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Research article

Commutators of Hardy-Cesàro operators on Morrey-Herz spaces with variable exponents

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Abstract: The aim of this paper is to establish some sufficient conditions for the boundedness of commutators of Hardy-Cesàro operators with symbols in central BMO spaces with variable exponent on some function spaces such as the local central Morrey, Herz, and Morrey-Herz spaces with variable exponents.

Keywords: Hardy-Cesàro operator; commutator; local central Morrey space; Herz space; Morrey-Herz space; block space; variable exponent
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1. Introduction

In 1984, the weighted Hardy-Littlewood average operator U_ψ was defined as follows by the authors [3].

$$U_\psi(f) = \int_0^1 \psi(t) f(tx)dt,\ x \in \mathbb{R}^n,$$

where $\psi : [0, 1] \to [0, \infty)$ is a measurable function and f is a measurable function on $\mathbb{R}^n$. When we choose $n = 1$ and $\psi \equiv 1$, the Hardy-Littlewood average operator U_ψ is reduced to the classical Hardy operator.

Next, Xiao [35] proved that U_ψ is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$\mathcal{A}_{n,p,\psi} := \int_0^1 t^{-n/p}\psi(t)dt < \infty.$$

Moreover,

$$\|U_\psi\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \mathcal{A}_{n,p,\psi}.$$

Besides, Xiao discovered that this operator is bounded on BMO spaces.

Now, we consider the Hardy-Cesàro operator, which is generalized of Hardy-Littlewood average operator, defined as follows.

Definition 1.1. *Let* $\psi : [0,1]^d \to [0,\infty)$ *be measurable function and* $s : [0,1]^d \to \mathbb{R}$ *be almost everywhere non-zero measurable function. The Hardy-Cesàro operator is defined by*

$$U_{\psi,s,d}(f)(x) = \int\limits_{[0,1]^d} \psi(t) f(s(t)x)dt,$$

for a measurable complex-valued function f *on* $\mathbb{R}^n$.

In case $d = 1$, the Hardy-Cesàro operator $U_{\psi,s,1}$ was studied by Chuong and Hung [8]. On the weighted Lebesgue and weighted BMO spaces, the authors provided sufficient and necessary conditions for the boundedness of $U_{\psi,s,1}$.

Furthermore, the theory of commutators is crucial in the investigation of the regularity of solutions to partial differential equations. For the natural extension, the commutators of Coifman-Rochberg-Weiss type of Hardy-Cesàro operators are discussed in this study as follows,

$$U^b_{\psi,s,d}(f) = bU_{\psi,s,d}(f) - U_{\psi,s,d}(bf) = \int_{[0,1]^d} \psi(t) f(s(t)x)\big(b(x) - b(s(t)x)\big)dt.$$

In case $d = 1$ and $s(t) = i_1(t) \equiv t$, the commutator of Hardy-Cesàro operator $U^b_{\psi,i_1,1}$ was researched by Fu et al. [15]. The authors demonstrated that $U^b_{\psi,i_1,1}$ is bounded on $L^p(\mathbb{R}^n)$ for all $b \in BMO(\mathbb{R}^n)$ if and only if

$$\int_0^1 t^{-n/p}\psi(t)\log\frac{2}{t}dt < \infty.$$

Recently, the commutators of the Hardy operator, Hardy-Cesàro operator, and Hausdorff operator have been extensively studied on the real field, p-adic field and Heisenberg group (see e.g., [6, 10, 16, 18, 27] and references therein for more details). As is well known, the theory of function spaces with variable exponents has some essential applications in the electronic fluid mechanics, recovery of graphics, elasticity, harmonic analysis, and partial differential equations (see e.g., [1, 2, 4, 5, 9, 12, 13, 19, 25, 26, 32–34, 36] and the references therein).

Motivated by the above results, the purpose of this paper is to give sufficient conditions for the boundedness of $U^b_{\psi,s,d}$ on the local central Morrey and Morrey-Herz spaces with variable exponents when the symbol functions belong to central BMO spaces with variable exponent. Moreover, through block decompositions, the boundedness of $U^b_{\psi,s,d}$ on the Herz spaces with variable exponents is also discussed. Finally, we establish the sufficient and necessary conditions for the boundedness of $U^b_{\psi,s,d}$ on the local central Morrey and Morrey-Herz spaces with constant exponents.

The following is the structure of our paper. In Section 2, we present necessary preliminaries on Lebesgue spaces, local central Morrey spaces, Herz spaces, Morrey-Herz spaces, and central BMO spaces with variable exponents. Our main results are given and proved in Section 3.

2. Preliminaries

Let us give the following symbols and notations before we state our results in the next section:

1) $B(a, r)$ denotes the ball centered at a with radius r for every $a \in \mathbb{R}$ and $r > 0$.

2) Given a measurable set Ω, let χ_Ω denote its characteristic function, $\chi_k = \chi_{C_k}$, $C_k = B_k \setminus B_{k-1}$ and $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, for all $k \in \mathbb{Z}$.

3) Let $\omega(\cdot)$ be a non-negative weighted function on $\mathbb{R}^n$ and a measurable set E. Then,

$$\omega(E) := \int_E \omega(x)dx.$$

4) We use $a \lesssim b$ to mean that there is a positive constant C, independent of the main parameters, such that $a \le Cb$. The symbol $a \simeq b$ means that both $a \lesssim b$ and $b \lesssim a$ hold.

Let us present the definition of the Lebesgue space with variable exponent (see e.g., [4, 12, 13] and the references therein).

Definition 2.1. *Let $\mathcal{P}_b(\mathbb{R}^n)$ be the set of all measurable functions $p(\cdot) : \mathbb{R} \to [1, \infty)$ such that*

$$1 < p_- \le p(x) \le p_+ < \infty, \text{ for all } x \in \mathbb{R}^n,$$

where $p_- = ess\, inf_{x\in\mathbb{R}^n} p(x)$ and $p_+ = ess\, sup_{x\in\mathbb{R}^n} p(x)$.

For $p(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$, the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of all complex-valued measurable functions f defined on $\mathbb{R}^n$ such that for some $\eta > 0$,

$$F_p(f/\eta) = \int\limits_{\mathbb{R}^n} \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx < \infty.$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space when equipped with the norm

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\eta > 0 : F_p\left(\frac{f}{\eta}\right) \le 1\right\}.$$

For $p \in \mathcal{P}_b(\mathbb{R}^n)$, we have the following inequalities, which are commonly utilized in the sequel.

$$\begin{aligned} &[i] \text{ If } F_p(f) \le C, \text{ then } \left\|f\right\|_{L^{p(\cdot)}} \le \max\{C^{\frac{1}{p_-}}, C^{\frac{1}{p_+}}\}, \text{ for all } f \in L^{p(\cdot)}(\mathbb{R}^n),\\ &[ii] \text{ If } F_p(f) \ge C, \text{ then } \left\|f\right\|_{L^{p(\cdot)}} \ge \min\{C^{\frac{1}{p_-}}, C^{\frac{1}{p_+}}\}, \text{ for all } f \in L^{p(\cdot)}(\mathbb{R}^n). \end{aligned} \tag{2.1}$$

The space $\mathcal{P}_\infty(\mathbb{R}^n)$ is defined by the set of all measurable functions $p(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$ and there exists a constant p_∞ such that

$$p_\infty = \lim_{|x|\to\infty} p(x).$$

The set $L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n \setminus \{0\})$ consists of all measurable functions f on $\mathbb{R}^n \setminus \{0\}$ satisfying $f.\chi_K \in L^{p(\cdot)}(\mathbb{R}^n)$ for any compact set $K \subset \mathbb{R}^n \setminus \{0\}$.

Let $\mathbf{C}^{\log}_0(\mathbb{R}^n)$ denote the set of all log-Hölder continuous functions $\alpha(\cdot)$ satisfying at the origin

$$|\alpha(x) - \alpha(0)| \le \frac{C^\alpha_0}{\log\left(e + \frac{1}{|x|}\right)}, \text{ for all } x \in \mathbb{R}^n.$$

Denote by $\mathbf{C}_\infty^{\log}(\mathbb{R}^n)$ the set of all log-Hölder continuous functions $\alpha(\cdot)$ satisfying at infinity

$$|\alpha(x)-\alpha_\infty| \le \frac{C_\infty^\alpha}{\log(e+|x|)}, \ \text{ for all } \ x\in\mathbb{R}^n.$$

Next, we would like to give the definition of Herz spaces, Morrey-Herz spaces (see [25, 32] for more details) and local central Morrey spaces.

Definition 2.2. *Let $p\in(0,\infty), q(\cdot)\in\mathcal{P}_b(\mathbb{R}^n)$, $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ with $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$ and $\omega(\cdot)$ be a non-negative weighted function on $\mathbb{R}^n$. The nonhomogeneous Herz space $K_{q(\cdot)}^{\alpha(\cdot),p}(\omega)$ is defined by*

$$K_{q(\cdot)}^{\alpha(\cdot),p}(\omega)=\left\{f\in L_{loc}^{q(\cdot)}(\mathbb{R}^n\setminus\{0\}):\|f\|_{K_{q(\cdot)}^{\alpha(\cdot),p}(\omega)}<\infty\right\}.$$

Here

$$\|f\|_{K_{q(\cdot)}^{\alpha(\cdot),p}(\omega)}=\left(\|\omega(B_0)^{\alpha(\cdot)/n}f\chi_{B_0}\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p+\sum_{k=1}^\infty\|\omega(B_k)^{\alpha(\cdot)/n}f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right)^{1/p}.$$

Definition 2.3. *Let $p\in(0,\infty), q(\cdot)\in\mathcal{P}_b(\mathbb{R}^n)$, $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ with $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$ and $\omega(\cdot)$ be a non-negative weighted function on $\mathbb{R}^n$. The homogeneous Herz space $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\omega)$ is defined by*

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\omega)=\left\{f\in L_{loc}^{q(\cdot)}(\mathbb{R}^n\setminus\{0\}):\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\omega)}<\infty\right\}.$$

Here

$$\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\omega)}=\Big(\sum_{k=-\infty}^\infty\|\omega(B_k)^{\alpha(\cdot)/n}f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\Big)^{1/p}.$$

Definition 2.4. *Assume that $\lambda\in[0,\infty)$, $p\in(0,\infty)$, $q(\cdot)\in\mathcal{P}_b(\mathbb{R}^n)$ and $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ with $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. The Morrey–Herz space $M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$ is defined by*

$$M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)=\left\{f\in L_{loc}^{q(\cdot)}(\mathbb{R}^n\setminus\{0\}):\|f\|_{M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}<\infty\right\}.$$

Here

$$\|f\|_{M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}=\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}\|2^{k\alpha(\cdot)}f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right)^{1/p}.$$

Theorem 2.1 (Proposition 2.5 in [25])**.** *If $\lambda\in[0,\infty)$, $p\in(0,\infty)$, $q(\cdot)\in\mathcal{P}_b(\mathbb{R}^n)$ and $\alpha\in L^\infty(\mathbb{R}^n)\cap\mathbf{C}_0^{log}(\mathbb{R}^n)\cap\mathbf{C}_\infty^{log}(\mathbb{R}^n)$, then we obtain*

$$\|f\|_{M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}\approx\max\left\{\sup_{k_0\in\mathbb{Z}^-}\mathcal{M}_{1,k_0},\sup_{k_0\in\mathbb{N}}\Big(\mathcal{M}_{2,k_0}+\mathcal{M}_{3,k_0}\Big)\right\}.$$

Here

$$\mathcal{M}_{1,k_0}=2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p}\|f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right)^{1/p},\mathcal{M}_{2,k_0}=2^{-k_0\lambda}\left(\sum_{k=-\infty}^{-1}2^{k\alpha(0)p}\|f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right)^{1/p},$$

$$\mathcal{M}_{3,k_0}=2^{-k_0\lambda}\left(\sum_{k=0}^{k_0}2^{k\alpha_\infty p}\|f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right)^{1/p}.$$

We get the following result from the definition of Morrey-Herz spaces with variable exponents and Proposition 2.5 in [25].

Lemma 2.1. *Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$, $p \in (0,\infty)$ and $\lambda \in [0,\infty)$. If $\alpha(\cdot)$ is log-Hölder continuous both at the origin and at infinity, then*

$$\left\|f\chi_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \lesssim 2^{j(\lambda-\alpha(0))}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}, \text{ for all } j \in \mathbb{Z}^-,$$
$$\left\|f\chi_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \lesssim 2^{j(\lambda-\alpha_\infty)}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}, \text{ for all } j \in \mathbb{N}.$$

Remark 2.1. *If $\omega \equiv 1$, then $K^{\alpha(\cdot),p}_{q(\cdot)}(\omega)$=$K^{\alpha(\cdot)}_{q(\cdot),p}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\omega) = \dot{K}^{\alpha(\cdot)}_{q(\cdot),p}(\mathbb{R}^n)$ are defined in [1]. If $\lambda = 0$, then $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot)}_{q(\cdot),p}(\mathbb{R}^n)$. When $\alpha(\cdot)$ is constant, we have $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n) = M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ (see [20]) . If both $\alpha(\cdot)$ and $q(\cdot)$ are constant, then $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n) = M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$, $K^{\alpha(\cdot)}_{q(\cdot),p}(\mathbb{R}^n) = K^{\alpha}_{q,p}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot)}_{q(\cdot),p}(\mathbb{R}^n) = \dot{K}^{\alpha}_{q,p}(\mathbb{R}^n)$ are classical Morrey–Herz spaces (see e.g. [22, 24]).*

It is well known that the local Morrey space [14, 29] has some important applications to the partial differential equations. As a natural extension, we define the local central Morrey space.

Definition 2.5. *Let $\lambda \in \mathbb{R}$, $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$. The local central Morrey space $\dot{B}^{q(\cdot),\lambda}_{\text{loc}}(\mathbb{R}^n)$ is defined by*

$$\dot{B}^{q(\cdot),\lambda}_{\text{loc}}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n) : \|f\|_{B^{q(\cdot),\lambda}_{\text{loc}}(\mathbb{R}^n)} < \infty\}.$$

Here

$$\|f\|_{\dot{B}^{q(\cdot),\lambda}_{\text{loc}}(\mathbb{R}^n)} = \sup_{k\le 0 \text{ and } k\in\mathbb{Z}} \frac{\|f\|_{L^{q(\cdot)}(B_k)}}{|B_k|^\lambda \|1\|_{L^{q(\cdot)}(B_k)}}.$$

If $q(\cdot)$ is constant, then we denote $\dot{M}^{q,\lambda}_{\text{loc}}(\mathbb{R}^n) := \dot{B}^{q(\cdot),\lambda}_{\text{loc}}(\mathbb{R}^n)$. Moreover, we recall the central Morrey space as follows.

Definition 2.6. *Assume that $q \in (1,\infty)$, $\lambda \in \mathbb{R}$. The central Morrey space $\dot{M}^{q,\lambda}(\mathbb{R}^n)$ is defined by*

$$\dot{M}^{q,\lambda}(\mathbb{R}^n) = \left\{f \in L^q_{loc}(\mathbb{R}^n) : \|f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n)} < \infty\right\}, \text{ where } \|f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n)} = \sup_{k\in\mathbb{Z}} \frac{1}{|B_k|^{1/q+\lambda}} \|f\|_{L^q(B_k)}.$$

Let us recall to define the central BMO spaces with variable exponent (see [30]).

Definition 2.7. *Let $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$. The variable exponent central bounded mean oscillation space $C\dot{M}O^{q(\cdot)}(\mathbb{R}^n)$ is defined as the set of all functions $f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ such that*

$$\left\|f\right\|_{C\dot{M}O^{q(\cdot)}(\mathbb{R}^n)} = \sup_{r>0} \frac{\|f - f_{B(0,r)}\|_{L^{q(\cdot)}(B(0,r))}}{\|1\|_{L^{q(\cdot)}(B(0,r))}} < \infty, \text{ where } f_{B(0,r)} = \frac{1}{|B(0,r)|}\int_{B(0,r)} f(x)dx.$$

Definition 2.8. *The space $BMO(\mathbb{R}^n)$ consists of all locally integrable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that*

$$\left\|f\right\|_{\text{BMO}(\mathbb{R}^n)} = \sup_B \frac{1}{|B|}\int_B |f(x) - f_B|dx < \infty,$$

where the supremum is taken over balls $B \subset \mathbb{R}^n$.

Remark 2.2. *If $q(\cdot)$ is constant, then $C\dot{M}O^{q(\cdot)}(\mathbb{R}^n) = C\dot{M}O^{q}(\mathbb{R}^n)$ is defined in [23]. The space $C\dot{M}O^{q}(\mathbb{R}^n)$ is a local version of the space $BMO(\mathbb{R}^n)$ at the origin. Furthermore, $BMO(\mathbb{R}^n) \subsetneq C\dot{M}O^{q}(\mathbb{R}^n)$.*

Definition 2.9. *Let $\beta \in (0,1]$. The Lipschitz space $\mathrm{Lip}^{\beta}(\mathbb{R}^n)$ is defined as the set of all functions $f : \mathbb{R}^n \to \mathbb{C}$ satisfying $\|f\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)} < \infty$, where*

$$\|f\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)} := \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\beta}}.$$

Definition 2.10. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The Hardy-Littlewood maximal operator M is defined as follows.*

$$M(f)(x) = \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)|dy.$$

The set $\mathfrak{B}(\mathbb{R}^n)$ consists of all measurable functions $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$ satisfying that the operator M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$.

Lemma 2.2. (see Lemmas 1 and 2 in paper [20]) *Let $q(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$.*

(i) Then we have a positive constant $\delta \in (0,1)$ such that

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta} \text{ and } \frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \lesssim \frac{|B|}{|S|}, \text{ for all balls } B \text{ in } \mathbb{R}^n \text{ and all measurable subsets } S \subset B.$$

(ii) Then we obtain

$$\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \simeq |B_k|, \text{ for all } k \in \mathbb{Z}.$$

Let us present the notation of the central block.

Definition 2.11. *Let $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n), \alpha(\cdot) \in L^{\infty}(\mathbb{R}^n) \cap \mathbf{C}_0^{log}(\mathbb{R}^n) \cap \mathbf{C}_{\infty}^{log}(\mathbb{R}^n)$, and $\alpha(0)$, $\alpha_{\infty} \in (0,\infty)$. A measurable function $b(x)$ is a central $(\alpha(\cdot), q(\cdot), \omega)$-block if there exists $k \in \mathbb{Z}$ such that*

(i) $\mathrm{supp}(b) \subset B_k$,

(ii) $\|b\|_{L^{q(\cdot)}(\mathbb{R}^n)} \lesssim \omega(B_k)^{-\alpha_k/n}$ *with* $\alpha_k = \begin{cases} \alpha(0), & \text{if } k < 0 \\ \alpha_{\infty}, & \text{otherwise}. \end{cases}$

Definition 2.12. *Let $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n), \alpha(\cdot) \in L^{\infty}(\mathbb{R}^n) \cap \mathbf{C}_{\infty}^{log}(\mathbb{R}^n)$, and $\alpha_{\infty} \in (0,\infty)$. A measurable function $b(x)$ is a central $(\alpha(\cdot), q(\cdot), \omega)$-block of restricted type if there exists $k \in \mathbb{N}$ such that*

(i) $\mathrm{supp}(b) \subset B_k$,

(ii) $\|b\|_{L^{q(\cdot)}(\mathbb{R}^n)} \lesssim \omega(B_k)^{-\alpha_{\infty}/n}$.

From the results in [24, 31], we give the following decomposition theorems.

Theorem 2.2. *Let $p \in (0,1]$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\alpha(\cdot) \in L^{\infty}(\mathbb{R}^n) \cap \mathbf{C}_{\infty}^{log}(\mathbb{R}^n) \cap \mathbf{C}_0^{log}(\mathbb{R}^n)$ and $\alpha(0), \alpha_{\infty} \in (0,\infty)$, $\omega(x) = |x|^{\beta}$ for $\beta \in (-n,\infty)$. The following two statements are equivalent:*

(i) $f \in \dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\omega)$.

(ii) *f can be represented by* $f = \sum_{k\in\mathbb{Z}} \lambda_k b_k$, *where* $\sum_{k\in\mathbb{Z}} |\lambda_k|^p < \infty$ *and each* b_k *is a central* $(\alpha(\cdot), q(\cdot), \omega)$-*block with the support in* B_k. *Additionally,*

$$\|f\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\omega)} \approx inf\left\{\sum_{k\in\mathbb{Z}} |\lambda_k|^p\right\}^{1/p},$$

where the infimum is taken over all decomposition of f as above.

Proof. Firstly, let us prove (i) infers (ii). For any $f \in \dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\omega)$,

$$f(x) = \sum_{k\in\mathbb{Z}} \lambda_k b_k(x), \text{ with } \lambda_k = \|\omega(B_k)^{\alpha(\cdot)/n} f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \text{ and } b_k(x) = \frac{f(x)\chi_k(x)}{\|\omega(B_k)^{\alpha(\cdot)/n} f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}}.$$

It is clear to see that $\sup(b_k) \subset B_k$. On the other hand, by assuming $\alpha(\cdot) \in \mathbf{C}^{\log}_{\infty}(\mathbb{R}^n) \cap \mathbf{C}^{\log}_{0}(\mathbb{R}^n)$ and using Step 1 and Step 4 in the proof of Theorem 3 in the paper [21],

$$\omega(B_k)^{\alpha(x)} \simeq \begin{cases} \omega(B_k)^{\alpha_\infty}, \text{ if } k \geq 0 \text{ and } x \in C_k, \\ \omega(B_k)^{\alpha(0)}, \text{ if } k < 0 \text{ and } x \in C_k. \end{cases} \tag{2.2}$$

Thus, for each $k \in \mathbb{Z}$,

$$\|b_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \lesssim \omega(B_k)^{-\alpha_k/n} \text{ with } \alpha_k = \begin{cases} \alpha(0), \text{ if } k < 0, \\ \alpha_\infty, \text{ otherwise.} \end{cases}$$

Consequently, for any $k \in \mathbb{Z}$, b_k is a central $(\alpha(\cdot), q(\cdot), \omega)$ - block with support contained in B_k and

$$\sum_{k\in\mathbb{Z}} |\lambda_k|^p = \sum_{k\in\mathbb{Z}} \|\omega(B_k)^{\alpha(\cdot)/n} f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} = \|f\|^p_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\omega)} < \infty.$$

Next, let us prove (ii) infers (i). Let $f(x) = \sum_{j=0}^{\infty} \lambda_j b_j(x)$ be a decomposition of function f which satisfies the assumption (ii) of this theorem. For any $k \in \mathbb{Z}$, by applying the Minkowski inequality,

$$\|f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq \sum_{j\geq k} |\lambda_j| \|b_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Hence, by $p \in (0, 1]$ and the inequality (2.2),

$$\begin{aligned} \|f\|^p_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\omega)} &\leq \sum_{k=-\infty}^{-1} \omega(B_k)^{\alpha(0)p/n}\Big(\sum_{j\geq k} |\lambda_j|^p \|b_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big) + \sum_{k=0}^{\infty} \omega(B_k)^{\alpha_\infty p/n}\Big(\sum_{j\geq k} |\lambda_j|^p \|b_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big) \\ &:= T_1 + T_2. \end{aligned} \tag{2.3}$$

To estimate T_1, by $\alpha(0), \alpha_\infty \in (0, \infty)$ and $\omega(x) = |x|^\beta$,

$$T_1 \lesssim \sum_{k=-\infty}^{-1} \omega(B_k)^{\alpha(0)p/n}\Big(\sum_{j=k}^{-1} |\lambda_j|^p \omega(B_j)^{-\alpha(0)p/n}\Big)1 + \sum_{k=-\infty}^{-1} \omega(B_k)^{\alpha(0)p/n}\Big(\sum_{j=0}^{\infty} |\lambda_j|^p \omega(B_j)^{-\alpha_\infty p/n}\Big)$$

$$\lesssim \sum_{k=-\infty}^{-1}\sum_{j=k}^{-1}|\lambda_j|^p 2^{(k-j)(n+\beta)\alpha(0)p/n} + \sum_{k=-\infty}^{-1} 2^{k(n+\beta)\alpha(0)p/n}\Big(\sum_{j=0}^{\infty}|\lambda_j|^p 2^{-j(n+\beta)\alpha_\infty p/n}\Big)$$
$$\lesssim \sum_{j=-\infty}^{-1}\sum_{k=-\infty}^{j}|\lambda_j|^p 2^{(k-j)(n+\beta)\alpha(0)p/n} + \sum_{j=0}^{\infty}|\lambda_j|^p \lesssim \sum_{j\in\mathbb{Z}}|\lambda_j|^p < \infty. \tag{2.4}$$

Now, by $\alpha_\infty \in (0,\infty)$, we estimate T_2 as follows.

$$T_2 \lesssim \sum_{k=0}^{\infty}\omega(B_k)^{\alpha_\infty p/n}\Big(\sum_{j\geq k}|\lambda_j|^p\omega(B_j)^{-\alpha_\infty p/n}\Big) \leq \sum_{k=0}^{\infty}\sum_{j\geq k}|\lambda_j|^p 2^{(k-j)(\beta+n)\alpha_\infty p/n}$$
$$= \sum_{j=0}^{\infty}\sum_{k\leq j}|\lambda_j|^p 2^{(k-j)(\beta+n)\alpha_\infty p/n} = \sum_{j=0}^{\infty}|\lambda_j|^p\sum_{k\leq j}2^{(k-j)(\beta+n)\alpha_\infty p/n} \lesssim \sum_{j=0}^{\infty}|\lambda_j|^p < \infty.$$

Hence, by (2.3) and (2.4), we imply $f \in \dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\omega)$ and finish the proof of Theorem 2.2.

By the similar argument as in the proof of Theorem 2.2 and the definition of the nonhomogeneous Herz space $K^{\alpha(\cdot),p}_{q(\cdot)}(\omega)$, we obtain the following theorem.

Theorem 2.3. *Let $p \in (0,1]$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathbf{C}^{log}_\infty(\mathbb{R}^n)$ and $\alpha_\infty \in (0,\infty)$, $\omega = |x|^\beta$ with $\beta \in (-n,\infty)$. The following two statements are equivalent:*

(i) $f \in K^{\alpha(\cdot),p}_{q(\cdot)}(\omega)$.

(ii) *f can be represented by $f = \sum_{k=0}^{\infty}\lambda_k b_k$, where $\sum_{k=0}^{\infty}|\lambda_k|^p < \infty$ and each b_k is a central $(\alpha(\cdot), q(\cdot), \omega)$-block of restricted type with the support in B_k. Moreover,*

$$\|f\|_{K^{\alpha(\cdot),p}_{q(\cdot)}(\omega)} \approx inf\left\{\sum_{k=0}^{\infty}|\lambda_k|^p\right\}^{1/p},$$

where the infimum is taken over all decomposition of f as above.

Remark 2.3. *We observe the differences between Theorems 2.2 and 2.3. In more detail, Theorem 2.2 presents that the central blocks $\{b_k\}_{k\in\mathbb{Z}}$ build the homogeneous Herz space $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\omega)$. Meanwhile, Theorem 2.3 shows that the central blocks of restricted type $\{b_k\}_{k\in\mathbb{N}}$ decompose the nonhomogeneous Herz space $K^{\alpha(\cdot),p}_{q(\cdot)}(\omega)$.*

3. Main results and their proofs

For simplicity of notation, we set

$$\mathcal{K}_{q,max}(t) = \max\{|s(t)|^{-n/q_+}, |s(t)|^{-n/q_-}\}, \mathbb{V}_+ = \{t \in [0,1]^d : |s(t)| > 1\}, \mathbb{V}_- = \{t \in [0,1]^d : |s(t)| \leq 1\},$$
$$\mathcal{P}_{h,s,\psi}(\mathbb{R}^n) = \{u \in \mathcal{P}_b(\mathbb{R}^n) : u(s^{-1}(t)\cdot) = u(\cdot), \textit{for almost everywhere } t \in supp(\psi)\}.$$

In 2011, Tang et al. [28] obtained a sufficient condition on $\psi(t)$ for the boundedness of $U^b_{\psi,i_1,1}$ on the Morrey-Herz spaces with constant exponents when the symbol b belongs to a Lipschitz space. In more detail, they proved the following result.

Theorem 3.1. [28, Theorem 1.1] *Let $\psi : [0,1] \to [0,\infty)$ be a measurable function, $\beta \in (0,1)$, $b \in \mathrm{Lip}^{\beta}(\mathbb{R}^n)$, $1 \le q_2 \le q_1 < \infty$, $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\lambda \in [0,\infty)$. If*

$$\mathcal{B} = \int_0^1 \psi(t) t^{-(\alpha_1-\lambda-n/q_1)} dt < \infty,$$

then $U^b_{\psi,i_1,1}$ is bounded from $M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{R}^n)$ to $M\dot{K}^{\alpha_2,\lambda}_{p,q_2}(\mathbb{R}^n)$, where $\alpha_1 = \alpha_2 + \beta + n(1/q_2 - 1/q_1)$.

Accordingly, the authors [7] give a sufficient condition on $\psi(t)$ and $s(t)$ such that $U^b_{\psi,s,1}$ is bounded on the weighted Morrey-Herz spaces.

By considering all weights to be constant, Theorem 4.2 in [18] reduces the below theorem.

Theorem 3.2. *Let $q, q_1, r_1 \in (1,\infty)$ and $\lambda \in (-1/q_1, 0)$ such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{r_1}$.*

(i) *If both $\mathcal{C} = \int_{[0,1]^d} \psi(t)|s(t)|^{n\lambda} dt$ and $\mathcal{D} = \int_{[0,1]^d} \psi(t)|s(t)|^{n\lambda} |\log|s(t)|| dt$ are finite then for any $b \in C\dot{M}O^{r_1}(\mathbb{R}^n)$ then $U^b_{\psi,s,d}$ is bounded from $\dot{M}^{q_1,\lambda}(\mathbb{R}^n)$ to $\dot{M}^{q,\lambda}(\mathbb{R}^n)$.*

(ii) *If for any $b \in C\dot{M}O^{r_1}(\mathbb{R}^n)$, $U^b_{\psi,s,d}$ is bounded from $\dot{M}^{q_1,\lambda}(\mathbb{R}^n)$ to $\dot{M}^{q,\lambda}(\mathbb{R}^n)$, then $\mathcal{D}$ is finite.*

Very recently, Dung and Thuy [11] consider the commutator of the Hausdorff operator, which is generalized of the commutator of Hardy-Cesàro operator. The authors establish the boundedness for the commutators of Hausdorff operators on the weighted Herz-type Hardy spaces with symbols in central BMO spaces and Lipschitz spaces.

As a natural development, we need to study the boundedness of commutators of Hardy-Cesàro operators with symbols in central BMO spaces with variable exponent on some function spaces. Theorems 3.3–3.5 partially solved the above problem.

Our first main result is presented as follows.

Theorem 3.3. *Let $q, q_1 \in \mathcal{P}_{h,s,\psi}(\mathbb{R}^n)$, $r_1 \in \mathcal{P}_{h,s,\psi}(\mathbb{R}^n) \cap \mathfrak{B}(\mathbb{R}^n)$, $\lambda, \lambda_1 \in \mathbb{R}$ such that*

$$\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{r_1(\cdot)}, \tag{3.1}$$

$$\lambda_1 = \lambda + \frac{1}{q_-} - \frac{1}{r_{1+}} - \frac{1}{q_{1+}}. \tag{3.2}$$

If $b \in C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)$ and

$$\mathcal{A}_1 = \int_{[0,1]^d} \psi(t) \mathcal{K}_{q_1,max}(t) \phi(t) |s(t)|^{n\lambda_1} \max\{|s(t)|^{n/q_{1+}}, |s(t)|^{n/q_{1-}}\} dt < \infty,$$

where $\phi(t) = 1 + \log_2|2s(t)|\chi_{\mathbb{V}_+}(t) + \log_2\left|\frac{1}{s(t)}\right|\chi_{\mathbb{V}_-}(t) + \mathcal{K}_{r_1,max}(t)\big(|s(t)|^n\chi_{\mathbb{V}_+}(t) + \chi_{\mathbb{V}_-(t)}\big)$, then $U^b_{\psi,s,d}$ is bounded from $\dot{B}^{q_1(\cdot),\lambda_1}_{\mathrm{loc}}(\mathbb{R}^n)$ to $\dot{B}^{q(\cdot),\lambda}_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover,

$$\|U^b_{\psi,s,d}\|_{\dot{B}^{q_1(\cdot),\lambda_1}_{\mathrm{loc}}(\mathbb{R}^n) \to \dot{B}^{q(\cdot),\lambda}_{\mathrm{loc}}(\mathbb{R}^n)} \lesssim \mathcal{A}_1 \|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}.$$

Proof. Firstly, we prove the following inequality

$$\|U^b_{\psi,s,d}(f)\|_{L^{q(\cdot)}(B_k)} \lesssim \mathbb{U}.\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)} \|1\|_{L^{r_1(\cdot)}(B_k)}, \tag{3.3}$$

for any $k \in \mathbb{Z}$, where $m-1 < \log_2|s(t)| \le m$ and $\mathbb{U} = \int_{[0,1]^d} \psi(t)\mathcal{K}_{q_1,max}(t)\phi(t)\|f\|_{L^{q_1(\cdot)}(B_{k+m})}dt.$

Indeed, by using the Minkowski inequality and the Hölder inequality,

$$\left\|U^b_{\psi,s,d}(f)\right\|_{L^{q(\cdot)}(B_k)} \lesssim \int\limits_{[0,1]^d} \psi(t)\|b(\cdot)-b(s(t)\cdot)\|_{L^{r_1(\cdot)}(B_k)}\|f(s(t)\cdot)\|_{L^{q_1(\cdot)}(B_k)}dt. \tag{3.4}$$

Next, it is easy to see that

$$\begin{aligned}\|b(\cdot)-b(s(t)\cdot)\|_{L^{r_1(\cdot)}(B_k)} &\le \|b(\cdot)-b_{B_k}\|_{L^{r_1(\cdot)}(B_k)} + \|b_{B_k}-b_{B_{k+m}}\|_{L^{r_1(\cdot)}(B_k)} + \|b(s(t)\cdot)-b_{B_{k+m}}\|_{L^{r_1(\cdot)}(B_k)}\\ &:= J_1+J_2+J_3.\end{aligned} \tag{3.5}$$

From the definition of the space $C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)$, it follows

$$J_1 \le \|1\|_{L^{r_1(\cdot)}(B_k)}\left\|b\right\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}. \tag{3.6}$$

On the other hand, by the Hölder inequality and Lemma 2.2.ii, for all $k \in \mathbb{Z}$,

$$\begin{aligned}\left|b_{B_k}-b_{B_{k+1}}\right| &\le \frac{1}{|B_k|}\int\limits_{B_k}\left|b(x)-b_{B_{k+1}}\right|dx \lesssim \frac{\|1\|_{L^{r_1'(\cdot)}(B_{k+1})}}{|B_{k+1}|}\|b-b_{B_{k+1}}\|_{L^{r_1(\cdot)}(B_{k+1})}\\ &\le \frac{\|1\|_{L^{r_1'(\cdot)}(B_{k+1})}\|1\|_{L^{r_1(\cdot)}(B_{k+1})}}{|B_{k+1}|}\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)} \lesssim \|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

If $m \ge 1$ then

$$\left|b_{B_k}-b_{B_{k+m}}\right| \le \left|b_{B_k}-b_{B_{k+1}}\right| + \cdots + \left|b_{B_{k+m-1}}-b_{B_{k+m}}\right| \lesssim m\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)} \le \log_2|2s(t)|.\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}.$$

Otherwise,

$$\left|b_{B_k}-b_{B_{k+m}}\right| \lesssim -m\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)} \le \log_2\left|\frac{1}{s(t)}\right|\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}.$$

Thus

$$J_2 \le \|1\|_{L^{r_1(\cdot)}(B_k)}\left|b_{B_k}-b_{B_{k+m}}\right| \lesssim \|1\|_{L^{r_1(\cdot)}(B_k)}\Big(\log_2|2s(t)|\chi_{\mathbb{V}_+}(t)+\log_2\left|\frac{1}{s(t)}\right|\chi_{\mathbb{V}_-}(t)\Big)\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}. \tag{3.7}$$

Now, we will estimate J_3. By using the formula for change of variables,

$$\begin{aligned}F_{r_1}\Big((b(s(t)\cdot)-b_{B_{k+m}})\chi_{B_k}/\eta\Big) &= \int\limits_{B_k}\Big(\frac{|b(s(t)x)-b_{B_{k+m}}|}{\eta}\Big)^{r_1(x)}dx = \int\limits_{s(t)B_k}\Big(\frac{|b(z)-b_{B_{k+m}}|}{\eta}\Big)^{r_1(z)}|s(t)|^{-n}dz\\ &\le \int\limits_{B_{k+m}}\Big(\max\{|s(t)|^{-n/r_{1+}},|s(t)|^{-n/r_{1-}}\}\frac{|b(z)-b_{B_{k+m}}|}{\eta}\Big)^{r_1(z)}dz.\end{aligned}$$

This gives $J_3 \le \mathcal{K}_{r_1,max}(t)\|b(\cdot)-b_{B_{k+m}}\|_{L^{r_1(\cdot)}(B_{k+m})} \le \|1\|_{L^{r_1(\cdot)}(B_k)}\mathcal{K}_{r_1,max}(t)\frac{\|1\|_{L^{r_1(\cdot)}(B_{k+m})}}{\|1\|_{L^{r_1(\cdot)}(B_k)}}\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}$. Besides, by using Lemma 2.2.i, there exists a positive constant $\delta \in (0,1)$ such that

$$\frac{\|1\|_{L^{r_1(\cdot)}(B_{k+m})}}{\|1\|_{L^{r_1(\cdot)}(B_k)}} \lesssim \begin{cases} \dfrac{|B_{k+m}|}{|B_k|}, \text{ if } m \ge 1 \\ \Big(\dfrac{|B_{k+m}|}{|B_k|}\Big)^{\delta}, \text{ otherwise} \end{cases} \lesssim \begin{cases} 2^{mn}, \text{ if } m \ge 1 \\ 2^{mn\delta}, \text{ otherwise.} \end{cases}$$

From these,

$$J_3 \lesssim \|1\|_{L^{r_1(\cdot)}(B_k)}\mathcal{K}_{r_1,max}(t)\Big(|s(t)|^n\chi_{\mathbb{V}_+}(t) + \chi_{\mathbb{V}_-}(t)\Big)\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}. \tag{3.8}$$

On the other hand,

$$\begin{aligned} F_{q_1}\Big(f(s(t)\cdot)\chi_{B_k}/\eta\Big) &= \int_{B_k}\Big(\frac{|f(s(t)x)|}{\eta}\Big)^{q_1(x)}dx = \int_{s(t)B_k}\Big(\frac{|f(z)|}{\eta}\Big)^{q_1(z)}|s(t)|^{-n}dz \\ &\le \int_{B_{k+m}}\Big(\max\{|s(t)|^{-n/q_{1+}}, |s(t)|^{-n/q_{1-}}\}\frac{|f(z)|}{\eta}\Big)^{q_1(z)}dz. \end{aligned}$$

Thus $\|f(s(y)\cdot)\|_{L^{q_1(\cdot)}(B_k)} \le \mathcal{K}_{q_1,max}(t)\|f\|_{L^{q_1(\cdot)}(B_{k+m})}$. Hence, together with inequalities (3.4) and (3.6)–(3.8), we obtain that the inequality (3.3) holds.

For any $k \le 0$ and $k \in \mathbb{Z}$, by (3.3),

$$\frac{\|U^b_{\psi,s,d}(f)\|_{L^{q(\cdot)}(B_k)}}{|B_k|^{\lambda}\|1\|_{L^{q(\cdot)}(B_k)}} \lesssim \|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\Bigg(\int_{[0,1]^d}\psi(t)\mathcal{K}_{q_1,max}(t)\phi(t)\eta_{k,m}dt\Bigg)\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}_{\rm loc}(\mathbb{R}^n)},$$

where $\eta_{k,m} = \dfrac{\|1\|_{L^{r_1(\cdot)}(B_k)}|B_{k+m}|^{\lambda_1}\|1\|_{L^{q_1(\cdot)}(B_{k+m})}}{|B_k|^{\lambda}\|1\|_{L^{q(\cdot)}(B_k)}}$. By (3.2), $2^{m-1} < |s(t)| \le 2^m$ and (2.1),

$$\begin{aligned} \eta_{k,m} &\lesssim \frac{2^{\max\{kn/r_{1+},kn/r_{1-}\}}2^{(k+m)n\lambda_1}2^{\max\{(k+m)n/q_{1+},(k+m)n/q_{1-}\}}}{2^{kn\lambda}2^{\min\{kn/q_+,kn/q_-\}}} \\ &\le 2^{k\big(\min\{n/r_{1+},n/r_{1-}\}+n\lambda_1+\min\{n/q_{1+},n/q_{1-}\}-n\lambda-\max\{n/q_+,n/q_-\}\big)}2^{mn\lambda_1}2^{\max\{mn/q_{1+},mn/q_{1-}\}} \\ &\lesssim |s(t)|^{n\lambda_1}\max\{|s(t)|^{n/q_{1+}}, |s(t)|^{n/q_{1-}}\}. \end{aligned}$$

Hence,

$$\|U^b_{\psi,s,d}(f)\|_{\dot{B}^{q(\cdot),\lambda}_{\rm loc}(\mathbb{R}^n)} \lesssim \mathcal{A}_1\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}_{\rm loc}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 3.3. □

Let us give the boundedness for the commutators of Hardy-Cesàro operators on the Morrey-Herz spaces with variable exponents.

Theorem 3.4. *Let $p, \lambda \in (0,\infty)$, $q, q_1 \in \mathcal{P}_{h,s,\psi}(\mathbb{R}^n)$, $r_1 \in \mathcal{P}_{h,s,\psi}(\mathbb{R}^n) \cap \mathfrak{B}(\mathbb{R}^n)$, $\alpha, \alpha_1 \in L^{\infty}(\mathbb{R}^n) \cap \mathbf{C}^{log}_0(\mathbb{R}^n) \cap \mathbf{C}^{log}_{\infty}(\mathbb{R}^n)$ such that*

$$\alpha(0) \ge \max\{\frac{-n}{r_{1+}} + \alpha_1(0), \frac{-n}{r_{1+}} + \alpha_{1\infty}\}, \tag{3.9}$$

$$\alpha_\infty \leq \min\{\frac{-n}{r_{1-}} + \alpha_1(0), \frac{-n}{r_{1-}} + \alpha_{1\infty}\}. \tag{3.10}$$

Assume that $b \in C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)$ *and the condition* (3.1) *in Theorem 3.3 holds. If*

$$\mathcal{A}_2 = \int_{[0,1]^d} \psi(t)\mathcal{K}_{q_1,max}(t)\phi(t)\max\{|s(t)|^{\lambda-\alpha_1(0)}, |s(t)|^{\lambda-\alpha_{1\infty}}\}dt < \infty,$$

where $\phi(t)$ *is given in Theorem 3.3, then* $U^b_{\psi,s,d}$ *is bounded from* $M\dot{K}^{\alpha_1(\cdot),\lambda}_{p,q_1(\cdot)}(\mathbb{R}^n)$ *to* $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$*. Moreover,*

$$\|U^b_{\psi,s,d}\|_{M\dot{K}^{\alpha_1(\cdot),\lambda}_{p,q_1(\cdot)}(\mathbb{R}^n)\to M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} \lesssim \mathcal{A}_2\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}.$$

Proof. By estimating as in (3.3) and using (2.1), we have

$$\begin{aligned}
&\|U^b_{\psi,s,d}(f)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\|1\|_{L^{r_1(\cdot)}(B_k)}\int_{[0,1]^d}\psi(t)\mathcal{K}_{q_1,max}(t)\phi(t)\Big(\|f\chi_{k+m-1}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} + \|f\chi_{k+m}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)dt \\
&\lesssim 2^{\rho_1}\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\int_{[0,1]^d}\psi(t)\mathcal{K}_{q_1,max}(t)\phi(t)\Big(\|f\chi_{k+m-1}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} + \|f\chi_{k+m}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)dt. \qquad (3.11)
\end{aligned}$$

for any $k \in \mathbb{Z}$, where $m-1 < \log_2|s(t)| \leq m$ and $\rho_1 = \max\{kn/r_{1+}, kn/r_{1-}\}$.

On the other hand, by Lemma 2.1,

$$\begin{aligned}
\Big\|f\chi_{k+m-1}\Big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} &\lesssim 2^{\max\{(k+m-1)(\lambda-\alpha_1(0)),(k+m-1)(\lambda-\alpha_{1\infty})\}}\|f\|_{M\dot{K}^{\alpha_1(\cdot),\lambda}_{p,q_1(\cdot)}(\mathbb{R}^n)} \\
&= 2^{\rho_2}\max\{2^{(m-1)(\lambda-\alpha_1(0))}, 2^{(m-1)(\lambda-\alpha_{1\infty})}\}\|f\|_{M\dot{K}^{\alpha_1(\cdot),\lambda}_{p,q_1(\cdot)}(\mathbb{R}^n)} \\
&\lesssim 2^{\rho_2}\max\Big\{|s(t)|^{\lambda-\alpha_1(0)}, |s(t)|^{\lambda-\alpha_{1\infty}}\Big\}\|f\|_{M\dot{K}^{\alpha_1(\cdot),\lambda}_{p,q_1(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

where $\rho_2 := \max\{k(\lambda-\alpha_1(0)), k(\lambda-\alpha_{1\infty})\}$.

Similarly,

$$\Big\|f\chi_{k+m}\Big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \lesssim 2^{\rho_2}\max\{|s(t)|^{\lambda-\alpha_1(0)}, |s(t)|^{\lambda-\alpha_{1\infty}}\}\|f\|_{M\dot{K}^{\alpha_1(\cdot),\lambda}_{p,q_1(\cdot)}(\mathbb{R}^n)}.$$

Thus, by (3.11),

$$\Big\|U^b_{\psi,s,d}(f)\chi_k\Big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \lesssim 2^{\rho_1+\rho_2}\mathcal{A}_2\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha_1(\cdot),\lambda}_{p,q_1(\cdot)}(\mathbb{R}^n)}. \tag{3.12}$$

Note that,

$$\rho_1 + \rho_2 = \begin{cases} k(\frac{n}{r_{1+}} + \min\{\lambda-\alpha_1(0), \lambda-\alpha_{1\infty}\}), \text{ if } k < 0, \\ k(\frac{n}{r_{1-}} + \max\{\lambda-\alpha_1(0), \lambda-\alpha_{1\infty}\}), \text{ otherwise.} \end{cases} \tag{3.13}$$

Now, by applying Proposition 2.5 in [25], we compose

$$\Big\|U^b_{\psi,s,d}(f)\Big\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} \lesssim \max\Big\{\sup_{k_0\in\mathbb{Z}^-}\mathcal{U}_1, \sup_{k_0\in\mathbb{N}}(\mathcal{U}_2+\mathcal{U}_3)\Big\}. \tag{3.14}$$

where

$$\mathcal{U}_1 = 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p}\left\|U^b_{\psi,s,d}(f)\chi_k\right\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{1/p}, \mathcal{U}_2 = 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p}\left\|U^b_{\psi,s,d}(f)\chi_k\right\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{1/p},$$

$$\mathcal{U}_3 = 2^{-k_0\lambda}\left(\sum_{k=0}^{k_0} 2^{k\alpha_\infty p}\left\|U^b_{\psi,s,d}(f)\chi_k\right\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{1/p}.$$

By using (3.12) and (3.13),

$$\begin{aligned}\mathcal{U}_1 &\leq \mathcal{A}_2\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p}2^{(\rho_1+\rho_2)p}\Big)^{1/p}\left\|f\right\|_{M\dot{K}^{\alpha_1(\cdot),\lambda}_{p,q_1(\cdot)}(\mathbb{R}^n)}\\ &= \mathcal{A}_2\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{kp.\min\{\frac{n}{r_{1+}}+\lambda-\alpha_1(0)+\alpha(0),\frac{n}{r_{1+}}+\lambda-\alpha_{1\infty}+\alpha(0)\}}\Big)^{1/p}\left\|f\right\|_{M\dot{K}^{\alpha_1(\cdot),\lambda}_{p,q_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

In the view of the inequality $\min\{\frac{n}{r_{1+}}+\lambda-\alpha_1(0)+\alpha(0),\frac{n}{r_{1+}}+\lambda-\alpha_{1\infty}+\alpha(0)\} > 0$, it is clear to see that

$$\mathcal{U}_1 \lesssim \mathcal{A}_2 2^{k_0\min\{\frac{n}{r_{1+}}-\alpha_1(0)+\alpha(0),\frac{n}{r_{1+}}-\alpha_{1\infty}+\alpha(0)\}}\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha_1(\cdot),\lambda}_{p,q_1(\cdot)}(\mathbb{R}^n)}. \tag{3.15}$$

By applying an argument similar to the above, we also have

$$\mathcal{U}_2 \lesssim \mathcal{A}_2 2^{-k_0\lambda}\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha_1(\cdot),\lambda}_{p,q_1(\cdot)}(\mathbb{R}^n)}. \tag{3.16}$$

Next, by (3.12) and (3.13), $\mathcal{U}_3$ is estimated as follows.

$$\begin{aligned}\mathcal{U}_3 &\lesssim \mathcal{A}_2 2^{-k_0\lambda}\Big(\sum_{k=0}^{k_0} 2^{k\alpha_\infty p}2^{(\rho_1+\rho_2)p}\Big)^{1/p}\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha_1(\cdot),\lambda}_{p,q_1(\cdot)}(\mathbb{R}^n)}\\ &= \mathcal{A}_2 2^{-k_0\lambda}\Big(\sum_{k=0}^{k_0} 2^{kp.\max\{\frac{n}{r_{1-}}+\lambda-\alpha_1(0)+\alpha_\infty,\frac{n}{r_{1-}}+\lambda-\alpha_{1\infty}+\alpha_\infty\}}\Big)^{1/p}\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\left\|f\right\|_{M\dot{K}^{\alpha_1(\cdot),\lambda}_{p,q_1(\cdot)}(\mathbb{R}^n)}\\ &\lesssim \mathcal{A}_2\left(2^{k_0\max\{\frac{n}{r_{1-}}-\alpha_1(0)+\alpha_\infty,\frac{n}{r_{1-}}-\alpha_{1\infty}+\alpha_\infty\}}+2^{-k_0\lambda}\right)\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha_1(\cdot),\lambda}_{p,q_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Hence, by (3.14)–(3.16), $\lambda > 0$, (3.9) and (3.10), we obtain

$$\|U^b_{\psi,s,d}\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} \lesssim \mathcal{A}_2\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha_1(\cdot),\lambda}_{p,q_1(\cdot)}(\mathbb{R}^n)}.$$

Therefore, the proof of Theorem 3.4 is completed.

Let us state the central BMO space with variable exponent estimate for $U^b_{\psi,s,d}$ on the Herz spaces with variable exponents.

Theorem 3.5. *Let* $p \in (0,1]$, $q, q_1 \in \mathcal{P}_{h,s,\psi}(\mathbb{R}^n)$, $r_1 \in \mathcal{P}_{h,s,\psi}(\mathbb{R}^n)\cap\mathfrak{B}(\mathbb{R}^n)$, $\alpha \in L^\infty(\mathbb{R}^n)\cap\mathbf{C}^{log}_0(\mathbb{R}^n)\cap\mathbf{C}^{log}_\infty(\mathbb{R}^n)$ *with* $\alpha(0) = \alpha_\infty \in (0,\infty)$. *Assume that* $\omega = |x|^\beta$, $v = |x|^\zeta$ *with* $\beta, \zeta \in (-n,\infty)$ *and* $\zeta = \beta - \frac{n^2}{\alpha_\infty r_{1-}}$. *Let*

$b \in C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)$ *and the condition* (3.1) *in Theorem 3.3 holds.*

(i) *If* $p = 1$ *and*

$$\mathcal{A}_3 = \int_{[0,1]^d} \psi(t)\mathcal{K}_{q_1,max}(t)\phi(t)K_{r_1,max}(t)|s(t)|^{\xi}dt < \infty,$$

where $\xi = n/r_{1-} - \alpha_\infty - \beta\alpha_\infty/n$ *and* $\phi(t)$ *is given in Theorem 3.3, then*

$$\|U^b_{\psi,s,d}\|_{K^{\alpha(\cdot),1}_{q_1(\cdot)}(\omega)\to \dot{K}^{\alpha(\cdot),1}_{q(\cdot)}(v)} \lesssim \mathcal{A}_3\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}.$$

(ii) *If* $0 < p < 1$, $\sigma > (1-p)/p$ *and*

$$\mathcal{A}_4 = \int_{[0,1]^d} \psi(t)\mathcal{K}_{q_1,max}(t)\phi(t)K_{r_1,max}(t)|s(t)|^{\xi}\Big((\log_2|s(t)|+1)^{\sigma}\chi_{\mathbb{V}_+}(t) + \big|\log_2|s(t)|\big|^{\sigma}\chi_{\mathbb{V}_-}(t)\Big)dt < \infty,$$

then

$$\|U^b_{\psi,s,d}\|_{K^{\alpha(\cdot),p}_{q_1(\cdot)}(\omega)\to \dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(v)} \lesssim \mathcal{A}_4\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}.$$

Proof. Let $f \in K^{\alpha(\cdot),p}_{q_1(\cdot)}(\omega)$. By Theorem 2.3, we compose

$$f = \sum_{k=0}^{\infty}\lambda_k b_k,$$

where $\Big(\sum\limits_{k=0}^{\infty}|\lambda_k|^p\Big)^{1/p} \lesssim \|f\|_{K^{\alpha(\cdot),p}_{q_1(\cdot)}(\omega)}$, and for $k \in \mathbb{N}$, b_k is a central $(\alpha(\cdot), q_1(\cdot), \omega)$-block of restricted type such that $\operatorname{supp}(b_k) \subset B_k$ and $\|b_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \lesssim \omega(B_k)^{-\alpha_\infty/n}$.

Consequently,

$$|U^b_{\psi,s,d}(f)(x)| \le \sum_{k=0}^{\infty}|\lambda_k|\widetilde{U}^b_{\psi,s,d}(b_k)(x) \text{ with } \widetilde{U}^b_{\psi,s,d}(b_k)(x) = \int_{[0,1]^d}\psi(t)|.|b(x) - b(s(t)x)|.|b_k(s(t)x)|dt.$$

We remark that the following inequality is true.

$$\|\widetilde{U}^b_{\psi,s,d}(b_k)\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(v)} \lesssim \begin{cases} \mathcal{A}_3\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}, & p = 1, \\ \mathcal{A}_4\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}, & p \in (0,1) \text{ and } \sigma > (1-p)/p. \end{cases} \tag{3.17}$$

Indeed, we consider

$$\widetilde{U}^b_{\psi,s,d}(b_k)(x) = \sum_{j\in\mathbb{Z}} u^b_{\psi,s,d,j}(b_k)(x), \text{ where } u^b_{\psi,s,d,j}(b_k)(x) = \int_{V_j}\psi(t)|b(x) - b(s(t)x)||b_k(s(t)x)|dt,$$

with $V_j = \{t \in [0,1]^d : 2^{-j} < |s(t)| \le 2^{-j+1}\}$. Combining this with $\operatorname{supp}(b_k) \subset B_k$,

$$\operatorname{supp}(u^b_{\psi,s,d,j}(b_k)) \subset B_{k+j}. \tag{3.18}$$

Note that,

$$\alpha_\ell = \alpha(0) = \alpha_\infty, \text{ for all } \ell \in \mathbb{Z}. \tag{3.19}$$

Thus, by estimating as in (3.3),

$$\begin{aligned}\|u^b_{\psi,s,d,j}(b_k)\|_{L^{q(\cdot)}(B_{k+j})} &\lesssim \|1\|_{L^{r_1(\cdot)}(B_{k+j})}\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\int_{V_j}\psi(t)\mathcal{K}_{q_1,max}(t)\phi(t)\|b_k\|_{L^{q_1(\cdot)}(B_{k+1})}dt\\ &\lesssim 2^{\rho_3}\omega(B_{k+j})^{-\alpha_{k+j}/n}\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\int_{V_j}\psi(t)\mathcal{K}_{q_1,max}(t)\phi(t)\Big(\frac{\omega(B_{k+j})}{\omega(B_k)}\Big)^{\alpha_k/n}dt,\end{aligned}$$

with $\rho_3 = \max\{(k+j)n/r_{1+}, (k+j)n/r_{1-}\}$. From the definition of ρ_3 and $k \in \mathbb{N}$,

$$\begin{aligned}2^{\rho_3} &\lesssim \max\{2^{kn/r_{1+}}, 2^{kn/r_{1-}}\}\max\{2^{jn/r_{1+}}, 2^{jn/r_{1-}}\}\\ &\lesssim 2^{kn/r_{1-}}\max\{|s(t)|^{-n/r_{1+}}, |s(t)|^{-n/r_{1-}}\}\\ &= 2^{(k+j)n/r_{1-}}\mathcal{K}_{r_1,max}(t)|s(t)|^{n/r_{1-}},\end{aligned}$$

for all $k \in \mathbb{N}$. Besides, by the relation (3.19) and $\omega(x) = |x|^\beta$, $v(x) = |x|^\zeta$ with $\zeta = \beta - \frac{n^2}{\alpha_\infty r_{1-}}$,

$$2^{\frac{(k+j)n}{r_{1-}}}\omega(B_{k+j})^{-\alpha_{k+j}/n} \simeq v(B_{k+j})^{-\alpha_{k+j}/n} \text{ and } \Big(\frac{\omega(B_{k+j})}{\omega(B_k)}\Big)^{\alpha_k/n} \simeq |s(t)|^{-\alpha_\infty(n+\beta)/n}.$$

These lead to that

$$\|u^b_{\psi,s,d,j}(b_k)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \lesssim u_j\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}v(B_{k+j})^{-\alpha_{k+j}/n}. \tag{3.20}$$

Here

$$u_j = \int_{V_j}\psi(t)\mathcal{K}_{q_1,max}(t)\phi(t)K_{r_1,max}(t)|s(t)|^\xi dt.$$

By setting

$$g^b_{\psi,s,d,j}(b_k) = \begin{cases}\dfrac{u^b_{\psi,s,d,j}(b_k)}{u_j\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}}, & \text{if } u_j\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)} \neq 0,\\ 0, & \text{otherwise.}\end{cases}$$

Thus

$$\widetilde{U}^p_{\psi,s,d}(b_k) = \sum_{j\in\mathbb{Z}} u_j\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}g^b_{\psi,s,d,j}(b_k).$$

By (3.18) and (3.20), for $j \in \mathbb{Z}$, the function $g^b_{\psi,s,d,j}(b_k)$ is a central $(\alpha(\cdot), q(\cdot), v)$-block. Combining this with Theorem 2.2,

$$\|\widetilde{U}^b_{\psi,s,d}(b_k)\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(v)} \lesssim \|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\Big(\sum_{j\in\mathbb{Z}}|u_j|^p\Big)^{1/p}.$$

Case 1: $p = 1$, we have $\sum_{j\in\mathbb{Z}}|u_j| = \mathcal{A}_3$.

Case 2: $p \in (0,1)$ and $\sigma > (1-p)/p$, by the Hölder inequality,

$$\begin{aligned}\Big(\sum_{j\in\mathbb{Z}}|u_j|^p\Big)^{1/p} &\lesssim \sum_{j\in\mathbb{Z}\setminus\{0\}}|j|^\sigma|u_j| + u_0\\ &\lesssim \sum_{j\in\mathbb{Z}}\int_{V_j}\psi(t)\mathcal{K}_{q_1,max}(t)\phi(t)K_{r_1,max}(t)|s(t)|^\xi\Big((\log_2|s(t)|+1)^\sigma\chi_{\mathbb{V}_+}(t) + \big|\log_2|s(t)|\big|^\sigma\chi_{\mathbb{V}_-}(t)\Big)dt\end{aligned}$$

$$= \mathcal{A}_4.$$

From these, the inequality (3.17) is achieved. Hence,

$$\|U^b_{\psi,s,d}(f)\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(v)} \leq \Big(\sum_{k=0}^{\infty}|\lambda_k|^p\|\widetilde{U}^b_{\psi,s,d}(b_k)\|^p_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(v)}\Big)^{1/p} \lesssim \begin{cases} \mathcal{A}_3\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\|f\|_{K^{\alpha(\cdot),p}_{q_1(\cdot)}(\omega)}, \ p=1, \\ \mathcal{A}_4\|b\|_{C\dot{M}O^{r_1(\cdot)}(\mathbb{R}^n)}\|f\|_{K^{\alpha(\cdot),p}_{q_1(\cdot)}(\omega)}, \ p\in(0,1), \\ \text{and}\ \sigma > (1-p)/p. \end{cases}$$

As a consequence, the proof of this theorem is obtained. □

When all of $q(\cdot), q_1(\cdot), r_1(\cdot), \alpha(\cdot)$ and $\alpha_1(\cdot)$ are constant, we give the sufficient and necessary conditions for the boundedness of $U^b_{\psi,s,d}$ on the local central Morrey spaces $\dot{M}^{q_1,\lambda}_{\rm loc}(\mathbb{R}^n)$ and Morrey-Herz spaces $M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{R}^n)$.

Theorem 3.6. *Let $\mathbb{V}_+$ be a null set, $q, q_1, r_1 \in (1,\infty)$ and $\lambda\in\mathbb{R}$ such that*

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{r_1}. \tag{3.21}$$

(i) *If $b \in C\dot{M}O^{r_1}(\mathbb{R}^n)$ and*

$$\mathcal{A}_1^* = \int_{[0,1]^d}\psi(t)\Big(1+\log_2\Big|\frac{1}{s(t)}\Big| + |s(t)|^{-n/r_1}\Big)|s(t)|^{n\lambda}dt < \infty,$$

then $U^b_{\psi,s,d}$ is bounded from $\dot{M}^{q_1,\lambda}_{\rm loc}(\mathbb{R}^n)$ to $\dot{M}^{q,\lambda}_{\rm loc}(\mathbb{R}^n)$. Moreover,

$$\|U^b_{\psi,s,d}\|_{\dot{M}^{q_1,\lambda}_{\rm loc}(\mathbb{R}^n)\to\dot{M}^{q,\lambda}_{\rm loc}(\mathbb{R}^n)} \lesssim \mathcal{A}_1^*\|b\|_{C\dot{M}O^{r_1}(\mathbb{R}^n)}.$$

(ii) *Conversely, if $U^b_{\psi,s,d}$ is bounded from $\dot{M}^{q_1,\lambda}_{\rm loc}(\mathbb{R}^n)$ to $\dot{M}^{q,\lambda}_{\rm loc}(\mathbb{R}^n)$ for all $b\in C\dot{M}O^{r_1}(\mathbb{R}^n)$, then*

$$\mathcal{A}_1^{**} = \int_{[0,1]^d}\psi(t)\log_2\Big|\frac{1}{s(t)}\Big||s(t)|^{n\lambda}dt \lesssim \|U^b_{\psi,s,d}\|_{\dot{M}^{q_1,\lambda}_{\rm loc}(\mathbb{R}^n)\to\dot{M}^{q,\lambda}_{\rm loc}(\mathbb{R}^n)}.$$

Proof. (i) By Theorem 3.3 with the null set $\mathbb{V}_+$, we immediately complete the proof of part (i).

(ii) Conversely, suppose $U^b_{\psi,s,d}$ is bounded from $\dot{M}^{q_1,\lambda}_{\rm loc}(\mathbb{R}^n)$ to $\dot{M}^{q,\lambda}_{\rm loc}(\mathbb{R}^n)$ for all $b\in C\dot{M}O^{r_1}(\mathbb{R}^n)$. We choose

$$b(x) = \log_2|x| \text{ and } f(x) = |x|^{n\lambda}.$$

From Example 7.1.3 in [17], $b\in C\dot{M}O^{r_1}(\mathbb{R}^n)$. Besides, $\|f\|_{\dot{M}^{q_1,\lambda}_{\rm loc}(\mathbb{R}^n)} \simeq \sup\limits_{k\leq 0 \text{ and } k\in\mathbb{Z}}\dfrac{2^{k(n\lambda+n/q_1)}}{2^{kn(\lambda+1/q_1)}} = 1$. By a similar argument, we also have $\|f\|_{\dot{M}^{q,\lambda}_{\rm loc}(\mathbb{R}^n)} \simeq \|f\|_{\dot{M}^{q_1,\lambda}_{\rm loc}(\mathbb{R}^n)}$. Moreover, by choosing b and f as above,

$$U^b_{\psi,s,d}(f)(x) = \mathcal{A}_1^{**}f(x).$$

Thus,

$$\|U^b_{\psi,s,d}\|_{\dot{M}^{q_1,\lambda}_{\rm loc}(\mathbb{R}^n)\to\dot{M}^{q,\lambda}_{\rm loc}(\mathbb{R}^n)} \geq \frac{\|U^b_{\psi,s,d}(f)\|_{\dot{M}^{q,\lambda}_{\rm loc}(\mathbb{R}^n)}}{\|f\|_{\dot{M}^{q_1,\lambda}_{\rm loc}(\mathbb{R}^n)}} = \mathcal{A}_1^{**}\frac{\|f\|_{\dot{M}^{q,\lambda}_{\rm loc}(\mathbb{R}^n)}}{\|f\|_{\dot{M}^{q_1,\lambda}_{\rm loc}(\mathbb{R}^n)}} \simeq \mathcal{A}_1^{**},$$

which finishes the proof of part (ii).

Theorem 3.7. *Let $\mathbb{V}_+$ be a null set, $p, \lambda \in (0,\infty)$, $q, q_1, r_1 \in (1,\infty)$, $\alpha, \alpha_1 \in \mathbb{R}$ such that $\alpha = -n/r_1 + \alpha_1$. Assume that the condition* (3.21) *in Theorem 3.6 holds.*

(i) *If $b \in C\dot{M}O^{r_1}(\mathbb{R}^n)$ and*

$$\mathcal{A}_2^* = \int_{[0,1]^d} \psi(t)\Big(1 + \log_2 \left|\frac{1}{s(t)}\right| + |s(t)|^{-n/r_1}\Big)|s(t)|^{\lambda-\alpha_1-n/q_1} dt < \infty,$$

then $U^b_{\psi,s,d}$ is bounded from $M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{R}^n)$ to $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$. Moreover,

$$\|U^b_{\psi,s,d}\|_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{R}^n)\to M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \lesssim \mathcal{A}_2^* \|b\|_{C\dot{M}O^{r_1}(\mathbb{R}^n)}.$$

(ii) *Conversely, if $U^b_{\psi,s,d}$ is bounded from $M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{R}^n)$ to $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ for all $b \in C\dot{M}O^{r_1}(\mathbb{R}^n)$, then*

$$\mathcal{A}_2^{**} = \int_{[0,1]^d} \psi(t)\log_2 \left|\frac{1}{s(t)}\right| |s(t)|^{\lambda-\alpha_1-n/q_1} dt \lesssim \|U^b_{\psi,s,d}\|_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{R}^n)\to M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}.$$

Proof. (i) By Theorem 3.4 with the null set $\mathbb{V}_+$, we obtain the proof of part (i).

(ii) Now, we will give the proof of part (ii). Assume that $U^b_{\psi,s,d}$ is bounded from $M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{R}^n)$ to $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ for all $b \in C\dot{M}O^{r_1}(\mathbb{R}^n)$. By choosing b as in the proof of Theorem 3.6 and $g(x) = |x|^{\lambda-\alpha_1-n/q_1}$. For any $k \in \mathbb{Z}$,

$$\|g\chi_k\|_{L^{q_1}(\mathbb{R}^n)} = \Big(\int_{C_k} |x|^{q_1\lambda - q_1\alpha_1 - n} dx\Big)^{1/q_1} \simeq 2^{k(\lambda-\alpha_1)}.$$

Hence, by $\lambda \in (0,\infty)$,

$$\|g\|_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{R}^n)} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{kp\alpha_1}\|g\chi_k\|^p_{L^{q_1}(\mathbb{R}^n)}\right)^{1/p} \simeq \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{kp\lambda}\right)^{1/p} \simeq 1.$$

By a similar estimation and $\alpha_1 + n/q_1 = \alpha + n/q$, one has $\|g\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \simeq \|g\|_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{R}^n)}$. Consequently, by $U^b_{\psi,s,d}(g)(x) = \mathcal{A}_2^{**} g(x)$, we deduce

$$\|U^b_{\psi,s,d}\|_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{R}^n)\to M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \geq \frac{\|U^b_{\psi,s,d}(g)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}}{\|g\|_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{R}^n)}} = \mathcal{A}_2^{**}\frac{\|g\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}}{\|g\|_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{R}^n)}} \simeq \mathcal{A}_2^{**}.$$

This ends the proof of Theorem 3.7.

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Conflict of interest

The authors declare no conflict of interest.

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