Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(10): 19026–19056.
DOI: 10.3934/math.20221046
Received: 21 July 2022
Revised: 19 August 2022
Accepted: 22 August 2022
Published: 29 August 2022

Research article

Applying faster algorithm for obtaining convergence, stability, and data dependence results with application to functional-integral equations

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Abstract: The goal of this manuscript is to create a new faster iterative algorithm than the previous writing's sober algorithms. In the setting of Banach spaces, this algorithm is used to analyze convergence, stability, and data-dependence results. Basic numerical examples are also provided to highlight the behavior and effectiveness of our approach. Ultimately, the proposed approach is used to solve the functional Volterra-Fredholm integral problem as an application.

Keywords: convergence result; fixed point; data-dependent result; Ξ-stable; numerical experiment; functional integral equation
Mathematics Subject Classification: 47H10, 47A56, 65R20

1. Introduction

Throughout this paper, we assume that Δ is a non-empty, closed and convex subset (CCS) of a Banach space (BS) Λ, $\mathbb{R}^+$ is the set of nonnegative real numbers and $\mathbb{N}$ is the set of natural numbers. In addition, the symbol $\rightharpoonup$ refers to the weak convergence and $\longrightarrow$ to the strong convergence.

The set of all fixed points (FPs) for an operator $\Xi : \Delta \to \Delta$ is denoted by $\Upsilon(\Xi)$, which is defined by the point $\nu \in \Delta$ such that the equation $\nu = \Xi\nu$ is satisfied.

Let $\Xi : \Delta \to \Delta$ be a self-mapping, then Ξ is called:

(1) Contraction if there exists a constant $\alpha \in [0, 1)$ such that $d(\Xi\nu, \Xi\varpi) \leq \alpha d(\nu, \varpi)$.
(1) Nonexoansive if $d(\Xi\nu, \Xi\varpi) \leq d(\nu, \varpi)$, for all $\nu, \varpi \in \Delta$.

Clearly, the contraction mapping is nonexpansive when $\alpha = 1$.

FP techniques are applied in many solid applications due to their ease and smoothness such as optimization theory, approximation theory, fractional derivative, dynamic theory, and game theory. This is the reason why researchers are attracted to this technique. Also, this technique plays a significant role not only in the above applications but also in nonlinear analysis and many other engineering sciences. One of the important trends of FP methods is the study of the behavior and performance of algorithms that contribute greatly to real-world applications.

One of the well-established principles of the *FP* theory is Banach's contraction principle (*BCP*). This principle is significant as a source of existence and uniqueness theorem in various parts of science. *BCP* depends on the Picard one-step iteration, which is given by:

$$\nu_{i+1} = \Xi\nu_i, \ \forall i \geq 1,$$

where Ξ is a contraction mapping defined on a complete metric space (*MS*). When the existence of the *FP* theorem is guaranteed in the setting of complete *MS, BCP* is not well applied to nonexpansive mapping because Picard's iteration gives poor results for the convergence of *FP*. So, many authors tended to create many iterative methods for approximating *FPs* in terms of improving the performance and convergence behavior of algorithms for nonexpansive mappings. Moreover, data-dependent results and the stability results with respect to Ξ via these methods have been introduced. For more details, we refer to some iterative methods such as the iteration of Mann [1], Ishikawa [2], Noor [3], Argawal et al. [4], Abbas and Nazir [5]. In addition, SP iteration [6], S^*-iteration [7], CR-iteration [8], Normal-S iteration [9], Picard-S iteration [10], Thakur iteration [11], M-iteration [12], M^*-iteration [13], Garodia and Uddin iteration [14], two-step Mann iteration [15]. Also, for more applications involved in iteration methods, see Hasanen et al. [16, 17], and many others.

Assume that $\{\eta_i\}$ and $\{\gamma_i\}$ are nonnegative sequences in $[0, 1]$. The algorithms below are known as S algorithm [4], Picard-S algorithm [10], Thakur algorithm [11] and K^*-algorithm [18], respectively:

$$\begin{cases} \nu_\circ \in \Delta, \\ \varpi_i = (1-\eta_i)\nu_i + \eta_i\Xi\nu_i, & \forall i \geq 1. \\ \nu_{i+1} = (1-\gamma_i)\nu_i + \gamma_i\Xi\varpi_i, \end{cases} \tag{1.1}$$

$$\begin{cases} \nu_\circ \in \Delta, \\ \varpi_i = (1-\eta_i)\nu_i + \eta_i\Xi\nu_i \\ \wp_i = (1-\gamma_i)\nu_i + \gamma_i\Xi\varpi_i, & \forall i \geq 1. \\ \nu_{i+1} = \Xi\wp_i, \end{cases} \tag{1.2}$$

$$\begin{cases} \nu_\circ \in \Delta, \\ \varpi_i = (1-\eta_i)\nu_i + \eta_i\Xi\nu_i, \\ \wp_i = \Xi\left((1-\gamma_i)\nu_i + \gamma_i\varpi_i\right), & \forall i \geq 1. \\ \nu_{i+1} = \Xi\wp_i, \end{cases} \tag{1.3}$$

$$\begin{cases} \nu_\circ \in \Delta, \\ \varpi_i = (1-\alpha_i)\nu_i + \alpha_i\Xi\nu_i, \\ \wp_i = \Xi\left((1-\gamma_i)\varpi_i + \gamma_i\Xi\varpi_i\right), & \forall i \geq 1. \\ \nu_{i+1} = \Xi\wp_i, \end{cases} \tag{1.4}$$

In 2014, Gursoy and Karakaya [10] presented the iterative method (1.2) and called it the Picard-S iteration. They proved numerically and analytically that the Picard-S iteration converges faster than

Picard, Mann, Ishikawa, Noor, SP, CR, S, S*, Abbas and Nazir, normal-S and two-step Mann iteration procedures for almost contraction mappings (*ACMs*).

In 2016, Thakur et al. [11] illustrated that the iteration (1.3) converges faster than Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iteration for Suzuki generalized nonexpansive mappings (*SGNMs*) by a numerical example.

Recently, Ullah and Arshad [18] gave K^*-algorithm (1.4) and proved that K^*-algorithm (1.4) converges faster than S algorithm (1.1), Picard-S algorithm (1.2) and Thakur algorithm (1.3) for *SGNMs*. Moreover, they noted that the Picard-S iteration (1.2) and Thakur iteration (1.3) have the same rate of convergence.

On the other hand, nonlinear integral equations are used to describe mathematical models arising from mathematical physics, engineering, economics, biology, etc [19]. In particular, Volterra-Friedholm equations arise from boundary value problems and mathematical modeling of the spatiotemporal evolution of the epidemic. For various biological models, see [20, 21]. Recently, a large number of researchers turned to solve nonlinear integral equations with the involvement of iterative methods, for example, see [22–26].

Based on the works mentioned above, in this manuscript, we construct a new algorithm to get a better affinity rate of *ACMs* and *SGNMs* as follows:

$$\begin{cases} \nu_\circ \in \Delta, \\ \varpi_i = (1-\alpha_i)\nu_i + \alpha_i \Xi \nu_i, \\ \wp_i = \Xi\left((1-\eta_i)\varpi_i + \eta_i \Xi \varpi_i\right), \\ \Im_i = \Xi\left((1-\gamma_i)\wp_i + \gamma_i \Xi \wp_i\right), \\ \nu_{i+1} = \Xi \Im_i, \end{cases} \tag{1.5}$$

for each $i \geq 1$, where α_i, η_i and γ_i are sequences in $[0, 1]$.

Our work is organized as follows: In section 2, we give some definitions, propositions, and lemmas, which facilitates the reader's palatability of our results. In section 3, the performance and convergence rate of our algorithms are analyzed analytically and we found that the convergence rate is satisfactory for *ACMs* in a *BS*. Moreover, the weak and strong convergence of the proposed algorithm is discussed for *SGNMs* in the context of *UCBSs* in section 4. In section 5, we proved that our new iterative algorithm is G-stable. Further, data-dependence results for *ACMs* under our iterative scheme (1.5) are studied in section6. In addition, we presented two examples to show that our method is faster than the iteration schemes (1.1)–(1.4) in section 7. Ultimately, in section 8, the proposed algorithm is implicated to find the solution to the Volterra-Fredholm integral equation. In section 9, the conclusion and future works are derived.

2. Preliminaries

This part is intended to give some definitions, propositions and lemmas that will comfort the reader in understanding our manuscript and will be useful in the sequel.

Definition 2.1. *A mapping* $\Xi : \Lambda \to \Lambda$ *is called SGNM if*

$$\frac{1}{2}\|\nu - \Xi\nu\| \leq \|\nu - \varpi\| \Rightarrow \|\Xi\nu - \Xi\varpi\| \leq \|\nu - \varpi\|, \ \forall \nu, \varpi \in \Lambda.$$

Definition 2.2. *A BS Λ is called a uniformly convex if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that for $\nu, \varpi \in \Lambda$ satisfying $\|\nu\| \leq 1$, $\|\varpi\| \leq 1$ and $\|\nu - \varpi\| > \epsilon$, we get $\left\|\frac{\nu+\varpi}{2}\right\| < 1 - \delta$.*

Definition 2.3. *A BS Λ is called satisfy Opial's condition if for any sequence $\{\nu_i\}$ in Λ so that $\nu_i \rightharpoonup \nu \in \Lambda$, implies*

$$\limsup_{i\to\infty} \|\nu_i - \nu\| < \limsup_{i\to\infty} \|\nu_i - \varpi\|, \text{ for all } \varpi \in \Lambda \text{ with } \nu \neq \varpi.$$

Definition 2.4. *Let $\{\nu_i\}$ be a bounded sequence in a BS Λ. For $\nu \in \Delta \subset \Lambda$, we set*

$$R(\nu, \{\nu_i\}) = \limsup_{i\to\infty} \|\nu_i - \nu\|.$$

The asymptotic radius of $\{\nu_i\}$ relative to Λ is defined by

$$R(\Lambda, \{\nu_i\}) = \inf\{R(\nu, \{\nu_i\}) : \nu \in \Lambda\}.$$

The asymptotic center of $\{\nu_i\}$ relative to Λ is defined by

$$Q(\Lambda, \{\nu_i\}) = \{\nu \in \Lambda : R(\nu, \{\nu_i\}) = R(\Lambda, \{\nu_i\})\}.$$

It should be noted that, $Q(\Lambda, \{\nu_i\})$ consists of exactly one point in a UCBS.

Definition 2.5. *[27] A mapping $\Xi : \Lambda \to \Lambda$ is called ACM, if there exists $\theta \in (0,1)$ and some constant $\ell \geq 0$ so that*

$$\|\Xi\nu - \Xi\varpi\| \leq \theta \|\nu - \varpi\| + \ell \|\nu - \Xi\nu\|, \ \forall \nu, \varpi \in \Lambda. \tag{2.1}$$

Definition 2.6. *[28] Suppose that the real sequences $\{\eta_i\}$ and $\{\gamma_i\}$ converge to η and γ, respectively. Suppose also there is*

$$\varkappa = \lim_{i\to\infty} \frac{\|\eta_i - \eta\|}{\|\gamma_i - \gamma\|}.$$

Then, we say that

(1) the sequence $\{\eta_i\}$ converges faster to η than $\{\gamma_i\}$ does to γ, if $\varkappa = 0$,
(2) $\{\eta_i\}$ and $\{\gamma_i\}$ have the same rate of convergence, if $\varkappa \in (0, \infty)$.

Definition 2.7. *[28] Suppose that $\Xi, \widehat{\Xi} : \Delta \to \Delta$ are given operators. An operator $\widehat{\Xi}$ is called an approximate operator for Ξ, if for some $\epsilon > 0$, we get*

$$\left\|\Xi\eta - \widehat{\Xi}\eta\right\| \leq \epsilon, \ \forall \eta \in \Delta.$$

Definition 2.8. *[29] Let $\daleth : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function satisfies $\daleth(0) = 0$ and $\daleth(s) > 0$ for all $s > 0$. A mapping $\Xi : \Lambda \to \Lambda$ is called satisfy a condition (I) if*

$$\|\nu - \Xi\nu\| \geq \daleth\left(d\left(\nu, \Upsilon(\Xi)\right)\right), \ \forall \nu \in \Lambda,$$

where $d(\nu, \Upsilon(\Xi)) = \inf\{\|\nu - \zeta\| : \zeta \in \Upsilon(\Xi)\}$.

Proposition 2.1. *[30] Let $\Xi : \Lambda \to \Lambda$ be a given map. If Ξ is*

(1) nonexpansive mapping, then it is SGNM.

(2) SGNM with a non-empty FP set, then it is quasi-nonexpansive mapping.
(3) SGNEM, then it is satisfied the following inequality

$$\|\nu - \Xi\varpi\| \leq 3\,\|\Xi\nu - \nu\| + \|\nu - \varpi\|\,,\ \forall \nu, \varpi \in \Lambda.$$

Lemma 2.1. *[30] Let Δ be a subset of a BS Λ, which satisfies Opial's condition and $\Xi : \Delta \to \Delta$ be a SGNM. If $\{\nu_i\} \rightharpoonup \zeta$ and $\lim_{i\to\infty} \|\Xi\varpi_i - \nu_i\| = 0$, then $\Xi\zeta = \zeta$, i.e., $I - \Xi$ is demiclosed at zero.*

Lemma 2.2. *[30] Let Δ be a weakly compact convex subset of a BS Λ with the Opial's property. If $\Xi : \Delta \to \Delta$ is a SGNM, then Ξ possess a FP.*

Lemma 2.3. *[28] Assume that $\{\varphi_i\}$ and $\{\varphi_i^*\}$ is nonnegative real sequences satisfy the inequality below:*

$$\varphi_{i+1} \leq (1 - z_i)\varphi_i + \varphi_i^*,$$

where $z_i \in (0, 1)$ for each $i \geq 1$, $\sum_{i=0}^{\infty} z_i = \infty$ and $\lim_{i\to\infty} \frac{\varphi_i^*}{z_i} = 0$, then $\lim_{i\to\infty} \varphi_i = 0$.

Lemma 2.4. *[31] Let $\{\varphi_i\}$ and $\{\varphi_i^*\}$ be nonnegative real sequences satisfy the inequality below:*

$$\varphi_{i+1} \leq (1 - z_i)\varphi_i + z_i\varphi_i^*,$$

where $z_i \in (0, 1)$ for each $i \geq 1$, $\sum_{i=0}^{\infty} z_i = \infty$ and $\varphi_i^* \geq 0$, then

$$0 \leq \limsup_{i\to\infty} \varphi_i \leq \limsup_{i\to\infty} \varphi_i^*.$$

Lemma 2.5. *[32] Assume that Λ is a UCBS and $\{\xi_i\}$ is a sequence satisfies $0 < n \leq \xi_i \leq n^* < 1$, for all $i \in \mathbb{N}$. Assume also $\{\varpi_i\}$ and $\{\nu_i\}$ are two sequences in Λ so that $\limsup_{i\to\infty}\{\varpi_i\} \leq \rho$, $\limsup_{i\to\infty}\{\nu_i\} \leq \rho$ and $\limsup_{i\to\infty} \|\xi\varpi_i + (1 - \xi)\nu_i\| = \rho$ for some $\rho \geq 0$. Then $\lim_{i\to\infty} \|\varpi_i - \nu_i\| = 0$.*

3. Convergence rate

In this section, we discuss the rate of convergence of our iterative algorithm for ACMs.

Theorem 3.1. *Assume that Λ is a BS and Δ is a closed convex subset (CCS) of Λ. Let $\Xi : \Lambda \to \Lambda$ be a mapping satisfying (2.1) with $\Upsilon(\Xi) \neq \emptyset$. Suppose that $\{\nu_i\}$ is the iterative sequence generated by (1.5) with $\{\alpha_i\}, \{\eta_i\}, \{\gamma_i\} \in [0, 1]$ such that $\sum_{i=0}^{\infty} \gamma_i = \infty$. Then $\{\nu_i\} \longrightarrow \zeta \in \Upsilon(\Xi)$.*

Proof. Consider $\zeta \in \Upsilon(\Xi)$, then from (1.5), we get

$$\begin{aligned}
\|\varpi_i - \zeta\| &= \|(1 - \alpha_i)\nu_i + \alpha_i\Xi\nu_i - \zeta\| \\
&= \|(1 - \alpha_i)(\nu_i - \zeta) + \alpha_i(\Xi\nu_i - \zeta)\| \\
&\leq (1 - \alpha_i)\|\nu_i - \zeta\| + \alpha_i\|\Xi\nu_i - \zeta\| \\
&\leq (1 - \alpha_i)\|\nu_i - \zeta\| + \theta\alpha_i\|\nu_i - \zeta\| \\
&= (1 - (1 - \theta)\alpha_i)\|\nu_i - \zeta\|\,. \qquad (3.1)
\end{aligned}$$

Using (1.5) and (3.1), we have

$$\begin{aligned}\|\wp_i-\zeta\| &= \|\Xi\left((1-\eta_i)\varpi_i+\eta_i\Xi\varpi_i\right)-\zeta\|\\ &= \|\Xi\zeta-\Xi\left((1-\eta_i)\varpi_i+\eta_i\Xi\varpi_i\right)\|\\ &\le \theta\,\|\zeta-((1-\eta_i)\varpi_i+\eta_i\Xi\varpi_i)\|+\ell\,\|\zeta-\Xi\zeta\|\\ &= \theta\,\|(1-\eta_i)\,(\varpi_i-\zeta)+\eta_i\,(\Xi\varpi_i-\zeta)\|\\ &\le \theta\left[(1-\eta_i)\,\|\varpi_i-\zeta\|+\theta\eta_i\,\|\varpi_i-\zeta\|\right]\\ &\le \theta\left[1-(1-\theta)\eta_i\right]\|\varpi_i-\zeta\|\\ &\le \theta\,(1-(1-\theta)\eta_i)\,(1-(1-\theta)\alpha_i)\,\|\nu_i-\zeta\|\,. \end{aligned}\tag{3.2}$$

Similarly, from (1.5) and (3.2), one can write

$$\begin{aligned}\left\|\Im_i-\zeta\right\| &\le \theta\,(1-(1-\theta)\gamma_i)\,\|\wp_i-\zeta\|\\ &\le \theta^2\,(1-(1-\theta)\gamma_i)\,(1-(1-\theta)\eta_i)\,(1-(1-\theta)\alpha_i)\,\|\nu_i-\zeta\|\,. \end{aligned}\tag{3.3}$$

It follows from (1.5) and (3.3) that

$$\begin{aligned}\|\nu_{i+1}-\zeta\| &= \left\|\Xi\Im_i-\zeta\right\|\\ &\le \theta\left\|\Im_i-\zeta\right\|\\ &\le \theta^3\,(1-(1-\theta)\gamma_i)\,(1-(1-\theta)\eta_i)\,(1-(1-\theta)\alpha_i)\,\|\nu_i-\zeta\|\,. \end{aligned}\tag{3.4}$$

Because $\theta\in(0,1)$ and $\eta_i,\alpha_i\in[0,1]$, for all $i\ge 1$, then $(1-(1-\theta)\eta_i)\,(1-(1-\theta)\alpha_i)<1$. Hence (3.4) reduces to

$$\|\nu_{i+1}-\zeta\|\le\theta^3\,(1-(1-\theta)\gamma_i)\,\|\nu_i-\zeta\|\,.\tag{3.5}$$

It follows from (3.5) that

$$\begin{aligned}\|\nu_{i+1}-\zeta\| &\le \theta^3\,(1-(1-\theta)\gamma_i)\,\|\nu_i-\zeta\|\\ &\le \theta^3\,(1-(1-\theta)\gamma_{i-1})\,\|\nu_{i-1}-\zeta\|\\ &\ \ \vdots\\ \|\nu_1-\zeta\| &\le \theta^3\,(1-(1-\theta)\gamma_0)\,\|\nu_0-\zeta\|\,. \end{aligned}\tag{3.6}$$

From (3.6), we have

$$\|\nu_{i+1}-\zeta\|\le\theta^{3(i+1)}\,\|\nu_0-\zeta\|\prod_{u=0}^{i}(1-(1-\theta)\gamma_u)\,.\tag{3.7}$$

Again, since $\theta\in(0,1)$ and $\gamma_u\in[0,1]$, for all $u\ge 1$, then $(1-(1-\theta)\gamma_u)<1$. It is known that $1-\nu\le e^{-\nu}$ for each $\nu\in[0,1]$, then by (3.7), we obtain

$$\|\nu_{i+1}-\zeta\|\le\frac{\theta^{3(i+1)}}{e^{(1-\theta)\sum\limits_{u=0}^{\infty}\gamma_u}}\,\|\nu_0-\zeta\|\,.\tag{3.8}$$

Taking the limit as $i\to\infty$ in (3.8), we have $\lim\limits_{i\to\infty}\|\nu_i-\zeta\|=0$, that is $\{\nu_i\}\longrightarrow\zeta\in\Upsilon(\Xi)$.

For the uniqueness. Let $\zeta,\zeta^*\in\Upsilon(\Xi)$ such that $\zeta\neq\zeta^*$, from the definition of Ξ, we can write

$$\|\zeta-\zeta^*\|=\|\Xi\zeta-\Xi\zeta^*\|\le\theta\,\|\zeta-\zeta^*\|+\ell\,\|\zeta-\Xi\zeta\|=\theta\,\|\zeta-\zeta^*\|<\|\zeta-\zeta^*\|\,,$$

a contrary. Hence $\zeta=\zeta^*$. □

The next theorem shows that our iteration (1.5) converges faster than the iteration (1.4) in the sense of Berinde [28].

Theorem 3.2. *Let Δ be a CCS of a BS Λ and $\Xi : \Lambda \to \Lambda$ be a mapping satisfies (2.1) with $\Upsilon(\Xi) \neq \emptyset$. Consider $\{\nu_i\}$ is the iterative sequence generated by the algorithm (1.5) with $\{\alpha_i\}, \{\eta_i\}, \{\gamma_i\} \in [0, 1]$ such that $0 < \gamma \leq \gamma_i \leq 1$, for all $i \geq 1$. Then the sequence $\{\nu_i\}$ converges faster to ν than the iterative scheme (1.4).*

Proof. It follows from (3.7) and the hypothesis $0 < \gamma \leq \gamma_i \leq 1$ that

$$\begin{aligned} \|\nu_{i+1} - \zeta\| &\leq \theta^{3(i+1)} \|\nu_0 - \zeta\| \prod_{u=0}^{i} (1 - (1 - \theta)\gamma_u) \\ &= \theta^{3(i+1)} \|\nu_0 - \zeta\| (1 - (1 - \theta)\gamma)^{i+1} . \end{aligned}$$

Analogously, the iterative process (1.4) ([18], Theorem 3.2) takes the form

$$\|l_{i+1} - \zeta\| \leq \theta^{2(i+1)} \|l_0 - \zeta\| \prod_{u=0}^{i} (1 - (1 - \theta)\gamma_u) . \tag{3.9}$$

Because $\gamma \leq \gamma_i \leq 1$, for some $\gamma > 0$ and all $i \geq 1$, then (3.9) can be written as

$$\begin{aligned} \|l_{i+1} - \zeta\| &\leq \theta^{2(i+1)} \|l_0 - \zeta\| \prod_{u=0}^{i} (1 - (1 - \theta)\gamma_u) \\ &= \theta^{2(i+1)} \|l_0 - \zeta\| (1 - (1 - \theta)\gamma)^{i+1} . \end{aligned}$$

Put

$$\zeta = \theta^{3(i+1)} \|\nu_0 - \zeta\| (1 - (1 - \theta)\gamma)^{i+1} ,$$

and

$$\widehat{\zeta} = \theta^{2(i+1)} \|l_0 - \zeta\| (1 - (1 - \theta)\gamma)^{i+1} .$$

Define

$$\varrho_i = \frac{\zeta}{\widehat{\zeta}} = \frac{\theta^{3(i+1)} \|\nu_0 - \zeta\| (1 - (1 - \theta)\gamma)^{i+1}}{\theta^{2(i+1)} \|l_0 - \zeta\| (1 - (1 - \theta)\gamma)^{i+1}} = \theta^{i+1}.$$

Taking the limit as $i \to \infty$, we have $\lim_{i\to\infty} \varrho_i = 0$. This means that $\{\nu_i\}$ converges faster than $\{l_i\}$ to ν. □

4. Weak and strong convergence

This part has been enriched to obtain some convergence results of our iteration procedure (1.5) for *SGNMs* in the setting of *UCBSs*.

We begin with the proof of the following lemmas:

Lemma 4.1. *Let Δ be a CCS of a BS Λ and $\Xi : \Lambda \to \Lambda$ be SGNM with $\Upsilon(\Xi) \neq \emptyset$. If $\{\nu_i\}$ is the iterative sequence given by the algorithm (1.5), then $\lim_{i\to\infty} \|\nu_i - \zeta\|$ exists for each $\zeta \in \Upsilon(\Xi)$.*

Proof. Let $\zeta \in \Upsilon(\Xi)$ and $\delta \in \Delta$. Clearly, from Proposition 2.1 (2), every SGNM is quasi-nonexpansive mapping, so

$$\frac{1}{2}\|\zeta - \Xi\zeta\| = 0 \leq \|\zeta - \delta\| \Rightarrow \|\Xi\zeta - \Xi\delta\| \leq \|\zeta - \delta\|.$$

Now, using (1.5), we obtain

$$\begin{aligned} \|\varpi_i - \zeta\| &= \|(1-\alpha_i)\nu_i + \alpha_i \Xi \nu_i - \zeta\| \\ &\leq (1-\alpha_i)\|\nu_i - \zeta\| + \alpha_i \|\Xi\nu_i - \zeta\| \\ &\leq (1-\alpha_i)\|\nu_i - \zeta\| + \alpha_i \|\nu_i - \zeta\| \\ &= \|\nu_i - \zeta\|. \end{aligned} \tag{4.1}$$

It follows from (1.5) and (4.1) that

$$\begin{aligned} \|\wp_i - \zeta\| &= \|\Xi\left((1-\eta_i)\varpi_i + \eta_i \Xi \varpi_i\right) - \zeta\| \\ &\leq \|(1-\eta_i)\varpi_i + \eta_i \Xi\varpi_i - \zeta\| \\ &\leq (1-\eta_i)\|\varpi_i - \zeta\| + \eta_i \|\Xi\varpi_i - \zeta\| \\ &\leq (1-\eta_i)\|\varpi_i - \zeta\| + \eta_i \|\varpi_i - \zeta\| \\ &= \|\varpi_i - \zeta\| \\ &\leq \|\nu_i - \zeta\|. \end{aligned} \tag{4.2}$$

Similarly, from (1.5) and (4.2), one can write

$$\left\|\Im_i - \zeta\right\| = \|\Xi\left((1-\gamma_i)\wp_i + \gamma_i \Xi \wp_i\right) - \zeta\| \leq \|\wp_i - \zeta\| \leq \|\nu_i - \zeta\|.$$

At the last, using (1.5) and (4.3), we get

$$\|\nu_{i+1} - \zeta\| = \left\|\Xi\Im_i - \zeta\right\| \leq \left\|\Im_i - \zeta\right\| \leq \|\nu_i - \zeta\|.$$

This leads to the sequence $\{\|\nu_i - \zeta\|\}$ is bounded and nondecreasing for all $\zeta \in \Upsilon(\Xi)$. Hence $\lim_{i\to\infty}\|\nu_i - \zeta\|$ exists. □

Lemma 4.2. *Let Δ be a non-empty CCS of a UCBS Λ and $\Xi : \Lambda \to \Lambda$ be SGNM. If $\{\nu_i\}$ is the iterative sequence defined by the algorithm (1.5). Then $\Upsilon(\Xi) \neq \emptyset$ if and only if $\{\nu_i\}$ is bounded and* $\lim_{i\to\infty}\|\Xi\nu_i - \nu_i\| = 0$.

Proof. Assume that $\Upsilon(\Xi) \neq \emptyset$ and take $\zeta \in \Upsilon(\Xi)$. Based on Lemma 4.1, $\lim_{i\to\infty}\|\nu_i - \zeta\|$ exists and $\{\nu_i\}$ is bounded. Put

$$\lim_{i\to\infty}\|\nu_i - \zeta\| = \varphi. \tag{4.3}$$

Applying (4.3) in (4.1) and taking lim sup, we can write

$$\limsup_{i\to\infty}\|\varpi_i - \zeta\| \leq \limsup_{i\to\infty}\|\nu_i - \zeta\| = \varphi.$$

According to Proposition 2.1 (2), we have

$$\limsup_{i\to\infty}\|\Xi\nu_i - \zeta\| \leq \limsup_{i\to\infty}\|\nu_i - \zeta\| = \varphi. \tag{4.4}$$

Again, using (1.5) and (4.1)–(4.3), we obtain

$$\begin{aligned}
\|\nu_{i+1}-\zeta\| &= \left\|\Xi\Im_i-\zeta\right\| \\
&\leq \|\Xi\left((1-\gamma_i)\wp_i+\gamma_i\Xi\wp_i\right)-\zeta\| \\
&\leq \|(1-\gamma_i)\wp_i+\gamma_i\Xi\wp_i-\zeta\| \\
&\leq (1-\gamma_i)\|\wp_i-\zeta\|+\gamma_i\|\Xi\wp_i-\zeta\| \\
&\leq (1-\gamma_i)\|\wp_i-\zeta\|+\|\wp_i-\zeta\| \\
&= \|\wp_i-\zeta\| \\
&= \|\Xi\left((1-\eta_i)\varpi_i+\eta_i\Xi\varpi_i\right)-\zeta\| \\
&\leq \|(1-\eta_i)\varpi_i+\eta_i\Xi\varpi_i-\zeta\| \\
&\leq (1-\eta_i)\|\varpi_i-\zeta\|+\eta_i\|\Xi\varpi_i-\zeta\| \\
&\leq (1-\eta_i)\|\nu_i-\zeta\|+\eta_i\|\varpi_i-\zeta\| \\
&= \|\nu_i-\zeta\|-\eta_i\|\nu_i-\zeta\|+\eta_i\|\varpi_i-\zeta\|.
\end{aligned}$$

Thus, we have

$$\frac{\|\nu_{i+1}-\zeta\|-\|\nu_i-\zeta\|}{\eta_i}\leq\|\varpi_i-\zeta\|-\|\nu_i-\zeta\|. \tag{4.5}$$

Since $\eta_i\in[0,1]$, then by (4.5), we get

$$\|\nu_{i+1}-\zeta\|-\|\nu_i-\zeta\|\leq\frac{\|\nu_{i+1}-\zeta\|-\|\nu_i-\zeta\|}{\eta_i}\leq\|\varpi_i-\zeta\|-\|\nu_i-\zeta\|,$$

which implies that $\|\nu_{i+1}-\zeta\|\leq\|\varpi_i-\zeta\|$. Then from (4.3), we have

$$\varphi\leq\liminf_{i\to\infty}\|\varpi_i-\zeta\|. \tag{4.6}$$

It follows from (4.4) and (4.6) that

$$\begin{aligned}
\varphi &= \lim_{i\to\infty}\|\varpi_i-\zeta\|=\lim_{i\to\infty}\|(1-\alpha_i)\nu_i+\alpha_i\Xi\nu_i-\zeta\| \\
&= \lim_{i\to\infty}\|(1-\alpha_i)(\nu_i-\zeta)+\alpha_i(\Xi\nu_i-\zeta)\| \\
&= \lim_{i\to\infty}\|\alpha_i(\Xi\nu_i-\zeta)+(1-\alpha_i)(\nu_i-\zeta)\|. \qquad (4.7)
\end{aligned}$$

From (4.3), (4.4), (4.7) and Lemma 2.5, we have $\lim_{i\to\infty}\|\Xi\nu_i-\nu_i\|=0$.

Conversely, suppose that $\{\nu_i\}$ is bounded and $\lim\limits_{i\to\infty}\|\Xi\nu_i-\nu_i\|=0$. Suppose also $\Xi\zeta\in Q(\Lambda,\{\nu_i\})$, then by Definition 2.4 and Proposition 2.1 (3), we can write

$$\begin{aligned}
R(\Xi\zeta,\{\nu_i\}) &= \limsup_{i\to\infty}\|\nu_i-\Xi\zeta\| \\
&\leq \limsup_{i\to\infty}\left(3\|\Xi\nu_i-\nu_i\|+\|\nu_i-\zeta\|\right) \\
&= \limsup_{i\to\infty}\|\nu_i-\zeta\|=R(\zeta,\{\nu_i\}),
\end{aligned}$$

which leads to $\zeta\in Q(\Lambda,\{\nu_i\})$. Since $Q(\Lambda,\{\nu_i\})$ is singleton and Λ is a uniformly convex, thus, we get $\Xi\zeta=\zeta$. □

Theorem 4.1. *Let Λ, Δ and Ξ be as in Lemma 4.2. If Λ satisfies Opial's condition and $\Upsilon(\Xi) \neq \emptyset$, then the sequence $\{\nu_i\}$ iterated by (1.5) converges weakly to a FP of Ξ, that is $\{\nu_i\} \rightharpoonup \zeta \in \Upsilon(\Xi)$.*

Proof. Let $\zeta \in \Upsilon(\Xi)$, based on Lemma 4.1, $\lim_{i\to\infty} \|\nu_i - \zeta\|$ exists.

Now, we prove that $\{\nu_i\}$ has a weak sequential limit in $\Upsilon(\Xi)$. Let $\{\nu_{i_j}\}$ and $\{\nu_{i_k}\}$ be subsequences of $\{\nu_i\}$ so that $\{\nu_{i_j}\} \rightharpoonup \nu$ and $\{\nu_{i_k}\} \rightharpoonup \nu^*$ for each $\nu, \nu^* \in \Delta$. Using Lemma 4.2, we obtain $\lim_{i\to\infty} \|\Xi\nu_i - \nu_i\| = 0$. From Lemma 2.1 and $I - \Xi$ is demiclosed at zero, we have $(I - \Xi)\nu = 0 \Rightarrow \Xi\nu = \nu$, analogously, $\Xi\nu^* = \nu^*$.

For the uniqueness. Let $\nu \neq \nu^*$, then by Opial's property, we find that

$$\begin{aligned} \lim_{i\to\infty} \|\nu_i - \nu\| &= \lim_{j\to\infty} \left\|\nu_{i_j} - \nu\right\| < \lim_{j\to\infty} \left\|\nu_{i_j} - \nu^*\right\| \\ &= \lim_{i\to\infty} \|\nu_i - \nu^*\| = \lim_{k\to\infty} \left\|\nu_{i_k} - \nu^*\right\| \\ &< \lim_{k\to\infty} \left\|\nu_{i_k} - \nu\right\| = \lim_{i\to\infty} \|\nu_i - \nu\|, \end{aligned}$$

a contradiction, so $\nu = \nu^*$ and $\{\nu_i\} \rightharpoonup \zeta \in \Upsilon(\Xi)$. □

Theorem 4.2. *Let Λ be a UCBS and Δ be a non-empty compact convex subset of Λ. Assume that $\Xi : \Delta \to \Delta$ is SGNM and $\{\nu_i\}$ is the iterative sequence given by (1.5). Then $\{\nu_i\} \longrightarrow \zeta \in \Upsilon(\Xi)$.*

Proof. According to Lemmas 2.2 and 4.2, we have $\Upsilon(\Xi) \neq \emptyset$ and $\lim_{i\to\infty} \|\Xi\nu_i - \nu_i\| = 0$. The compactness of Δ implies that there is a subsequence $\{\nu_{i_k}\}$ of $\{\nu_i\}$ so that $\nu_{i_k} \to \zeta$ for some $\zeta \in \Delta$. Using Proposition 2.1 (3), one can get

$$\left\|\nu_{i_k} - \Xi\zeta\right\| \leq 3\left\|\nu_{i_k} - \Xi\nu_{i_k}\right\| + \left\|\nu_{i_k} - \zeta\right\|, \ \forall i \geq 1.$$

Passing $k \to \infty$, we obtain $\Xi\zeta = \zeta$, that is $\zeta \in \Upsilon(\Xi)$. Also, by Lemma 4.1, we get $\lim_{i\to\infty} \|\nu_i - \zeta\|$ exists for all $\zeta \in \Upsilon(\Xi)$, thus $\nu_i \to \zeta$ strongly. □

Theorem 4.3. *Let Λ, Δ and Ξ be described as Lemma 4.2 and $\{\nu_i\}$ be an iterative sequence defined by (1.5). Then*

$$\{\nu_i\} \longrightarrow \zeta \in \Upsilon(\Xi) \Leftrightarrow \liminf_{i\to\infty} d(\nu_i, \Upsilon(\Xi)) = 0,$$

where $d(\nu, \Upsilon(\Xi)) = \inf\{\|\nu - \zeta\| : \zeta \in \Upsilon(\Xi)\}$.

Proof. A necessary condition is clear. Let $\liminf_{i\to\infty} d(\nu_i, \Upsilon(\Xi)) = 0$, by Lemma 4.1, we get $\lim_{i\to\infty} \|\nu_i - \zeta\|$ exists for all $\zeta \in \Upsilon(\Xi)$, this implies that $\liminf_{i\to\infty} d(\nu_i, \Upsilon(\Xi))$ exists. From our assumption $\liminf_{i\to\infty} d(\nu_i, \Upsilon(\Xi)) = 0$, we have $\lim_{i\to\infty} d(\nu_i, \Upsilon(\Xi)) = 0$.

Next, we shall show that the sequence $\{\nu_i\}$ is a Cauchy in Δ. Because $\liminf_{i\to\infty} d(\nu_i, \Upsilon(\Xi)) = 0$, then for a given $\epsilon > 0$, there is $i_0 \geq 1$ so that, for each $i, j \geq i_0$, we obtain

$$d(\nu_i, \Upsilon(\Xi)) \leq \frac{\epsilon}{2} \text{ and } d(\nu_j, \Upsilon(\Xi)) \leq \frac{\epsilon}{2}.$$

Hence, we get

$$\left\|\nu_i - \nu_j\right\| \leq \|\nu_i - \Upsilon(\Xi)\| + \left\|\Upsilon(\Xi) - \nu_j\right\| = d(\nu_i, \Upsilon(\Xi)) + d(\nu_j, \Upsilon(\Xi) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

This proves that $\{\nu_i\}$ is a Cauchy sequence in Δ. As Δ is closed, then there is $\widehat{\nu} \in \Delta$ so that $\lim_{i\to\infty} \nu_i = \widehat{\nu}$. Since $\lim_{i\to\infty} d(\nu_i, \Upsilon(\Xi)) = 0$, yields $\lim_{i\to\infty} d(\widehat{\nu}, \Upsilon(\Xi)) = 0$. Thus, $\widehat{\nu} \in \Upsilon(\Xi)$ as Δ is closed. This finishes the proof. □

Theorem 4.4. *Let Λ, Δ and Ξ be defined in Lemma 4.2 and $\{\nu_i\}$ be an iterative sequence generated by (1.5). If Ξ fulfills the condition (I), then $\{\nu_i\} \longrightarrow \zeta \in \Upsilon(\Xi)$.*

Proof. It follows from Lemma 4.2 that

$$\lim_{i\to\infty} \|\Xi\nu_i - \nu_i\| = 0. \tag{4.8}$$

Based on the condition (I) in Definition 2.8 and using (4.8), we observe that

$$\lim_{i\to\infty} \daleth(d(\nu_i, \Upsilon(\Xi))) \leq \lim_{i\to\infty} \|\Xi\nu_i - \nu_i\| = 0,$$

which leads to $\lim_{i\to\infty} \daleth(d(\nu_i, \Upsilon(\Xi))) = 0$. From the definition of $\daleth$, we get $\lim_{i\to\infty} d(\nu_i, \Upsilon(\Xi)) = 0$. Applying Theorem 4.3, we conclude that $\{\nu_i\} \longrightarrow \zeta \in \Upsilon(\Xi)$. □

5. Stability theorem

In this part, we show that our iteration process (1.5) is Ξ-stable.

Theorem 5.1. *Let Λ be a BS and Δ be a CCS of Λ. Suppose that $\Xi : \Lambda \to \Lambda$ is a self-mapping satisfies (2.1) and $\{\nu_i\}$ is the iterative sequence iterated by (1.5) with $\{\alpha_i\}, \{\eta_i\}, \{\gamma_i\} \in [0,1]$ so that $\sum_{i=0}^{\infty} \gamma_i = \infty$. Then the algorithm (1.5) is Ξ-stable.*

Proof. Let $\{\sigma_i\}$ be an arbitrary sequence in Δ and assume that the sequence of our iteration (1.5) can be written as $\nu_{i+1} = f(\Xi, \nu_i)$, which converges to a unique point ζ and that $\phi_i = \|\sigma_{i+1} - f(\Xi, \sigma_i)\|$. In order to show that Ξ is stable, we must prove that $\lim_{i\to\infty} \phi_i = 0$ if and only if $\lim_{i\to\infty} \sigma_i = \zeta$.

Assume that $\lim_{i\to\infty} \phi_i = 0$, then from (1.5) and (3.5), we get

$$\begin{aligned}
& \|\sigma_{i+1} - \zeta\| \\
= \; & \|\sigma_{i+1} - f(\Xi, \sigma_i) + f(\Xi, \sigma_i) - \zeta\| \\
\leq \; & \|\sigma_{i+1} - f(\Xi, \sigma_i)\| + \|f(\Xi, \sigma_i) - \zeta\| \\
= \; & \phi_i + \|f(\Xi, \sigma_i) - \zeta\| \\
= \; & \phi_i + \left\| \Xi\left(\Xi\left(\begin{array}{c} (1-\gamma_i)\left[\Xi\left(\begin{array}{c} (1-\eta_i)\left[(1-\alpha_i)\sigma_i + \alpha_i\Xi\sigma_i\right] \\ +\eta_i\Xi\left[(1-\alpha_i)\sigma_i + \alpha_i\Xi\sigma_i\right] \end{array}\right)\right] \\ +\gamma_i\Xi\left[\Xi\left(\begin{array}{c} (1-\eta_i)\left[(1-\alpha_i)\sigma_i + \alpha_i\Xi\sigma_i\right] \\ +\eta_i\Xi\left[(1-\alpha_i)\sigma_i + \alpha_i\Xi\sigma_i\right] \end{array}\right)\right] \end{array}\right)\right) - \zeta \right\| \\
= \; & \theta^3\left(1 - (1-\theta)\gamma_i\right)\|\sigma_i - \zeta\| + \phi_i,
\end{aligned}$$

for $i \geq 1$, set

$$\varphi_i = \|\sigma_i - \zeta\|, \; z_i = (1-\theta)\gamma_i \in (0,1) \text{ and } \varphi_i^* = \phi_i.$$

Because $\lim_{i\to\infty}\phi_i = 0$, then $\lim_{i\to\infty}\frac{\varphi_i^*}{z_i} = \lim_{i\to\infty}\frac{\phi_i}{z_i} = 0$. Thus, all requirements of Lemma 2.3 are fulfilled. Hence $\lim_{i\to\infty}\|\sigma_i - \zeta\| = 0$, that is $\lim_{i\to\infty}\sigma_i = \zeta$.

Conversely, assume that $\lim_{i\to\infty}\sigma_i = \zeta$, then, we get

$$\begin{aligned}
\phi_i &= \|\sigma_{i+1} - f(\Xi,\sigma_i)\| \\
&= \|\sigma_{i+1} - \zeta + \zeta - f(\Xi,\sigma_i)\| \\
&\leq \|\sigma_{i+1} - \zeta\| + \|\zeta - f(\Xi,\sigma_i)\| \\
&\leq \|\sigma_{i+1} - \zeta\| + \theta^3\left(1-(1-\theta)\gamma_i\right)\|\sigma_i - \zeta\|,
\end{aligned}$$

taking the limit, we have $\lim_{i\to\infty}\phi_i = 0$. This finishes the proof. □

The following example support Theorem 5.1.

Example 5.1. *Consider $\Lambda = [0,1]$ and $\Xi\nu = \frac{\nu}{2}$. Clearly, 0 is a FP of Ξ. To prove Ξ satisfies the condition (2.1), let $\theta = \frac{1}{2}$ and for each $\ell \geq 0$, we get*

$$\begin{aligned}
\|\Xi\nu - \Xi\varpi\| - \theta\|\nu-\varpi\| - \ell\|\nu - \Xi\nu\| &= \frac{1}{2}\|\nu-\varpi\| - \frac{1}{2}\|\nu-\varpi\| - \ell\left\|\nu - \frac{\nu}{2}\right\| \\
&= -\frac{\ell}{2}\nu \leq 0, \text{ for all } \nu,\varpi\in\Lambda.
\end{aligned}$$

Now, we illustrate that our algorithm (1.5) is Ξ-stable. Suppose that $\alpha_i = \eta_i = \gamma_i = \frac{1}{i+1}$ and $\nu_0 \in [0,1]$, then we obtain

$$\begin{aligned}
\varpi_i &= (1-\frac{1}{i+1})\nu_i + \frac{1}{i+1}\left(\frac{\nu_i}{2}\right) = \left(1 - \frac{1}{2(i+1)}\right)\nu_i, \\
\wp_i &= \frac{1}{2}\left((1-\frac{1}{i+1})\varpi_i + \frac{1}{i+1}\left(\frac{\varpi_i}{2}\right)\right) = \frac{1}{2}\left(1 - \frac{1}{(i+1)} + \frac{1}{4(i+1)^2}\right)\nu_i, \\
\Im_i &= \Xi\left((1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i\right) = \frac{1}{4}\left(1 - \frac{3}{2(i+1)} + \frac{3}{4(i+1)^2} - \frac{1}{8(i+1)^3}\right)\nu_i, \\
\nu_{i+1} &= \Xi\Im_i = \frac{1}{8}\left(1 - \frac{3}{2(i+1)} + \frac{3}{4(i+1)^2} - \frac{1}{8(i+1)^3}\right)\nu_i \\
&= \left(1 - \left(\frac{7}{8} + \frac{3}{2(i+1)} - \frac{3}{4(i+1)^2} + \frac{1}{8(i+1)^3}\right)\right)\nu_i.
\end{aligned}$$

Take $\sigma_i = \frac{7}{8} + \frac{3}{2(i+1)} - \frac{3}{4(i+1)^2} + \frac{1}{8(i+1)^3}$. It is clear that $\sigma_i \in (0,1)$ for all $i\in\mathbb{N}$, $\sum_{i=0}^{\infty}\sigma_i = 0$. According to Lemma 2.3, $\lim_{i\to\infty}\nu_i = 0$. Choose $\hbar_i = \frac{1}{i+2}$, we get

$$\begin{aligned}
\phi_i &= \|\hbar_{i+1} - f(\Xi,\hbar_i)\| \\
&\left\| \frac{1}{i+3} - \frac{1}{4}\left(\begin{array}{c} (1-\frac{1}{i+1})\left[\left(\begin{array}{c} (1-\frac{1}{i+1})\left[(1-\frac{1}{i+1})\hbar_i + \frac{1}{i+1}\frac{\hbar_i}{2}\right] \\ +\frac{1}{2(i+1)}\left[(1-\frac{1}{i+1})\hbar_i + \frac{1}{i+1}\frac{\hbar_i}{2}\right] \end{array}\right)\right] \\ +\frac{1}{2(i+1)}\left[\left(\begin{array}{c} (1-\frac{1}{i+1})\left[(1-\frac{1}{i+1})\hbar_i + \frac{1}{i+1}\frac{\hbar_i}{2}\right] \\ +\frac{1}{2(i+1)}\left[(1-\frac{1}{i+1})\hbar_i + \frac{1}{i+1}\frac{\hbar_i}{2}\right] \end{array}\right)\right] \end{array}\right)\right\|
\end{aligned}$$

$$= \left\| \frac{1}{i+3} - \frac{1}{4} \left(\begin{array}{c} (1 - \frac{1}{i+1}) \left[\left(1 - \frac{1}{(i+1)} + \frac{1}{4(i+1)^2}\right) \hbar_i \right] \\ + \frac{1}{2(i+1)} \left[\left(1 - \frac{1}{(i+1)} + \frac{1}{4(i+1)^2}\right) \hbar_i \right] \end{array} \right) \right\|$$

$$= \left\| \frac{1}{i+3} - \left(\frac{1}{4} - \frac{3}{4(i+1)} + \frac{3}{16(i+1)^2} - \frac{1}{32(i+1)^3} \right) \hbar_i \right\|$$

$$= \left\| \frac{1}{i+3} - \left(\frac{1}{4(i+2)} - \frac{3}{4(i+1)(i+2)} + \frac{3}{16(i+1)^2(i+2)} - \frac{1}{32(i+1)^3(i+2)} \right) \right\|,$$

when $i \to \infty$, we have $\lim_{i\to\infty} \phi_i = 0$. This implies that our iterative scheme is Ξ-stable with respect to Ξ.

6. Data-dependence theorem

In this part, we present the data-dependence result for the operator Ξ satisfies the inequality (2.1) via our iterative algorithm (1.5).

Theorem 6.1. *Let $\widehat{\Xi}$ be an approximation operator for a mapping Ξ fulfills (2.1). Suppose that $\{\nu_i\}$ is the iterative sequence given by (1.5) for Ξ. Define an iterative sequence $\{\widehat{\nu_i}\}$ for $\widehat{\Xi}$ as follows:*

$$\begin{cases} \widehat{\nu}_\circ \in \Delta, \\ \widehat{\varpi}_i = (1-\alpha_i)\widehat{\nu_i} + \alpha_i \widehat{\Xi}\widehat{\nu_i}, \\ \widehat{\wp}_i = \widehat{\Xi}\left((1-\eta_i)\widehat{\varpi}_i + \eta_i \widehat{\Xi}\widehat{\varpi}_i\right), \\ \widehat{\Im}_i = \widehat{\Xi}\left((1-\gamma_i)\widehat{\wp}_i + \gamma_i \widehat{\Xi}\widehat{\wp}_i\right), \\ \widehat{\nu}_{i+1} = \widehat{\Xi}\widehat{\Im}_i, \end{cases} \tag{6.1}$$

for all $i \geq 1$, where $\{\alpha_i\}$, $\{\eta_i\}$ and $\{\gamma_i\}$ are sequences in $[0,1]$ such that

(p_1) *for all $i \geq 1$, $2\gamma_i \geq 1$,*
(p_2) $\sum_{i=0}^{\infty} \gamma_i = \infty$.

If $\Xi\zeta = \zeta$ and $\widehat{\Xi}\widehat{\zeta} = \widehat{\zeta}$ so that $\lim_{i\to\infty} \widehat{\nu_i} = \widehat{\zeta}$, then

$$\left\| \zeta - \widehat{\zeta} \right\| \leq \frac{14\epsilon}{(1-\theta)},$$

for any fixed number $\epsilon > 0$.

Proof. It follows from (2.1), (1.5), (6.1) and Definition 2.7 that

$$\begin{aligned} \left\| \varpi_i - \widehat{\varpi}_i \right\| &= \left\| (1-\alpha_i)\nu_i + \alpha_i \Xi \nu_i - \left((1-\alpha_i)\widehat{\nu_i} + \alpha_i \widehat{\Xi}\widehat{\nu_i}\right) \right\| \\ &= \left\| (1-\alpha_i)(\nu_i - \widehat{\nu_i}) + \alpha_i \left(\Xi\nu_i - \widehat{\Xi}\widehat{\nu_i}\right) \right\| \\ &\leq (1-\alpha_i) \left\| \nu_i - \widehat{\nu_i} \right\| + \alpha_i \left(\left\| \Xi\nu_i - \Xi\widehat{\nu_i} \right\| + \left\| \Xi\widehat{\nu_i} - \widehat{\Xi}\widehat{\nu_i} \right\| \right) \\ &\leq (1-\alpha_i) \left\| \nu_i - \widehat{\nu_i} \right\| + \alpha_i\theta \left\| \nu_i - \widehat{\nu_i} \right\| + \alpha_i \ell \left\| \nu_i - \Xi\nu_i \right\| + \alpha_i \epsilon \\ &= (1-(1-\theta)\alpha_i) \left\| \nu_i - \widehat{\nu_i} \right\| + \alpha_i \ell \left\| \nu_i - \Xi\nu_i \right\| + \alpha_i \epsilon. \end{aligned} \tag{6.2}$$

Again, using (2.1), (1.5), (6.1) and Definition 2.7, we have

$$\begin{aligned}
\left\|\wp_i - \widehat{\wp}_i\right\| &= \left\|\Xi\left((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i\right) - \widehat{\Xi}\left((1-\eta_i)\widehat{\varpi}_i + \eta_i\widehat{\Xi}\widehat{\varpi}_i\right)\right\| \\
&\leq \left\|\Xi\left((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i\right) - \Xi\left((1-\eta_i)\widehat{\varpi}_i + \eta_i\widehat{\Xi}\widehat{\varpi}_i\right)\right\| \\
&\quad + \left\|\Xi\left((1-\eta_i)\widehat{\varpi}_i + \eta_i\widehat{\Xi}\widehat{\varpi}_i\right) - \widehat{\Xi}\left((1-\eta_i)\widehat{\varpi}_i + \eta_i\widehat{\Xi}\widehat{\varpi}_i\right)\right\| \\
&\leq \theta\Big((1-\eta_i)\left\|\varpi_i - \widehat{\varpi}_i\right\| + \eta_i\left\|\Xi\varpi_i - \widehat{\Xi}\widehat{\varpi}_i\right\|\Big) \\
&\quad + \ell\left[(1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i - \Xi\left((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i\right)\right] + \epsilon \\
&\leq \theta\Big((1-\eta_i)\left\|\varpi_i - \widehat{\varpi}_i\right\| + \eta_i\Big(\left\|\Xi\varpi_i - \Xi\widehat{\varpi}_i\right\| + \left\|\Xi\widehat{\varpi}_i - \widehat{\Xi}\widehat{\varpi}_i\right\|\Big)\Big) \\
&\quad + \ell\left[(1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i - \Xi\left((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i\right)\right] + \epsilon \\
&\leq \theta\Big((1-\eta_i)\left\|\varpi_i - \widehat{\varpi}_i\right\| + \eta_i\theta\left\|\varpi_i - \widehat{\varpi}_i\right\| + \eta_i\ell\left\|\varpi_i - \Xi\varpi_i\right\| + \epsilon\Big) \\
&\quad + \ell\left[(1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i - \Xi\left((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i\right)\right] + \epsilon \\
&= \theta\left(1-(1-\theta)\eta_i\right)\left\|\varpi_i - \widehat{\varpi}_i\right\| + \theta\eta_i\ell\left\|\varpi_i - \Xi\varpi_i\right\| + \theta\epsilon \\
&\quad + \ell\left[(1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i - \Xi\left((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i\right)\right] + \epsilon. \qquad (6.3)
\end{aligned}$$

Applying (6.2) on (6.3), we get

$$\begin{aligned}
\left\|\wp_i - \widehat{\wp}_i\right\| &= \theta\left(1-(1-\theta)\eta_i\right)\Big[(1-(1-\theta)\alpha_i)\left\|\nu_i - \widehat{\nu_i}\right\| + \alpha_i\ell\left\|\nu_i - \Xi\nu_i\right\| + \alpha_i\epsilon\Big] \\
&\quad + \theta\eta_i\ell\left\|\varpi_i - \Xi\varpi_i\right\| + \theta\epsilon + \epsilon \\
&\quad + \ell\left[(1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i - \Xi\left((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i\right)\right] \\
&= \theta\left(1-(1-\theta)\eta_i\right)\left(1-(1-\theta)\alpha_i\right)\left\|\nu_i - \widehat{\nu_i}\right\| \\
&\quad + \theta\left(1-(1-\theta)\eta_i\right)\alpha_i\ell\left\|\nu_i - \Xi\nu_i\right\| + \theta\alpha_i\epsilon + \theta\eta_i\alpha_i\epsilon\left(\theta-1\right) \\
&\quad + \theta\eta_i\ell\left\|\varpi_i - \Xi\varpi_i\right\| + \theta\epsilon + \epsilon \\
&\quad + \ell\left[(1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i - \Xi\left((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i\right)\right] \\
&= \theta\left(1-(1-\theta)\eta_i\right)\left(1-(1-\theta)\alpha_i\right)\left\|\nu_i - \widehat{\nu_i}\right\| \\
&\quad + \theta\left(1-(1-\theta)\eta_i\right)\alpha_i\ell\left\|\nu_i - \Xi\nu_i\right\| + \theta\eta_i\ell\left\|\varpi_i - \Xi\varpi_i\right\| \\
&\quad + \ell\left[(1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i - \Xi\left((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i\right)\right] \\
&\quad + \theta\alpha_i\epsilon + \theta\eta_i\alpha_i\epsilon\left(\theta-1\right) + \theta\epsilon + \epsilon. \qquad (6.4)
\end{aligned}$$

Similar to (6.3), one can write

$$\begin{aligned}
\left\|\mathfrak{I}_i - \widehat{\mathfrak{I}}_i\right\| &\leq \theta\left(1-(1-\theta)\gamma_i\right)\left\|\wp_i - \widehat{\wp}_i\right\| + \theta\gamma_i\ell\left\|\wp_i - \Xi\wp_i\right\| + \theta\epsilon \\
&\quad + \ell\left[(1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i - \Xi\left((1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i\right)\right] + \epsilon. \qquad (6.5)
\end{aligned}$$

Applying (6.4) on (6.5), we have

$$\begin{aligned}
\left\|\mathfrak{I}_i - \widehat{\mathfrak{I}}_i\right\| &\leq \theta\left(1-(1-\theta)\gamma_i\right)\Big\{\theta\left(1-(1-\theta)\eta_i\right)\left(1-(1-\theta)\alpha_i\right)\left\|\nu_i - \widehat{\nu_i}\right\| \\
&\quad \theta\left(1-(1-\theta)\eta_i\right)\alpha_i\ell\left\|\nu_i - \Xi\nu_i\right\| + \theta\eta_i\ell\left\|\varpi_i - \Xi\varpi_i\right\| \\
&\quad + \ell\left[(1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i - \Xi\left((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\theta\alpha_i\epsilon + \theta\eta_i\alpha_i\epsilon(\theta-1) + \theta\epsilon + \epsilon\} \\
&+\theta\gamma_i\ell\|\wp_i - \Xi\wp_i\| + \theta\epsilon + \epsilon \\
&+\ell\left[(1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i - \Xi((1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i)\right] \\
= \;& \theta^2(1-(1-\theta)\gamma_i)(1-(1-\theta)\eta_i)(1-(1-\theta)\alpha_i)\left\|\nu_i - \widehat{\nu_i}\right\| \\
&+\theta^2(1-(1-\theta)\gamma_i)(1-(1-\theta)\eta_i)\alpha_i\ell\|\nu_i - \Xi\nu_i\| \\
&+\theta^2(1-(1-\theta)\gamma_i)\eta_i\ell\|\varpi_i - \Xi\varpi_i\| \\
&+\theta(1-(1-\theta)\gamma_i)\ell\left[(1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i - \Xi((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i)\right] \\
&+\theta^2\alpha_i\epsilon - \theta^2\gamma_i\alpha_i\epsilon + \theta^3\gamma_i\alpha_i\epsilon + \theta^2\eta_i\alpha_i\epsilon(\theta-1) - \theta^2\gamma_i\eta_i\alpha_i\epsilon(\theta-1) \\
&+\theta^3\gamma_i\eta_i\alpha_i\epsilon(\theta-1) + \theta^2\epsilon - \theta^2\gamma_i\epsilon + \theta^3\gamma_i\epsilon + \theta\epsilon - \theta\gamma_i\epsilon + \theta^2\gamma_i\epsilon \\
&+\theta\gamma_i\ell\|\wp_i - \Xi\wp_i\| + \theta\epsilon + \epsilon \\
&+\ell\left[(1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i - \Xi((1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i)\right],
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\left\|\Im_i - \widehat{\Im}_i\right\| \le \;& \theta^2(1-(1-\theta)\gamma_i)(1-(1-\theta)\eta_i)(1-(1-\theta)\alpha_i)\left\|\nu_i - \widehat{\nu_i}\right\| \\
&+\theta^2(1-(1-\theta)\gamma_i)(1-(1-\theta)\eta_i)\alpha_i\ell\|\nu_i - \Xi\nu_i\| \\
&+\theta^2(1-(1-\theta)\gamma_i)\eta_i\ell\|\varpi_i - \Xi\varpi_i\| \\
&+\theta(1-(1-\theta)\gamma_i)\ell\left[(1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i - \Xi((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i)\right] \\
&+\ell\left[(1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i - \Xi((1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i)\right] \\
&+\theta\gamma_i\ell\|\wp_i - \Xi\wp_i\| + \theta^2\alpha_i\epsilon + \theta^2\gamma_i\alpha_i\epsilon(\theta-1) + \theta^2\eta_i\alpha_i\epsilon(\theta-1) \\
&+\theta^2\gamma_i\eta_i\alpha_i\epsilon(\theta-1)^2 + \theta^2\epsilon + \theta^2\gamma_i\epsilon(\theta-1) + 2\theta\epsilon + \theta\gamma_i\epsilon(\theta-1) + \epsilon. \qquad (6.6)
\end{aligned}
$$

Finally, from (2.1), (1.5) and (6.6), we get

$$
\begin{aligned}
\left\|\nu_{i+1} - \widehat{\nu}_{i+1}\right\| = \;& \left\|\Xi\Im_i - \widehat{\Xi}\widehat{\Im}_i\right\| \\
= \;& \left\|\Xi\Im_i - \Xi\widehat{\Im}_i + \Xi\widehat{\Im}_i - \widehat{\Xi}\widehat{\Im}_i\right\| \\
\le \;& \left\|\Xi\Im_i - \Xi\widehat{\Im}_i\right\| + \left\|\Xi\widehat{\Im}_i - \widehat{\Xi}\widehat{\Im}_i\right\| \\
\le \;& \theta\left\|\Im_i - \widehat{\Im}_i\right\| + \ell\left\|\Im_i - \Xi\Im_i\right\| + \epsilon \\
\le \;& \theta^3(1-(1-\theta)\gamma_i)(1-(1-\theta)\eta_i)(1-(1-\theta)\alpha_i)\left\|\nu_i - \widehat{\nu_i}\right\| \\
&+\theta^3(1-(1-\theta)\gamma_i)(1-(1-\theta)\eta_i)\alpha_i\ell\|\nu_i - \Xi\nu_i\| \\
&+\theta^3(1-(1-\theta)\gamma_i)\eta_i\ell\|\varpi_i - \Xi\varpi_i\| \\
&+\theta^2(1-(1-\theta)\gamma_i)\ell\left[(1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i - \Xi((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i)\right] \\
&+\theta\ell\left[(1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i - \Xi((1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i)\right] \\
&+\theta^2\gamma_i\ell\|\wp_i - \Xi\wp_i\| + \ell\left\|\Im_i - \Xi\Im_i\right\| + \theta^3\alpha_i\epsilon + \theta^3\gamma_i\alpha_i\epsilon(\theta-1) \\
&+\theta^3\eta_i\alpha_i\epsilon(\theta-1) + \theta^3\gamma_i\eta_i\alpha_i\epsilon(\theta-1)^2 + \theta^3\epsilon + \theta^3\gamma_i\epsilon(\theta-1) + 2\theta^2\epsilon \\
&+\theta^2\gamma_i\epsilon(\theta-1) + \theta\epsilon + \epsilon. \qquad (6.7)
\end{aligned}
$$

Since $\alpha_i, \eta_i, \gamma_i \in [0,1]$ and $\theta \in (0,1)$, we conclude that

$$\begin{cases} (1-(1-\theta)\alpha_i) < 1, \\ (1-(1-\theta)\eta_i) < 1, \\ (1-(1-\theta)\gamma_i) < 1, \\ (\theta - 1) < 0, \\ \theta^3, \theta^2, \theta < 1, \\ \theta^3\alpha_i, \theta^3\gamma_i, \theta^3\eta_i < 1. \end{cases} \tag{6.8}$$

Applying (6.8) in (6.7), we have

$$\begin{aligned} \left\| \nu_{i+1} - \widehat{\nu}_{i+1} \right\| \leq\ & (1-(1-\theta)\gamma_i) \left\| \nu_i - \widehat{\nu}_i \right\| + \ell \left\| \nu_i - \Xi\nu_i \right\| + \ell \left\| \varpi_i - \Xi\varpi_i \right\| \\ & + \ell \left[(1-\eta_i)\varpi_i + \eta_i \Xi\varpi_i - \Xi\left((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i \right) \right] \\ & + \ell \left[(1-\gamma_i)\wp_i + \gamma_i \Xi\wp_i - \Xi\left((1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i \right) \right] \\ & + \ell \left\| \Im_i - \Xi\Im_i \right\| + \gamma_i \ell \left\| \wp_i - \Xi\wp_i \right\| + 2\gamma_i\epsilon + 6\epsilon. \end{aligned}$$

From the axiom $2\gamma_i \geq 1$, we can obtain

$$\begin{aligned} & \left\| \nu_{i+1} - \widehat{\nu}_{i+1} \right\| \\ \leq\ & (1-(1-\theta)\gamma_i) \left\| \nu_i - \widehat{\nu}_i \right\| + 2\gamma_i\ell \left\| \nu_i - \Xi\nu_i \right\| \\ & + 2\gamma_i\ell \left\| \varpi_i - \Xi\varpi_i \right\| + 2\gamma_i\ell \left\| \Im_i - \Xi\Im_i \right\| + \gamma_i\ell \left\| \wp_i - \Xi\wp_i \right\| \\ & + 2\gamma_i\ell \left[(1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i - \Xi\left((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i \right) \right] \\ & + 2\gamma_i\ell \left[(1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i - \Xi\left((1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i \right) \right] \\ & + 2\gamma_i\epsilon + 12\gamma_i\epsilon \\ =\ & (1-(1-\theta)\gamma_i) \left\| \nu_i - \widehat{\nu}_i \right\| \\ & + \gamma_i(1-\theta) \times \left\{ \frac{2\ell \left\| \nu_i - \Xi\nu_i \right\| + 2\ell \left\| \varpi_i - \Xi\varpi_i \right\| + 2\ell \left\| \Im_i - \Xi\Im_i \right\| + \ell \left\| \wp_i - \Xi\wp_i \right\|}{(1-\theta)} \right. \\ & + \frac{2\ell \left[(1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i - \Xi\left((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i \right) \right]}{(1-\theta)} \\ & \left. + \frac{2\ell \left[(1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i - \Xi\left((1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i \right) \right] + 14\epsilon}{(1-\theta)} \right\}. \end{aligned}$$

Put $\varphi_i = \left\| \nu_i - \widehat{\nu}_i \right\|$, $z_i = (1-\theta)\gamma_i \in (0,1)$ and

$$\begin{aligned} \varphi_i^* =\ & \left\{ \frac{2\ell \left\| \nu_i - \Xi\nu_i \right\| + 2\ell \left\| \varpi_i - \Xi\varpi_i \right\| + 2\ell \left\| \Im_i - \Xi\Im_i \right\| + \ell \left\| \wp_i - \Xi\wp_i \right\|}{(1-\theta)} \right. \\ & + \frac{2\ell \left[(1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i - \Xi\left((1-\eta_i)\varpi_i + \eta_i\Xi\varpi_i \right) \right]}{(1-\theta)} \\ & \left. + \frac{2\ell \left[(1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i - \Xi\left((1-\gamma_i)\wp_i + \gamma_i\Xi\wp_i \right) \right] + 14\epsilon}{(1-\theta)} \right\}. \end{aligned}$$

We know that from Theorem 3.1, $\lim_{i\to\infty} \nu_i = \zeta$ and since $\Xi\zeta = \zeta$, we have

$$\begin{aligned}\lim_{i\to\infty}\|\nu_i-\Xi\nu_i\| &= \lim_{i\to\infty}\|\varpi_i-\Xi\varpi_i\| \\ &= \lim_{i\to\infty}\left\|\Im_i-\Xi\Im_i\right\| = \lim_{i\to\infty}\|\wp_i-\Xi\wp_i\| \\ &= \lim_{i\to\infty}\left[(1-\eta_i)\varpi_i+\eta_i\Xi\varpi_i-\Xi\left((1-\eta_i)\varpi_i+\eta_i\Xi\varpi_i\right)\right] \\ &= \lim_{i\to\infty}\left[(1-\gamma_i)\wp_i+\gamma_i\Xi\wp_i-\Xi\left((1-\gamma_i)\wp_i+\gamma_i\Xi\wp_i\right)\right]=0.\end{aligned}$$

Hence, from Lemma 2.4, we get

$$0\leq\limsup_{i\to\infty}\left\|\nu_i-\widehat{\nu_i}\right\|\leq\limsup_{i\to\infty}\frac{14\epsilon}{(1-\theta)}. \tag{6.9}$$

Since $\lim_{i\to\infty}\nu_i=\zeta$ and by our axiom $\lim_{i\to\infty}\widehat{\nu_i}=\widehat{\zeta}$, the inequality (6.9) leads to

$$\left\|\zeta-\widehat{\zeta}\right\|\leq\frac{14\epsilon}{(1-\theta)}.$$

This completes the proof. □

7. Numerical examples

The following example supports the analytical results obtained from Theorem 3.2 and studies the performance and speed of our algorithm compared with the previous algorithms.

Example 7.1. *Let* $\Lambda=\mathbb{R}$, $\Delta=[0,50]$, *and* $\Xi:\Delta\to\Delta$ *be a mapping defined by*

$$\Xi(\nu)=\sqrt{\nu^2-9\nu+54}.$$

Clearly, 6.0000 *is a FP of the mapping* Ξ. *Take* $\alpha_i=\eta_i=\gamma_i=\frac{1}{5(i+2)}$, *with different initial values. Then we obtain the following tables (see Tables 1–3) and graph (see Figures 1–3) for comparison of the various iterative methods.*

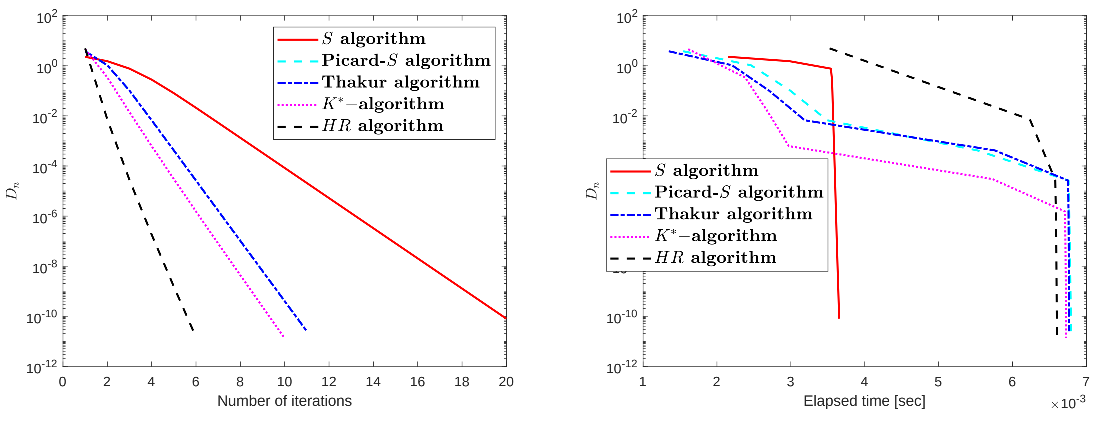

Figure 1. Graphically comparison of the proposed algorithm (*HR* **algorithm**) when $\nu_\circ=1$.

Table 1. Example 7.1: Numerical effectiveness comparison of the proposed algorithm (HR algorithm) when $\nu_\circ = 1$.

Iter (n)	S algorithm	Picard-S algorithm	Thakur algorithm	K^*-algorithm	HR algorithm
1	8.70091704981746	7.16921920849454	7.16914772374443	6.36059443309194	6.00727524833131
2	7.16526232112099	6.10977177957558	6.10975923118057	6.01468466644452	6.00002780412529
3	6.39088232469395	6.00714403607375	6.00714316980988	6.00065450058063	6.00000018328059
4	6.10931564922512	6.00044721072023	6.00044715630748	6.00003168526056	6.00000000153606
5	6.02823416956883	6.00002793225376	6.00002792885454	6.00000161316022	6.00000000001474
6	6.00711636371271	6.00000174471605	6.00000174450373	6.00000008491151	6.00000000000015
7	6.00178220920879	6.00000010899371	6.00000010898045	6.00000000457683	
8	6.00044563525887	6.00000000680959	6.00000000680876	6.00000000025113	
9	6.00011139089900	6.00000000042547	6.00000000042542	6.00000000001397	
10	6.00002784178933	6.00000000002659	6.00000000002658	6.00000000000079	
11	6.00000695905778	6.00000000000166	6.00000000000166		
12	6.00000173945939				
13	6.00000043479852				
14	6.00000010868515				
15	6.00000002716811				
16	6.00000000679132				
17	6.00000000169767				
18	6.00000000042438				
19	6.00000000010609				
20	6.00000000002652				

Note: CPU time in seconds, respectively: 0.0036537, 0.0067923, 0.0067674, 0.0067261, 0.0066002.

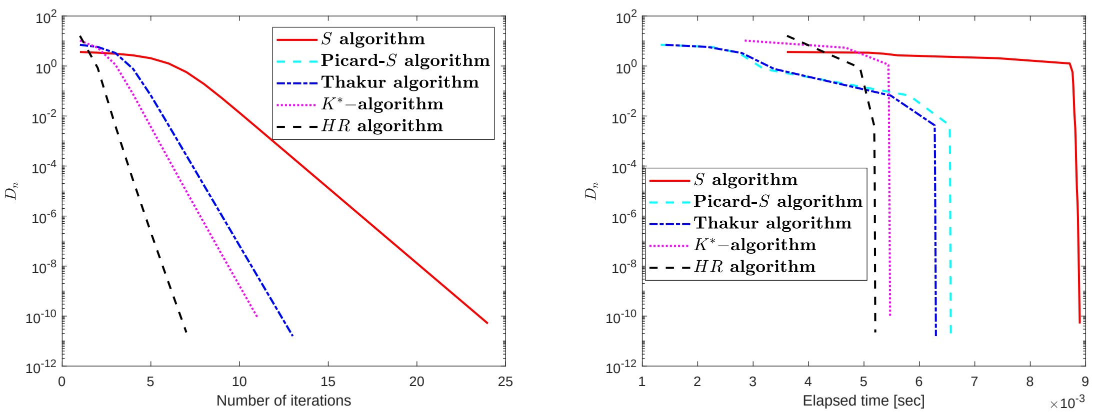

Figure 2. Graphically comparison of the proposed algorithm (HR **algorithm**) when $\nu_\circ = 23$.

Table 2. Example 7.1: Numerical effectiveness comparison of the proposed algorithm (*HR* **algorithm**) when $\nu_\circ = 23$.

Iter (n)	S algorithm	Picard-S algorithm	Thakur algorithm	K^*-algorithm	HR algorithm
1	19.3563152555029	15.9518056335603	15.9517798586745	12.5877267284391	6.85064626172682
2	15.9377280808459	10.1547429779848	10.1547063746371	7.21310921795745	6.00404110670722
3	12.8216267389821	6.83637415013381	6.83635232023217	6.07773436689889	6.00002666859578
4	10.1453665006161	6.07061076131832	6.07060799598213	6.00385998151440	6.00000022350839
5	8.09904894091384	6.00453182122915	6.00453163699128	6.00019677562843	6.00000000214472
6	6.83342864338475	6.00028356649349	6.00028355493982	6.00001035833149	6.00000000002245
7	6.26044908931579	6.00001771655983	6.00001771583789	6.00000055832830	6.00000000000025
8	6.07032543085564	6.00000110688254	6.00000110683744	6.00000003063582	
9	6.01796113338512	6.00000006915944	6.00000006915662	6.00000000170455	
10	6.00451434416413	6.00000000432139	6.00000000432122	6.00000000009591	
11	6.00112994220417	6.00000000027003	6.00000000027002	6.00000000000545	
12	6.00028253511929	6.00000000001687	6.00000000001687		
13	6.00007062920329	6.00000000000105	6.00000000000105		
14	6.00001765533615				
15	6.00000441334114				
16	6.00000110322227				
17	6.00000027578013				
18	6.00000006893931				
19	6.00000001723354				
20	6.00000000430809				
21	6.00000000107696				
22	6.00000000026923				
23	6.00000000006730				
24	6.00000000001683				

Note: CPU time in seconds, respectively: 0.0088935, 0.0065673, 0.0062978, 0.0054715, 0.0052041.

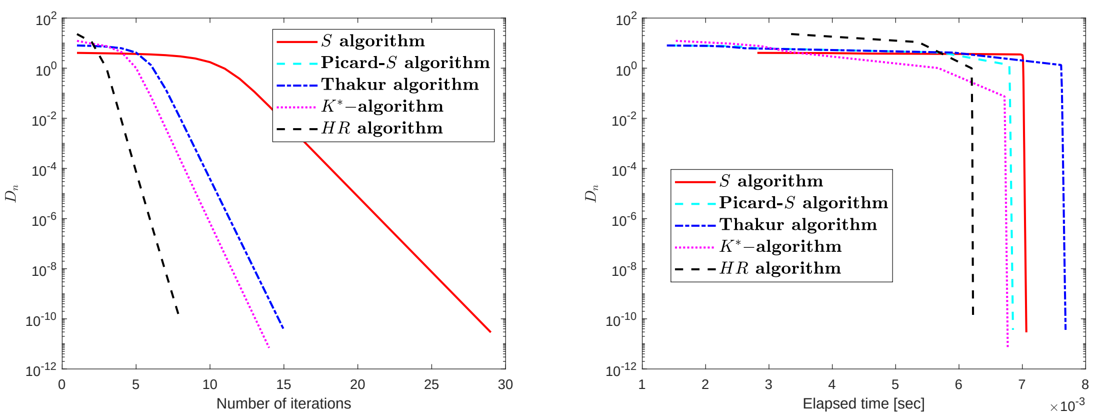

Figure 3. Graphically comparison of proposed algorithm (*HR* **algorithm**) when $\nu_\circ = 41$.

Table 3. Example 7.1: Numerical effectiveness comparison of the proposed algorithm (*HR* **algorithm**) when $\nu_\circ = 41$.

Iter (n)	S **algorithm**	**Picard-S algorithm**	**Thakur algorithm**	K^***-algorithm**	*HR* **algorithm**
1	36.9195393902310	32.9359459295575	32.9359410372494	28.6777050430159	18.0010100671820
2	32.9185209048623	25.1854713159820	25.1854637821918	19.1184050275537	6.99692674147933
3	28.9966652255498	17.9433658908193	17.9433563898048	11.5396125686070	6.00870649597241
4	25.1701649172411	11.6863804499820	11.6863697408416	7.09098245036886	6.00007316294814
5	21.4670877470757	7.49707008913586	7.49706191231517	6.07877285288080	6.00000070206652
6	17.9313757495734	6.15612940775542	6.15612775011077	6.00426075582864	6.00000000734890
7	14.6320375697776	6.01035659987680	6.01035647945727	6.00023000499555	6.00000000008173
8	11.6781275774842	6.00064966715110	6.00064965955411	6.00001262154899	6.00000000000095
9	9.23371552492475	6.00004060231534	6.00004060184040	6.00000070225384	
10	7.49350575600870	6.00000253705577	6.00000253702609	6.00000003951159	
11	6.53524954796329	6.00000015853311	6.00000015853125	6.00000000224357	
12	6.15563769964487	6.00000000990656	6.00000000990645	6.00000000012838	
13	6.04078294734902	6.00000000061907	6.00000000061906	6.00000000000739	
14	6.01032406840519	6.00000000003869	6.00000000003869	6.00000000000043	
15	6.00258903649963	6.00000000000242	6.00000000000242		
16	6.00064771546926				
17	6.00016194664418				
18	6.00004048534517				
19	6.00001012070523				
20	6.00000253001219				
21	6.00000063246434				
22	6.00000015810715				
23	6.00000003952473				
24	6.00000000988071				
25	6.00000000247007				
26	6.00000000061749				
27	6.00000000015437				
28	6.00000000003859				
29	6.00000000000965				

Note: CPU time in seconds, respectively: 0.0070634, 0.0068518, 0.0076826, 0.0067694, 0.006222.

In the next example, we consider a mapping Ξ as SGNM but not nonexpansive and we will show under certain conditions that our algorithm (1.5) is superior in behavior to some of the leading iterative methods in the previous literature in terms of convergence speed.

Example 7.2. *Consider a mapping* $\Xi : [0,1] \to [0,1]$ *described by*

$$\Xi(\nu) = \begin{cases} 1-\nu, & if\ \nu \in [0, \frac{1}{14}), \\ \frac{\nu+13}{14}, & if\ \nu \in [\frac{1}{14}, 1]. \end{cases}$$

Now, we illustrate that Ξ *is a SGNM but not nonexpansive. Set* $\nu = \frac{7}{100}$ *and* $\varpi = \frac{1}{14}$, *we get*

$$\|\Xi\nu - \Xi\varpi\| = |\Xi\nu - \Xi\varpi| = \left|1 - \nu - \left(\frac{\varpi + 13}{14}\right)\right| = \left|\frac{93}{100} - \frac{183}{196}\right| = \frac{9}{2450},$$

and

$$\|\nu - \varpi\| = |\nu - \varpi| = \frac{1}{700}.$$

It follows that $\|\Xi\nu - \Xi\varpi\| = \frac{9}{2450} > \frac{1}{700} = \|\nu - \varpi\|$. *Thus* Ξ *is not nonexpansive mapping. To prove that* Ξ *is a SGNM, we consider the cases below:*

(1) If $\nu \in [0, \frac{1}{14})$, *then*

$$\frac{1}{2}\|\nu - \Xi\nu\| = \frac{1}{2}|\nu - (1-\nu)| = \frac{1-2\nu}{2} \in (\frac{3}{7}, \frac{1}{2}].$$

For $\frac{1}{2}\|\nu - \Xi\nu\| \leq \|\nu - \varpi\|$, *we must obtain* $\frac{1-2\nu}{2} \leq |\nu - \varpi|$. *Clearly* $\varpi < \nu$ *impossible. So, we must take* $\varpi > \nu$. *Thus,* $\frac{1-2\nu}{2} \leq \varpi - \nu$, *which yields* $\varpi \geq \frac{1}{2}$ *and hence* $\varpi \in [\frac{1}{2}, 1]$. *Now,*

$$\|\Xi\nu - \Xi\varpi\| = \left|\frac{\varpi + 13}{14} - (1-\nu)\right| = \left|\frac{\varpi + 14\nu - 1}{14}\right| < \frac{1}{14},$$

and

$$\|\nu - \varpi\| = \left|\frac{1}{14} - \frac{1}{2}\right| = \frac{3}{7}.$$

Hence

$$\frac{1}{2}\|\nu - \Xi\nu\| \leq \|\nu - \varpi\| \text{ implies } \|\Xi\nu - \Xi\varpi\| < \frac{1}{14} < \frac{3}{7} = \|\nu - \varpi\|.$$

(2) If $\nu \in [\frac{1}{14}, 1]$, *then*

$$\frac{1}{2}\|\nu - \Xi\nu\| = \frac{1}{2}\left|\frac{\nu + 13}{14} - \nu\right| = \frac{13 - 13\nu}{28} \in [0, \frac{169}{392}].$$

For $\frac{1}{2}\|\nu - \Xi\nu\| \leq \|\nu - \varpi\|$, *we have* $\frac{13-13\nu}{28} \leq |\nu - \varpi|$, *which leads to the following possibilities:*

(*i*) *When* $\nu < \varpi$, *we have*

$$\frac{13 - 13\nu}{28} \leq \varpi - \nu \Longrightarrow \varpi \geq \frac{13 + 15\nu}{28} \Longrightarrow \varpi \in [\frac{197}{392}, 1] \subset [\frac{1}{14}, 1].$$

So

$$\|\Xi\nu - \Xi\varpi\| = \left|\frac{\nu + 13}{14} - \frac{\varpi + 13}{14}\right| = \frac{1}{14}|\nu - \varpi| \leq |\nu - \varpi|.$$

Hence

$$\frac{1}{2}\|\nu - \Xi\nu\| \leq \|\nu - \varpi\| \Rightarrow \|\Xi\nu - \Xi\varpi\| \leq \|\nu - \varpi\|.$$

(*ii*) *When* $\nu > \varpi$, *we get*

$$\frac{13 - 13\nu}{28} \leq \nu - \varpi \Longrightarrow \varpi \leq \frac{41\nu - 13}{28} \Longrightarrow \varpi \in [\frac{-141}{392}, 1].$$

Because $\varpi \in [0, 1]$ *and* $\varpi \leq \frac{41\nu-13}{28}$, *we can write* $\nu \geq \frac{28\varpi+13}{41} \Rightarrow \nu \in [\frac{13}{41}, 1]$.

It should be noted that the case of $\nu \in [\frac{13}{41}, 1]$ *and* $\varpi \in [\frac{1}{14}, 1]$ *is similar to case (i), so, we will discuss when* $\nu \in [\frac{13}{41}, 1]$ *and* $\varpi \in [0, \frac{1}{14})$. *So, we have*

$$\|\Xi\nu - \Xi\varpi\| = \left|\frac{\nu + 13}{14} - (1-\varpi)\right| = \left|\frac{\nu + 14\varpi - 1}{14}\right| < \frac{1}{14},$$

and $\|\nu-\varpi\| = |\nu-\varpi| > \left|\frac{13}{41}-\frac{1}{14}\right| = \frac{141}{574} > \frac{1}{14}$. *Thus* $\frac{1}{2}\|\nu-\Xi\nu\| \leq \|\nu-\varpi\| \Rightarrow \|\Xi\nu-\Xi\varpi\| \leq \|\nu-\varpi\|$. *Hence* Ξ *is SGNM.*

Now, we will discuss the behavior of the iterative scheme (1.5) and illustrate that it is faster than S, Tharkur and K^* *iteration procedure by using different control conditions* $\alpha_i = \eta_i = \gamma_i = \frac{i}{(i+1)}$.

Remark 7.1. *The effectiveness and success of the iterative method are measured by two main factors: The time and the number of repetitions. When obtaining strong convergence in a short time using the least possible repetitions, saves effort and time in many problems of optimization and variational inequalities. Based on what has been shown from the tables (see Tables 4 and 5) and figures (see Figures 4 and 5), it is clear that our method is successful and the behavior of our algorithm is satisfactory compared to some sober iterations in this direction.*

Table 4. Example 7.2: Numerical effectiveness comparison of proposed algorithm (*HR* **algorithm**) when $\nu_\circ = 0.30$.

Iter (n)	S **algorithm**	**Picard-**S **algorithm**	**Thakur algorithm**	K^***-algorithm**	HR **algorithm**
1	0.918925619834711	0.992629601803156	0.992629601803156	0.999385287890171	0.999999491973463
2	0.992659381189810	0.999939333728841	0.999939333728841	0.999997438874483	0.999999999938058
3	0.999334187674034	0.999999499765345	0.999999499765345	0.999999986205653	0.999999999999984
4	0.999939559648030	0.999999995871843	0.999999995871843	0.999999999916912	
5	0.999994510972627	0.999999999965918	0.999999999965918	0.999999999999467	
6	0.999999501367829	0.999999999999719	0.999999999999719		
7	0.999999954695558				
8	0.999999995883263				
9	0.999999999625887				
10	0.999999999966000				
11	0.999999999996910				

Note: CPU time in seconds, respectively: 0.0021438, 0.0086125, 0.0079631, 0.0063799, 0.0054917.

Table 5. Example 7.2: Numerical effectiveness comparison of proposed algorithm (*HR* **algorithm**) when $\nu_\circ = 0.30$.

Iter (n)	S **algorithm**	**Picard-**S **algorithm**	**Thakur algorithm**	K^***-algorithm**	HR **algorithm**
1	0.981983471074380	0.998362133734035	0.998362133734035	0.999863397308927	0.999999887105214
2	0.998368751375513	0.999986518606409	0.999986518606409	0.999999430860996	0.999999999986235
3	0.999852041705341	0.999999888836743	0.999999888836743	0.999999996934589	0.999999999999996
4	0.999986568810673	0.999999999082632	0.999999999082632	0.999999999981536	
5	0.999998780216139	0.999999999992426	0.999999999992426	0.999999999999881	
6	0.999999889192851	0.999999999999938	0.999999999999938		
7	0.999999989932346				
8	0.999999999085170				
9	0.999999999916864				
10	0.999999999992444				

Note: CPU time in seconds, respectively: 0.0053331, 0.0078199, 0.0048301, 0.005278, 0.0001471.

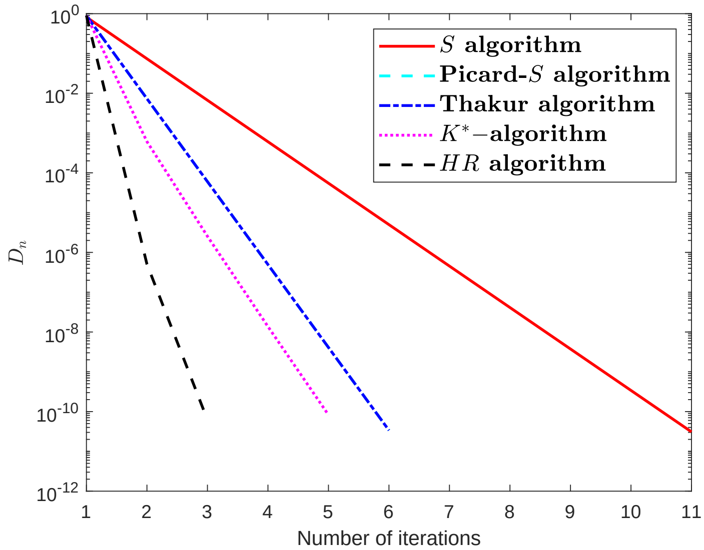

Figure 4. Graphically comparison of proposed algorithm (*HR* **algorithm**) when $\nu_\circ = 0.30$.

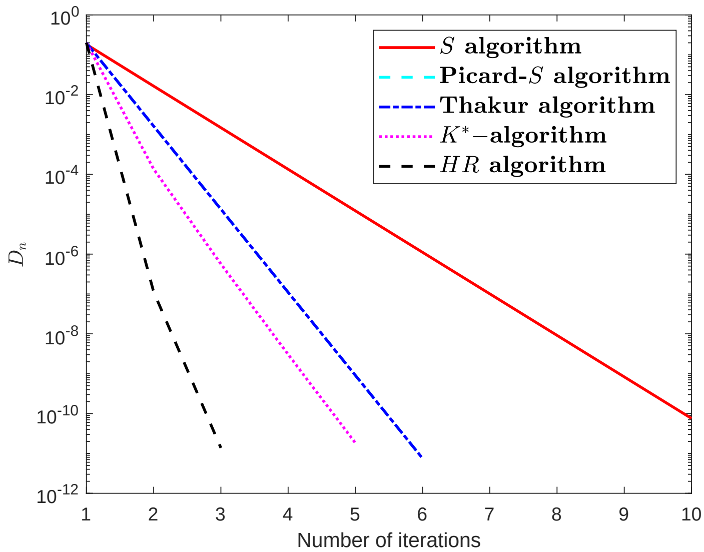

Figure 5. Graphically comparison of proposed algorithm (*HR* **algorithm**) when $\nu_\circ = 0.80$.

8. Solve Volterra-Fredholm integral equation

In this part, we apply our algorithm (1.5) to solve Volterra-Fredholm integral equation which was suggested by Lungu and Rus [23].

Consider the following problem:

$$\xi(\nu, \varpi) = \aleph(\nu, \varpi, z(\xi(\nu, \varpi)) + \int_0^{\nu} \int_0^{\varpi} \mho\,(\nu, \varpi, \kappa^*, \tau^*, \xi(\kappa^*, \tau^*))\, d\kappa^* d\tau^*, \tag{8.1}$$

for all $\nu, \varpi \in \mathbb{R}_+$. Assume that $(\Gamma, |.|)$ is a *BS*, $s > 0$ and

$$\chi_s = \left\{\xi \in C\left(\mathbb{R}_+^2, \Gamma\right) : \text{ there is } U(\xi) > 0 \text{ so that } |\xi(\nu, \varpi)|\, e^{-s(\nu+\varpi)} \leq U(\xi)\right\}.$$

Define the norm on χ_s as follows:

$$\|\xi\|_s = \sup_{\nu,\varpi\in\mathbb{R}_+} \left(|\xi(\nu, \varpi)|\, e^{-s(\nu+\varpi)}\right).$$

It follows from the paper [33] that $(\chi_s, \|\xi\|_s)$ is a *BS*.

The following theorem helps us for proving our main result in this section.

Theorem 8.1. *[23] Assume that the postulates below are satisfied:*

(P_i) $\aleph \in C\left(\mathbb{R}_+^2 \times \Gamma, \Gamma\right)$ *and* $\mho \in C\left(\mathbb{R}_+^4 \times \Gamma, \Gamma\right)$;

(P_{ii}) *There are* $z : \chi_s \to \chi_s$ *and* $\pi_z > 0$ *so that*

$$|z(\xi(\nu, \varpi)) - z(\xi^*(\nu, \varpi))| \leq \pi_z \|\xi - \xi^*\|\, e^{s(\nu+\varpi)},$$

for all $\nu, \varpi \in \mathbb{R}_+$ *and* $\xi, \xi^* \in \chi_s$;

(P_{iii}) *For all* $\nu, \varpi \in \mathbb{R}_+$ *and* $c, c^* \in \Gamma$, *there is* $\pi_\aleph > 0$ *so that*

$$|\aleph(\nu, \varpi, c) - \aleph(\nu, \varpi, c^*)| \leq \pi_\aleph\, |c - c^*|\,;$$

(P_{iv}) *For all* $\nu, \varpi, \kappa^*, \tau^* \in \mathbb{R}_+$ *and* $c, c^* \in \Gamma$, *there is* $\pi_\mho(\nu, \varpi, \kappa^*, \tau^*) > 0$ *so that*

$$|\mho(\nu, \varpi, \kappa^*, \tau^*, c) - \mho(\nu, \varpi, \kappa^*, \tau^*, c^*)| \leq \pi_\mho(\nu, \varpi, \kappa^*, \tau^*)\, |c - c^*|\,;$$

(P_v) $\pi_\mho \in C\left(\mathbb{R}_+^4, \mathbb{R}_+\right)$ *and*

$$\int_0^\nu \int_0^\varpi \pi_\mho(\nu, \varpi, \kappa^*, \tau^*)\, e^{s(\kappa^*+\tau^*)} d\kappa^* d\tau^* \leq \pi e^{s(\kappa^*+\tau^*)},$$

for all $\nu, \varpi \in \mathbb{R}_+$;

(P_{vi}) $\pi_z \pi_\aleph + \pi < 1$.

Then the problem (8.1) has a unique solution $\zeta \in \chi_s$ *and the iterative sequence*

$$\xi_{i+1}(\nu, \varpi) = \aleph(\nu, \varpi, z(\xi(\nu, \varpi)) + \int_0^\nu \int_0^\varpi \mho(\nu, \varpi, \kappa^*, \tau^*, \xi_i(\kappa^*, \tau^*))\, d\kappa^* d\tau^*,$$

for all $i \geq 1$ *converges uniformly to* ζ.

Now, after the above hypotheses, we can present our main theorem as follows:

Theorem 8.2. *Let* $\{\nu_i\}$ *be an iterative sequence generated by (1.5) with sequences* $\{\alpha_i\}, \{\eta_i\}, \{\gamma_i\} \in [0, 1]$ *so that* $\sum_{i=0}^{\infty} \gamma_i = \infty$. *If the postulates* $(P_i)-(P_{vi})$ *of Theorem 8.1 hold. Then the problem (8.1) has a unique solution* $\zeta \in \chi_s$ *and the intended algorithm (1.5) converges strongly to* ζ.

Proof. Let $\{\nu_i\}$ be an iterative sequence generated by (1.5) and define the operator $H : \chi_s \to \chi_s$ by

$$H(\xi(\nu,\varpi)) = \aleph(\nu,\varpi,z(\xi(\nu,\varpi)) + \int_0^{\nu}\int_0^{\varpi} \mho(\nu,\varpi,\kappa^*,\tau^*,\xi(\kappa^*,\tau^*))\, d\kappa^* d\tau^*.$$

We shall prove that $\lim_{i\to\infty} \nu_i = 0$. Based on (1.5), we get

$$\|\nu_{i+1} - \zeta\| = \sup_{\nu,\varpi\in\mathbb{R}_+} \left(\left| H(\Im_i(\nu,\varpi)) - H(\zeta(\nu,\varpi)) \right| e^{-s(\nu+\varpi)} \right).$$

Now,

$$\begin{aligned}
& \left| H(\Im_i(\nu,\varpi)) - H(\zeta(\nu,\varpi)) \right| \\
\leq\ & \left| \aleph(\nu,\varpi,z(\Im_i(\nu,\varpi))) - \aleph(\nu,\varpi,z(\zeta(\nu,\varpi)) \right| \\
& + \left| \int_0^{\nu}\int_0^{\varpi} \mho(\nu,\varpi,\kappa^*,\tau^*,\Im_i(\kappa^*,\tau^*))\, d\kappa^* d\tau^* - \int_0^{\nu}\int_0^{\varpi} \mho(\nu,\varpi,\kappa^*,\tau^*,\zeta(\kappa^*,\tau^*))\, d\kappa^* d\tau^* \right| \\
\leq\ & \pi_{\aleph} \left| z(\Im_i(\nu,\varpi)) - z(\zeta(\nu,\varpi)) \right| \\
& + \int_0^{\nu}\int_0^{\varpi} \left| \mho(\nu,\varpi,\kappa^*,\tau^*,\Im_i(\kappa^*,\tau^*)) - \mho(\nu,\varpi,\kappa^*,\tau^*,\zeta(\kappa^*,\tau^*)) \right| d\kappa^* d\tau^* \\
\leq\ & \pi_{\aleph}\pi_z \left\| \Im_i - \zeta \right\|_s e^{s(\nu+\varpi)} + \int_0^{\nu}\int_0^{\varpi} \pi_{\mho}(\nu,\varpi,\kappa^*,\tau^*) \left| \Im_i(\kappa^*,\tau^*) - \zeta(\kappa^*,\tau^*) \right| d\kappa^* d\tau^* \\
\leq\ & \pi_{\aleph}\pi_z \left\| \Im_i - \zeta \right\|_s e^{s(\nu+\varpi)} + \pi \left\| \Im_i - \zeta \right\|_s e^{s(\nu+\varpi)} \\
=\ & (\pi_{\aleph}\pi_z + \pi) \left\| \Im_i - \zeta \right\|_s e^{s(\nu+\varpi)}.
\end{aligned}$$

Thus,

$$\|\nu_{i+1} - \zeta\|_s \leq (\pi_{\aleph}\pi_z + \pi) \left\| \Im_i - \zeta \right\|_s. \tag{8.2}$$

Again

$$\left\| \Im_i - \zeta \right\|_s = \sup_{\nu,\varpi\in\mathbb{R}_+} \left(\left| H((1-\gamma_i)\wp_i + \gamma_i H\wp_i)(\nu,\varpi) - H(\zeta(\nu,\varpi)) \right| e^{-s(\nu+\varpi)} \right),$$

and

$$\begin{aligned}
& \left| H((1-\gamma_i)\wp_i + \gamma_i H\wp_i)(\nu,\varpi) - H(\zeta(\nu,\varpi)) \right| \\
\leq\ & \left| \aleph(\nu,\varpi,z((1-\gamma_i)\wp_i + \gamma_i H\wp_i)(\nu,\varpi))) - \aleph(\nu,\varpi,z(\zeta(\nu,\varpi)) \right| \\
& + \left| \int_0^{\nu}\int_0^{\varpi} \mho(\nu,\varpi,\kappa^*,\tau^*,((1-\gamma_i)\wp_i + \gamma_i H\wp_i)(\kappa^*,\tau^*))\, d\kappa^* d\tau^* \right. \\
& \left. - \int_0^{\nu}\int_0^{\varpi} \mho(\nu,\varpi,\kappa^*,\tau^*,\zeta(\kappa^*,\tau^*))\, d\kappa^* d\tau^* \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \pi_{\aleph}\left|z\left(\left((1-\gamma_i)\wp_i+\gamma_i H\wp_i\right)(\kappa^*,\tau^*)\right)-z(\zeta(\nu,\varpi)\right| \\
&\quad+\int_0^{\nu}\int_0^{\varpi}\left|\mho\left(\nu,\varpi,\kappa^*,\tau^*,\left((1-\gamma_i)\wp_i+\gamma_i H\wp_i\right)(\kappa^*,\tau^*)\right)\right. \\
&\quad\left.-\mho\left(\nu,\varpi,\kappa^*,\tau^*,\zeta(\kappa^*,\tau^*)\right)\right|d\kappa^* d\tau^* \\
&\leq \pi_{\aleph}\pi_z\left\|(1-\gamma_i)\wp_i+\gamma_i H\wp_i-\zeta\right\|_s e^{s(\nu+\varpi)} \\
&\quad+\pi\left\|(1-\gamma_i)\wp_i+\gamma_i H\wp_i-\zeta\right\|_s e^{s(\nu+\varpi)} \\
&= (\pi_{\aleph}\pi_z+\pi)\left\|(1-\gamma_i)\wp_i+\gamma_i H\wp_i-\zeta\right\|_s e^{s(\nu+\varpi)}.
\end{aligned}$$

Thus

$$\left\|\Im_i-\zeta\right\|_s \leq (\pi_{\aleph}\pi_z+\pi)\left\|(1-\gamma_i)\wp_i+\gamma_i H\wp_i-\zeta\right\|_s. \tag{8.3}$$

Similarly, one can write

$$\left\|\wp_i-\zeta\right\|_s \leq (\pi_{\aleph}\pi_z+\pi)\left\|(1-\eta_i)\varpi_i+\eta_i H\varpi_i-\zeta\right\|_s. \tag{8.4}$$

Since

$$\begin{aligned}
\left\|(1-\eta_i)\varpi_i+\eta_i H\varpi_i-\zeta\right\|_s &= \left\|((1-\eta_i)\varpi_i-\zeta)+\eta_i\left(H\varpi_i-\zeta\right)\right\|_s \\
&\leq (1-\eta_i)\left\|\varpi_i-\zeta\right\|_s+\eta_i\left\|H\varpi_i-\zeta\right\|_s.
\end{aligned} \tag{8.5}$$

Now

$$\left\|H\varpi_i-\zeta\right\|_s=\sup_{\nu,\varpi\in\mathbb{R}_+}\left(\left|H\left(\varpi_i(\nu,\varpi)\right)-H\left(\zeta(\nu,\varpi)\right)\right|e^{-s(\nu+\varpi)}\right),$$

and

$$\begin{aligned}
&\left|H\left(\varpi_i(\nu,\varpi)\right)-H\left(\zeta(\nu,\varpi)\right)\right| \\
&\leq \left|\aleph(\nu,\varpi,z(\varpi_i(\nu,\varpi)))-\aleph(\nu,\varpi,z(\zeta(\nu,\varpi))\right| \\
&\quad+\left|\int_0^{\nu}\int_0^{\varpi}\mho\left(\nu,\varpi,\kappa^*,\tau^*,\varpi_i(\kappa^*,\tau^*)\right)d\kappa^* d\tau^*-\int_0^{\nu}\int_0^{\varpi}\mho\left(\nu,\varpi,\kappa^*,\tau^*,\zeta(\kappa^*,\tau^*)\right)d\kappa^* d\tau^*\right| \\
&\leq \pi_{\aleph}\left|z(\varpi_i(\nu,\varpi))-z(\zeta(\nu,\varpi))\right| \\
&\quad+\int_0^{\nu}\int_0^{\varpi}\left|\mho\left(\nu,\varpi,\kappa^*,\tau^*,\varpi_i(\kappa^*,\tau^*)\right)-\mho\left(\nu,\varpi,\kappa^*,\tau^*,\zeta(\kappa^*,\tau^*)\right)\right|d\kappa^* d\tau^* \\
&\leq \pi_{\aleph}\pi_z\left\|\varpi_i-\zeta\right\|_s e^{s(\nu+\varpi)}+\int_0^{\nu}\int_0^{\varpi}\pi_{\mho}\left(\nu,\varpi,\kappa^*,\tau^*\right)\left|\varpi_i(\kappa^*,\tau^*)-\zeta(\kappa^*,\tau^*)\right|d\kappa^* d\tau^* \\
&\leq \pi_{\aleph}\pi_z\left\|\varpi_i-\zeta\right\|_s e^{s(\nu+\varpi)}+\pi\left\|\varpi_i-\zeta\right\|_s e^{s(\nu+\varpi)} \\
&= (\pi_{\aleph}\pi_z+\pi)\left\|\varpi_i-\zeta\right\|_s e^{s(\nu+\varpi)}.
\end{aligned}$$

Thus

$$\left\|H\varpi_i-\zeta\right\|_s \leq (\pi_{\aleph}\pi_z+\pi)\left\|\varpi_i-\zeta\right\|_s. \tag{8.6}$$

Applying (8.6) in (8.5), we have

$$\begin{aligned}
& \|(1-\eta_i)\varpi_i + \eta_i H\varpi_i - \zeta\|_s \\
\leq\ & (1-\eta_i)\|\varpi_i - \zeta\|_s + \eta_i(\pi_{\aleph}\pi_z + \pi)\|\varpi_i - \zeta\|_s e^{s(\nu+\varpi)} \\
=\ & (1-\eta_i(1-(\pi_{\aleph}\pi_z + \pi))\|\varpi_i - \zeta\|_s .
\end{aligned} \tag{8.7}$$

Using (8.4) and (8.7), we get

$$\|\wp_i - \zeta\|_s \leq (\pi_{\aleph}\pi_z + \pi)(1-\eta_i(1-(\pi_{\aleph}\pi_z + \pi))\|\varpi_i - \zeta\|_s . \tag{8.8}$$

By the same manner, (8.3) can be written as

$$\left\|\Im_i - \zeta\right\|_s \leq (\pi_{\aleph}\pi_z + \pi)(1-\gamma_i(1-(\pi_{\aleph}\pi_z + \pi))\|\wp_i - \zeta\|_s . \tag{8.9}$$

From (8.8) in (8.9), we find that

$$\left\|\Im_i - \zeta\right\|_s \leq (\pi_{\aleph}\pi_z + \pi)^2(1-\gamma_i(1-(\pi_{\aleph}\pi_z + \pi)) \times (1-\eta_i(1-(\pi_{\aleph}\pi_z + \pi))\|\varpi_i - \zeta\|_s . \tag{8.10}$$

Applying (8.10) in (8.2), we get

$$\|\nu_{i+1} - \zeta\|_s \leq (\pi_{\aleph}\pi_z + \pi)^3(1-\gamma_i(1-(\pi_{\aleph}\pi_z + \pi)) \times (1-\eta_i(1-(\pi_{\aleph}\pi_z + \pi))\|\varpi_i - \zeta\|_s . \tag{8.11}$$

Using (1.5) and similar to (8.6), we obtain that

$$\begin{aligned}
\|\varpi_i - \zeta\|_s &= \|(1-\alpha_i)\nu_i + \alpha_i H\nu_i - \zeta\|_s \\
&\leq (1-\alpha_i)\|\nu_i - \zeta\|_s + \alpha_i\|H\nu_i - \zeta\|_s \\
&\leq (1-\alpha_i)\|\nu_i - \zeta\|_s + \alpha_i(\pi_{\aleph}\pi_z + \pi)\|\nu_i - \zeta\|_s \\
&= (1-(1-\alpha_i(\pi_{\aleph}\pi_z + \pi))\|\nu_i - \zeta\|_s .
\end{aligned} \tag{8.12}$$

Substituting from (8.12) in (8.11), we find that

$$\begin{aligned}
\|\nu_{i+1} - \zeta\|_s \leq\ & (\pi_{\aleph}\pi_z + \pi)^3(1-\gamma_i(1-(\pi_{\aleph}\pi_z + \pi)) \\
\times\ & (1-\eta_i(1-(\pi_{\aleph}\pi_z + \pi))(1-(1-\alpha_i(\pi_{\aleph}\pi_z + \pi))\|\nu_i - \zeta\|_s .
\end{aligned}$$

Since $\alpha_i, \eta_i \in [0,1]$ and $\pi_{\aleph}\pi_z + \pi < 1$, then we have

$$\|\nu_{i+1} - \zeta\|_s \leq (1-\gamma_i(1-(\pi_{\aleph}\pi_z + \pi))\|\nu_i - \zeta\|_s ,$$

by induction, we can write

$$\|\nu_{i+1} - \zeta\|_s \leq \|\nu_0 - \zeta\|_s \prod_{r}^{i}(1-\gamma_r(1-(\pi_{\aleph}\pi_z + \pi)) . \tag{8.13}$$

It follows from the postulate (P_{vi}) and $\gamma_r \in [0,1]$ that

$$1-\gamma_r(1-(\pi_{\aleph}\pi_z + \pi)) < 1.$$

From our information in classical analysis, we can write for $\nu \in [0,1]$, $1-\nu \leq e^{-\nu}$. Therefore, (8.13) take the form

$$\|\nu_{i+1} - \zeta\|_s \leq \|\nu_0 - \zeta\|_s e^{-[(1-\gamma_r(1-(\pi_{\aleph}\pi_z+\pi))]\sum_{r=0}^{i}\gamma_r},$$

which implies that $\lim_{i\to\infty}\|\nu_i - \zeta\|_s = 0$. This completes the proof. □

9. Conclusions and future works

It is well known that the efficiency and effectiveness of iterative methods are measured by two main factors. The first factor is the speed of convergence and the second is the number of repetitions, meaning if the convergence is faster with fewer repetitions, the method was successful in approximating the fixed points. So, in this article, we demonstrated analytically and numerically that our algorithm is better in conduct than a portion of the main iterative techniques in the past writing [4, 10, 11, 18] as far as convergence speed.

Also, the prevalence and speed of convergence, stability, and data dependence results were displayed in comparison graphs of calculations. Moreover, our approach was eventually supported by a solution to an integral problem as an application. In light of references [4, 10, 11, 18], our method is therefore successful or effective. Finally, as future works for this paper, we appointed the following:

(1) If we define a mapping Ξ in a Hilbert space Δ endowed with inner product space, we can find a common solution to the variational inequality problem by using our iteration (1.5). This problem can be stated as follows: find $\wp^* \in \Delta$ such that

$$\langle \Xi\wp^*, \wp - \wp^* \rangle \geq 0 \text{ for all } \wp \in \Delta,$$

where $\Xi : \Delta \to \Delta$ is a nonlinear mapping. Variational inequalities are an important and essential modeling tool in many fields such as engineering mechanics, transportation, economics, and mathematical programming, see [34, 35].

(2) We can generalize our algorithm to gradient and extra-gradient projection methods, these methods are very important for finding saddle points and solving many problems in optimization, see [36].

(3) We can accelerate the convergence of the proposed algorithm by adding shrinking projection and CQ terms. These methods stimulate algorithms and improve their performance to obtain strong convergence, for more details, see [37–39].

(4) If we consider the mapping Ξ as an $\alpha-$inverse strongly monotone and the inertial term is added to our algorithm, then we have the inertial proximal point algorithm. This algorithm is used in many applications such as monotone variational inequalities, image restoration problems, convex optimization problems, and split convex feasibility problems, see [40–43]. For more accuracy, these problems can be expressed as mathematical models such as machine learning and the linear inverse problem.

(5) We can also use our algorithm to solve second-order differential equations and fractional differential equations, where these equations can be converted into integral equations by Green's function. So it is easy to treat and solve with the same approach used in Part 8.

(6) We can try to determine the error of our present iteration.

Acknowledgments

M. Zayed extends her appreciation to the Deanship of Scientific Research at King Khalid University, Saudi Arabia. for supporting this work through research groups program grant R.G.P.2/207/43

Conflict of interest

The authors declare that they have no conflicts of interest.

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