_1)^{\alpha}}\left(m\left|\Phi'\left(\frac{\zeta}{m}\right)\right| - |\Phi'(\xi_1)|\right){}^{\alpha}I_{\xi_1^+}\Psi(\zeta)\right)$$
$$-\frac{C(\zeta - m\xi_1)^2}{m(\zeta-\xi_1)^{\alpha}}\left(\Gamma(\alpha+1)\,{}^{\alpha}I_{\xi_1^+}\Psi(\zeta) - \frac{\Gamma(2\alpha+1)\,{}^{\alpha}I_{\xi_1^+}\Psi(\zeta)}{(\zeta-\xi_1)^{\alpha}}\right) + \Lambda_{\xi_2}^{\zeta}(M^{\lambda,\rho,\theta,k,\eta}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\Psi;\Upsilon)$$
$$\left(\left(|\Phi'(\xi_2)|\Psi(\xi_2) - m\left|\Phi'\left(\frac{\zeta}{m}\right)\right|\Psi(\zeta)\right) - \frac{\Gamma(\alpha+1)}{(\xi_2-\zeta)^{\alpha}}\left(|\Phi'(\xi_2)| - m\left|\Phi'\left(\frac{\zeta}{m}\right)\right|\right){}^{\alpha}I_{\xi_2^-}\Psi(\zeta)\right)$$
$$-\frac{C(m\xi_2-\zeta)^2}{m(\xi_2-\zeta)^{\alpha}}\left(\frac{\Gamma(2\alpha+1)\,{}^{2\alpha}I_{\xi_2^-}\Psi(\zeta)}{(\xi_2-\zeta)^{\alpha}} - \Gamma(\alpha+1)\,{}^{\alpha}I_{\xi_2^-}\Psi(\zeta)\right).$$

Corollary 6. *If $(\alpha, m) = (1, 1)$ and $h(\zeta) = \zeta$ in (2.20), then the following inequality holds for strongly convex function:*

$$\left|\left({}^{\Upsilon}_{\Psi}\Omega^{\omega,\lambda,\rho,\theta,k,\eta}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi * \Psi\right)(\zeta;\varrho) + \left({}^{\Upsilon}_{\Psi}\Omega^{\omega,\lambda,\rho,\theta,k,\eta}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi * \Psi\right)(\zeta;\varrho)\right|$$
$$\leq \Lambda_{\zeta}^{\xi_1}(M^{\lambda,\rho,\theta,k,\eta}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\ \Psi;\Upsilon)\left(\left(|\Phi'(\zeta)| + |\Phi'(\xi_1)|\right)(\Psi(\zeta) - \Psi(\xi_1)) - C(\zeta-\xi_1)2I(\xi_1,\zeta,I_d\Psi) - (\xi_1+\zeta)I(\xi_1,\zeta,\Psi)\right)$$
$$+\Lambda_{\xi_2}^{\zeta}(M^{\lambda,\rho,\theta,k,\eta}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\Psi;\Upsilon)\Big(\left(|\Phi'(\xi_2)| + |\Phi'(\zeta)|\right)(\Psi(\xi_2) - \Psi(\zeta)) - C(\xi_2-\zeta)(2I(\zeta,\xi_2,I_d\Psi) - (\zeta+\xi_2)I(\zeta,\xi_2,\Psi))\Big).$$

For positive values of C all the results obtained in the aforementioned Remarks/Corollaries get their refinements.

3. Conclusions

This article investigates the bounds of fractional integral operators containing unified Mittag-Leffler function via strongly $(\alpha, h-m)$-convex function. The proved results are more generalized and refined as compared to existing inequalities in the literature. Also, they hold implicitly for various classes of functions such as (α, m)-convex, (s, m)-convex, $(h-m)$-convex, strongly convex, strongly (α, m)-convex and strongly $(h-m)$-convex functions.

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Conflict of interest

It is declared that authors have no competing interests.

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