AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(10): 18971–19017.
DOI: 10.3934/math.20221044
Received: 28 June 2022
Revised: 28 July 2022
Accepted: 03 August 2022
Published: 26 August 2022

Research article

Rough set models in a more general manner with applications

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Abstract: Several tools have been put forth to handle the problem of uncertain knowledge. Pawlak (1982) initiated the concept of rough set theory, which is a completely new tool for solving imprecision and vagueness (uncertainty). The main notions in this theory are the upper and lower approximations. One of the most important aims of this theory is to reduce the vagueness of a concept to uncertainty areas at their borders by decreasing the upper approximations and increasing the lower approximations. So, the object of this study is to propose four types of approximation spaces in rough set theory utilizing ideals and a new type of neighborhoods called "the intersection of maximal right and left neighborhoods". We investigate the master properties of the proposed approximation spaces and demonstrate that these spaces reduce boundary regions and improve accuracy measures. A comparative study of the present methods and the previous ones is given and shown that the current study is more general and accurate. The importance of the current paper is not only that it is introducing new kinds of approximation spaces relying mainly on ideals and a new type of neighborhoods which increases the accuracy measure and reduces the boundary region of subsets, but also that these approximation spaces are monotonic, which means that it can be successfully used to evaluate the uncertainty in the data. In the end of this paper, we provide a medical example of the heart attacks problem to show the efficiency of the current techniques in terms of approximation operators, accuracy measures, and monotonic property.

Keywords: ideals; maximal neighborhoods; approximation operators; accuracy measure; rough set
Mathematics Subject Classification: 03E99, 54A05, 68T30, 91B06

1. Introduction

The classical set theory deal with the kind of problems that are characterized by certainty, precision, perfection, exactness and specificity. However, there are many problems in the real-life inherently involve uncertainties, imprecision, ambiguity, vagueness, inconsistency and incompleteness. In particular, these problems arise in different fields such as computer network, economics, engineering, image processing, data mining, social, environmental and medical sciences. The classical set fail to deal with this kind of these problems. There are several tools to manipulate and understand these problems. For instance, in 1982, the world-known new concept called rough sets theory on the hand of Pawlak [28, 29]. It is considered as a powerful tool to solve uncertain data and address the issues of vagueness in knowledge and granularity in information systems. Methodology of this theory is based on the equivalence relations to classify the objects into three main areas known as lower, upper approximations and boundary regions. These areas in the classical rough set theory are expressed in terms of equivalence classes. The lower approximation of a set is the union of equivalence classes which are entirely included in the set while the upper approximation is the union of all equivalence classes which have nonempty intersection with the set. The boundary region is the difference between the upper and lower approximations. The accuracy of the set or ambiguous is depending on the boundary region is empty or not respectively. If the boundary of a set is nonempty, then this means that our knowledge about this set is not sufficient to define it precisely. Minimizing the boundary region is one of the pivotal goal for this theory. To extend the applications of this theory, many scholars replaced the equivalence relation by reflexive relation [2], similarity relation [15, 26, 30, 32] or binary relation [3, 31, 36–38].

The most efficacious tools to study the generalization of rough set theory are the neighborhoods systems. The central idea in this theory is the upper and lower approximations and they have been defined using different types of neighborhoods instead of equivalence classes such as left and right neighborhoods [7, 9, 34, 35], minimal left neighborhoods [4] and minimal right neighborhoods [5], the intersection of minimal left and right neighborhoods [23]. Afterwards, Abo-Tabl [1] defined the approximations by minimal right neighborhoods which determined by reflexive relations that form the base of topological space. In 2018, Dai et al. [14] presented new kind of neighborhoods, namely the maximal right neighborhoods which determined by similarity relations and they have been used to propose three new kinds of approximations. Dai et al.'s approximations [14] differed from Abo-Tabl's approximations [1] in that the corresponding upper and lower approximations, boundary regions, accuracy measures and roughness measures in two types of Dai et al.'s approximations [14] had monotonicity. Later on, Al-shami [8] embraced a new type of neighborhoods systems namely, the intersection of maximal right and left neighborhoods, and then used this type to present new approximations. These approximations improved the accuracy measures more than Dai et al.'s approximations [14]. Also, Al-shami's [8] accuracy measures preserved the monotonic property under any arbitrary relation. Recently, two classes of neighborhood systems called containment neighborhoods [6] and subset neighborhoods [10] have been presented and studied.

An ideal is a nonempty collection of sets which is closed under hereditary property and the finite additivity [27, 33]. It is a completely new approach for modeling vagueness and uncertainty by reducing the boundary region and increasing the accuracy of the set which helped scholars to solve many life problems [13, 16–19, 21, 23, 24]. More recently, Hosny [20] expressed the main concepts of rough set via ideals and maximal right neighborhood deduced by binary relations. Hosny's approximations was considered as an extension of Al-shami's approximations [8] and Dai et al.'s approximations [14].

Although many scholars have published a lot of papers to develop many concepts of rough set content, yet there is up to now a lot of development for this set. The essential motivation of this paper is to improve the approximations and accuracy measure of a set by deleting some objects from the upper approximations of decision categories and/or adding new objects to the lower approximations. This matter can be achieved using the proposed technique "ideals and the intersection of maximal right and left neighborhoods deduced by binary relations". Another motivation is the desire of initiating approximation spaces keeping the monotonic property, which helps to evaluate the uncertainty in the given data.

This paper is divided as follows: After the introduction, Section 2 recalls the necessary results and concepts which required in the sequel to this work. In Section 3, four methods are suggested to express the main concepts of rough set via ideals and the intersection of maximal right and left neighborhoods which generated by binary relations. The relevant properties of these methods are scrutinized. Moreover, it is shown that the boundary of a subset decreases and the accuracy increases as the ideal increases. More importantly, it is proved that three types of these methods are monotonic. In Section 4, we compare among the suggested approximations and prove that the third approach is the best one. Then, we show that the current methods produce better approximations and higher accuracy values than their counterparts introduced in [8, 11, 14, 20]. Meanwhile, Section 5 shows the overall benefits of the current study by introducing medical application in the decision making problems. We computed the approximations, the boundary and the accuracy measure using the proposed method and the previous approaches [14, 20, 23] to explain the significance of the suggested techniques in decision making. The results are shown that Kandil et al.'s methods [23] are not monotonic, so it cannot be used to evaluate the uncertainty in the data. Whereas, the current methods are solved the trouble and imperfection of Kandil et al.'s methods [23]. Additionally, the present methods are reduced the boundary region and improved the accuracy measure in the comparison with Dai et al.'s methods [14]. Consequently, they handle any ambiguity in symptoms of the diseases and this automatically help the medical staff. The conclusions of this manuscript and remarks for future work are given in Section 6.

2. Preliminaries

Definition 2.1. *[28] Let R be an equivalence relation on a universe U and $[u]_R$ be the equivalence class containing $u \in U$. The lower approximation $\underline{apr}(A)$ and upper approximation $\overline{apr}(A)$ of a subset A of U are defined by*

$$\underline{apr}(A) = \{u \in U : [u]_R \subseteq A\}.$$
$$\overline{apr}(A) = \{u \in U : [u]_R \cap A \neq \phi\}.$$

These approximations satisfy the following properties:

($\mathcal{L}_1$) $\underline{apr}(A^c) = [\overline{apr}(A)]^c$, where A^c is the complement of A.

($\mathcal{L}_2$) $\underline{apr}(U) = U$.

($\mathcal{L}_3$) $\underline{apr}(\phi) = \phi$.

($\mathcal{L}_4$) $\underline{apr}(A) \subseteq A$.

($\mathcal{L}_5$) $\underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B)$.

($\mathcal{L}_6$) $\underline{apr}(A \cup B) \supseteq \underline{apr}(A) \cup \underline{apr}(B)$.

($\mathcal{L}_7$) $A \subseteq B \Rightarrow \underline{apr}(A) \subseteq \underline{apr}(B)$.

($\mathcal{L}_8$) $\underline{apr}(\underline{apr}(A)) = \underline{apr}(A)$.

($\mathcal{L}_9$) $\overline{apr}(A) \subseteq \underline{apr}(\overline{apr}(A))$.

($\mathcal{U}_1$) $\overline{apr}(A^c) = [\underline{apr}(A)]^c$.

($\mathcal{U}_2$) $\overline{apr}(U) = U$.

($\mathcal{U}_3$) $\overline{apr}(\phi) = \phi$.

($\mathcal{U}_4$) $A \subseteq \overline{apr}(A)$.

($\mathcal{U}_5$) $\overline{apr}(A \cup B) = \overline{apr}(A) \cup \overline{apr}(B)$.

($\mathcal{U}_6$) $\overline{apr}(A \cap B) \subseteq \overline{apr}(A) \cap \overline{apr}(B)$.

($\mathcal{U}_7$) $A \subseteq B \Rightarrow \overline{apr}(A) \subseteq \overline{apr}(B)$.

($\mathcal{U}_8$) $\overline{apr}(\overline{apr}(A)) = \overline{apr}(A)$.

($\mathcal{U}_9$) $\overline{apr}(\underline{apr}(A)) \subseteq \underline{apr}(A)$.

Definition 2.2. *[28] Let R be an equivalence relation on a universe U. Then accuracy measure $Acc_R(A)$ of any nonempty subset A is defined as follows: $Acc_R(A) = \frac{|\underline{apr}(A)|}{|\overline{apr}(A)|}$.*

If R_1 and R_2 are equivalence relations on a universe U such that $R_1 \subseteq R_2$. Then the approximations induced from these relations have the monotonic property if $Acc_{R_2}(A) \leq Acc_{R_1}(A)$.

Definition 2.3. *[5, 25] The following types of neighborhoods are defined for each element u of the universe U with respect to any binary relation R on U.*

(i) *$< u > R$ means the intersection of all right neighborhoods containing u, i.e., $< u > R = \cap\{pR : u \in pR\}$.*

(ii) *$R < u >$ means the intersection of all left neighborhoods containing u, i.e., $R < u >= \cap\{Rp : u \in Rp\}$.*

(iii) *$R < u > R =< u > R \cap R < u >$.*

Definition 2.4. *[8, 14] The following types of neighborhoods are defined for each element u of the universe U with respect to any binary relation R on U.*

(i) *$< u > \breve{R}$ means the union of all right neighborhoods containing u, i.e., $< u > \breve{R} = \cup\{pR : u \in pR\}$.*

(ii) *$\breve{R} < u >$ means the union of all left neighborhoods containing u, i.e., $\breve{R} < u >= \cup\{Rp : u \in Rp\}$.*

(iii) *$\breve{R} < u > \breve{R} = \breve{R} < u > \cap < u > \breve{R}$.*

Theorem 2.5. *[8] Let R_1, R_2 be binary relations on the universe U such that $R_1 \subseteq R_2$. Then $\breve{R}_1 < u > \breve{R}_1 \subseteq \breve{R}_2 < u > \breve{R}_2, \forall u \in U$.*

We means by an approximation space the pair (U, R), where R is a binary relation $U \neq \phi$.

Definition 2.6. *[14] Let (U, R) be approximation space, where R is a similarity relation. The first type of the approximations (lower and upper), boundary region, measures (accuracy and roughness) of $A \subseteq U$ with respect to $< x > \breve{R}$ are respectively defined by:*
$\underline{apr}_R(A) = \{x \in U :< x > \breve{R} \subseteq A\}$.
$\overline{apr}_R(A) = \{x \in U :< x > \breve{R} \cap A \neq \phi\}$.
$Boundary_R(A) = \overline{apr}_R(A) - \underline{apr}_R(A)$.
$Accuracy_R(A) = \frac{|\underline{apr}_R(A)|}{|\overline{apr}_R(A)|}, \overline{apr}_R(A) \neq \phi$.
$Roughness_R(A) = 1 - Accuracy_R(A)$.

Definition 2.7. *[14] Let (U, R) be approximation space, where R is a similarity relation. The second type of the approximations (lower and upper), boundary region, measures (accuracy and roughness) of $A \subseteq U$ with respect to $< x > \breve{R}$ are respectively defined by:*
$\underline{apr'}_R(A) = \cup\{< x > \breve{R} :< x > \breve{R} \subseteq A\}$.
$\overline{apr'}_R(A) = (\underline{apr'}_R(A^c))^c$.
$Boundary'_R(A) = \overline{apr'}_R(A) - \underline{apr'}_R(A)$.
$Accuracy'_R(A) = \frac{|\underline{apr'}_R(A)|}{|\overline{apr'}_R(A)|}, \overline{apr'}_R(A) \neq \phi$.
$Roughness'_R(A) = 1 - Accuracy'_R(A)$.

Definition 2.8. *[14] Let (U, R) be approximation space, where R is a similarity relation. The third type of the approximations (lower and upper), boundary region, measures (accuracy and roughness) of $A \subseteq U$ with respect to $< x > \breve{R}$ are respectively defined by:*
$\overline{apr''}_R(A) = \cup\{< x > \breve{R} :< x > \breve{R} \cap A \neq \phi\}$.
$\underline{apr''}_R(A) = (\overline{apr''}_R(A^c))^c$.
$Boundary''_R(A) = \overline{apr''}_R(A) - \underline{apr''}_R(A)$.
$Accuracy''_R(A) = \frac{|\underline{apr''}_R(A)|}{|\overline{apr''}_R(A)|}, \overline{apr''}_R(A) \neq \phi$.
$Roughness''_R(A) = 1 - Accuracy''_R(A)$.

Definition 2.9. *[8] Let (U, R) be approximation space. The approximations (lower and upper), boundary region, measures (accuracy and roughness) of $A \subseteq U$ with respect to $\breve{R} < x > \breve{R}$ are respectively defined by:*
$R_{\bigstar\bigstar}(A) = \{x \in U : \breve{R} < x > \breve{R} \subseteq A\}$.
$R^{\bigstar\bigstar}(A) = \{x \in U : \breve{R} < x > \breve{R} \cap A \neq \phi\}$.
$B_R^{\bigstar\bigstar}(A) = R^{\bigstar\bigstar}(A) - R_{\bigstar\bigstar}(A)$.
$Acc_R^{\bigstar\bigstar}(A) = \frac{|R_{\bigstar\bigstar}(A) \cap A|}{|R^{\bigstar\bigstar}(A) \cup A|}$.
$Rough_R^{\bigstar\bigstar}(A) = 1 - Acc^{\bigstar\bigstar}(A)$.

Definition 2.10. *[22] An ideal $\mathcal{I}$ on a set $U \neq \phi$ is nonempty collection of subsets of U that is closed under finite unions and subsets; i.e., it satisfies the following conditions:*

(i) $A \in \mathcal{I}$ *and* $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,

(ii) $A \in \mathcal{I}$ *and* $B \subseteq A \Rightarrow B \in \mathcal{I}$.

Definition 2.11. *[23] For two ideals on a nonempty set* U*, the smallest collection generating by* $\mathcal{I}_1, \mathcal{I}_2$*, denoted by* $\mathcal{I}_1 \vee \mathcal{I}_2$*, is given by:*

$$\mathcal{I}_1 \vee \mathcal{I}_2 = \{G \cup F : G \in \mathcal{I}_1, F \in \mathcal{I}_2\}.$$

Proposition 2.12. *[23] If* $\mathcal{I}_1, \mathcal{I}_2$ *are two ideals on a nonempty set* U *and* A, B *are two subsets of* U*. Then, the collection* $\mathcal{I}_1 \vee \mathcal{I}_2$ *satisfies the following conditions:*

(i) $\mathcal{I}_1 \vee \mathcal{I}_2 \neq \phi$,

(ii) $A \in \mathcal{I}_1 \vee \mathcal{I}_2,\ B \subseteq A \Rightarrow B \in \mathcal{I}_1 \vee \mathcal{I}_2$,

(iii) $A, B \in \mathcal{I}_1 \vee \mathcal{I}_2 \Rightarrow A \cup B \in \mathcal{I}_1 \vee \mathcal{I}_2$.

It means that the collection $\mathcal{I}_1 \vee \mathcal{I}_2$ *is an ideal on* U*.*

We means by an ideal approximation space the triplet $(U, R, \mathcal{I})$, where R and $\mathcal{I}$ are respectively binary relation and ideal on $U \neq \phi$.

Definition 2.13. *[23] Let* $(U, R, \mathcal{I})$ *be an ideal approximation space. The lower, upper approximations, boundary and accuracy of* $A \subseteq U$ *are defined respectively by:*
$R_\bigstar(A) = \{x \in U : R < x > R \cap A^c \in \mathcal{I}\}$.
$R^\bigstar(A) = \{x \in U : R < x > R \cap A \notin \mathcal{I}\}$.
$B_R^\bigstar(A) = R^\bigstar(A) - R_\bigstar(A)$.
$Acc_R^\bigstar(A) = \frac{|R_\bigstar(A)|}{|R^\bigstar(A)|}, R^\bigstar(A) \neq \phi$.

Definition 2.14. *[20] Let* $(U, R, \mathcal{I})$ *be an ideal approximation space. The first kind of the generalized approximations (lower and upper), boundary region, measures (accuracy and roughness) of* $A \subseteq U$ *with respect to* $< x > \breve{R}$ *are respectively defined by:*
$\underline{apr}_R^{\mathcal{I}}(A) = \{x \in U :< x > \breve{R} \cap A^c \in \mathcal{I}\}$.
$\overline{apr}_R^{\mathcal{I}}(A) = \{x \in U :< x > \breve{R} \cap A \notin \mathcal{I}\}$.
$Boundary_R^{\mathcal{I}}(A) = \overline{apr}_R^{\mathcal{I}}(A) - \underline{apr}_R^{\mathcal{I}}(A)$.
$Accuracy_R^{\mathcal{I}}(A) = \frac{|\underline{apr}_R^{\mathcal{I}}(A)|}{|\overline{apr}_R^{\mathcal{I}}(A)|}, \overline{apr}_R^{\mathcal{I}}(A) \neq \phi$.
$Roughness_R^{\mathcal{I}}(A) = 1 - Accuracy_R^{\mathcal{I}}(A)$.

Definition 2.15. *[20] Let* $(U, R, \mathcal{I})$ *be an ideal approximation space. The second kind of the generalized approximations (lower and upper), boundary region, measures (accuracy and roughness) of* $A \subseteq U$ *with respect to* $< x > \breve{R}$ *are respectively defined by:*
$\underline{\underline{apr}}{}^{\mathcal{I}}_R(A) = \{x \in A :< x > \breve{R} \cap A^c \in \mathcal{I}\}$.
$\overline{\overline{apr}}{}^{\mathcal{I}}_R(A) = A \cup \overline{apr}_R^{\mathcal{I}}(A)$.
$\underline{Boundary}_R^{\mathcal{I}}(A) = \overline{\overline{apr}}{}^{\mathcal{I}}_R(A) - \underline{\underline{apr}}{}^{\mathcal{I}}_R(A)$.
$\underline{Accuracy}_R^{\mathcal{I}}(A) = \frac{|\underline{\underline{apr}}{}^{\mathcal{I}}_R(A)|}{|\overline{\overline{apr}}{}^{\mathcal{I}}_R(A)|}, \overline{\overline{apr}}{}^{\mathcal{I}}_R(A) \neq \phi$.
$\underline{Roughness}_R^{\mathcal{I}}(A) = 1 - \underline{Accuracy}_R^{\mathcal{I}}(A)$.

Definition 2.16. *[20] Let $(U, R, \mathcal{I})$ be an ideal approximation space. The third kind of the generalized approximations (lower and upper), boundary region, measures (accuracy and roughness) of $A \subseteq U$ with respect to $< x > \breve{R}$ are respectively defined by:*
$\underline{apr'}^{\mathcal{I}}_R(A) = \cup\{< x > \breve{R} :< x > \breve{R} \cap A^c \in \mathcal{I}\}$.
$\overline{apr'}^{\mathcal{I}}_R(A) = (\underline{apr'}^{\mathcal{I}}_R(A^c))^c$.
$Boundary'^{\mathcal{I}}_R(A) = \overline{apr'}^{\mathcal{I}}_R(A) - \underline{apr'}^{\mathcal{I}}_R(A)$.
$Accuracy'^{\mathcal{I}}_R(A) = \frac{|\underline{apr'}^{\mathcal{I}}_R(A)|}{|\overline{apr'}^{\mathcal{I}}_R(A)|}, \overline{apr'}^{\mathcal{I}}_R(A) \neq \phi$.
$Roughness'^{\mathcal{I}}_R(A) = 1 - Accuracy'^{\mathcal{I}}_R(A)$.

Definition 2.17. *[20] Let $(U, R, \mathcal{I})$ be an ideal approximation space. The fourth kind of the generalized approximations (lower and upper), boundary region, measures (accuracy and roughness) of $A \subseteq U$ with respect to $< x > \breve{R}$ are respectively defined by:*
$\overline{apr''}^{\mathcal{I}}_R(A) = \cup\{< x > \breve{R} :< x > \breve{R} \cap A \notin \mathcal{I}\}$.
$\underline{apr''}^{\mathcal{I}}_R(A) = (\overline{apr''}^{\mathcal{I}}_R(A^c))^c$.
$Boundary''^{\mathcal{I}}_R(A) = \overline{apr''}^{\mathcal{I}}_R(A) - \underline{apr''}^{\mathcal{I}}_R(A)$.
$Accuracy''^{\mathcal{I}}_R(A) = \frac{|\underline{apr''}^{\mathcal{I}}_R(A)|}{|\overline{apr''}^{\mathcal{I}}_R(A)|}, \overline{apr''}_R(A) \neq \phi$.
$Roughness''^{\mathcal{I}}_R(A) = 1 - Accuracy''^{\mathcal{I}}_R(A)$.

Definition 2.18. *[11] Let $(U, R, \mathcal{I})$ be an ideal approximation space. The first kind of the generalized approximations (lower and upper), boundary region, measures (accuracy and roughness) of $\phi \neq A \subseteq U$ with respect to $\breve{R} < x >$ are respectively defined by:*
$\underline{apr\bigstar}^{\mathcal{I}}_R(A) = \{x \in U : \breve{R} < x > \cap A^c \in \mathcal{I}\}$.
$\overline{apr\bigstar}^{\mathcal{I}}_R(A) = \{x \in U : \breve{R} < x > \cap A \notin \mathcal{I}\}$.
$Boundary\bigstar^{\mathcal{I}}_R(A) = \overline{apr\bigstar}^{\mathcal{I}}_R(A) - \underline{apr\bigstar}^{\mathcal{I}}_R(A)$.
$Accuracy\bigstar^{\mathcal{I}}_R(A) = \frac{|\underline{apr\bigstar}^{\mathcal{I}}_R(A) \cap A|}{|\overline{apr\bigstar}^{\mathcal{I}}_R(A) \cup A|}$.
$Roughness\bigstar^{\mathcal{I}}_R(A) = 1 - Accuracy\bigstar^{\mathcal{I}}_R(A)$.

Definition 2.19. *[11] Let $(U, R, \mathcal{I})$ be an ideal approximation space. The second kind of the generalized approximations (lower and upper), boundary region, measures (accuracy and roughness) of $\phi \neq A \subseteq U$ with respect to $< x > \breve{R}$ are respectively defined by:*
$\underline{\underline{\underline{apr\bigstar}}}^{\mathcal{I}}_R(A) = \{x \in A : \breve{R} < x > \cap A^c \in \mathcal{I}\}$.
$\overline{\overline{\overline{apr\bigstar}}}^{\mathcal{I}}_R(A) = A \cup \overline{apr\bigstar}^{\mathcal{I}}_R(A)$.
$\underline{Boundary\bigstar}^{\mathcal{I}}_R(A) = \overline{\overline{apr\bigstar}}^{\mathcal{I}}_R(A) - \underline{\underline{apr\bigstar}}^{\mathcal{I}}_R(A)$.
$\underline{Accuracy\bigstar}^{\mathcal{I}}_R(A) = \frac{|\underline{\underline{\underline{apr\bigstar}}}^{\mathcal{I}}_R(A)|}{|\overline{\overline{\overline{apr\bigstar}}}^{\mathcal{I}}_R(A)|}$.
$\underline{Roughness\bigstar}^{\mathcal{I}}_R(A) = 1 - \underline{Accuracy\bigstar}^{\mathcal{I}}_R(A)$.

Definition 2.20. *[11] Let $(U, R, \mathcal{I})$ be an ideal approximation space. The third kind of the generalized approximations (lower and upper), boundary region, measures (accuracy and roughness) of $\phi \neq A \subseteq U$ with respect to $\breve{R} < x >$ are respectively defined by:*
$\underline{apr'\bigstar}^{\mathcal{I}}_{R}(A) = \underset{x \in U}{\cup}\{\breve{R} < x >: \breve{R} < x > \cap A^c \in \mathcal{I}\}$.
$\overline{apr'\bigstar}^{\mathcal{I}}_{R}(A) = (\underline{apr'\bigstar}^{\mathcal{I}}_{R}(A^c))^c$.
$Boundary'\bigstar^{\mathcal{I}}_{R}(A) = \overline{apr'\bigstar}^{\mathcal{I}}_{R}(A) - \underline{apr'\bigstar}^{\mathcal{I}}_{R}(A)$.
$Accuracy'\bigstar^{\mathcal{I}}_{R}(A) = \frac{|\underline{apr'\bigstar}^{\mathcal{I}}_{R}(A) \cap A|}{|\overline{apr'\bigstar}^{\mathcal{I}}_{R}(A) \cup A|}$.
$Roughness'\bigstar^{\mathcal{I}}_{R}(A) = 1 - Accuracy'\bigstar^{\mathcal{I}}_{R}(A)$.

Definition 2.21. *[11] Let $(U, R, \mathcal{I})$ be an ideal approximation space. The fourth kind of the generalized approximations (lower and upper), boundary region, measures (accuracy and roughness) of $\phi \neq A \subseteq U$ with respect to $\breve{R} < x >$ are respectively defined by:*
$\overline{apr''\bigstar}^{\mathcal{I}}_{R}(A) = \underset{x \in U}{\cup}\{\breve{R} < x >: \breve{R} < x > \cap A \notin \mathcal{I}\}$.
$\underline{apr''\bigstar}^{\mathcal{I}}_{R}(A) = (\overline{apr''\bigstar}^{\mathcal{I}}_{R}(A^c))^c$.
$Boundary''\bigstar^{\mathcal{I}}_{R}(A) = \overline{apr''\bigstar}^{\mathcal{I}}_{R}(A) - \underline{apr''\bigstar}^{\mathcal{I}}_{R}(A)$.
$Accuracy''\bigstar^{\mathcal{I}}_{R}(A) = \frac{|\underline{apr''\bigstar}^{\mathcal{I}}_{R}(A) \cap A|}{|\overline{apr''\bigstar}^{\mathcal{I}}_{R}(A) \cup A|}$.
$Roughness''\bigstar^{\mathcal{I}}_{R}(A) = 1 - Accuracy''\bigstar^{\mathcal{I}}_{R}(A)$.

3. Some new rough set models induced from $\breve{R} < x > \breve{R}$-neighborhoods and ideals

We dedicate this main section of the manuscript to display four novel kinds of rough set models. They are generated using the concepts of $\breve{R} < x > \breve{R}$-neighborhoods and ideals. We study their main properties and provide various illustrative examples.

3.1. The first method of the improvement of the approximations and accuracy measure of a rough set

Definition 3.1. *Let $(U, R, \mathcal{I})$ be an ideal approximation space. The first kind of the improvement of approximations (lower and upper), boundary region, measures (accuracy and roughness) of $\phi \neq A \subseteq U$ with respect to $\breve{R} < x > \breve{R}$ are respectively defined by:*
$\underline{LOW}^{\mathcal{I}}_{R}(A) = \{x \in U : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}\}$.
$\overline{UPP}^{\mathcal{I}}_{R}(A) = \{x \in U : \breve{R} < x > \breve{R} \cap A \notin \mathcal{I}\}$.
$BND^{\mathcal{I}}_{R}(A) = \overline{UPP}^{\mathcal{I}}_{R}(A) - \underline{LOW}^{\mathcal{I}}_{R}(A)$.
$ACC^{\mathcal{I}}_{R}(A) = \frac{|\underline{LOW}^{\mathcal{I}}_{R}(A) \cap A|}{|\overline{UPP}^{\mathcal{I}}_{R}(A) \cup A|}$.
$Rough^{\mathcal{I}}_{R}(A) = 1 - ACC^{\mathcal{I}}_{R}(A)$.

Proposition 3.2. *Let $(U, R, \mathcal{I})$ and $(U, R, \mathcal{J})$ be two ideal approximation spaces and let $A, B \subseteq U$. Then, we have next properties.*

(i) $\overline{UPP}^{\mathcal{I}}_{R}(\phi) = \phi$.

(ii) $A \subseteq B \Rightarrow \overline{UPP}_R^{\mathcal{I}}(A) \subseteq \overline{UPP}_R^{\mathcal{I}}(B)$.

(iii) $\overline{UPP}_R^{\mathcal{I}}(A \cap B) \subseteq \overline{UPP}_R^{\mathcal{I}}(A) \cap \overline{UPP}_R^{\mathcal{I}}(B)$.

(iv) $\overline{UPP}_R^{\mathcal{I}}(A \cup B) = \overline{UPP}_R^{\mathcal{I}}(A) \cup \overline{UPP}_R^{\mathcal{I}}(B)$.

(v) $\overline{UPP}_R^{\mathcal{I}}(A) = (\underline{LOW}_R^{\mathcal{I}}(A^c))^c$.

(vi) *if* $A \in \mathcal{I}$, *then* $\overline{UPP}_R^{\mathcal{I}}(A) = \phi$.

(vii) *if* $\mathcal{I} \subseteq \mathcal{J}$, *then* $\overline{UPP}_R^{\mathcal{J}}(A) \subseteq \overline{UPP}_R^{\mathcal{I}}(A)$.

(viii) *if* $\mathcal{I} = P(U)$, *then* $\overline{UPP}_R^{\mathcal{I}}(A) = \phi$.

(ix) $\overline{UPP}_R^{\mathcal{I} \cap \mathcal{J}}(A) = \overline{UPP}_R^{\mathcal{I}}(A) \cup \overline{UPP}_R^{\mathcal{J}}(A)$.

(x) $\overline{UPP}_R^{\mathcal{I} \vee \mathcal{J}}(A) = \overline{UPP}_R^{\mathcal{I}}(A) \cap \overline{UPP}_R^{\mathcal{J}}(A)$.

Proof.

(i)

$$\begin{aligned}\overline{UPP}_R^{\mathcal{I}}(\phi) &= \{x \in U : \breve{R} < x > \breve{R} \cap \phi \notin \mathcal{I}\}.\\ &= \phi.\end{aligned}$$

(ii) Let $x \in \overline{UPP}_R^{\mathcal{I}}(A)$. Then, $\breve{R} < x > \breve{R} \cap A \notin \mathcal{I}$. Since, $A \subseteq B$ and $\mathcal{I}$ is an ideal. Thus, $\breve{R} < x > \breve{R} \cap B \notin \mathcal{I}$. Therefore, $x \in \overline{UPP}_R^{\mathcal{I}}(B)$. Hence, $\overline{UPP}_R^{\mathcal{I}}(A) \subseteq \overline{UPP}_R^{\mathcal{I}}(B)$.

(iii) Immediately by part (ii).

(iv) $\overline{UPP}_R^{\mathcal{I}}(A) \cup \overline{UPP}_R^{\mathcal{I}}(B) \subseteq \overline{UPP}_R^{\mathcal{I}}(A \cup B)$ by part (ii). Let $x \in \overline{UPP}_R^{\mathcal{I}}(A \cup B)$. Then, $\breve{R} < x > \breve{R} \cap (A \cup B) \notin \mathcal{I}$. It follows that $((\breve{R} < x > \breve{R} \cap A) \cup (\breve{R} < x > \breve{R} \cap B)) \notin \mathcal{I}$. Therefore, $\breve{R} < x > \breve{R} \cap A \notin I$ or $\breve{R} < x > \breve{R} \cap B \notin \mathcal{I}$, that means $x \in \overline{UPP}_R^{\mathcal{I}}(A)$ or $x \in \overline{UPP}_R^{\mathcal{I}}(B)$. Then, $x \in \overline{UPP}_R^{\mathcal{I}}(A) \cup \overline{UPP}_R^{\mathcal{I}}(B)$. Thus, $\overline{UPP}_R^{\mathcal{I}}(A \cup B) \subseteq \overline{UPP}_R^{\mathcal{I}}(A) \cup \overline{UPP}_R^{\mathcal{I}}(B)$. Hence, $\overline{UPP}_R^{\mathcal{I}}(A \cup B) = \overline{UPP}_R^{\mathcal{I}}(A) \cup \overline{UPP}_R^{\mathcal{I}}(B)$.

(v)

$$\begin{aligned}(\underline{LOW}_R^{\mathcal{I}}(A^c))^c &= (\{x \in U : \breve{R} < x > \breve{R} \cap A \in \mathcal{I}\})^c.\\ &= \{x \in U : \breve{R} < x > \breve{R} \cap A \notin \mathcal{I}\}.\\ &= \overline{UPP}_R^{\mathcal{I}}(A).\end{aligned}$$

(vi) Straightforward by Definition 3.1.

(vii) Let $x \in \overline{UPP}_R^{\mathcal{J}}(A)$. Then, $\breve{R} < x > \breve{R} \cap A \notin \mathcal{J}$. Since, $\mathcal{I} \subseteq \mathcal{J}$. Thus, $\breve{R} < x > \breve{R} \cap A \notin \mathcal{I}$. Therefore, $x \in \overline{UPP}_R^{\mathcal{I}}(A)$. Hence, $\overline{UPP}_R^{\mathcal{J}}(A) \subseteq \overline{UPP}_R^{\mathcal{I}}(A)$.

(viii) Straightforward by Definition 3.1.

(ix)

$$\begin{aligned}\overline{UPP}_R^{\mathcal{I}\cap\mathcal{J}}(A) &= \{x \in U : \breve{R} < x > \breve{R} \cap A \notin \mathcal{I} \cap \mathcal{J}\}.\\ &= \{x \in U : \breve{R} < x > \breve{R} \cap A \notin \mathcal{I}\} \; or \; \{x \in U : \breve{R} < x > \breve{R} \cap A \notin \mathcal{J}\}.\\ &= \{x \in U : \breve{R} < x > \breve{R} \cap A \notin \mathcal{I}\} \cup \{x \in U : \breve{R} < x > \breve{R} \cap A \notin \mathcal{J}\}.\\ &= \overline{UPP}_R^{\mathcal{I}}(A) \cup \overline{UPP}_R^{\mathcal{J}}(A).\end{aligned}$$

(x)

$$\begin{aligned}\overline{UPP}_R^{\mathcal{I}\vee\mathcal{J}}(A) &= \{x \in U : \breve{R} < x > \breve{R} \cap A \notin \mathcal{I} \vee \mathcal{J}\}.\\ &= \{x \in U : \breve{R} < x > \breve{R} \cap A \notin \mathcal{I} \cup \mathcal{J}\}.\\ &= \{x \in U : \breve{R} < x > \breve{R} \cap A \notin \mathcal{I}\} \; and \; \{x \in U : \breve{R} < x > \breve{R} \cap A \notin \mathcal{J}\}.\\ &= \{x \in U : \breve{R} < x > \breve{R} \cap A \notin \mathcal{I}\} \cap \{x \in U : \breve{R} < x > \breve{R} \cap A \notin \mathcal{J}\}.\\ &= \overline{UPP}_R^{\mathcal{I}}(A) \cap \overline{UPP}_R^{\mathcal{J}}(A).\end{aligned}$$

Proposition 3.3. *Let $(U, R, \mathcal{I})$ and $(U, R, \mathcal{J})$ be two ideal approximation spaces and let $A, B \subseteq U$. Then, we have next properties.*

(i) $\underline{LOW}_R^{\mathcal{I}}(U) = U$.

(ii) $A \subseteq B \Rightarrow \underline{LOW}_R^{\mathcal{I}}(A) \subseteq \underline{LOW}_R^{\mathcal{I}}(B)$.

(iii) $\underline{LOW}_R^{\mathcal{I}}(A) \cup \underline{LOW}_R^{\mathcal{I}}(B) \subseteq \underline{LOW}_R^{\mathcal{I}}(A \cup B)$.

(iv) $\underline{LOW}_R^{\mathcal{I}}(A \cap B) = \underline{LOW}_R^{\mathcal{I}}(A) \cap \underline{LOW}_R^{\mathcal{I}}(B)$.

(v) $\underline{LOW}_R^{\mathcal{I}}(A) = (\overline{UPP}_R^{\mathcal{I}}(A^c))^c$.

(vi) *if $A^c \in \mathcal{I}$, then $\underline{LOW}_R^{\mathcal{I}}(A) = U$.*

(vii) *if $\mathcal{I} \subseteq \mathcal{J}$, then $\underline{LOW}_R^{\mathcal{I}}(A) \subseteq \underline{LOW}_R^{\mathcal{J}}(A)$.*

(viii) *if $\mathcal{I} = P(U)$, then $\underline{LOW}_R^{\mathcal{I}}(A) = U$.*

(ix) $\underline{LOW}_R^{\mathcal{I}\cap\mathcal{J}}(A) = \underline{LOW}_R^{\mathcal{I}}(A) \cap \underline{LOW}_R^{\mathcal{J}}(A)$.

(x) $\underline{LOW}_R^{\mathcal{I}\vee\mathcal{J}}(A) = \underline{LOW}_R^{\mathcal{I}}(A) \cup \underline{LOW}_R^{\mathcal{J}}(A)$.

Proof.

(i)

$$\begin{aligned}\underline{LOW}_R^{\mathcal{I}}(U) &= \{x \in U : \breve{R} < x > \breve{R} \cap \phi \in \mathcal{I}\}.\\ &= U.\end{aligned}$$

(ii) Let $x \in \underline{LOW}_R^{\mathcal{I}}(A)$. Then, $\breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}$. Since, $B^c \subseteq A^c$ and $\mathcal{I}$ is an ideal. Thus, $\breve{R} < x > \breve{R} \cap B^c \in \mathcal{I}$. Therefore, $x \in \underline{LOW}_R^{\mathcal{I}}(B)$. Hence, $\underline{LOW}_R^{\mathcal{I}}(A) \subseteq \underline{LOW}_R^{\mathcal{I}}(B)$.

(iii) Immediately by part (ii).

(iv) $\underline{LOW}_R^{\mathcal{I}}(A) \cap \underline{LOW}_R^{\mathcal{I}}(B) \supseteq \underline{LOW}_R^{\mathcal{I}}(A \cap B)$ by part (ii). Let $x \in \underline{LOW}_R^{\mathcal{I}}(A) \cap \underline{LOW}_R^{\mathcal{I}}(B)$. Then, $\breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}$ and $\breve{R} < x > \breve{R} \cap B^c \in \mathcal{I}$. It follows that $(\breve{R} < x > \breve{R} \cap (A^c \cup B^c)) \in \mathcal{I}$. So, $(\breve{R} < x > \breve{R} \cap (A \cap B)^c) \in \mathcal{I}$. Therefore, $x \in \underline{LOW}_R^{\mathcal{I}}(A \cap B)$. Thus, $\underline{LOW}_R^{\mathcal{I}}(A) \cap \underline{LOW}_R^{\mathcal{I}}(B) \subseteq \underline{LOW}_R^{\mathcal{I}}(A \cap B)$. Hence, $\underline{LOW}_R^{\mathcal{I}}(A) \cap \underline{LOW}_R^{\mathcal{I}}(B) = \underline{LOW}_R^{\mathcal{I}}(A \cap B)$.

(v)

$$\begin{aligned}(\overline{UPP}_R^{\mathcal{I}}(A^c))^c &= (\{x \in U : \breve{R} < x > \breve{R} \cap A^c \notin \mathcal{I}\})^c.\\ &= \{x \in U : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}\}.\\ &= \underline{LOW}_R^{\mathcal{I}}(A).\end{aligned}$$

(vi) Straightforward by Definition 3.1.

(vii) Let $x \in \underline{LOW}_R^{\mathcal{I}}(A)$. Then, $\breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}$. Since, $\mathcal{I} \subseteq \mathcal{J}$. Thus, $\breve{R} < x > \breve{R} \cap A^c \in \mathcal{J}$. Therefore, $x \in \underline{LOW}_R^{\mathcal{I}}(A)$. Hence, $\underline{LOW}_R^{\mathcal{I}}(A) \subseteq \underline{LOW}_R^{\mathcal{J}}(A)$.

(viii) Straightforward by Definition 3.1.

(ix)

$$\begin{aligned}\underline{LOW}_R^{\mathcal{I}\cap\mathcal{J}}(A) &= \{x \in U : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{I} \cap \mathcal{J}\}.\\ &= \{x \in U : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}\} \textit{ and } \{x \in U : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{J}\}.\\ &= \{x \in U : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}\} \cap \{x \in U : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{J}\}.\\ &= \underline{LOW}_R^{\mathcal{I}}(A) \cap \underline{LOW}_R^{\mathcal{J}}(A).\end{aligned}$$

(x)

$$\begin{aligned}\underline{LOW}_R^{\mathcal{I}\vee\mathcal{J}}(A) &= \{x \in U : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{I} \vee \mathcal{J}\}.\\ &= \{x \in U : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{I} \cup \mathcal{J}\}.\\ &= \{x \in U : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}\} \textit{ or } \{x \in U : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{J}\}.\\ &= \{x \in U : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}\} \cup \{x \in U : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{J}\}.\\ &= \underline{LOW}_R^{\mathcal{I}}(A) \cup \underline{LOW}_R^{\mathcal{J}}(A).\end{aligned}$$

Remark 3.4. *In Propositions 3.2 and 3.3 the converse of parts (ii), (iii), (vi), (vii), and (viii) is false, in general as illustrated by the next example.*

Example 3.5. **(i)** *Let* $U = \{u_1, u_2, u_3, u_4\}, \mathcal{I} = \{\phi, \{u_1\}, \{u_2\}, \{u_3\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_2, u_3\}, \{u_1, u_2, u_3\}\}$ *and* $R = \{(u_1, u_2), (u_1, u_3), (u_2, u_3), (u_2, u_4), (u_3, u_1), (u_3, u_4)\}$ *be a binary relation defined on U. By calculations, we obtain* $\breve{R} < u_1 > \breve{R} = \{u_1\}, \breve{R} < u_2 > \breve{R} = \breve{R} < u_3 > \breve{R} = \{u_2, u_3\}, \breve{R} < u_4 > \breve{R} = \phi$. *For part (ii), take* $A = \{u_1, u_2\}$ *and* $B = \{u_1, u_4\}$*, then*

(a) $\overline{UPP}_R^{\mathcal{I}}(A) = \overline{UPP}_R^{\mathcal{I}}(B) = \phi$. *Therefore,* $\overline{UPP}_R^{\mathcal{I}}(A) \subseteq \overline{UPP}_R^{\mathcal{I}}(B)$*, but* $A \nsubseteq B$.

(b) $\underline{LOW}_R^{\mathcal{I}}(A) = \underline{LOW}_R^{\mathcal{I}}(B) = U$. *Therefore,* $\underline{LOW}_R^{\mathcal{I}}(A) \subseteq \underline{LOW}_R^{\mathcal{I}}(B)$*, but* $A \nsubseteq B$.

(ii) *Let* $U = \{u_1, u_2, u_3, u_4\}, \mathcal{J} = \{\phi, \{u_2\}\}, \mathcal{I} = \{\phi, \{u_1\}\}$ *and* $R = \{(u_2, u_2), (u_3, u_3), (u_4, u_4)\}$ *be a binary relation defined on* U*. By calculations, we obtain* $\breve{R} < u_1 > \breve{R} = \phi, \breve{R} < u_2 > \breve{R} = \{u_2\}, \breve{R} < u_3 > \breve{R} = \{u_3\}, \breve{R} < u_4 > \breve{R} = \{u_4\}$.

For part (vi), take

- **(a)** $A = \{u_1, u_2\}$, *then* $\overline{UPP}_R^{\mathcal{J}}(A) = \phi$*. Therefore,* $\overline{UPP}_R^{\mathcal{J}}(A) = \phi$*, but* $A \notin \mathcal{J}$.
- **(b)** $A = \{u_3, u_4\}$, *then* $\underline{LOW}_R^{\mathcal{J}}(A) = U$*. Therefore,* $\underline{LOW}_R^{\mathcal{J}}(A) = U$*, but* $A^c \notin \mathcal{J}$.

For part (vii), take

- **(a)** $A = \{u_1, u_2\}$, *then* $\overline{UPP}_R^{\mathcal{I}}(A) = \{u_2\}, \overline{UPP}_R^{\mathcal{J}}(A) = \phi$*. Therefore,* $\overline{UPP}_R^{\mathcal{J}}(A) \subseteq \overline{UPP}_R^{\mathcal{I}}(A)$*, but* $\mathcal{I} \nsubseteq \mathcal{J}$.
- **(b)** $A = \{u_3, u_4\}$, *then* $\underline{LOW}_R^{\mathcal{I}}(A) = \{u_1, u_3, u_4\}, \underline{LOW}_R^{\mathcal{J}}(A) = U$*. Therefore,* $\underline{LOW}_R^{\mathcal{I}}(A) \subseteq \underline{LOW}_R^{\mathcal{J}}(A)$*, but* $\mathcal{I} \nsubseteq \mathcal{J}$.

For part (viii), take

- **(a)** $A = \{u_1, u_2\}$, *then* $\overline{UPP}_R^{\mathcal{J}}(A) = \phi$*, but* $\mathcal{J} \neq P(U)$.
- **(b)** $A = \{u_3, u_4\}$, *then* $\underline{LOW}_R^{\mathcal{J}}(A) = U$*, but* $\mathcal{J} \neq P(U)$.

(iii) *Let* $U = \{u_1, u_2, u_3, u_4\}, \mathcal{I} = \{\phi, \{u_4\}\}$ *and* $R = \Delta \cup \{(u_2, u_1), (u_3, u_1), (u_4, u_1)\}$ *be a binary relation defined on* U*, (where* Δ *is the identity relation and equal to* $\{(u_1, u_1), (u_2, u_2), (u_3, u_3), (u_4, u_4)\}$*). By calculations, we obtain* $\breve{R} < u_1 > \breve{R} = U, \breve{R} < u_2 > \breve{R} = \{u_1, u_2\}, \breve{R} < u_3 > \breve{R} = \{u_1, u_3\}, \breve{R} < u_4 > \breve{R} = \{u_1, u_4\}$*. For part (iii), take* $A = \{u_1, u_4\}, B = \{u_2, u_3\}$*. Hence,*

- **(a)** $A \cap B = \phi$, *then* $\overline{UPP}_R^{\mathcal{I}}(A) = U, \overline{UPP}_R^{\mathcal{I}}(B) = \{u_1, u_2, u_3\}, \overline{UPP}_R^{\mathcal{I}}(A \cap B) = \phi$*. Therefore,* $\overline{UPP}_R^{\mathcal{I}}(A) \cap \overline{UPP}_R^{\mathcal{I}}(B) = \{u_1, u_2, u_3\} \neq \phi = \overline{UPP}_R^{\mathcal{I}}(A \cap B)$.
- **(b)** $A \cup B = U$, *then* $\underline{LOW}_R^{\mathcal{I}}(A) = \{u_4\}, \underline{LOW}_R^{\mathcal{I}}(B) = \phi, \overline{UPP}_R^{\mathcal{I}}(A \cup B) = U$*. Therefore,* $\underline{LOW}_R^{\mathcal{I}}(A) \cup \underline{LOW}_R^{\mathcal{I}}(B) = \{u_4\} \neq U = \underline{LOW}_R^{\mathcal{I}}(A \cup B)$.

Remark 3.6. *In the first type of improvement of ideal approximation space, some properties of Pawlak approximation space are not satisfied.*

(i) *In Example 3.5 (i) take*

- **(a)** $A = \{u_1, u_2\}$, *then* $\overline{UPP}_R^{\mathcal{I}}(A) = \phi$*. Hence,* $A \nsubseteq \overline{UPP}_R^{\mathcal{I}}(A)$.
- **(b)** $A = \{u_1, u_2\}$, *then* $\underline{LOW}_R^{\mathcal{I}}(A) = U$*. Hence,* $\underline{LOW}_R^{\mathcal{I}}(A) \nsubseteq A$.
- **(c)** $A = U$, *then* $\overline{UPP}_R^{\mathcal{I}}(U) = \phi$*. Hence,* $\overline{UPP}_R^{\mathcal{I}}(U) \neq U$.
- **(d)** $A = \phi$, *then* $\underline{LOW}_R^{\mathcal{I}}(\phi) = U$*. Hence,* $\underline{LOW}_R^{\mathcal{I}}(\phi) \neq \phi$.

(ii) *In Example 3.5 (iii) take*

- **(a)** $A = \{u_2, u_3\}$, *then* $\overline{UPP}_R^{\mathcal{I}}(A) = \{u_1, u_2, u_3\}, \overline{UPP}_R^{\mathcal{I}}(\overline{UPP}_R^{\mathcal{I}}(A)) = U$*. Hence,* $\overline{UPP}_R^{\mathcal{I}}(A) \neq \overline{UPP}_R^{\mathcal{I}}(\overline{UPP}_R^{\mathcal{I}}(A))$.
- **(b)** $A = \{u_1, u_4\}$, *then* $\underline{LOW}_R^{\mathcal{I}}(A) = \{u_4\}, \underline{LOW}_R^{\mathcal{I}}(\underline{LOW}_R^{\mathcal{I}}(A)) = \phi$*. Hence,* $\underline{LOW}_R^{\mathcal{I}}(A) \neq \underline{LOW}_R^{\mathcal{I}}(\underline{LOW}_R^{\mathcal{I}}(A))$.

Example 3.7. *Let* $U = \{u_1, u_2, u_3, u_4\}, \mathcal{I} = \{\phi, \{u_1\}\}$ *and* $R = \Delta \cup \{(u_1, u_2), (u_2, u_1), (u_2, u_3), (u_3, u_1),$ $(u_3, u_2), (u_4, u_1), (u_4, u_2)\}$ *be a binary relation defined on* U*. By calculations, we obtain* $\breve{R} < u_1 > \breve{R} = \breve{R} < u_2 > \breve{R} = U, \breve{R} < u_3 > \breve{R} = \{u_1, u_2, u_3\}, \breve{R} < u_4 > \breve{R} = \{u_1, u_2, u_4\}$*. Then*

(i) *If* $A = \{u_3\}$*, then* $\overline{UPP}^{\mathcal{I}}_R(A) = \{u_1, u_2, u_3\}$ *and* $\underline{LOW}^{\mathcal{I}}_R(\overline{UPP}^{\mathcal{I}}_R(A)) = \{u_3\}$*. Hence,* $\overline{UPP}^{\mathcal{I}}_R(A) \nsubseteq \underline{LOW}^{\mathcal{I}}_R(\overline{UPP}^{\mathcal{I}}_R(A))$.

(ii) *If* $A = \{u_1, u_2, u_4\}$*, then* $\underline{LOW}^{\mathcal{I}}_R(A) = \{u_4\}$ *and* $\overline{UPP}^{\mathcal{I}}_R(\underline{LOW}^{\mathcal{I}}_R(A)) = \{u_1, u_2, u_4\}$*. Hence,* $\overline{UPP}^{\mathcal{I}}_R(\underline{LOW}^{\mathcal{I}}_R(A)) \nsubseteq \underline{LOW}^{\mathcal{I}}_R(A)$.

Proposition 3.8. *Let* $(U, R, \mathcal{I})$ *be an ideal approximation space and* $\phi \neq A \subseteq U$*. Then*

(i) $0 \leq ACC^{\mathcal{I}}_R(A) \leq 1$.

(ii) $ACC^{\mathcal{I}}_R(U) = 1$.

Proof. We prove (i) only and (ii) is straightforward. Since, $\phi \neq A \subseteq U$, then $\overline{UPP}^{\mathcal{I}}_R(A) \cup A \neq \phi$. Hence, $\phi \subseteq \underline{LOW}^{\mathcal{I}}_R(A) \cap A \subseteq \overline{UPP}^{\mathcal{I}}_R(A) \cup A$. Therefore, $0 \leq |\underline{LOW}^{\mathcal{I}}_R(A) \cap A| \leq |\overline{UPP}^{\mathcal{I}}_R(A) \cup A|$. So, $0 \leq \frac{|\underline{LOW}^{\mathcal{I}}_R(A) \cap A|}{|\overline{UPP}^{\mathcal{I}}_R(A) \cup A|} \leq 1$. It means that, $0 \leq ACC^{\mathcal{I}}_R(A) \leq 1$.

Theorem 3.9. *Let* $(U, R, \mathcal{I})$ *and* $(U, R, \mathcal{J})$ *be ideal approximation spaces such that* $\mathcal{I} \subseteq \mathcal{J}$*. Then for each nonempty subset A of U we have the next results.*

(i) $BND^{\mathcal{J}}_R(A) \subseteq BND^{\mathcal{I}}_R(A)$.

(ii) $ACC^{\mathcal{I}}_R(A) \leq ACC^{\mathcal{J}}_R(A)$.

(iii) $Rough^{\mathcal{J}}_R(A) \leq Rough^{\mathcal{I}}_R(A)$.

Proof.

(i) Let $x \in BND^{\mathcal{J}}_R(A)$. Then, $x \in \overline{UPP}^{\mathcal{J}}_R(A) - \underline{LOW}^{\mathcal{J}}_R(A)$. So, $x \in \overline{UPP}^{\mathcal{J}}_R(A)$ and $x \in (\underline{LOW}^{\mathcal{J}}_R(A))^c$. Hence, $x \in \overline{UPP}^{\mathcal{I}}_R(A)$ and $x \in (\underline{LOW}^{\mathcal{I}}_R(A))^c$ by Propositions 3.2 and 3.3 part (vii). It follows that $x \in BND^{\mathcal{I}}_R(A)$. Therefore, $BND^{\mathcal{J}}_R(A) \subseteq BND^{\mathcal{I}}_R(A)$.

(ii)

$$\begin{aligned} ACC^{\mathcal{I}}_R(A) &= \frac{|\underline{LOW}^{\mathcal{I}}_R(A) \cap A|}{|\overline{UPP}^{\mathcal{I}}_R(A) \cup A|}. \\ &\leq \frac{|\underline{LOW}^{\mathcal{J}}_R(A) \cap A|}{|\overline{UPP}^{\mathcal{J}}_R(A) \cup A|}. \\ &= ACC^{\mathcal{J}}_R(A). \end{aligned}$$

(iii) Straightforward by (ii).

Remark 3.10. **(i)** *Example 3.5 (ii) shows that the converse of part (i) in Theorem 3.9 is not necessary to be true in general. To elucidate that, take,* $A = \{u_3, u_4\}$*. Then* $BND^{\mathcal{J}}_R(A) = \phi \subseteq \phi = BND^{\mathcal{I}}_R(A)$*, but* $\mathcal{I} \nsubseteq \mathcal{J}$.

(ii) *Example 3.5 (iii) shows that the converse of parts (ii) and (iii) in Theorem 3.9 is not necessary to be true in general. To elucidate that, take,* $\mathcal{J} = \{\phi, \{u_1\}\}, A = \{u_1, u_4\}$. *Then*

(a) $ACC_R^{\mathcal{I}}(A) = \frac{1}{4} < \frac{1}{2} = ACC_R^{\mathcal{J}}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

(b) $Rough_R^{\mathcal{J}}(A) = \frac{1}{2} < \frac{3}{4} = Rough_R^{\mathcal{I}}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

Theorem 3.11. *Let* $\phi \neq A \subseteq U$, $\mathcal{I}$ *be an ideal on* U *and* R_1, R_2 *be two binary relations on* U. *If* $R_1 \subseteq R_2$, *then*

(i) $\overline{UPP}_{R_1}^{\mathcal{I}}(A) \subseteq \overline{UPP}_{R_2}^{\mathcal{I}}(A)$.

(ii) $\underline{LOW}_{R_2}^{\mathcal{I}}(A) \subseteq \underline{LOW}_{R_1}^{\mathcal{I}}(A)$.

(iii) $BND_{R_1}^{\mathcal{I}}(A) \subseteq BND_{R_2}^{\mathcal{I}}(A)$.

(iv) $ACC_{R_2}^{\mathcal{I}}(A) \leq ACC_{R_1}^{\mathcal{I}}(A)$.

(v) $Rough_{R_1}^{\mathcal{I}}(A) \leq Rough_{R_2}^{\mathcal{I}}(A)$.

Proof.

(i) Let $x \in \overline{UPP}_{R_1}^{\mathcal{I}}(A)$. Then, $\breve{R}_1 < x > \breve{R}_1 \cap A \notin \mathcal{I}$. Since, $\breve{R}_1 < x > \breve{R}_1 \subseteq \breve{R}_2 < x > \breve{R}_2$ (by Theorem 2.5 [8]). It follows that $\breve{R}_2 < x > \breve{R}_2 \cap A \notin \mathcal{I}$. Thus, $x \in \overline{UPP}_{R_2}^{\mathcal{I}}(A)$. Hence, $\overline{UPP}_{R_1}^{\mathcal{I}}(A) \subseteq \overline{UPP}_{R_2}^{\mathcal{I}}(A)$.

(ii) Let $x \in \underline{LOW}_{R_2}^{\mathcal{I}}(A)$. Then, $\breve{R}_2 < x > \breve{R}_2 \cap A^c \in \mathcal{I}$. Since, $\breve{R}_1 < x > \breve{R}_1 \subseteq \breve{R}_2 < x > \breve{R}_2$ (by Theorem 2.5 [8]). It follows that $\breve{R}_1 < x > \breve{R}_1 \cap A^c \in \mathcal{I}$. Thus, $x \in \underline{LOW}_{R_1}^{\mathcal{I}}(A)$. Hence, $\underline{LOW}_{R_2}^{\mathcal{I}}(A) \subseteq \underline{LOW}_{R_1}^{\mathcal{I}}(A)$.

(iii) Let $x \in BND_{R_1}^{\mathcal{I}}(A)$. Then, $x \in \overline{UPP}_{R_1}^{\mathcal{I}}(A) - \underline{LOW}_{R_1}^{\mathcal{I}}(A)$. So, $x \in \overline{UPP}_{R_1}^{\mathcal{I}}(A)$ and $x \in (\underline{LOW}_{R_1}^{\mathcal{I}}(A))^c$. Thus, $x \in \overline{UPP}_{R_2}^{\mathcal{I}}(A)$ and $x \in (\underline{LOW}_{R_2}^{\mathcal{I}}(A))^c$ by parts (i) and (ii). Hence, $x \in BND_{R_2}^{\mathcal{I}}(A)$. Therefore, $BND_{R_1}^{\mathcal{I}}(A) \subseteq BND_{R_2}^{\mathcal{I}}(A)$.

(iv)

$$\begin{aligned} ACC_{R_2}^{\mathcal{I}}(A) &= |\frac{\underline{LOW}_{R_2}^{\mathcal{I}}(A) \cap A|}{\overline{UPP}_{R_2}^{\mathcal{I}}(A) \cup A|}. \\ &\leq \frac{|\underline{LOW}_{R_1}^{\mathcal{I}}(A) \cap A|}{|\overline{UPP}_{R_1}^{\mathcal{I}}(A) \cup A|}. \\ &= ACC_{R_1}^{\mathcal{I}}(A). \end{aligned}$$

(v) Straightforward by (iv).

In Theorem 3.11 the inclusion and less than relations can not be replaced by an equality relation as the next example elucidates.

Example 3.12. *Let $U = \{u_1, u_2, u_3, u_4\}$, $\mathcal{I} = \{\phi, \{u_2\}, \{u_3\}, \{u_4\}, \{u_2, u_3\}, \{u_2, u_4\}, \{u_3, u_4\}, \{u_2, u_3, u_4\}\}$ and $R_1 = \Delta \cup \{(u_1, u_2), (u_2, u_1)\}$, $R_2 = \Delta \cup \{(u_1, u_2), (u_2, u_1), (u_1, u_3), (u_3, u_1)\}$ be two similarity relation defined on U thus $\breve{R}_1 < u_1 > \breve{R}_1 = \breve{R}_1 < u_2 > \breve{R}_1 = \{u_1, u_2\}$, $\breve{R}_1 < u_3 > \breve{R}_1 = \{u_3\}$, $\breve{R}_1 < u_4 > \breve{R}_1 = \{u_4\}$, $\breve{R}_2 < u_1 > \breve{R}_2 = \breve{R}_2 < u_2 > \breve{R}_2 = \breve{R}_2 < u_3 > \breve{R}_2 = \{u_1, u_2, u_3\}$, $\breve{R}_2 < u_4 > \breve{R}_2 = \{u_4\}$. Take*

(i) *$A = \{u_1, u_4\}$, then*

1. $\overline{UPP}^{\mathcal{I}}_{R_1}(A) = \{u_1, u_2\} \neq \{u_1, u_2, u_3\} = \overline{UPP}^{\mathcal{I}}_{R_2}(A)$.
2. $ACC^{\mathcal{I}}_{R_1}(A) = \frac{2}{3} \neq \frac{1}{2} = ACC^{\mathcal{I}}_{R_2}(A)$.
3. $Rough^{\mathcal{I}}_{R_2}(A) = \frac{1}{2} \neq \frac{1}{3} = Rough^{\mathcal{I}}_{R_1}(A)$.

(ii) *$A = \{u_2, u_3\}$, then $\underline{LOW}^{\mathcal{I}}_{R_1}(A) = \{u_3, u_4\} \neq \{u_4\} = \underline{LOW}^{\mathcal{I}}_{R_2}(A)$.*

3.2. The second method of improvement of the approximations and accuracy measure of a rough set

Definition 3.13. *Let $(U, R, \mathcal{I})$ be an ideal approximation space. The second kind of the improvement of approximations (lower and upper), boundary region, measures (accuracy and roughness) of $\phi \neq A \subseteq U$ with respect to $\breve{R} < x > \breve{R}$ are respectively defined by:*

$\underline{\underline{LOW}}^{\mathcal{I}}_{R}(A) = \{x \in A : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}\}$.

$\overline{\overline{UPP}}^{\mathcal{I}}_{R}(A) = A \cup \overline{UPP}^{\mathcal{I}}_{R}(A)$.

$\underline{BND}^{\mathcal{I}}_{R}(A) = \overline{\overline{UPP}}^{\mathcal{I}}_{R}(A) - \underline{\underline{LOW}}^{\mathcal{I}}_{R}(A)$.

$\underline{ACC}^{\mathcal{I}}_{R}(A) = \frac{|\underline{\underline{LOW}}^{\mathcal{I}}_{R}(A)|}{|\overline{\overline{UPP}}^{\mathcal{I}}_{R}(A)|}$.

$\underline{Rough}^{\mathcal{I}}_{R}(A) = 1 - \underline{ACC}^{\mathcal{I}}_{R}(A)$.

Proposition 3.14. *Let $(U, R, \mathcal{I})$ and $(U, R, \mathcal{J})$ be two ideal approximation spaces and let $A, B \subseteq U$. Then, we have next properties.*

(i) *$A \subseteq \overline{\overline{UPP}}^{\mathcal{I}}_{R}(A)$ equality hold if $A = \phi$ or U.*

(ii) *$A \subseteq B \Rightarrow \overline{\overline{UPP}}^{\mathcal{I}}_{R}(A) \subseteq \overline{\overline{UPP}}^{\mathcal{I}}_{R}(B)$.*

(iii) *$\overline{\overline{UPP}}^{\mathcal{I}}_{R}(A) \subseteq \overline{\overline{UPP}}^{\mathcal{I}}_{R}(\overline{\overline{UPP}}^{\mathcal{I}}_{R}(A))$.*

(iv) *$\overline{\overline{UPP}}^{\mathcal{I}}_{R}(A \cap B) \subseteq \overline{\overline{UPP}}^{\mathcal{I}}_{R}(A) \cap \overline{\overline{UPP}}^{\mathcal{I}}_{R}(B)$.*

(v) *$\overline{\overline{UPP}}^{\mathcal{I}}_{R}(A \cup B) = \overline{\overline{UPP}}^{\mathcal{I}}_{R}(A) \cup \overline{\overline{UPP}}^{\mathcal{I}}_{R}(B)$.*

(vi) *$\overline{\overline{UPP}}^{\mathcal{I}}_{R}(A) = (\underline{\underline{LOW}}^{\mathcal{I}}_{R}(A^c))^c$.*

(vii) *if $A \in \mathcal{I}$, then $\overline{\overline{UPP}}^{\mathcal{I}}_{R}(A) = A$.*

(viii) *if $\mathcal{I} \subseteq \mathcal{J}$, then $\overline{\overline{UPP}}^{\mathcal{J}}_{R}(A) \subseteq \overline{\overline{UPP}}^{\mathcal{I}}_{R}(A)$.*

(ix) *if $\mathcal{I} = P(U)$, then $\overline{\overline{UPP}}^{\mathcal{I}}_{R}(A) = A$.*

(x) $\overline{\overline{UPP}}_R^{\mathcal{I}\cap\mathcal{J}}(A) = \overline{\overline{UPP}}_R^{\mathcal{I}}(A) \cup \overline{\overline{UPP}}_R^{\mathcal{J}}(A)$.

(xi) $\overline{\overline{UPP}}_R^{\mathcal{I}\vee\mathcal{J}}(A) = \overline{\overline{UPP}}_R^{\mathcal{I}}(A) \cap \overline{\overline{UPP}}_R^{\mathcal{J}}(A)$.

Proof. Similar to Proposition 3.2.

Proposition 3.15. *Let $(U, R, \mathcal{I})$ and $(U, R, \mathcal{J})$ be two ideal approximation spaces and let $A, B \subseteq U$. Then, we have next properties.*

(i) $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) \subseteq A$ *equality hold if* $A = \phi$ *or* U.

(ii) $A \subseteq B \Rightarrow \underline{\underline{LOW}}_R^{\mathcal{I}}(A) \subseteq \underline{\underline{LOW}}_R^{\mathcal{I}}(B)$.

(iii) $\underline{\underline{LOW}}_R^{\mathcal{I}}(\underline{\underline{LOW}}_R^{\mathcal{I}}(A)) \subseteq \underline{\underline{LOW}}_R^{\mathcal{I}}(A)$.

(iv) $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) \cup \underline{\underline{LOW}}_R^{\mathcal{I}}(B) \subseteq \underline{\underline{LOW}}_R^{\mathcal{I}}(A \cup B)$.

(v) $\underline{\underline{LOW}}_R^{\mathcal{I}}(A \cap B) = \underline{\underline{LOW}}_R^{\mathcal{I}}(A) \cap \underline{\underline{LOW}}_R^{\mathcal{I}}(B)$.

(vi) $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) = (\overline{\overline{UPP}}_R^{\mathcal{I}}(A^c))^c$.

(vii) *if* $A^c \in \mathcal{I}$, *then* $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) = A$.

(viii) *if* $\mathcal{I} \subseteq \mathcal{J}$, *then* $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) \subseteq \underline{\underline{LOW}}_R^{\mathcal{J}}(A)$.

(ix) *if* $\mathcal{I} = P(U)$, *then* $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) = A$.

(x) $\underline{\underline{LOW}}_R^{\mathcal{I}\cap\mathcal{J}}(A) = \underline{\underline{LOW}}_R^{\mathcal{I}}(A) \cap \underline{\underline{LOW}}_R^{\mathcal{J}}(A)$.

(xi) $\underline{\underline{LOW}}_R^{\mathcal{I}\vee\mathcal{J}}(A) = \underline{\underline{LOW}}_R^{\mathcal{I}}(A) \cup \underline{\underline{LOW}}_R^{\mathcal{J}}(A)$.

Proof. Similar to Proposition 3.3.

Remark 3.16. **(i)** *Example 3.5 (i) also shows that the converse of parts (vii) and (ix) in Propositions 3.14 and 3.15 is not necessarily to be true in general.*
For part (vii), take

(a) $A = \{u_1, u_3, u_4\}$, *then* $\overline{\overline{UPP}}_R^{\mathcal{I}}(A) = A$, *but* $A \notin \mathcal{I}$.

(b) $A = \{u_2\}$, *then* $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) = A$, *but* $A^c \notin \mathcal{I}$.

For part (ix), take

(a) $A = \{u_1, u_3, u_4\}$, *then* $\overline{\overline{UPP}}_R^{\mathcal{I}}(A) = A$, *but* $\mathcal{I} \neq P(U)$.

(b) $A = \{u_2\}$, *then* $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) = A$, *but* $\mathcal{I} \neq P(U)$.

(ii) *Example 3.5 (ii) also shows that the converse of part (viii) in Propositions 3.14 and 3.15 is not necessarily to be true in general. Take*

(a) $A = \{u_1, u_2\}$, *then* $\overline{\overline{UPP}}_R^{\mathcal{J}}(A) = \{u_1, u_2\} \subseteq \{u_1, u_2\} = \overline{\overline{UPP}}_R^{\mathcal{I}}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

(b) $A = \{u_3, u_4\}$, *then* $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) = \{u_3, u_4\} \subseteq \{u_3, u_4\} = \underline{\underline{LOW}}_R^{\mathcal{J}}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

(iii) *Example 3.5 (iii) also shows that the inclusion of parts (ii)-(iv) in Propositions 3.14 and 3.15 can not be replaced by the equality in general.*
For part (ii), take $A = \{u_2\}, B = \{u_1\}$, *then*

(a) $\overline{\overline{UPP}}_R^{\mathcal{I}}(A) = \{u_1, u_2\} \subseteq U = \overline{\overline{UPP}}_R^{\mathcal{I}}(B)$, *but* $A \nsubseteq B$.

(b) $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) = \phi \subseteq \phi = \underline{\underline{LOW}}_R^{\mathcal{I}}(B)$, *but* $A \nsubseteq B$.

For part (iii), take

(a) $A = \{u_2, u_3\}$, *then* $\overline{\overline{UPP}}_R^{\mathcal{I}}(A) = \{u_1, u_2, u_3\}, \overline{\overline{UPP}}_R^{\mathcal{I}}(\overline{\overline{UPP}}_R^{\mathcal{I}}(A)) = U$. *Therefore,* $\overline{\overline{UPP}}_R^{\mathcal{I}}(A) = \{u_1, u_2, u_3\} \neq U = \overline{\overline{UPP}}_R^{\mathcal{I}}(\overline{\overline{UPP}}_R^{\mathcal{I}}(A))$.

(b) $A = \{u_1, u_4\}$, *then* $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) = \{u_4\}, \underline{\underline{LOW}}_R^{\mathcal{I}}(\underline{\underline{LOW}}_R^{\mathcal{I}}(A)) = \phi$. *Therefore,* $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) = \{u_4\} \neq \phi = \underline{\underline{LOW}}_R^{\mathcal{I}}(\underline{\underline{LOW}}_R^{\mathcal{I}}(A))$.

For part (iv), take $A = \{u_1, u_4\}, B = \{u_2, u_3\}$, *then*

(a) $A \cap B = \phi$. *Hence,* $\overline{\overline{UPP}}_R^{\mathcal{I}}(A) = U, \overline{\overline{UPP}}_R^{\mathcal{I}}(B) = \{u_1, u_2, u_3\}$. *Therefore,* $\overline{\overline{UPP}}_R^{\mathcal{I}}(A) \cap \overline{\overline{UPP}}_R^{\mathcal{I}}(B) = \{u_1, u_2, u_3\} \neq \phi = \overline{\overline{UPP}}_R^{\mathcal{I}}(A \cap B)$.

(b) $A \cup B = U$. *Hence,* $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) = \{u_4\}, \underline{\underline{LOW}}_R^{\mathcal{I}}(B) = \phi$. *Therefore,* $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) \cup \underline{\underline{LOW}}_R^{\mathcal{I}}(B) = \{u_4\} \neq U = \underline{\underline{LOW}}_R^{\mathcal{I}}(A \cup B)$.

Remark 3.17. *In the second type of improvement of ideal approximation space, some properties of Pawlak approximation space are not satisfied.*

(i) *In Example 3.5 (i) take*

(a) $A = \{u_1\} \in \mathcal{I}$, *then* $\overline{\overline{UPP}}_R^{\mathcal{I}}(A) = A$. *Hence, if* $A \in \mathcal{I} \nRightarrow \overline{\overline{UPP}}_R^{\mathcal{I}}(A) = \phi$.

(b) $A^c = \{u_1\} \in \mathcal{I}$, *then* $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) = A$. *Hence, if* $A^c \in \mathcal{I} \nRightarrow \underline{\underline{LOW}}_R^{\mathcal{I}}(A) = U$.

(ii) *In Example 3.5 (ii) take*

(a) $\mathcal{J} = P(U), A = \{u_1, u_3\}$, *then* $\overline{\overline{UPP}}_R^{\mathcal{J}}(A) = A$. *Hence, if* $\mathcal{J} = P(U) \nRightarrow \overline{\overline{UPP}}_R^{\mathcal{J}}(A) = \phi$.

(b) $\mathcal{J} = P(U), A = \{u_2, u_4\}$, *then* $\underline{\underline{LOW}}_R^{\mathcal{J}}(A) = A$. *Hence, if* $\mathcal{J} = P(U) \nRightarrow \underline{\underline{LOW}}_R^{\mathcal{J}}(A) = U$.

(iii) *In Example 3.7 take*

(a) $A = \{u_3\}$, *then* $\overline{\overline{UPP}}_R^{\mathcal{I}}(A) = \{u_1, u_2, u_3\}, \underline{\underline{LOW}}_R^{\mathcal{I}}(\overline{\overline{UPP}}_R^{\mathcal{I}}(A)) = \{u_3\}$. *Therefore,* $\overline{\overline{UPP}}_R^{\mathcal{I}}(A) = \{u_1, u_2, u_3\} \nsubseteq \{u_3\} = \underline{\underline{LOW}}_R^{\mathcal{I}}(\overline{\overline{UPP}}_R^{\mathcal{I}}(A))$.

(b) $A = \{u_1, u_2, u_4\}$, *then* $\underline{\underline{LOW}}^{\mathcal{I}}_R(A) = \{u_4\}, \overline{\overline{UPP}}^{\mathcal{I}}_R(\underline{\underline{LOW}}^{\mathcal{I}}_R(A)) = \{u_1, u_2, u_4\}$. *Therefore,* $\overline{\overline{UPP}}^{\mathcal{I}}_R(\underline{\underline{LOW}}^{\mathcal{I}}_R(A)) = \{u_1, u_2, u_4\} \nsubseteq \{u_4\} = \underline{\underline{LOW}}^{\mathcal{I}}_R(A)$.

Proposition 3.18. *Let* $(U, R, \mathcal{I})$ *be an ideal approximation space and* $\phi \neq A \subseteq U$. *Then*

(i) $0 \leq \underline{ACC}^{\mathcal{I}}_R(A) \leq 1$.

(ii) $\underline{ACC}^{\mathcal{I}}_R(U) = 1$.

Proof. It is similar to Proposition 3.8.

Theorem 3.19. *Let* $(U, R, \mathcal{I})$ *and* $(U, R, \mathcal{J})$ *be ideal approximation spaces such that* $\mathcal{I} \subseteq \mathcal{J}$. *Then for each nonempty subset A of U we have the next results.*

(i) $\underline{BND}^{\mathcal{J}}_R(A) \subseteq \underline{BND}^{\mathcal{I}}_R(A)$.

(ii) $\underline{ACC}^{\mathcal{I}}_R(A) \leq \underline{ACC}^{\mathcal{J}}_R(A)$.

(iii) $\underline{Rough}^{\mathcal{J}}_R(A) \leq \underline{Rough}^{\mathcal{I}}_R(A)$.

Proof. Similar to the proof of Theorem 3.9.

Remark 3.20. *Example 3.5 (ii) shows that the converse of parts (i) and (ii) in Theorem 3.19 is not necessary to be true in general. Take,* $A = \{u_2, u_4\}$, *then*

(i) $\underline{BND}^{\mathcal{J}}_R(A) = \phi \subseteq \phi = \underline{BND}^{\mathcal{I}}_R(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

(ii) $\underline{ACC}^{\mathcal{J}}_R(A) = 1 \leq 1 = \underline{ACC}^{\mathcal{I}}_R(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

(iii) $\underline{Rough}^{\mathcal{J}}_R(A) = 0 \leq 0 = \underline{Rough}^{\mathcal{I}}_R(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

Theorem 3.21. *Let* $\phi \neq A \subseteq U$, $\mathcal{I}$ *be an ideal on U and* R_1, R_2 *be two binary relations on U. If* $R_1 \subseteq R_2$, *then*

(i) $\overline{\overline{UPP}}^{\mathcal{I}}_{R_1}(A) \subseteq \overline{\overline{UPP}}^{\mathcal{I}}_{R_2}(A)$.

(ii) $\underline{\underline{LOW}}^{\mathcal{I}}_{R_2}(A) \subseteq \underline{\underline{LOW}}^{\mathcal{I}}_{R_1}(A)$.

(iii) $\underline{BND}^{\mathcal{I}}_{R_1}(A) \subseteq \underline{BND}^{\mathcal{I}}_{R_2}(A)$.

(iv) $\underline{ACC}^{\mathcal{I}}_{R_2}(A) \leq \underline{ACC}^{\mathcal{I}}_{R_1}(A)$.

(v) $\underline{Rough}^{\mathcal{I}}_{R_1}(A) \leq \underline{Rough}^{\mathcal{I}}_{R_2}(A)$.

Proof. Similar to Theorem 3.11.

Remark 3.22. *Example 3.12 shows that the inclusion and the less than in Theorem 3.21 can not be replaced by equality relation in general. Take* $A = \{u_1, u_4\}$, *then*

(i) $\overline{\overline{UPP}}^{\mathcal{I}}_{R_1}(A) = \{u_1, u_2, u_4\} \neq U = \overline{\overline{UPP}}^{\mathcal{I}}_{R_2}(A)$.

(ii) $\underline{BND}^{\mathcal{I}}_{R_1}(A) = \{u_2\} \neq \{u_2, u_3\} = \underline{BND}^{\mathcal{I}}_{R_2}(A)$.

(iii) $\underline{ACC}^{\mathcal{I}}_{R_1}(A) = \frac{2}{3} \neq \frac{1}{2} = \underline{ACC}^{\mathcal{I}}_{R_2}(A)$.

(iv) $\underline{Rough}^{\mathcal{I}}_{R_1}(A) = \frac{1}{3} \neq \frac{1}{2} = \underline{Rough}^{\mathcal{I}}_{R_2}(A)$.

3.3. The third method of the improvement of the approximations and accuracy measure of a rough set

Definition 3.23. *Let $(U, R, \mathcal{I})$ be an ideal approximation space. The third kind of the improvement of approximations (lower and upper), boundary region, measures (accuracy and roughness) of $\phi \neq A \subseteq U$ with respect to $\breve{R} < x > \breve{R}$ are respectively defined by:*

$\underline{LOW'}^{\mathcal{I}}_{R}(A) = \bigcup_{x \in U}\{\breve{R} < x > \breve{R} : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}\}$.

$\overline{UPP'}^{\mathcal{I}}_{R}(A) = (\underline{LOW'}^{\mathcal{I}}_{R}(A^c))^c$.

$BND'^{\mathcal{I}}_{R}(A) = \overline{UPP'}^{\mathcal{I}}_{R}(A) - \underline{LOW'}^{\mathcal{I}}_{R}(A)$.

$ACC'^{\mathcal{I}}_{R}(A) = \frac{|\underline{LOW'}^{\mathcal{I}}_{R}(A) \cap A|}{|\overline{UPP'}^{\mathcal{I}}_{R}(A) \cup A|}$.

$Rough'^{\mathcal{I}}_{R}(A) = 1 - ACC'^{\mathcal{I}}_{R}(A)$.

Proposition 3.24. *Let $(U, R, \mathcal{I})$ and $(U, R, \mathcal{J})$ be two ideal approximation spaces and let $A, B \subseteq U$. Then, we have next properties.*

(i) $A \subseteq B \Rightarrow \underline{LOW'}^{\mathcal{I}}_{R}(A) \subseteq \underline{LOW'}^{\mathcal{I}}_{R}(B)$.

(ii) $\underline{LOW'}^{\mathcal{I}}_{R}(A) \cup \underline{LOW'}^{\mathcal{I}}_{R}(B) \subseteq \underline{LOW'}^{\mathcal{I}}_{R}(A \cup B)$.

(iii) $\underline{LOW'}^{\mathcal{I}}_{R}(A \cap B) \subseteq \underline{LOW'}^{\mathcal{I}}_{R}(A) \cap \underline{LOW'}^{\mathcal{I}}_{R}(B)$.

(iv) $\underline{LOW'}^{\mathcal{I}}_{R}(A) = (\overline{UPP'}^{\mathcal{I}}_{R}(A^c))^c$.

(v) *if $\mathcal{I} \subseteq \mathcal{J}$, then $\underline{LOW'}^{\mathcal{I}}_{R}(A) \subseteq \underline{LOW'}^{\mathcal{J}}_{R}(A)$.*

(vi) $\underline{LOW'}^{\mathcal{I} \cap \mathcal{J}}_{R}(A) = \underline{LOW'}^{\mathcal{I}}_{R}(A) \cap \underline{LOW'}^{\mathcal{J}}_{R}(A)$.

Proof.

(i) Let $A \subseteq B$ and $x \in \underline{LOW'}^{\mathcal{I}}_{R}(A)$. Then, $\exists\ y \in U$ such that $x \in \breve{R} < y > \breve{R} \cap A^c \in \mathcal{I}$. Hence, $x \in \breve{R} < y > \breve{R} \cap B^c \in \mathcal{I}$ (by $B^c \subseteq A^c$, and the properties of ideal). Thus, $x \in \underline{LOW'}^{\mathcal{I}}_{R}(B)$. Therefore, $\underline{LOW'}^{\mathcal{I}}_{R}(A) \subseteq \underline{LOW'}^{\mathcal{I}}_{R}(B)$.

(ii) Immediately by part (i).

(iii) Immediately by part (i).

(iv) Straightforward by Definition 3.23.

(v) Let $\mathcal{I} \subseteq \mathcal{J}$ and $x \in \underline{LOW'}^{\mathcal{I}}_{R}(A)$. Then, $\exists\ y \in U$ such that $x \in \breve{R} < y > \breve{R} \cap A^c \in \mathcal{I} \subseteq \mathcal{J}$. So, $x \in \underline{LOW'}^{\mathcal{J}}_{R}(A)$, and hence $\underline{LOW'}^{\mathcal{I}}_{R}(A) \subseteq \underline{LOW'}^{\mathcal{J}}_{R}(A)$.

(vi)

$$\begin{aligned}\underline{LOW'}^{\mathcal{I} \cap \mathcal{J}}_{R}(A) &= \bigcup_{x \in U}\{\breve{R} < x > \breve{R} : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{I} \cap \mathcal{J}\}.\\ &= (\bigcup_{x \in U}\{\breve{R} < x > \breve{R} : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}\})\ \textit{and}\ (\bigcup_{x \in U}\{\breve{R} < x > \breve{R} : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{J}\}).\\ &= (\bigcup_{x \in U}\{\breve{R} < x > \breve{R} : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}\}) \cap (\bigcup_{x \in U}\{\breve{R} < x > \breve{R} : \breve{R} < x > \breve{R} \cap A^c \in \mathcal{J}\}).\\ &= \underline{LOW'}^{\mathcal{I}}_{R}(A) \cap \underline{LOW'}^{\mathcal{J}}_{R}(A).\end{aligned}$$

Proposition 3.25. *Let $(U, R, \mathcal{I})$ and $(U, R, \mathcal{J})$ be two ideal approximation spaces and let $A, B \subseteq U$. Then, we have next properties.*

(i) $A \subseteq B \Rightarrow \overline{UPP'}^{\mathcal{I}}_R(A) \subseteq \overline{UPP'}^{\mathcal{I}}_R(B)$.

(ii) $\overline{UPP'}^{\mathcal{I}}_R(A \cap B) \subseteq \overline{UPP'}^{\mathcal{I}}_R(A) \cap \overline{UPP'}^{\mathcal{I}}_R(B)$.

(iii) $\overline{UPP'}^{\mathcal{I}}_R(A) \cup \overline{UPP'}^{\mathcal{I}}_R(B) \subseteq \overline{UPP'}^{\mathcal{I}}_R(A \cup B)$.

(iv) $\overline{UPP'}^{\mathcal{I}}_R(A) = (\underline{LOW'}^{\mathcal{I}}_R(A^c))^c$.

(v) *if* $\mathcal{I} \subseteq \mathcal{J}$, *then* $\overline{UPP'}^{\mathcal{J}}_R(A) \subseteq \overline{UPP'}^{\mathcal{I}}_R(A)$.

(vi) $\overline{UPP'}^{\mathcal{I}\cap\mathcal{J}}_R(A) = \overline{UPP'}^{\mathcal{I}}_R(A) \cup \overline{UPP'}^{\mathcal{J}}_R(A)$.

Proof.

(i) Let $A \subseteq B$. Thus, $B^c \subseteq A^c$, and hence $\underline{LOW'}^{\mathcal{I}}_R(B^c) \subseteq \underline{LOW'}^{\mathcal{I}}_R(A^c)$ (by No. (1) in Proposition 3.24). So, $(\underline{LOW'}^{\mathcal{I}}_R(A^c))^c \subseteq (\underline{LOW'}^{\mathcal{I}}_R(B^c))^c$. Consequently, $\overline{UPP'}^{\mathcal{I}}_R(A) \subseteq \overline{UPP'}^{\mathcal{I}}_R(B)$.

(ii) Immediately by part (i).

(iii) Immediately by part (i).

(iv) Straightforward by Definition 3.23.

(v) Let $\mathcal{I} \subseteq \mathcal{J}$ and $x \in \overline{UPP'}^{\mathcal{J}}_R(A)$. Then, $x \in (\underline{LOW'}^{\mathcal{J}}_R(A^c))^c \subseteq (\underline{LOW'}^{\mathcal{I}}_R(A^c))^c$, (by No. (5) in Proposition 3.24). Thus, $x \in (\underline{LOW'}^{\mathcal{I}}_R(A^c))^c = \overline{UPP'}^{\mathcal{I}}_R(A)$. Therefore, $\overline{UPP'}^{\mathcal{J}}_R(A) \subseteq \overline{UPP'}^{\mathcal{I}}_R(A)$.

(vi)

$$
\begin{aligned}
\overline{UPP'}^{\mathcal{I}\cap\mathcal{J}}_R(A) &= (\underline{LOW'}^{\mathcal{I}\cap\mathcal{J}}_R(A^c))^c.\\
&= (\underline{LOW'}^{\mathcal{I}}_R(A^c) \cap \underline{LOW'}^{\mathcal{J}}_R(A^c))^c (\textit{by No. } (6) \textit{ in Proposition } 3.24).\\
&= (\underline{LOW'}^{\mathcal{I}}_R(A^c))^c \cup (\underline{LOW'}^{\mathcal{J}}_R(A^c))^c.\\
&= \overline{UPP'}^{\mathcal{I}}_R(A) \cup \overline{UPP'}^{\mathcal{J}}_R(A).
\end{aligned}
$$

Remark 3.26. **(i)** *Example 3.5 (i) shows that the converse of part (i) in Propositions 3.24 and 3.25 is not necessarily to be true in general. Take*

- **(a)** $A = \{u_1\}, B = \{u_4\}$, *then* $\overline{UPP'}^{\mathcal{I}}_R(A) = \overline{UPP'}^{\mathcal{I}}_R(B) = \{u_4\}$. *Therefore,* $\overline{UPP'}^{\mathcal{I}}_R(A) \subseteq \overline{UPP'}^{\mathcal{I}}_R(B)$, *but* $A \nsubseteq B$.
- **(b)** $A = \{u_2\}, B = \{u_1, u_3, u_4\}$, *then* $\underline{LOW'}^{\mathcal{I}}_R(A) = \underline{LOW'}^{\mathcal{I}}_R(B) = \{u_1, u_2, u_3\}$. *Therefore,* $\underline{LOW'}^{\mathcal{I}}_R(A) \subseteq \underline{LOW'}^{\mathcal{I}}_R(B)$, *but* $A \nsubseteq B$.

(ii) *Example 3.5 (iii) shows that the inclusion of part (ii) in Propositions 3.24 and 3.25 can not be replaced by equality relation in general. Take* $A = \{u_1, u_4\}, B = \{u_2, u_3\}$, *then*

(a) $\overline{UPP'}^{\mathcal{I}}_{R}(A) = U, \overline{UPP'}^{\mathcal{I}}_{R}(B) = B, \overline{UPP'}^{\mathcal{I}}_{R}(A \cap B) = \phi$. *Therefore,* $\overline{UPP'}^{\mathcal{I}}_{R}(A) \cap \overline{UPP'}^{\mathcal{I}}_{R}(B) = B \neq \phi = \overline{UPP'}^{\mathcal{I}}_{R}(A \cap B)$.

(b) $\underline{LOW'}^{\mathcal{I}}_{R}(A) = A, \underline{LOW'}^{\mathcal{I}}_{R}(B) = \phi, \underline{LOW'}^{\mathcal{I}}_{R}(A \cup B) = U$. *Therefore,* $\underline{LOW'}^{\mathcal{I}}_{R}(A) \cup \underline{LOW'}^{\mathcal{I}}_{R}(B) = A \neq U = \underline{LOW'}^{\mathcal{I}}_{R}(A \cup B)$.

(iii)

Example 3.27. *Let* $U = \{u_1, u_2, u_3, u_4\}, \mathcal{I} = \{\phi, \{u_1\}\}$ *and* $R = \{(u_1, u_1), (u_1, u_3), (u_1, u_4), (u_2, u_1), (u_2, u_2), (u_2, u_3), (u_3, u_1), (u_3, u_2), (u_4, u_1), (u_4, u_2)\}$ *be a binary relation defined on U thus* $\breve{R} < u_1 > \breve{R} = \breve{R} < u_3 > \breve{R} = U, \breve{R} < u_2 > \breve{R} = \{u_1, u_2, u_3\}, \breve{R} < u_4 > \breve{R} = \{u_1, u_3, u_4\}$. *This example shows that the inclusion of part (iii) in Propositions 3.24 and 3.25 can not be replaced by equality relation in general. Take*

(a) $A = \{u_1, u_3, u_4\}, B = \{u_1, u_2, u_3\}, A \cap B = \{u_1, u_3\}$, *then* $\underline{LOW'}^{\mathcal{I}}_{R}(A) = A, \underline{LOW'}^{\mathcal{I}}_{R}(B) = B, \underline{LOW'}^{\mathcal{I}}_{R}(A \cap B) = \phi$. *Therefore,* $\underline{LOW'}^{\mathcal{I}}_{R}(A) \cap \underline{LOW'}^{\mathcal{I}}_{R}(B) = \{u_1, u_3\} \neq \phi = \underline{LOW'}^{\mathcal{I}}_{R}(A \cap B)$.

(b) $A = \{u_2\}, B = \{u_4\}, A \cup B = \{u_2, u_4\}$, *then* $\overline{UPP'}^{\mathcal{I}}_{R}(A) = A, \overline{UPP'}^{\mathcal{I}}_{R}(B) = B, \overline{UPP'}^{\mathcal{I}}_{R}(A \cup B) = U$. *Therefore,* $\overline{UPP'}^{\mathcal{I}}_{R}(A) \cup \overline{UPP'}^{\mathcal{I}}_{R}(B) = \{u_2, u_4\} \neq U = \overline{UPP'}^{\mathcal{I}}_{R}(A \cup B)$.

(iv) *Example 3.5 (ii) shows that the converse of part (v) in Propositions 3.24 and 3.25 is not necessarily to be true in general. Take*

(a) $A = \{u_1, u_2\}$, *then* $\overline{UPP'}^{\mathcal{I}}_{R}(A) = \{u_1, u_2\}, \overline{UPP'}^{\mathcal{J}}_{R}(A) = \{u_1\}$. *Therefore,* $\overline{UPP'}^{\mathcal{J}}_{R}(A) \subseteq \overline{UPP'}^{\mathcal{I}}_{R}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

(b) $A = \{u_1, u_2, u_3\}$, *then* $\underline{LOW'}^{\mathcal{I}}_{R}(A) = \{u_2, u_3\}, \underline{LOW'}^{\mathcal{J}}_{R}(A) = \{u_2, u_3\}$. *Therefore,* $\underline{LOW'}^{\mathcal{I}}_{R}(A) \subseteq \underline{LOW'}^{\mathcal{J}}_{R}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

Remark 3.28. *In the third type of improvement of ideal approximation space, some properties of Pawlak approximation space are not satisfied.*

(i) *In Example 3.5 (i) take*

(a) $A = \{u_1\}$, *then* $\overline{UPP'}^{\mathcal{I}}_{R}(A) = \{u_4\}$. *Hence,* $A \nsubseteq \overline{UPP'}^{\mathcal{I}}_{R}(A)$.

(b) $A = \{u_2\}$, *then* $\underline{LOW'}^{\mathcal{I}}_{R}(A) = \{u_1, u_2, u_3\}$. *Hence,* $\underline{LOW'}^{\mathcal{I}}_{R}(A) \nsubseteq A$.

(c) $A = U$, *then* $\overline{UPP'}^{\mathcal{I}}_{R}(U) = \{u_4\}$. *Hence,* $\overline{UPP'}^{\mathcal{I}}_{R}(U) \neq U$.

(d) $A = \phi$, *then* $\underline{LOW'}^{\mathcal{I}}_{R}(\phi) = \{u_1, u_2, u_3\}$. *Hence,* $\underline{LOW'}^{\mathcal{I}}_{R}(\phi) \neq \phi$.

(ii)

Example 3.29. *Let* $U = \{u_1, u_2, u_3, u_4\}, \mathcal{I} = \{\phi, \{u_1\}\}$ *and* $R = \{(u_1, u_1)\}$ *be a binary relation defined on U thus* $\breve{R} < u_1 > \breve{R} = \{u_1\}, \breve{R} < u_2 > \breve{R} = \breve{R} < u_3 > \breve{R} = \breve{R} < u_4 > \breve{R} = \phi$. *Take*

(a) $A = U$, *then* $\underline{LOW'}^{\mathcal{I}}_{R}(U) = \{u_1\}$. *Hence,* $\underline{LOW'}^{\mathcal{I}}_{R}(U) \neq U$.

(b) $A = \phi$, *then* $\overline{UPP'}^{\mathcal{I}}_{R}(\phi) = \{u_2, u_3, u_4\}$. *Hence,* $\overline{UPP'}^{\mathcal{I}}_{R}(\phi) \neq \phi$.

(iii) *In Example 3.7 if*

(a) $A = \{u_3\}$, *then* $\overline{UPP'}^{\mathcal{I}}_{R}(A) = A, \underline{LOW'}^{\mathcal{I}}_{R}(\overline{UPP'}^{\mathcal{I}}_{R}(A)) = \phi$. *Hence,* $\overline{UPP'}^{\mathcal{I}}_{R}(A) \nsubseteq \underline{LOW'}^{\mathcal{I}}_{R}(\overline{UPP'}^{\mathcal{I}}_{R}(A))$.

(b) $A = \{u_1, u_2, u_4\}$, *then* $\underline{LOW'}^{\mathcal{I}}_{R}(A) = A, \overline{UPP'}^{\mathcal{I}}_{R}(\underline{LOW'}^{\mathcal{I}}_{R}(A)) = U$. *Hence,* $\overline{UPP'}^{\mathcal{I}}_{R}$ $(\underline{LOW'}^{\mathcal{I}}_{R}(A)) \nsubseteq \underline{LOW'}^{\mathcal{I}}_{R}(A)$.

(iv) *In Example 3.29 take*

(a) $A = \{u_2, u_3, u_4\}$, *then* $A^c \in \mathcal{I}$, *then* $\underline{LOW'}^{\mathcal{I}}_{R}(A) = \{u_1\}$. *Hence, if* $A^c \in \mathcal{I} \nRightarrow \underline{LOW'}^{\mathcal{I}}_{R}(A) = U$ *or* A.

(b) $A = \{u_1\} \in \mathcal{I}$, *then* $\overline{UPP'}^{\mathcal{I}}_{R}(A) = \{u_2, u_3, u_4\}$. *Hence, if* $A \in \mathcal{I} \nRightarrow \overline{UPP'}^{\mathcal{I}}_{R}(A) = \phi$ *or* A.

(c) $A = \{u_2, u_3, u_4\}, \mathcal{I} = P(U)$, *then* $\underline{LOW'}^{\mathcal{I}}_{R}(A) = \{u_1\}$. *Hence, if* $\mathcal{I} = P(U) \nRightarrow \underline{LOW'}^{\mathcal{I}}_{R}(A) = U$, *or* A.

(d) $A = \{u_1\}, \mathcal{I} = P(U)$, *then* $\overline{UPP'}^{\mathcal{I}}_{R}(A) = \{u_2, u_3, u_4\}$. *Hence, if* $\mathcal{I} = P(U) \nRightarrow \overline{UPP'}^{\mathcal{I}}_{R}(A) = \phi$, *or* A.

Proposition 3.30. *Let* $(U, R, \mathcal{I})$ *be an ideal approximation space and* $\phi \neq A \subseteq U$. *Then*

(i) $0 \leq ACC'^{\mathcal{I}}_{R}(A) \leq 1$.

(ii) $\underline{ACC}^{\mathcal{I}}_{R}(U) = 1$.

Proof. It is similar to Proposition 3.8.

Theorem 3.31. *Let* $(U, R, \mathcal{I})$ *and* $(U, R, \mathcal{J})$ *be ideal approximation spaces such that* $\mathcal{I} \subseteq \mathcal{J}$. *Then for each nonempty subset A of U we have the next results.*

(i) $BND'^{\mathcal{J}}_{R}(A) \subseteq BND'^{\mathcal{I}}_{R}(A)$.

(ii) $ACC'^{\mathcal{I}}_{R}(A) \leq ACC'^{\mathcal{J}}_{R}(A)$.

(iii) $Rough'^{\mathcal{J}}_{R}(A) \leq Rough'^{\mathcal{I}}_{R}(A)$.

Proof. Similar to Theorem 3.9.

Remark 3.32. *Example 3.5 (ii) shows that the converse of parts (i) and (ii) in Theorem 3.31 is not necessary to be true in general. Take,* $A = \{u_3, u_4\}$. *Then*

(i) $BND'^{\mathcal{J}}_{R}(A) = \{u_1\} \subseteq \{u_1\} = BND'^{\mathcal{I}}_{R}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

(ii) $ACC'^{\mathcal{I}}_{R}(A) = \frac{2}{3} \leq \frac{2}{3} = ACC'^{\mathcal{J}}_{R}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

(iii) $Rough'^{\mathcal{J}}_{R}(A) = \frac{1}{3} \leq \frac{1}{3} = Rough'^{\mathcal{J}}_{R}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

The third type of improvement of ideal approximation space does not have the monotonicity with respect to the its approximations (lower and upper), boundary region, measures (accuracy and roughness). The the next example clarifies this matter.

Example 3.33. *Let* $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}, \mathcal{I} = \{\phi, \{u_1\}\}$, R_1, R_2 *be two binary relations on* U *where*

$$R_1 = \Delta \cup \{(u_1, u_3), (u_3, u_1), (u_3, u_7), (u_4, u_6), (u_5, u_7), (u_6, u_4), (u_7, u_3), (u_7, u_5)\},$$

$$R_2 = R_1 \cup \{(u_1, u_4), (u_1, u_5), (u_2, u_6), (u_4, u_1), (u_5, u_1), (u_6, u_2)\}.$$

Thus, $\breve{R}_1 < u_1 > \breve{R}_1 = \{u_1, u_3, u_7\}, \breve{R}_1 < u_2 > \breve{R}_1 = \{u_2\}, \breve{R}_1 < u_3 > \breve{R}_1 = \breve{R}_1 < u_7 > \breve{R}_1 = \{u_1, u_3, u_5, u_7\}, \breve{R}_1 < u_4 > \breve{R}_1 = \breve{R}_1 < u_6 > \breve{R}_1 = \{u_4, u_6\}, \breve{R}_1 < u_5 > \breve{R}_1 = \{u_3, u_5, u_7\}, \breve{R}_2 < u_1 > \breve{R}_2 = \{u_1, u_3, u_4, u_5, u_6, u_7\}, \breve{R}_2 < u_2 > \breve{R}_2 = \{u_2, u_4, u_6\}, \breve{R}_2 < u_3 > \breve{R}_2 = \breve{R}_2 < u_5 > \breve{R}_2 = \{u_1, u_3, u_4, u_5, u_7\}, \breve{R}_2 < u_4 > \breve{R}_2 = \{u_1, u_2, u_3, u_4, u_5, u_6\}, \breve{R}_2 < u_6 > \breve{R}_2 = \{u_1, u_2, u_4, u_6\}, \breve{R}_2 < u_7 > \breve{R}_2 = \{u_1, u_3, u_5, u_7\}$. *Take*

(i) $A = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, *then* $\underline{LOW'}^{\mathcal{I}}_{R_1}(A) = \{u_2, u_4, u_6\}, \underline{LOW'}^{\mathcal{I}}_{R_2}(A) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. *Therefore,* $\underline{LOW'}^{\mathcal{I}}_{R_1}(A) \not\supseteq \underline{LOW'}^{\mathcal{I}}_{R_2}(A)$.

(ii) $A = \{u_7\}$, *then* $\overline{UPP'}^{\mathcal{I}}_{R_1}(A) = \{u_1, u_3, u_5, u_7\}, \overline{UPP'}^{\mathcal{I}}_{R_2}(A) = \{u_7\}$. *Therefore,* $\overline{UPP'}^{\mathcal{I}}_{R_1}(A) \not\subseteq \overline{UPP'}^{\mathcal{I}}_{R_2}(A)$.

(iii) $A = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, *then* $\underline{LOW'}^{\mathcal{I}}_{R_1}(A) = \{u_2, u_4, u_6\}, \overline{UPP'}^{\mathcal{I}}_{R_1}(A) = U, \underline{LOW'}^{\mathcal{I}}_{R_2}(A) = A, \overline{UPP'}^{\mathcal{I}}_{R_2}(A) = U$. *Therefore,*

- **(a)** $BND'^{\mathcal{I}}_{R_1}(A) = \{u_1, u_3, u_5, u_7\} \not\subseteq \{u_7\} = BND'^{\mathcal{I}}_{R_2}(A)$.
- **(b)** $ACC'^{\mathcal{I}}_{R_1}(A) = \frac{3}{7} < \frac{6}{7} = ACC'^{\mathcal{I}}_{R_2}(A)$.
- **(c)** $Rough'^{\mathcal{I}}_{R_1}(A) = \frac{4}{7} > \frac{1}{7} = Rough'^{\mathcal{I}}_{R_2}(A)$.

Although, $R_1 \subseteq R_2$.

3.4. The fourth method of the improvement of the approximations and accuracy measure of a rough set

Definition 3.34. *Let* $(U, R, \mathcal{I})$ *be an ideal approximation space. The fourth kind of the improvement of approximations (lower and upper), boundary region, measures (accuracy and roughness) of* $\phi \neq A \subseteq U$ *with respect to* $\breve{R} < x > \breve{R}$ *are respectively defined by:*

$\overline{UPP''}^{\mathcal{I}}_{R}(A) = \bigcup_{x \in U} \{\breve{R} < x > \breve{R} : \breve{R} < x > \breve{R} \cap A \notin \mathcal{I}\}$.

$\underline{LOW''}^{\mathcal{I}}_{R}(A) = (\overline{UPP''}^{\mathcal{I}}_{R}(A^c))^c$.

$BND''^{\mathcal{I}}_{R}(A) = \overline{UPP''}^{\mathcal{I}}_{R}(A) - \underline{LOW''}^{\mathcal{I}}_{R}(A)$.

$ACC''^{\mathcal{I}}_{R}(A) = \frac{|\underline{LOW''}^{\mathcal{I}}_{R}(A) \cap A|}{|\overline{UPP''}^{\mathcal{I}}_{R}(A) \cup A|}$.

$Rough''^{\mathcal{I}}_{R}(A) = 1 - ACC''^{\mathcal{I}}_{R}(A)$.

Proposition 3.35. *Let* $(U, R, \mathcal{I})$ *and* $(U, R, \mathcal{J})$ *be two ideal approximation spaces and let* $A, B \subseteq U$. *Then, we have next properties.*

(i) $\overline{UPP''}^{\mathcal{I}}_{R}(\phi) = \phi$.

(ii) $A \subseteq B \Rightarrow \overline{UPP''}^{\mathcal{I}}_{R}(A) \subseteq \overline{UPP''}^{\mathcal{I}}_{R}(B)$.

(iii) $\overline{UPP''}^{\mathcal{I}}_{R}(A \cap B) \subseteq \overline{UPP''}^{\mathcal{I}}_{R}(A) \cap \overline{UPP''}^{\mathcal{I}}_{R}(B)$.

(iv) $\overline{UPP''}^{\mathcal{I}}_{R}(A \cup B) = \overline{UPP''}^{\mathcal{I}}_{R}(A) \cup \overline{UPP''}^{\mathcal{I}}_{R}(B)$.

(v) $\overline{UPP''}^{\mathcal{I}}_{R}(A) = (\underline{LOW''}^{\mathcal{I}}_{R}(A^c))^c$.

(vi) *if* $A \in \mathcal{I}$, *then* $\overline{UPP''}^{\mathcal{I}}_{R}(A) = \phi$.

(vii) *if* $\mathcal{I} \subseteq \mathcal{J}$, *then* $\overline{UPP''}^{\mathcal{J}}_{R}(A) \subseteq \overline{UPP''}^{\mathcal{I}}_{R}(A)$.

(viii) *if* $\mathcal{I} = P(U)$, *then* $\overline{UPP''}^{\mathcal{I}}_{R}(A) = \phi$.

(ix) $\overline{UPP''}^{\mathcal{I}\cap\mathcal{J}}_{R}(A) = \overline{UPP''}^{\mathcal{I}}_{R}(A) \cup \overline{UPP''}^{\mathcal{J}}_{R}(A)$.

(x) $\overline{UPP''}^{\mathcal{I}\vee\mathcal{J}}_{R}(A) = \overline{UPP''}^{\mathcal{I}}_{R}(A) \cap \overline{UPP''}^{\mathcal{J}}_{R}(A)$.

Proof.

(i) $\overline{UPP''}^{\mathcal{I}}_{R}(\phi) = \cup\{\breve{R} < x > \breve{R} : \breve{R} < x > \breve{R} \cap \phi \notin \mathcal{I}\} = \phi$.

(ii) Let $A \subseteq B$ and $x \in \overline{UPP''}^{\mathcal{I}}_{R}(A)$. Then, $\exists\ y \in U$ such that $x \in< y > \breve{R}$ and $\breve{R} < y > \breve{R} \cap A \notin \mathcal{I}$. Thus, $\breve{R} < y > \breve{R} \cap B \notin \mathcal{I}$. So, $x \in \overline{UPP''}^{\mathcal{I}}_{R}(B)$. Consequently, $\overline{UPP''}^{\mathcal{I}}_{R}(A) \subseteq \overline{UPP''}^{\mathcal{I}}_{R}(B)$.

(iii) Immediately by part (ii).

(iv)

$$\begin{aligned}\overline{UPP''}^{\mathcal{I}}_{R}(A \cup B) &= \bigcup_{x\in U}\{\breve{R} < x > \breve{R} : \breve{R} < x > \breve{R} \cap (A \cup B) \notin \mathcal{I}\}.\\ &= (\bigcup_{x\in U}\{\breve{R} < x > \breve{R} : \breve{R} < x > \breve{R} \cap A \notin \mathcal{I}\}) \cup (\bigcup_{x\in U}\{\breve{R} < x > \breve{R} : \breve{R} < x > \breve{R} \cap B \notin \mathcal{I}\}).\\ &= (\bigcup_{x\in U}\{\breve{R} < x > \breve{R} : \breve{R} < x > \breve{R} \cap A \notin \mathcal{I}\})\ or\ (\bigcup_{x\in U}\{\breve{R} < x > \breve{R} : \breve{R} < x > \breve{R} \cap B \notin \mathcal{I}\}).\\ &= \overline{UPP''}^{\mathcal{I}}_{R}(A) \cup \overline{UPP''}^{\mathcal{I}}_{R}(B).\end{aligned}$$

(v)

$$\begin{aligned}(\underline{LOW''}^{\mathcal{I}}_{R}(A^c))^c &= ((\overline{UPP''}^{\mathcal{I}}_{R}(A))^c)^c.\\ &= \overline{UPP''}^{\mathcal{I}}_{R}(A).\end{aligned}$$

(vi) Straightforward by Definition 3.34.

(vii) Let $\mathcal{I} \subseteq \mathcal{J}, x \in \overline{UPP''}^{\mathcal{J}}_{R}(A)$. Then, $\exists\ y \in U$ such that $x \in \breve{R} < y > \breve{R}$ and $\breve{R} < y > \breve{R} \cap A \notin \mathcal{J}$. Thus, $\breve{R} < y > \breve{R} \cap A \notin \mathcal{I}$ as $\mathcal{I} \subseteq \mathcal{J}$. So, $x \in \overline{UPP''}^{\mathcal{I}}_{R}(A)$. Hence, $\overline{UPP''}^{\mathcal{J}}_{R}(A) \subseteq \overline{UPP''}^{\mathcal{I}}_{R}(A)$.

(viii) Straightforward by Definition 3.34.

(ix)

$$\begin{aligned}\overline{UPP''}_R^{\mathcal{I}\cap\mathcal{J}}(A) &= \underset{x\in U}{\cup}\{\breve{R}<x>\breve{R} : \breve{R}<x>\breve{R}\cap A \notin \mathcal{I}\cap\mathcal{J}\}.\\ &= (\underset{x\in U}{\cup}\{\breve{R}<x>\breve{R} : \breve{R}<x>\breve{R}\cap A \notin \mathcal{I}\})\ or\ (\underset{x\in U}{\cup}\{\breve{R}<x>\breve{R} : \breve{R}<x>\breve{R}\cap A \notin \mathcal{J}\}).\\ &= (\underset{x\in U}{\cup}\{\breve{R}<x>\breve{R} : \breve{R}<x>\breve{R}\cap A \notin \mathcal{I}\})\cup(\underset{x\in U}{\cup}\{\breve{R}<x>\breve{R} : \breve{R}<x>\breve{R}\cap A \notin \mathcal{J}\}).\\ &= \overline{UPP''}_R^{\mathcal{I}}(A)\cup\overline{UPP''}_R^{\mathcal{J}}(A).\end{aligned}$$

(x)

$$\begin{aligned}\overline{UPP''}_R^{\mathcal{I}\vee\mathcal{J}}(A) &= \underset{x\in U}{\cup}\{\breve{R}<x>\breve{R} : \breve{R}<x>\breve{R}\cap A \notin \mathcal{I}\vee\mathcal{J}\}.\\ &= \underset{x\in U}{\cup}\{\breve{R}<x>\breve{R} : \breve{R}<x>\breve{R}\cap A \notin \mathcal{I}\cup\mathcal{J}\}.\\ &= (\underset{x\in U}{\cup}\{\breve{R}<x>\breve{R} : \breve{R}<x>\breve{R}\cap A \notin \mathcal{I}\})\ and\ (\underset{x\in U}{\cup}\{\breve{R}<x>\breve{R} : \breve{R}<x>\breve{R}\cap A \notin \mathcal{J}\}).\\ &= (\underset{x\in U}{\cup}\{\breve{R}<x>\breve{R} : \breve{R}<x>\breve{R}\cap A \notin \mathcal{I}\})\cap(\underset{x\in U}{\cup}\{\breve{R}<x>\breve{R} : \breve{R}<x>\breve{R}\cap A \notin \mathcal{J}\}).\\ &= \overline{UPP''}_R^{\mathcal{I}}(A)\cap\overline{UPP''}_R^{\mathcal{J}}(A).\end{aligned}$$

Proposition 3.36. *Let* $(U,R,\mathcal{I})$ *and* $(U,R,\mathcal{J})$ *be two ideal approximation spaces and let* $A,B\subseteq U$*. Then, we have next properties.*

(i) $\underline{LOW''}_R^{\mathcal{I}}(U)=U.$

(ii) $A\subseteq B\Rightarrow \underline{LOW''}_R^{\mathcal{I}}(A)\subseteq\underline{LOW''}_R^{\mathcal{I}}(B).$

(iii) $\underline{LOW''}_R^{\mathcal{I}}(A)\cup\underline{LOW''}_R^{\mathcal{I}}(B)\subseteq\underline{LOW''}_R^{\mathcal{I}}(A\cup B).$

(iv) $\underline{LOW''}_R^{\mathcal{I}}(A\cap B)=\underline{LOW''}_R^{\mathcal{I}}(A)\cap\underline{LOW''}_R^{\mathcal{I}}(B).$

(v) $\underline{LOW''}_R^{\mathcal{I}}(A)=(\overline{UPP''}_R^{\mathcal{I}}(A^c))^c.$

(vi) *if* $A^c\in\mathcal{I}$*, then* $\underline{LOW''}_R^{\mathcal{I}}(A)=U.$

(vii) *if* $\mathcal{I}\subseteq\mathcal{J}$*, then* $\underline{LOW''}_R^{\mathcal{I}}(A)\subseteq\underline{LOW''}_R^{\mathcal{J}}(A).$

(viii) *if* $\mathcal{I}=P(U)$*, then* $\underline{LOW''}_R^{\mathcal{I}}(A)=U.$

(ix) $\underline{LOW''}_R^{\mathcal{I}\cap\mathcal{J}}(A)=\underline{LOW''}_R^{\mathcal{I}}(A)\cap\underline{LOW''}_R^{\mathcal{J}}(A).$

(x) $\underline{LOW''}_R^{\mathcal{I}\vee\mathcal{J}}(A)=\underline{LOW''}_R^{\mathcal{I}}(A)\cup\underline{LOW''}_R^{\mathcal{J}}(A).$

Proof.

(i) $\underline{LOW''}_R^{\mathcal{I}}(U)=(\overline{UPP''}_R^{\mathcal{I}}(\phi))^c=\phi^c=U$ by Proposition 3.35 part (i).

(ii) Let $A \subseteq B$. Thus, $B^c \subseteq A^c$ and hence $\overline{UPP''}^{\mathcal{I}}_R(B^c) \subseteq \overline{UPP''}^{\mathcal{I}}_R(A^c)$ (by Proposition 3.35 part (ii)). Then, $(\overline{UPP''}^{\mathcal{I}}_R(A^c))^c \subseteq (\overline{UPP''}^{\mathcal{I}}_R(B^c))^c$. So, $\underline{LOW''}^{\mathcal{I}}_R(A) \subseteq \underline{LOW''}^{\mathcal{I}}_R(B)$.

(iii) Immediately by part (ii).

(iv)

$$\begin{aligned}\underline{LOW''}^{\mathcal{I}}_R(A \cap B) &= (\overline{UPP''}^{\mathcal{I}}_R(A \cap B)^c)^c.\\ &= (\overline{UPP''}^{\mathcal{I}}_R(A^c \cup B^c))^c.\\ &= (\overline{UPP''}^{\mathcal{I}}_R(A^c) \cup \overline{UPP''}^{\mathcal{I}}_R(B^c))^c(\ by\ No.\ (4)\ in\ Proposition\ 3.35).\\ &= (\overline{UPP''}^{\mathcal{I}}_R(A^c))^c \cap (\overline{UPP''}^{\mathcal{I}}_R(B^c))^c.\\ &= \underline{LOW''}^{\mathcal{I}}_R(A) \cap \underline{LOW''}^{\mathcal{I}}_R(B).\end{aligned}$$

(v) Straightforward by Definition 3.34.

(vi) Let $A^c \in \mathcal{I}$, then $\underline{LOW''}^{\mathcal{I}}_R(A) = (\overline{UPP''}^{\mathcal{I}}_R(A^c))^c = (\phi)^c = U$ by Proposition 3.35 part (vi).

(vii) Let $\mathcal{I} \subseteq \mathcal{J}$. Then, $\overline{UPP''}^{\mathcal{J}}_R(A^c) \subseteq \overline{UPP''}^{\mathcal{I}}_R(A^c)$ by Proposition 3.35 part (vii). Thus, $(\overline{UPP''}^{\mathcal{I}}_R(A^c))^c \subseteq (\overline{UPP''}^{\mathcal{J}}_R(A^c))^c$. Hence, $\underline{LOW''}^{\mathcal{I}}_R(A) \subseteq \underline{LOW''}^{\mathcal{J}}_R(A)$.

(viii) Let $\mathcal{I} = P(U)$, then $\underline{LOW''}^{\mathcal{I}}_R(A) = (\overline{UPP''}^{\mathcal{I}}_R(A^c))^c = (\phi)^c = U$ by Proposition 3.35 part (viii).

(ix)

$$\begin{aligned}\underline{LOW''}^{\mathcal{I}\cap\mathcal{J}}_R(A) &= (\overline{UPP''}^{\mathcal{I}\cap\mathcal{J}}_R(A^c))^c.\\ &= (\overline{UPP''}^{\mathcal{I}}_R(A^c) \cup \overline{UPP''}^{\mathcal{J}}_R(A^c))^c(\ by\ Proposition\ 3.35\ part\ (ix)).\\ &= (\overline{UPP''}^{\mathcal{I}}_R(A^c))^c \cap (\overline{UPP''}^{\mathcal{J}}_R(A^c))^c.\\ &= \underline{LOW''}^{\mathcal{I}}_R(A) \cap \underline{LOW''}^{\mathcal{J}}_R(A).\end{aligned}$$

(x)

$$\begin{aligned}\underline{LOW''}^{\mathcal{I}\vee\mathcal{J}}_R(A) &= (\overline{UPP''}^{\mathcal{I}\vee\mathcal{J}}_R(A^c))^c.\\ &= (\overline{UPP''}^{\mathcal{I}}_R(A^c) \cap \overline{UPP''}^{\mathcal{J}}_R(A^c))^c(\ by\ Proposition\ 3.35\ part\ (x)).\\ &= (\overline{UPP''}^{\mathcal{I}}_R(A^c))^c \cup (\overline{UPP''}^{\mathcal{J}}_R(A^c))^c.\\ &= \underline{LOW''}^{\mathcal{I}}_R(A) \cup \underline{LOW''}^{\mathcal{J}}_R(A).\end{aligned}$$

Remark 3.37. **(i)** *Example 3.5 (i) shows that the converse of part (ii) in Propositions 3.35 and 3.36 is not necessarily to be true in general. Take*

(a) $A = \{u_1\}, B = \{u_4\}$, *then* $\overline{UPP''}^{\mathcal{I}}_R(A) = \overline{UPP''}^{\mathcal{I}}_R(B) = \phi$. *Therefore,* $\overline{UPP''}^{\mathcal{I}}_R(A) \subseteq \overline{UPP''}^{\mathcal{I}}_R(B)$, *but* $A \nsubseteq B$.

(b) $A = \{u_2, u_3, u_4\}, B = \{u_1, u_2 u_3\}$, *then* $\underline{LOW''}^{\mathcal{I}}_R(A) = \underline{LOW''}^{\mathcal{I}}_R(B) = U$. *Therefore,* $\underline{LOW''}^{\mathcal{I}}_R(A) \subseteq \underline{LOW''}^{\mathcal{I}}_R(B)$, *but* $A \nsubseteq B$.

(ii) *Example 3.5 (ii) shows that the converse of parts (vi)-(viii) in Propositions 3.35 and 3.36 is not necessarily to be true in general.*
For part (vi) take

(a) $A = \{u_1, u_2\}$, *then* $\overline{UPP''}^{\mathcal{J}}_R(A) = \phi$. *Therefore,* $\overline{UPP''}^{\mathcal{J}}_R(A) = \phi$, *but* $A \notin \mathcal{J}$.

(b) $A = \{u_3, u_4\}$, *then* $\underline{LOW''}^{\mathcal{J}}_R(A) = U$. *Therefore,* $\underline{LOW''}^{\mathcal{J}}_R(A) = U$, *but* $A^c \notin \mathcal{J}$.

For part (vii) take

(a) $A = \{u_1, u_2\}$, *then* $\overline{UPP''}^{\mathcal{I}}_R(A) = \{u_2\}, \overline{UPP''}^{\mathcal{J}}_R(A) = \phi$. *Therefore,* $\overline{UPP''}^{\mathcal{J}}_R(A) \subseteq \overline{UPP''}^{\mathcal{I}}_R(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

(b) $A = \{u_3, u_4\}$, *then* $\underline{LOW''}^{\mathcal{I}}_R(A) = \{u_1, u_3, u_4\}, \underline{LOW''}^{\mathcal{J}}_R(A) = U$. *Therefore,* $\underline{LOW''}^{\mathcal{I}}_R(A) \subseteq \underline{LOW''}^{\mathcal{J}}_R(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

For part (viii) take

(a) $A = \{u_1, u_2\}$, *then* $\overline{UPP''}^{\mathcal{J}}_R(A) = \phi$, *but* $\mathcal{J} \neq P(U)$.

(b) $A = \{u_3, u_4\}$, *then* $\underline{LOW''}^{\mathcal{J}}_R(A) = U$, *but* $\mathcal{J} \neq P(U)$.

(iii) *Example 3.5 (iii) shows that the inclusion of part (iii) in Propositions 3.35 and 3.36 can not be replaced by equality relation in general. Take* $A = \{u_1, u_4\}, B = \{u_2, u_3\}$, *then*

(a) $\overline{UPP''}^{\mathcal{I}}_R(A) = \overline{UPP''}^{\mathcal{I}}_R(B) = U, \overline{UPP''}^{\mathcal{I}}_R(A \cap B) = \phi$. *Therefore,* $\overline{UPP''}^{\mathcal{I}}_R(A) \cap \overline{UPP''}^{\mathcal{I}}_R(B) = U \neq \phi = \overline{UPP'}^{\mathcal{I}}_R(A \cap B)$.

(b) $\underline{LOW''}^{\mathcal{I}}_R(A) = \underline{LOW''}^{\mathcal{I}}_R(B) = \phi, \underline{LOW''}^{\mathcal{I}}_R(A \cup B) = U$. *Therefore,* $\underline{LOW''}^{\mathcal{I}}_R(A) \cup \underline{LOW''}^{\mathcal{I}}_R(B) = \phi \neq U = \underline{LOW''}^{\mathcal{I}}_R(A \cup B)$.

Remark 3.38. *In the fourth type of improvement of ideal approximation space, some properties of Pawlak approximation space are not satisfied.*

(i) *In Example 3.5 (i) take*

(a) $A = \{u_1\}$, *then* $\overline{UPP''}^{\mathcal{I}}_R(A) = \phi$. *Hence,* $A \nsubseteq \overline{UPP''}^{\mathcal{I}}_R(A)$.

(b) $A = \{u_2, u_3, u_4\}$, *then* $\underline{LOW''}^{\mathcal{I}}_R(A) = U$. *Hence,* $\underline{LOW''}^{\mathcal{I}}_R(A) \nsubseteq A$.

(ii) *In Example 3.5 (ii) take*

(a) $A = U$, *then* $\overline{UPP''}^{\mathcal{I}}_R(U) = \{u_2, u_3, u_4\}$. *Hence,* $\overline{UPP''}^{\mathcal{I}}_R(U) \neq U$.

(b) $A = \phi$, *then* $\underline{LOW''}^{\mathcal{I}}_R(\phi) = \{u_1\}$. *Hence,* $\underline{LOW''}^{\mathcal{I}}_R(\phi) \neq \phi$.

Proposition 3.39. *Let* $(U, R, \mathcal{I})$ *be an ideal approximation space and* $\phi \neq A \subseteq U$. *Then*

(i) $0 \leq ACC''^{\mathcal{I}}_R(A) \leq 1$.

(ii) $ACC''^{\mathcal{I}}_{R}(U) = 1$.

Proof. It is similar to Proposition 3.8.

Theorem 3.40. *Let $(U, R, \mathcal{I})$ and $(U, R, \mathcal{J})$ be ideal approximation spaces such that $\mathcal{I} \subseteq \mathcal{J}$. Then for each nonempty subset A of U we have the next results.*

(i) $BND''^{\mathcal{J}}_{R}(A) \subseteq BND''^{\mathcal{I}}_{R}(A)$.

(ii) $ACC''^{\mathcal{I}}_{R}(A) \leq ACC''^{\mathcal{J}}_{R}(A)$.

(iii) $Rough''^{\mathcal{J}}_{R}(A) \leq Rough''^{\mathcal{I}}_{R}(A)$.

Proof. Similar to Theorem 3.9.

Remark 3.41. *Example 3.5 (ii) shows that the converse of parts (i) and (ii) in Theorem 3.40 is not necessary to be true in general. Take, $A = \{u_3, u_4\}$, then*

(i) $BND''^{\mathcal{J}}_{R}(A) = \phi \subseteq \phi = BND''^{\mathcal{I}}_{R}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

(ii) $ACC''^{\mathcal{I}}_{R}(A) = 1 \leq 1 = ACC''^{\mathcal{J}}_{R}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

(iii) $Rough''^{\mathcal{J}}_{R}(A) = 0 \leq 0 = Rough''^{\mathcal{I}}_{R}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

Theorem 3.42. *Let $\phi \neq A \subseteq U$, $\mathcal{I}$ be an ideal on U and R_1, R_2 be two binary relations on U. If $R_1 \subseteq R_2$, then*

(i) $\overline{UPP''}^{\mathcal{I}}_{R_1}(A) \subseteq \overline{UPP''}^{\mathcal{I}}_{R_2}(A)$.

(ii) $\underline{LOW''}^{\mathcal{I}}_{R_2}(A) \subseteq \underline{LOW''}^{\mathcal{I}}_{R_1}(A)$.

(iii) $BND''^{\mathcal{I}}_{R_1}(A) \subseteq BND''^{\mathcal{I}}_{R_2}(A)$.

(iv) $ACC''^{\mathcal{I}}_{R_2}(A) \leq ACC''^{\mathcal{I}}_{R_1}(A)$.

(v) $Rough''^{\mathcal{I}}_{R_1}(A) \leq Rough''^{\mathcal{I}}_{R_2}(A)$.

Proof.

(i) Let $x \in \overline{UPP''}^{\mathcal{I}}_{R_1}(A)$. Then, $\exists\, y \in U$ such that $x \in \breve{R}_1 < y > \breve{R}_1 \cap A \notin \mathcal{I}$. Since, $\breve{R}_1 < y > \breve{R}_1 \subseteq \breve{R}_2 < y > \breve{R}_2$ (by Theorem 2.5 [8]). It follows that $x \in \breve{R}_2 < y > \breve{R}_2 \cap A \notin \mathcal{I}$. Thus, $x \in \overline{UPP''}^{\mathcal{I}}_{R_2}(A)$. Hence, $\overline{UPP''}^{\mathcal{I}}_{R_1}(A) \subseteq \overline{UPP''}^{\mathcal{I}}_{R_2}(A)$.

(ii)

$$\begin{aligned} x \in \underline{LOW''}^{\mathcal{I}}_{R_2}(A) &= (\overline{UPP''}^{\mathcal{I}}_{R_2}(A^c))^c. \\ &\subseteq (\overline{UPP''}^{\mathcal{I}}_{R_1}(A^c))^c (\textit{ by part } (i)). \\ &= \underline{LOW''}^{\mathcal{I}}_{R_1}(A). \end{aligned}$$

(iii) Let $x \in BND''^{\mathcal{I}}_{R_1}(A)$. Then, $x \in \overline{UPP''}^{\mathcal{I}}_{R_1}(A) - \underline{LOW''}^{\mathcal{I}}_{R_1}(A)$. So, $x \in \overline{UPP''}^{\mathcal{I}}_{R_1}(A)$ and $x \in (\underline{LOW''}^{\mathcal{I}}_{R_1}(A))^c$. Thus, $x \in \overline{UPP''}^{\mathcal{I}}_{R_2}(A)$ and $x \in (\underline{LOW''}^{\mathcal{I}}_{R_2}(A))^c$ by parts (i) and (ii). Hence, $x \in BND''^{\mathcal{I}}_{R_2}(A)$. Therefore, $BND''^{\mathcal{I}}_{R_1}(A) \subseteq BND''^{\mathcal{I}}_{R_2}(A)$.

(iv)

$$\begin{aligned} ACC''^{\mathcal{I}}_{R_2}(A) &= \frac{|\underline{LOW''}^{\mathcal{I}}_{R_2}(A) \cap A|}{|\overline{UPP''}^{\mathcal{I}}_{R_2}(A) \cup A|} \\ &\leq \frac{|\underline{LOW''}^{\mathcal{I}}_{R_1}(A) \cap A|}{|\overline{UPP''}^{\mathcal{I}}_{R_1}(A) \cup A|} \\ &= ACC''^{\mathcal{I}}_{R_1}(A). \end{aligned}$$

(v) Straightforward by (iv).

Remark 3.43. *It can not be replaced the inclusion and the less than relations, in Theorem 3.42, by an equality relation as Example 3.12 shows.*

(i) *Take* $A = \{u_1, u_4\}$, *then*

- **(a)** $\overline{UPP''}^{\mathcal{I}}_{R_1}(A) = \{u_1, u_2\} \neq \{u_1, u_2, u_3\} = \overline{UPP''}^{\mathcal{I}}_{R_2}(A)$.
- **(b)** $ACC''^{\mathcal{I}}_{R_1}(A) = \frac{2}{3} \neq \frac{1}{2} = ACC''^{\mathcal{I}}_{R_2}(A)$.
- **(c)** $Rough''^{\mathcal{I}}_{R_1}(A) = \frac{1}{3} \neq \frac{1}{2} = Rough''^{\mathcal{I}}_{R_2}(A)$.

(ii) $A = \{u_2, u_3\}$, *then* $\underline{LOW''}^{\mathcal{I}}_{R_1}(A) = \{u_3, u_4\} \neq \{u_4\} = \underline{LOW''}^{\mathcal{I}}_{R_2}(A)$.

4. Comparison the proposed methods and their advantages compared to the previous ones

In this section, we first compare among the suggested approximations and prove that the third approach is the best one. Then, we show that the current methods produce better approximations and higher accuracy values than their counterparts introduced in [20]. Also, we demonstrate that the current methods are better than the methods defined by Al-shami [8] and Dai et al. [14].

4.1. Comparison the proposed methods in terms approximations and accuracy measures of subsets

Theorem 4.1. *Let* $(U, R, \mathcal{I})$ *be an ideal approximation space and let* $A \subseteq U$. *Then, we have the next properties.*

(i) $\overline{UPP}^{\mathcal{I}}_{R}(A) \subseteq \overline{\overline{UPP}}^{\mathcal{I}}_{R}(A)$.

(ii) $\underline{\underline{LOW}}^{\mathcal{I}}_{R}(A) \subseteq \underline{LOW}^{\mathcal{I}}_{R}(A)$.

(iii) $BND^{\mathcal{I}}_{R}(A) \subseteq \underline{BND}^{\mathcal{I}}_{R}(A)$.

(iv) $\underline{ACC}^{\mathcal{I}}_{R}(A) = ACC^{\mathcal{I}}_{R}(A)$.

(v) $\underline{Rough}^{\mathcal{I}}_R(A) = Rough^{\mathcal{I}}_R(A)$.

Proof. It directly follows from Definitions 3.1 and 3.13.

Remark 4.2. *It can not be replaced the inclusion and less than relations, in Theorem 4.1, by an equality relation in general. To illustrate this note consider Example 3.12 and take* $A = \{u_1, u_3, u_4\}$*. Then*

(i) $\overline{UPP}^{\mathcal{I}}_{R_1}(A) = \{u_1, u_2\} \neq U = \overline{\overline{UPP}}^{\mathcal{I}}_{R_1}(A)$.

(ii) $\underline{\underline{LOW}}^{\mathcal{I}}_{R_1}(A) = \{u_1, u_3, u_4\} \neq U = \underline{LOW}^{\mathcal{I}}_{R_1}(A)$.

(iii) $BND^{\mathcal{I}}_{R_1}(A) = \phi \neq \{u_2\} = \underline{BND}^{\mathcal{I}}_{R_1}(A)$.

Theorem 4.3. *Let* $(U, R, \mathcal{I})$ *be an ideal approximation space such that* R *is a reflexive relation. Then for each* $A \subseteq U$ *we have the next results.*

(i) $\underline{\underline{LOW}}^{\mathcal{I}}_R(A) \subseteq \underline{LOW}^{\mathcal{I}}_R(A) \subseteq \underline{LOW'}^{\mathcal{I}}_R(A)$.

(ii) $\overline{UPP'}^{\mathcal{I}}_R(A) \subseteq \overline{UPP}^{\mathcal{I}}_R(A) \subseteq \overline{\overline{UPP}}^{\mathcal{I}}_R(A)$.

(iii) $BND'^{\mathcal{I}}_R(A) \subseteq BND^{\mathcal{I}}_R(A) \subseteq \underline{BND}^{\mathcal{I}}_R(A)$.

(iv) $\underline{ACC}^{\mathcal{I}}_R(A) \leq ACC^{\mathcal{I}}_R(A) \leq ACC'^{\mathcal{I}}_R(A)$.

(v) $Rough'^{\mathcal{I}}_R(A) \leq Rough^{\mathcal{I}}_R(A) \leq \underline{Rough}^{\mathcal{I}}_R(A)$.

Proof.

(i) By Theorem 4.1, we have $\underline{\underline{LOW}}^{\mathcal{I}}_R(A) \subseteq \underline{LOW}^{\mathcal{I}}_R(A)$. To prove, $\underline{LOW}^{\mathcal{I}}_R(A) \subseteq \underline{LOW'}^{\mathcal{I}}_R(A)$. Let $x \in \underline{LOW}^{\mathcal{I}}_R(A)$, then $\breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}$. Hence, $\breve{R} < x > \breve{R} \subseteq \underline{LOW'}^{\mathcal{I}}_R(A)$. Since, R is a reflexive relation, thus $x \in \breve{R} < x > \breve{R} \subseteq \underline{LOW'}^{\mathcal{I}}_R(A)$. Therefore, $x \in \underline{LOW'}^{\mathcal{I}}_R(A)$.

(ii) To prove, $\overline{UPP'}^{\mathcal{I}}_R(A) \subseteq \overline{UPP}^{\mathcal{I}}_R$. Let $x \in \overline{UPP'}^{\mathcal{I}}_R(A) = (\underline{LOW'}_R(A^c))^c$, then $x \notin \underline{LOW'}_R(A^c)$. Hence, by Definition 3.23, we get $\breve{R} < x > \breve{R} \cap A \notin \mathcal{I}$. It follows that $x \in \overline{UPP}^{\mathcal{I}}_R(A)$. By Theorem 4.1, we have $\overline{UPP}^{\mathcal{I}}_R(A) \subseteq \overline{\overline{UPP}}^{\mathcal{I}}_R(A)$.

(iii)–(v) Straightforward from (i) and (ii).

Remark 4.4. *Example 3.5 (iii) shows that the inclusion and less than in Theorem 4.3 can not be replaced by equality relation in general. Take* $A = \{u_2, u_3\}$*, then* $\overline{UPP'}^{\mathcal{I}}_R(A) = \{u_2, u_3\} \subsetneq \{u_1, u_2, u_3\} = \overline{UPP}^{\mathcal{I}}_R$*. Moreover, Take* $A = \{u_1, u_4\}$*, then*

(i) $\underline{LOW}^{\mathcal{I}}_R(A) = \{u_4\} \subsetneq \{u_1, u_4\} = \underline{LOW'}^{\mathcal{I}}_R(A)$.

(ii) $BND'^{\mathcal{I}}_R(A) = \{u_2, u_3\} \subsetneq \{u_1, u_2, u_3\} = BND^{\mathcal{I}}_R(A)$.

(iii) $ACC^{\mathcal{I}}_R(A) = \frac{1}{4} \lneq \frac{1}{2} = ACC'^{\mathcal{I}}_R(A)$.

(iv) $Rough'^{\mathcal{I}}_R(A) = \frac{1}{2} \lneq \frac{3}{4} = Rough^{\mathcal{I}}_R(A)$.

Theorem 4.5. *Let $(U, R, \mathcal{I})$ be an ideal approximation space such that R is a reflexive relation. Then for each $A \subseteq U$ we have the next results.*

(i) $\underline{LOW''}^{\mathcal{I}}_R(A) \subseteq \underline{LOW}^{\mathcal{I}}_R(A) \subseteq \underline{LOW'}^{\mathcal{I}}_R(A)$.

(ii) $\overline{UPP'}^{\mathcal{I}}_R(A) \subseteq \overline{UPP}^{\mathcal{I}}_R \subseteq \overline{UPP''}^{\mathcal{I}}_R(A)$.

(iii) $BND'^{\mathcal{I}}_R(A) \subseteq BND^{\mathcal{I}}_R(A) \subseteq BND''^{\mathcal{I}}_R(A)$.

(iv) $ACC''^{\mathcal{I}}_R(A) \leq ACC^{\mathcal{I}}_R(A) \leq ACC'^{\mathcal{I}}_R(A)$.

(v) $Rough'^{\mathcal{I}}_R(A) \leq Rough^{\mathcal{I}}_R(A) \leq Rough''^{\mathcal{I}}_R(A)$.

Proof.

(i) By Theorem 4.3, we have $\underline{LOW}^{\mathcal{I}}_R(A) \subseteq \underline{LOW'}^{\mathcal{I}}_R(A)$. To prove, $\underline{LOW''}^{\mathcal{I}}_R(A) \subseteq \underline{LOW}^{\mathcal{I}}_R(A)$, let $x \in \underline{LOW''}^{\mathcal{I}}_R(A) = \overline{UPP''}^{\mathcal{I}}_R(A^c)^c$. Then, $x \notin \overline{UPP''}^{\mathcal{I}}_R(A^c)$. Thus, by Definition 3.34, $\breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}$. It follows that $\breve{R} < x > \breve{R} \subseteq \underline{LOW}^{\mathcal{I}}_R(A)$. Since, R is a reflexive relation, then $x \in \breve{R} < x > \breve{R} \subseteq \underline{LOW}^{\mathcal{I}}_R(A)$. Therefore, $x \in \underline{LOW}^{\mathcal{I}}_R(A)$.

(ii) By Theorem 4.3, we have $\overline{UPP'}^{\mathcal{I}}_R(A) \subseteq \overline{UPP}^{\mathcal{I}}_R(A)$. To prove $\overline{UPP}^{\mathcal{I}}_R(A) \subseteq \overline{UPP''}^{\mathcal{I}}_R(A)$, let $x \in \overline{UPP}^{\mathcal{I}}_R(A)$, then $\breve{R} < x > \breve{R} \cap A \notin \mathcal{I}$. It follows that $\breve{R} < x > \breve{R} \subseteq \overline{UPP''}^{\mathcal{I}}_R(A)$. Since, R is a reflexive relation, then $x \in \breve{R} < x > \breve{R} \subseteq \overline{UPP''}^{\mathcal{I}}_R(A)$. Therefore, $x \in \overline{UPP''}^{\mathcal{I}}_R(A)$.

(iii)–(v) Straightforward from (i) and (ii).

Remark 4.6. *It can not be replaced the inclusion and the less than relations, in Theorem 4.5, by an equality relation in general. To validate this matter consider Example 3.5 (iii) and take $A = \{u_2, u_3\}$. Then $\overline{UPP}^{\mathcal{I}}_R = \{u_1, u_2, u_3\} \subsetneq U = \overline{UPP''}^{\mathcal{I}}_R(A)$. Moreover, Take $A = \{u_1, u_4\}$, then*

(i) $\underline{LOW''}^{\mathcal{I}}_R(A) = \phi \subsetneq \{u_4\} = \underline{LOW}^{\mathcal{I}}_R(A)$.

(ii) $BND^{\mathcal{I}}_R(A) = \{u_1, u_2, u_3\} \subsetneq U = BND''^{\mathcal{I}}_R(A)$.

(iii) $ACC''^{\mathcal{I}}_R(A) = 0 \lneqq \frac{1}{4} = ACC^{\mathcal{I}}_R(A)$.

(iv) $Rough^{\mathcal{I}}_R(A) = \frac{3}{4} \lneqq 1 = Rough''^{\mathcal{I}}_R(A)$.

Remark 4.7. *It follows from Theorems 4.1 and 4.3 that the best method to improve the approximations and increase the accuracy values is that given in the third type in Subsection 3.3, since this type, compared to the other types, decreases (or cancels) the boundary region by decreasing the upper approximation and increasing the lower approximation, which means this type produces accuracy measures higher than the other types.*

In Example 3.5 (iii), we calculate lower and upper approximations, and boundary regions for all subsets of U in Table 1.

Table 1. Comparison between lower and upper approximations, and boundary regions by using the proposed methods.

A	First approach			Second approach			Third approach			Fourth approach		
	$\underline{LOW}_R^{\mathcal{I}}$	$\overline{UPP}_R^{\mathcal{I}}$	$BND_R^{\mathcal{I}}$	$\underline{\underline{LOW}}_R^{\mathcal{I}}$	$\overline{\overline{UPP}}_R^{\mathcal{I}}$	$\underline{BND}_R^{\mathcal{I}}$	$\underline{LOW'}_R^{\mathcal{I}}$	$\overline{UPP'}_R^{\mathcal{I}}$	$BND'^{\mathcal{I}}_R$	$\underline{LOW''}_R^{\mathcal{I}}$	$\overline{UPP''}_R^{\mathcal{I}}$	$BND''^{\mathcal{I}}_R$
$\{u_1\}$	$\{u_1, u_4\}$	U	$\{u_2, u_3\}$	$\{u_1\}$	U	$\{u_2, u_3, u_4\}$	U	U	ϕ	ϕ	U	U
$\{u_2\}$	ϕ	$\{u_1, u_2\}$	$\{u_1, u_2\}$	ϕ	$\{u_1, u_2\}$	$\{u_1, u_2\}$	ϕ	$\{u_2\}$	$\{u_2\}$	ϕ	U	U
$\{u_3\}$	ϕ	$\{u_1, u_3\}$	$\{u_1, u_3\}$	ϕ	$\{u_1, u_3\}$	$\{u_1, u_3\}$	ϕ	$\{u_3\}$	$\{u_3\}$	ϕ	U	U
$\{u_4\}$	ϕ	ϕ	ϕ	ϕ	$\{u_4\}$	$\{u_4\}$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1, u_2\}$	$\{u_2, u_4\}$	U	$\{u_1, u_3\}$	$\{u_2\}$	U	$\{u_1, u_3, u_4\}$	$\{u_1, u_2, u_4\}$	U	$\{u_3\}$	ϕ	U	U
$\{u_1, u_3\}$	$\{u_3, u_4\}$	U	$\{u_1, u_2\}$	$\{u_3\}$	U	$\{u_1, u_2, u_4\}$	$\{u_1, u_3, u_4\}$	U	$\{u_2\}$	ϕ	U	U
$\{u_1, u_4\}$	$\{u_4\}$	U	$\{u_1, u_2, u_3\}$	$\{u_4\}$	U	$\{u_1, u_2, u_3\}$	$\{u_1, u_4\}$	U	$\{u_2, u_3\}$	ϕ	U	U
$\{u_2, u_3\}$	ϕ	$\{u_1, u_2, u_3\}$	$\{u_1, u_2, u_3\}$	ϕ	$\{u_1, u_2, u_3\}$	$\{u_1, u_2, u_3\}$	ϕ	$\{u_2, u_3\}$	$\{u_2, u_3\}$	ϕ	U	U
$\{u_2, u_4\}$	ϕ	$\{u_1, u_2\}$	$\{u_1, u_2\}$	ϕ	$\{u_1, u_2, u_4\}$	$\{u_1, u_2, u_4\}$	ϕ	$\{u_2\}$	$\{u_2\}$	ϕ	U	U
$\{u_3, u_4\}$	ϕ	$\{u_1, u_3\}$	$\{u_1, u_3\}$	ϕ	$\{u_1, u_3, u_4\}$	$\{u_1, u_3, u_4\}$	ϕ	$\{u_3\}$	$\{u_3\}$	ϕ	U	U
$\{u_1, u_2, u_3\}$	U	U	ϕ	$\{u_1, u_2, u_3\}$	U	$\{u_4\}$	U	U	ϕ	U	U	ϕ
$\{u_1, u_2, u_4\}$	$\{u_2, u_4\}$	U	$\{u_1, u_3\}$	$\{u_2, u_4\}$	U	$\{u_1, u_3\}$	$\{u_1, u_2, u_4\}$	U	$\{u_3\}$	ϕ	U	U
$\{u_1, u_3, u_4\}$	$\{u_3, u_4\}$	U	$\{u_1, u_2\}$	$\{u_3, u_4\}$	U	$\{u_1, u_2\}$	$\{u_1, u_3, u_4\}$	U	$\{u_2\}$	ϕ	U	U
$\{u_2, u_3, u_4\}$	ϕ	$\{u_1, u_2, u_3\}$	$\{u_1, u_2, u_3\}$	ϕ	U	U	ϕ	ϕ	ϕ	ϕ	U	U
U	U	U	ϕ	U	U	ϕ	U	U	ϕ	U	U	ϕ

For example, take $\{u_1\}$, then the boundary by the first, second and third methods are $\{u_2, u_3\}, \{u_2, u_3, u_4\}$ and ϕ, respectively. Whereas, the boundary by the first, third and fourth methods are $\{u_2, u_3\}, \phi$ and U, respectively. Now, we calculate the accuracy values for all subsets of U in Table 2.

Table 2. Accuracy by using the proposed methods.

A	First approach	Second approach	Third approach	Fourth approach
$\{u_1\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0
$\{u_2\}$	0	0	0	0
$\{u_3\}$	0	0	0	0
$\{u_4\}$	0	0	0	0
$\{u_1, u_2\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0
$\{u_1, u_3\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0
$\{u_1, u_4\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0
$\{u_2, u_3\}$	0	0	0	0
$\{u_2, u_4\}$	0	0	0	0
$\{u_3, u_4\}$	0	0	0	0
$\{u_1, u_2, u_3\}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
$\{u_1, u_2, u_4\}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	0
$\{u_1, u_3, u_4\}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	0
$\{u_2, u_3, u_4\}$	0	0	0	0
U	1	1	1	1

In Tables 3 and 4, we compare the four approaches given in this manuscript in terms of satisfying Pawlak properties or not, where $\surd$ means that the property holds, while $\times$ means that the property does not hold.

Table 3. Comparison between the first and second methods according to the properties in Definition 2.1.

	The first method	The second method
$\mathcal{L}_1$	√	√
$\mathcal{L}_2$	√	√
$\mathcal{L}_3$	×	√
$\mathcal{L}_4$	×	√
$\mathcal{L}_5$	√	√
$\mathcal{L}_6$	√	√
$\mathcal{L}_7$	√	√
$\mathcal{L}_8$	×	×
$\mathcal{L}_9$	×	×
$\mathcal{U}_1$	√	√
$\mathcal{U}_2$	×	√
$\mathcal{U}_3$	√	√
$\mathcal{U}_4$	×	√
$\mathcal{U}_5$	√	√
$\mathcal{U}_6$	√	√
$\mathcal{U}_7$	√	√
$\mathcal{U}_8$	×	×
$\mathcal{U}_9$	×	×

Table 4. Comparison between the third and fourth methods according to the properties in Definition 2.1.

	The third method	The fourth method
$\mathcal{L}_1$	√	√
$\mathcal{L}_2$	×	√
$\mathcal{L}_3$	×	×
$\mathcal{L}_4$	×	×
$\mathcal{L}_5$	×	√
$\mathcal{L}_6$	√	√
$\mathcal{L}_7$	√	√
$\mathcal{U}_1$	√	√
$\mathcal{U}_2$	×	×
$\mathcal{U}_3$	×	√
$\mathcal{U}_4$	×	×
$\mathcal{U}_5$	×	√
$\mathcal{U}_6$	√	√
$\mathcal{U}_7$	√	√

4.2. Comparison the proposed methods with the previous ones

In this subsection, we elucidate the good performance and efficiency of the followed approaches compared to those given in [8, 11, 14, 20]. Also, we illustrate under what conditions the approaches presented in [8, 11, 14, 20] are identical with the current approaches.

The following result elucidates that the first type of our approaches is better than the approximation spaces given in [8] (see, Definition 2.9).

Theorem 4.8. *Let $(U, R, \mathcal{I})$ be an ideal approximation space and let $A \subseteq U$. Then*

(i) $\overline{UPP}_R^{\mathcal{I}}(A) \subseteq R^{\bigstar\bigstar}(A)$.

(ii) $R_{\bigstar\bigstar}(A) \subseteq \underline{LOW}_R^{\mathcal{I}}(A)$.

(iii) $BND_R^{\mathcal{I}}(A) \subseteq B_R^{\bigstar\bigstar}(A)$.

(iv) $ACC_R^{\mathcal{I}}(A) \leq Acc_R^{\bigstar\bigstar}(A)$.

(v) $Rough_R^{\bigstar\bigstar}(A) \leq Rough_R^{\mathcal{I}}(A)$.

Proof.

(i) Let $x \in \overline{UPP}_R^{\mathcal{I}}(A)$. Then, $\breve{R} < x > \breve{R} \cap A \notin \mathcal{I}$. Therefore, $\breve{R} < x > \breve{R} \cap A \neq \phi$. Hence, $x \in R^{\bigstar\bigstar}(A)$, which means that $\overline{UPP}_R^{\mathcal{I}}(A) \subseteq R^{\bigstar\bigstar}(A)$.

(ii) Let $x \in R_{\bigstar\bigstar}(A)$. Then, $\breve{R} < x > \breve{R} \subseteq A$. Therefore, $\breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}$. Hence, $x \in \underline{LOW}_R^{\mathcal{I}}(A)$, which means that $R_{\bigstar\bigstar}(A) \subseteq \underline{LOW}_R^{\mathcal{I}}(A)$.

(iii)–(v) Immediately by parts (i) and (ii).

Remark 4.9. *It can not be replaced the inclusion and the less than relations, in Theorem 4.8, by an equality relation in general. To validate this note consider Example 3.12 and take $A = \{u_1, u_4\}$. Then*

(i) $\overline{UPP}_{R_1}^{\mathcal{I}}(A) = \{u_1, u_2\} \neq \{u_1, u_2, u_4\} = R_1^{\bigstar\bigstar}(A)$.

(ii) $\underline{LOW}_{R_1}^{\mathcal{I}}(A) = U \neq \{u_4\} = R_{1\bigstar\bigstar}(A)$.

(iii) $BND_{R_1}^{\mathcal{I}}(A) = \phi \neq \{u_1, u_2\} = B_{R_1}^{\bigstar\bigstar}(A)$.

(iv) $ACC_{R_1}^{\mathcal{I}}(A) = \frac{2}{3} \neq \frac{1}{3} = Acc_{R_1}^{\bigstar\bigstar}(A)$.

(v) $Rough_{R_1}^{\mathcal{I}}(A) = \frac{1}{3} \neq \frac{2}{3} = Rough_{R_1}^{\bigstar\bigstar}(A)$.

In the next result, we prove that the first type of our approaches is more accurate than the approach displayed in [14] (see, Definition 2.6).

Theorem 4.10. *Let $(U, R, \mathcal{I})$ be an ideal approximation space such that R is a similarity relation. Then for each $A \subseteq U$ we have the next results.*

(i) $\overline{UPP}_R^{\mathcal{I}}(A) \subseteq \overline{apr}_R(A)$.

(ii) $\underline{apr}_R(A) \subseteq \underline{LOW}_R^{\mathcal{I}}(A)$.

(iii) $BND_R^{\mathcal{I}}(A) \subseteq Boundary_R(A)$.

(iv) $Accurcey_R(A) \leq ACC_R^{\mathcal{I}}(A)$.

(v) $Roughness_R(A) \leq Rough_R^{\mathcal{I}}(A)$.

Proof.

(i) Let $x \in \overline{UPP}_R^{\mathcal{I}}(A)$. Then, $\breve{R} < x > \breve{R} \cap A \notin \mathcal{I}$. Since, $\breve{R} < x > \breve{R} \cap A \subseteq < x > \breve{R} \cap A$. Therefore, $< x > \breve{R} \cap A \notin \mathcal{I}$. So, $< x > \breve{R} \cap A \neq \phi$. Hence, $x \in \overline{apr}_R(A)$, which means that $\overline{UPP}_R^{\mathcal{I}}(A) \subseteq \overline{apr}_R(A)$.

(ii) Let $x \in \underline{apr}_R(A)$. Then, $< x > \breve{R} \subseteq A$. So, $< x > \breve{R} \cap A^c \in \mathcal{I}$. Since, $\breve{R} < x > \breve{R} \cap A^c \subseteq < x > \breve{R} \cap A^c$. Therefore, $\breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}$. Hence, $x \in \underline{LOW}_R^{\mathcal{I}}(A)$, which means that $\underline{apr}_R(A) \subseteq \underline{LOW}_R^{\mathcal{I}}(A)$.

(iii)–(v) Immediately by parts (i) and (ii).

Remark 4.11. *It can not be replaced the inclusion and the less than relations, in Theorem 4.8, by an equality relation in general. To validate this note consider Example 3.12 and take* $A = \{u_1, u_4\}$. *Then*

(i) $\overline{UPP}_{R_1}^{\mathcal{I}}(A) = \{u_1, u_2\} \neq \{u_1, u_2, u_4\} = \overline{apr}_{R_1}(A)$.

(ii) $\underline{LOW}_{R_1}^{\mathcal{I}}(A) = U \neq \{u_4\} = \underline{apr}_{R_1}(A)$.

(iii) $BND_{R_1}^{\mathcal{I}}(A) = \phi \neq \{u_1, u_2\} = Boundary_{R_1}(A)$.

(iv) $ACC_{R_1}^{\mathcal{I}}(A) = 1 \neq \frac{1}{3} = Accuracy_{R_1}(A)$.

(v) $Rough_{R_1}^{\mathcal{I}}(A) = 0 \neq \frac{2}{3} = Roughness_{R_1}(A)$.

According to Theorems 4.8 and 4.10, it can be seen that the present methods reduce the boundary region by increasing the lower approximations and decreasing the upper approximations with the comparison of Dai et al.'s methods [14] and Al-shami's methods [8]. This means that the current approximation spaces are proper generalizations of Dai et al.'s approximations [14] and Al-shami's approximations [8].

One can easily prove the next two results, they show that Dai et al.'s approximations [14] and Al-shami's approximations [8] are special cases of the current approximations.

Proposition 4.12. *If the ideal $\mathcal{I}$ is the empty set and the binary relation R is a similarity relation, then the approximation spaces given herein and the approximation spaces given in [14] are identical.*

Proposition 4.13. *If the ideal $\mathcal{I}$ is the empty set, then the approximation spaces given herein and the approximation spaces given in [8] are identical.*

Now, we demonstrate that our approaches improve the rough set models introduced in [11].

Theorem 4.14. *Let $(U, R, \mathcal{I})$ be an ideal approximation space. Then*

(i) $\overline{UPP}_R^{\mathcal{I}}(A) \subseteq \overline{apr\bigstar}_R^{\mathcal{I}}(A)$.

(ii) $\underline{apr\bigstar}_R^{\mathcal{I}}(A) \subseteq \underline{LOW}_R^{\mathcal{I}}(A)$.

(iii) $BND_R^{\mathcal{I}}(A) \subseteq Boundary\bigstar_R^{\mathcal{I}}(A)$.

(iv) $Accuracy\bigstar^{\mathcal{I}}(A) \leq ACC_R^{\mathcal{I}}(A)$.

(v) $Rough_R^{\mathcal{I}}(A) \leq Roughness\bigstar_R^{\mathcal{I}}(A)$.

Proof.

(i) Let $x \in \overline{UPP}_R^{\mathcal{I}}(A)$. Then, $\breve{R} < x > \breve{R} \cap A \notin \mathcal{I}$. Therefore, $\breve{R} < x > \cap A \notin \mathcal{I}$. Thus, $x \in \overline{apr\bigstar}_R^{\mathcal{I}}(A)$. Hence, $\overline{UPP}_R^{\mathcal{I}}(A) \subseteq \overline{apr\bigstar}_R^{\mathcal{I}}(A)$.

(ii) Let $x \in \underline{apr\bigstar}_R^{\mathcal{I}}(A)$. Then, $\breve{R} < x > \cap A^c \in \mathcal{I}$. Therefore, $\breve{R} < x > \breve{R} \cap A^c \in \mathcal{I}$. Thus, $x \in \underline{LOW}_R^{\mathcal{I}}(A)$. Hence, $\underline{apr\bigstar}_R^{\mathcal{I}}(A) \subseteq \underline{LOW}_R^{\mathcal{I}}(A)$.

(iii)–(v) Immediately by parts (i) and (ii).

Remark 4.15. *Example 3.5 (i) shows that the inclusion and the less than in Theorem 4.14 can not be replaced by equality relation in general. Take* $A = \{u_2, u_3\}$ *and* $\mathcal{I} = \{\phi, \{u_1\}\}$*, then* $\overline{UPP}_R^{\mathcal{I}}(A) = \{u_2, u_3\} \neq \{u_1, u_2, u_3\} = \overline{apr\bigstar}_R^{\mathcal{I}}(A)$*. Moreover, take* $A = \{u_2, u_3\}$ *and* $\mathcal{I} = \{\phi, \{u_2\}\}$*, then*

(i) $\underline{LOW}_R^{\mathcal{I}}(A) = \{u_2, u_3, u_4\} \neq \{u_3, u_4\} = \underline{apr\bigstar}_R^{\mathcal{I}}(A)$.

(ii) $BND_R^{\mathcal{I}}(A) = \phi \neq \{u_2\} = Boundary\bigstar_R^{\mathcal{I}}(A)$.

(iii) $Accuracy\bigstar_R^{\mathcal{I}}(A) = \frac{1}{2} \neq 1 = ACC_R^{\mathcal{I}}(A)$.

(iv) $Rough_R^{\mathcal{I}}(A) = 0 \neq \frac{1}{2} = Roughness\bigstar_R^{\mathcal{I}}(A)$.

Theorem 4.16. *Let* $(U, R, \mathcal{I})$ *be an ideal approximation space. Then*

(i) $\overline{\overline{UPP}}_R^{\mathcal{I}}(A) \subseteq \overline{\overline{apr\bigstar}}_R^{\mathcal{I}}(A)$.

(ii) $\underline{\underline{apr\bigstar}}_R^{\mathcal{I}}(A) \subseteq \underline{\underline{LOW}}_R^{\mathcal{I}}(A)$.

(iii) $\underline{BND}_R^{\mathcal{I}}(A) \subseteq \underline{Boundary\bigstar}_R^{\mathcal{I}}(A)$.

(iv) $\underline{Accuracy\bigstar}_R^{\mathcal{I}}(A) \leq \underline{ACC}_R^{\mathcal{I}}(A)$.

(v) $\underline{Rough}_R^{\mathcal{I}}(A) \leq \underline{Roughness\bigstar}_R^{\mathcal{I}}(A)$.

Proof. The proof is similar to that of Theorem 4.14.

Remark 4.17. *In Example 3.5 (i), take* $\mathcal{I} = \{\phi, \{u_1\}\}$*, then it shows that the inclusion and the less than in Theorem 4.16 can not be replaced by an equality relation. Take* $A = \{u_1\}$*, then* $\underline{\underline{LOW}}_R^{\mathcal{I}}(A) = \{u_1\} \neq \phi = \underline{\underline{apr\bigstar}}_R^{\mathcal{I}}(A)$*. Moreover, if* $A = \{u_2, u_3, u_4\}$*, then*

(i) $\overline{\overline{apr\bigstar}}_R^{\mathcal{I}}(A) = U \neq \{u_2, u_3, u_4\} = \overline{\overline{UPP}}_R^{\mathcal{I}}(A)$.

(ii) $\underline{BND}_R^{\mathcal{I}}(A) = \phi \neq \{u_1\} = \underline{Boundary\bigstar}_R^{\mathcal{I}}(A)$.

(iii) $\underline{Accuracy\bigstar}_R^{\mathcal{I}}(A) = \frac{3}{4} \lneq 1 = \underline{ACC}_R^{\mathcal{I}}(A)$.

(iv) $\underline{Rough}^{\mathcal{I}}_{R}(A) = 0 \lneq \frac{1}{4} = \underline{Roughness\bigstar}^{\mathcal{I}}_{R}(A)$.

Remark 4.18. *It should be noted that, the third type in this work and the previous third type in [11] are not comparable as it is shown in the following example.*

Example 4.19. *Let* $U = \{u_1, u_2, u_3, u_4\}, \mathcal{I} = \{\phi, \{u_1\}, \{u_2\}, \{u_3\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_2, u_3\}, \{u_1, u_2, u_3\}\}$ *and* $R = \{(u_1, u_3), (u_2, u_1), (u_3, u_1), (u_3, u_2), (u_4, u_2), (u_4, u_3)\}$ *be a binary relation defined on* U. *By calculations, we obtain* $\breve{R} < u_1 > \breve{R} = \{u_1\}, \breve{R} < u_2 > \breve{R} = \breve{R} < u_3 > \breve{R} = \{u_2, u_3\}, \breve{R} < u_4 > \breve{R} = \phi$.

(i) *Take* $A = \{u_1\}$; *then*

- **(a)** $\underline{apr'\bigstar}^{\mathcal{I}}_{R}(A) = \{u_2, u_3\} \subsetneq \{u_1, u_2, u_3\} = \underline{LOW'}^{\mathcal{I}}_{R}(A)$.
- **(b)** $\overline{UPP'}^{\mathcal{I}}_{R_1}(A) = \{u_4\} \supsetneq \phi = \overline{apr'\bigstar}^{\mathcal{I}}_{R}(A)$.

(ii) *Take* $A = \{u_2, u_4\}$; *then*

- **(a)** $\underline{apr'\bigstar}^{\mathcal{I}}_{R}(A) = U \supsetneq \{u_1, u_2, u_3\} = \underline{LOW'}^{\mathcal{I}}_{R}(A)$.
- **(b)** $\overline{UPP'}^{\mathcal{I}}_{R_1}(A) = \{u_4\} \subsetneq \{u_1, u_4\} = \overline{apr'\bigstar}^{\mathcal{I}}_{R}(A)$.

Theorem 4.20. *Let* $(U, R, \mathcal{I})$ *be an ideal approximation space. Then*

(i) $\overline{UPP''}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr''\bigstar}^{\mathcal{I}}_{R}(A)$.

(ii) $\underline{apr''\bigstar}^{\mathcal{I}}_{R}(A) \subseteq \underline{LOW''}^{\mathcal{I}}_{R}(A)$.

(iii) $BND''^{\mathcal{I}}_{R}(A) \subseteq Boundary''\bigstar^{\mathcal{I}}_{R}(A)$.

(iv) $Accuracy''\bigstar^{\mathcal{I}}_{R}(A) \leq ACC''^{\mathcal{I}}_{R}(A)$.

(v) $Rough''^{\mathcal{I}}_{R}(A) \leq Roughness''\bigstar^{\mathcal{I}}_{R}(A)$.

Proof.

(i) Let $x \in \overline{UPP''}^{\mathcal{I}}_{R}(A)$. Then, $\exists\, y \in U, x \in \breve{R} < y > \breve{R}, \breve{R} < y > \breve{R} \cap A \notin \mathcal{I}$. Thus, $\exists\, y \in U, x \in \breve{R} < y >$ $, \breve{R} < y > \cap A \notin \mathcal{I}$. Hence, $x \in \overline{apr''}^{\mathcal{I}}_{R}(A)$. Therefore, $\overline{UPP''}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr''\bigstar}^{\mathcal{I}}_{R}(A)$.

(ii) Since, $\overline{UPP''}^{\mathcal{I}}_{R}(A^c) \subseteq \overline{apr''\bigstar}^{\mathcal{I}}_{R}(A^c)$ by (1). So, $(\overline{apr''\bigstar}^{\mathcal{I}}_{R}(A^c))^c \subseteq (\overline{UPP''}^{\mathcal{I}}_{R}(A^c))^c$. Hence, $\underline{apr''\bigstar}^{\mathcal{I}}_{R}(A) \subseteq \underline{LOW''}^{\mathcal{I}}_{R}(A)$ by Definitions 2.21 and 3.34.

(iii)–(v) Immediately by parts (i) and (ii).

Remark 4.21. *In Example 3.5 (i), take* $\mathcal{I} = \{\phi, \{u_2\}\}$, *then it shows that the inclusion and the less than in Theorem 4.35 can not be replaced by equality relation in general. Take* $A = \{u_1, u_4\}$, *then*

(i) $\overline{UPP''}^{\mathcal{I}}_{R}(A) = \{u_1\} \neq \{u_1, u_2, u_3\} = \overline{apr''\bigstar}^{\mathcal{I}}_{R}(A)$.

(ii) $\underline{apr''\bigstar}^{\mathcal{I}}_{R}(A) = \{u_4\} \neq \{u_1, u_4\} = \underline{LOW''}^{\mathcal{I}}_{R}(A)$.

(iii) $BND''^{\mathcal{I}}_{R}(A) = \phi \neq \{u_1, u_2, u_3\} = Boundary''\bigstar^{\mathcal{I}}_{R}(A)$.

(iv) $Accuracy''\bigstar^{\mathcal{I}}_{R}(A) = \frac{1}{4} \neq 1 = ACC''^{\mathcal{I}}_{R}(A)$.

(v) $Rough''^{\mathcal{I}}_{R}(A) = 0 \neq \frac{3}{4} = Roughness''\bigstar^{\mathcal{I}}_{R}(A)$.

In the remaining part of this subsection, we demonstrate that our approaches improve the rough set models introduced in [20]. First, we need the next remark.

Remark 4.22. *It is well known that the value of accuracy for every subset lies in the closed interval [0,1]. The formulations of accuracy values presented in [20] produce accuracy values greater than one in some cases, which is a contradiction. To demonstrates this matter, in Example 3.12 take*

(i) $A = \{u_1, u_4\}$, *then*

- **(a)** $Accuracy^{\mathcal{I}}_{R_1}(A) = 2 > 1$.
- **(b)** $Accuracy''^{\mathcal{I}}_{R_1}(A) = 2 > 1$.

(ii) $A = \{u_1, u_3, u_4\}$, *then* $Accuracy'^{\mathcal{I}}_{R_1}(A) = 2 > 1$.

According to Remark 4.22, we reformulate the three types of accuracy measures introduced in [20] as follows.

Definition 4.23. *Let R be a binary relation on a nonempty set U. For any subset $\phi \neq A \subseteq U$, the first, third and fourth kind of the accuracy of A according to R is redefined respectively by:*

$$Accuracy^{\bullet\mathcal{I}}_{R}(A) = \frac{|\underline{apr}^{\mathcal{I}}_{R}(A) \cap A|}{|\overline{apr}^{\mathcal{I}}_{R}(A) \cup A|}.$$

$$Accuracy^{\bullet\bullet\bullet\mathcal{I}}_{R}(A) = \frac{|\underline{apr'}^{\mathcal{I}}_{R}(A) \cap A|}{|\overline{apr'}^{\mathcal{I}}_{R}(A) \cup A|}.$$

$$Accuracy^{\bullet\bullet\bullet\bullet\mathcal{I}}_{R}(A) = \frac{|\underline{apr''}^{\mathcal{I}}_{R}(A) \cap A|}{|\overline{apr''}^{\mathcal{I}}_{R}(A) \cup A|}.$$

Proposition 4.24. *Let $(U, R, \mathcal{I})$ be an ideal approximation space. Then for each nonempty subset $A \subseteq U$ we have the next properties.*

(i) $0 \leq Accuracy^{\bullet\mathcal{I}}_{R}(A) \leq 1$.

(ii) $Accuracy^{\bullet\mathcal{I}}_{R}(U) = 1$.

(iii) $0 \leq Accuracy^{\bullet\bullet\bullet\mathcal{I}}_{R}(A) \leq 1$.

(iv) $Accuracy^{\bullet\bullet\bullet\mathcal{I}}_{R}(U) = 1$.

(v) $0 \leq Accuracy^{\bullet\bullet\bullet\bullet\mathcal{I}}_{R}(A) \leq 1$.

(vi) $Accuracy^{\bullet\bullet\bullet\bullet\mathcal{I}}_{R}(U) = 1$.

Proof. It is similar to Proposition 3.8.

Definition 4.25. *Let R be a binary relation on a nonempty set U. For any subset $\phi \neq A \subseteq U$, the first, third and fourth kind of the roughness of A according to R is redefined respectively by:*

$Roughness^{\bullet\mathcal{I}}_{R}(A) = 1 - Accuracy^{\bullet\mathcal{I}}_{R}(A)$.
$Roughness^{\bullet\bullet\bullet\mathcal{I}}_{R}(A) = 1 - Accuracy^{\bullet\bullet\bullet\mathcal{I}}_{R}(A)$.
$Roughness^{\bullet\bullet\bullet\bullet\mathcal{I}}_{R}(A) = 1 - Accuracy^{\bullet\bullet\bullet\bullet\mathcal{I}}_{R}(A)$.

Theorem 4.26. *Let $(U, R, \mathcal{I})$ be an ideal approximation space. Then*

(i) $\overline{UPP}^{\mathcal{I}}_R(A) \subseteq \overline{apr}^{\mathcal{I}}_R(A)$.

(ii) $\underline{apr}^{\mathcal{I}}_R(A) \subseteq \underline{LOW}^{\mathcal{I}}_R(A)$.

(iii) $BND^{\mathcal{I}}_R(A) \subseteq Boundary^{\mathcal{I}}_R(A)$.

(iv) $Accuracy^{\bullet^{\mathcal{I}}}(A) \leq ACC^{\mathcal{I}}_R(A)$.

(v) $Rough^{\mathcal{I}}_R(A) \leq Roughness^{\bullet^{\mathcal{I}}}_R(A)$.

Proof. It is similar to Theorem 4.14.

Remark 4.27. *In Example 3.5 (i) take $\mathcal{I} = \{\phi, \{u_2\}\}$, then it shows that the inclusion and the less than in Theorem 4.26 can not be replaced by equality relation in general. Take $A = \{u_2, u_3\}$, then*

(i) $\overline{UPP}^{\mathcal{I}}_R(A) = \{u_2, u_3\} \neq \{u_2, u_3, u_4\} = \overline{apr}^{\mathcal{I}}_R(A)$.

(ii) $\underline{LOW}^{\mathcal{I}}_R(A) = \{u_2, u_3, u_4\} \neq \{u_2\} = \underline{apr}^{\mathcal{I}}_R(A)$.

(iii) $BND^{\mathcal{I}}_R(A) = \phi \neq \{u_3, u_4\} = Boundary^{\mathcal{I}}_R(A)$.

(iv) $Accuracy^{\bullet^{\mathcal{I}}}_R(A) = \frac{1}{3} \neq 1 = ACC^{\mathcal{I}}_R(A)$.

(v) $Rough^{\mathcal{I}}_R(A) = 0 \neq \frac{2}{3} = Roughness^{\bullet^{\mathcal{I}}}_R(A)$.

Theorem 4.28. *Let $(U, R, \mathcal{I})$ be an ideal approximation space. Then*

(i) $\overline{\overline{UPP}}^{\mathcal{I}}_R(A) \subseteq \overline{\overline{apr}}^{\mathcal{I}}_R(A)$.

(ii) $\underline{\underline{apr}}^{\mathcal{I}}_R(A) \subseteq \underline{\underline{LOW}}^{\mathcal{I}}_R(A)$.

(iii) $\underline{BND}^{\mathcal{I}}_R(A) \subseteq \underline{Boundary}^{\mathcal{I}}_R(A)$.

(iv) $\underline{Accuracy}^{\mathcal{I}}_R(A) \leq \underline{ACC}^{\mathcal{I}}_R(A)$.

(v) $\underline{Rough}^{\mathcal{I}}_R(A) \leq \underline{Roughness}^{\mathcal{I}}_R(A)$.

Proof. The proof is similar to that of Theorem 4.26.

Remark 4.29. *In Example 3.5 (i), take $\mathcal{I} = \{\phi, \{u_2\}\}$, then it shows that the inclusion and the less than in Theorem 4.28 can not be replaced by an equality relation. Take $A = \{u_2, u_3, u_4\}$, then*

(i) $\overline{\overline{apr}}^{\mathcal{I}}_R(A) = U \neq \{u_2, u_3, u_4\} = \overline{\overline{UPP}}^{\mathcal{I}}_R(A)$.

(ii) $\underline{\underline{LOW}}^{\mathcal{I}}_R(A) = \{u_2, u_3, u_4\} \neq \{u_2, u_3\} = \underline{\underline{apr}}^{\mathcal{I}}_R(A)$.

(iii) $\underline{BND}^{\mathcal{I}}_R(A) = \phi \neq \{u_1, u_4\} = \underline{Boundary}^{\mathcal{I}}_R(A)$.

(iv) $\underline{Accuracy}^{\mathcal{I}}_R(A) = \frac{1}{2} \lneq 1 = \underline{ACC}^{\mathcal{I}}_R(A)$.

(v) $\underline{Rough}^{\mathcal{I}}_{R}(A) = 0 \lneq \frac{1}{2} = \underline{Roughness}^{\mathcal{I}}_{R}(A)$.

Theorem 4.30. *Let $(U, R, \mathcal{I})$ be an ideal approximation space such that R is a similarity relation. Then*

(i) $\underline{apr'}_{R}(A) \subseteq \underline{LOW'}^{\mathcal{I}}_{R}(A)$.

(ii) $\overline{UPP'}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr'}_{R}(A)$.

(iii) $BND'^{\mathcal{I}}_{R}(A) \subseteq Boundary'_{R}(A)$.

(iv) $Accuracy'_{R}(A) \leq ACC'^{\mathcal{I}}_{R}(A)$.

(v) $Rough'^{\mathcal{I}}_{R}(A) \leq Roughness'_{R}(A)$.

Proof.

(i) Let $x \in \underline{apr'}_{R}(A)$. Then, $\exists\ y \in U, x \in < y > \breve{R}, < y > \breve{R} \subseteq A$. Hence, $x \in \breve{R} < y > \breve{R}$, (as R is a symmetry relation), $\breve{R} < y > \breve{R} \cap A^c \in \mathcal{I}$. Therefore, $x \in \underline{LOW'}^{\mathcal{I}}_{R}(A)$. So, $\underline{apr'}_{R}(A) \subseteq \underline{LOW'}^{\mathcal{I}}_{R}(A)$.

(ii) Since, $\underline{apr'}_{R}(A^c) \subseteq \underline{LOW'}^{\mathcal{I}}_{R}(A^c)$ by (i). So, $(\underline{LOW'}^{\mathcal{I}}_{R}(A^c))^c \subseteq (\underline{apr'}_{R}(A^c))^c$. Hence, $\overline{UPP'}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr'}_{R}(A)$ by Definitions 3.23 and 2.7.

(iii)–(v) Immediately by parts (i) and (ii).

Remark 4.31. *It can not be replaced the inclusion and the less than relations, in Theorem 4.30, by an equality relation in general. To validate this note consider Example 3.12 and take $A = \{u_1, u_3, u_4\}$. Then*

(i) $\overline{UPP'}^{\mathcal{I}}_{R_1}(A) = \{u_1, u_2\} \neq U = \overline{apr'}_{R_1}(A)$.

(ii) $\underline{apr'}_{R_1}(A) = \{u_3, u_4\} \neq U = \underline{LOW'}^{\mathcal{I}}_{R_1}(A)$.

(iii) $BND'^{\mathcal{I}}_{R_1}(A) = \phi \neq \{u_1, u_2\} = Boundary'_{R_1}(A)$.

(iv) $Accuracy'_{R_1}(A) = \frac{1}{2} \lneq \frac{3}{4} = ACC'^{\mathcal{I}}_{R_1}(A)$.

(v) $Rough'^{\mathcal{I}}_{R_1}(A) = \frac{1}{4} \lneq \frac{1}{2} = Roughness'_{R_1}(A)$.

Remark 4.32. *It should be noted that, the third type in this work and the previous third type in [20] are not comparable. In Example 3.5 (i), take*

(i) $A = \{u_1\}$, *then*

- **(a)** $\underline{apr'}^{\mathcal{I}}_{R}(A) = \{u_2, u_3\} \subsetneq \{u_1, u_2, u_3\} = \underline{LOW'}^{\mathcal{I}}_{R}(A)$.
- **(b)** $\overline{UPP'}^{\mathcal{I}}_{R}(A) = \{u_4\} \supsetneq \phi = \overline{apr'}^{\mathcal{I}}_{R}(A)$.

(ii) $A = \{u_2, u_4\}$, *then*

- **(a)** $\underline{apr'}^{\mathcal{I}}_{R}(A) = U \supsetneq \{u_1, u_2, u_3\} = \underline{LOW'}^{\mathcal{I}}_{R_1}(A)$.

(b) $\overline{UPP'}^{\mathcal{I}}_{R}(A) = \{u_4\} \subsetneq \{u_1, u_4\} = \overline{apr'}^{\mathcal{I}}_{R}(A)$.

Theorem 4.33. *Let $(U, R, \mathcal{I})$ be an ideal approximation space such that R is a similarity relation. Then,*

(i) $\overline{UPP''}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr''}_{R}(A)$.

(ii) $\underline{apr''}_{R}(A) \subseteq \underline{LOW''}^{\mathcal{I}}_{R}(A)$.

(iii) $BND''^{\mathcal{I}}_{R}(A) \subseteq Boundary''_{R}(A)$.

(iv) $Accuracy''_{R}(A) \leq ACC''^{\mathcal{I}}_{R}(A)$.

(v) $Roughs''^{\mathcal{I}}_{R}(A) \leq Roughness''_{R}(A)$.

Proof.

(i) Let $x \in \overline{UPP''}^{\mathcal{I}}_{R}(A)$. Then, $\exists\ y \in U, x \in \breve{R} < y > \breve{R}, \breve{R} < y > \breve{R} \cap A \notin \mathcal{I}$. Thus, $\exists\ y \in U, x \in < y > \breve{R}, < y > \breve{R} \cap A \notin \mathcal{I}$. So, $\exists\ y \in U, x \in < y > \breve{R}, < y > \breve{R} \cap A \neq \phi$. Hence, $x \in \overline{apr''}_{R}(A)$. Therefore, $\overline{UPP''}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr''}_{R}(A)$.

(ii) Since, $\overline{UPP''}^{\mathcal{I}}_{R}(A^c) \subseteq \overline{apr''}_{R}(A^c)$ by (i). So, $(\overline{apr''}_{R}(A^c))^c \subseteq (\overline{UPP''}^{\mathcal{I}}_{R}(A^c))^c$. Hence, $\underline{apr''}_{R}(A) \subseteq \underline{LOW''}^{\mathcal{I}}_{R}(A)$ by Definitions 3.34 and 2.8.

(iii)–(v) Immediately by parts (i) and (ii).

Remark 4.34. *It can not be replaced the inclusion and the less than relations, in Theorem 4.33, by an equality relation in general. To validate this note consider Example 3.12 and take $A = \{u_1, u_3, u_4\}$. Then*

(i) $\overline{UPP''}^{\mathcal{I}}_{R_1}(A) = \{u_1, u_2\} \neq U = \overline{apr''}_{R_1}(A)$.

(ii) $\underline{apr''}_{R_1}(A) = \{u_3, u_4\} \neq U = \underline{LOW''}^{\mathcal{I}}_{R_1}(A)$.

(iii) $BND''^{\mathcal{I}}_{R_1}(A) = \phi \neq \{u_1, u_2\} = Boundary''_{R_1}(A)$.

(iv) $Accuracy''_{R_1}(A) = \frac{1}{2} \lneq \frac{3}{4} = ACC''^{\mathcal{I}}_{R_1}(A)$.

(v) $Rough''^{\mathcal{I}}_{R_1}(A) = \frac{1}{4} \lneq \frac{1}{2} = Roughness''_{R_1}(A)$.

Theorem 4.35. *Let $(U, R, \mathcal{I})$ be an ideal approximation space. Then*

(i) $\overline{UPP''}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr''}^{\mathcal{I}}_{R}(A)$.

(ii) $\underline{apr''}^{\mathcal{I}}_{R}(A) \subseteq \underline{LOW''}^{\mathcal{I}}_{R}(A)$.

(iii) $BND''^{\mathcal{I}}_{R}(A) \subseteq Boundary''^{\mathcal{I}}_{R}(A)$.

(iv) $Accuracy^{\bullet\bullet\bullet\bullet\mathcal{I}}_{R}(A) \leq ACC''^{\mathcal{I}}_{R}(A)$.

(v) $Rough''^{\mathcal{I}}_{R}(A) \leq Roughness^{\bullet\bullet\bullet\bullet\mathcal{I}}_{R}(A)$.

Proof. It's similar to Theorem 4.20.

Remark 4.36. *In Example 3.5 (i), take $\mathcal{I} = \{\phi, \{u_2\}\}$, then it shows that the inclusion and the less than in Theorem 4.35 can not be replaced by equality relation in general. Take $A = \{u_1, u_4\}$, then*

(i) $\overline{UPP''}^{\mathcal{I}}_R(A) = \{u_1\} \neq U = \overline{apr''}^{\mathcal{I}}_R(A)$.

(ii) $\underline{apr''}^{\mathcal{I}}_R(A) = \phi \neq \{u_1, u_4\} = \underline{LOW''}^{\mathcal{I}}_R(A)$.

(iii) $BND''^{\mathcal{I}}_R(A) = \phi \neq U = Boundary''^{\mathcal{I}}_R(A)$.

(iv) $Accuracy^{\bullet\bullet\bullet\bullet\mathcal{I}}_R(A) = 0 \neq 1 = ACC''^{\mathcal{I}}_R(A)$.

(v) $Rough''^{\mathcal{I}}_R(A) = 0 \neq 1 = Roughness^{\bullet\bullet\bullet\bullet\mathcal{I}}_R(A)$.

One can easily prove the next result, it shows that Hosny and Al-shami's approximations [11], Hosny's approximations [20] are a special case of the current approximations.

Proposition 4.37. *If the binary relation R is a symmetry relation, then the approximation spaces given herein and the approximation spaces given in [11] and [20] are identical.*

5. Medical application in decision-making of the heart attacks problem

In this section, we examine the real data of heart attacks using the proposed approach and show its good performance to analyzing the data and minimizing the boundary regions compared to some previous approaches. The data set in Table 5 is carried out at Al-Azhar University's cardiology department [12] (Hospital of Sayed Glal University, Cairo, Egypt). Table 5 represents the set of objects (patients) $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}\}$ and reduced to $\{u_1, u_2, u_3, u_4, u_5, u_8, u_9\}$ as the attributes in rows (objects) are identical. The study is included patients with different symptoms (the set of attributes) = {detailed history = a_1, physical examination = a_2, full labs = a_3, resting ECG = a_4, conventional echo assessment = a_5} and Decision of heart attacks is confirmed or ruled out = $\{d\}$ as shown in Table 5.

Table 5. The information's decisions data set.

Patients	a_1	a_2	a_3	a_4	a_5	d
$\{u_1\}$	yes	yes	yes	yes	no	yes
$\{u_2\}$	no	no	no	yes	yes	no
$\{u_3\}$	yes	yes	yes	yes	yes	yes
$\{u_4\}$	no	no	no	yes	no	no
$\{u_5\}$	yes	no	no	yes	yes	no
$\{u_8\}$	yes	yes	no	yes	yes	yes
$\{u_9\}$	yes	no	yes	yes	no	yes

Table 6. Similarities between symptoms of patients.

Patients	u_1	u_2	u_3	u_4	u_5	u_8	u_9
$\{u_1\}$	1	0.2	0.8	0.4	0.4	0.6	0.8
$\{u_2\}$	0.2	1	0.4	0.8	0.8	0.6	0.4
$\{u_3\}$	0.8	0.4	1	0.2	0.6	0.8	0.6
$\{u_4\}$	0.4	0.8	0.2	1	0.6	0.4	0.6
$\{u_5\}$	0.4	0.8	0.6	0.6	1	0.8	0.6
$\{u_8\}$	0.6	0.6	0.8	0.4	0.8	1	0.4
$\{u_9\}$	0.8	0.4	0.6	0.6	0.6	0.4	1

It is illustrated, in Table 6, the similarities between symptoms patients, where the similarity degree $v(u_i, u_j)$ between any two patients u_i, u_j is given by

$$v(u_i, u_j) = \frac{\sum_{k=1}^{s} a_k(u_i) = a_k(u_j)}{s}, i, j \in \{1, 2, 3, 4, 5, 8, 9\}, \tag{5.1}$$

where s is the number of conditions attributes. Now, we define the following similarity relations, let $u_i R_1 u_j \Leftrightarrow v(u_i, u_j) > 0.6$, and $u_i R_2 u_j \Leftrightarrow v(u_i, u_j) > 0.4$, Then, we computed the lower, upper approximations, boundary and the accuracy measure of A

(i) if $\mathcal{I} = \{\phi, \{u_9\}\}$ and $A = \{u_2, u_9\}$. Thus, the lower, upper approximations, boundary and the accuracy measure of A using R_1 in Kandil et al.'s methods [23] are $R_{1\bigstar}(A) = \{u_2\}, R_1{}^{\bigstar}(A) = \{u_2, u_4\}, B^{\bigstar}_{R_1}(A) = \{u_4\}, Acc^{\bigstar}_{R_1}(A) = \frac{1}{2}$, while these computations using R_2 are $R_{2\bigstar}(A) = \{u_9\}, R_2{}^{\bigstar}(A) = \{u_2\}, B^{\bigstar}_{R_2}(A) = \{u_2\}, Acc^{\bigstar}_{R_2}(A) = 1$. Therefore,

(a) $R_{1\bigstar}(A) = \{u_2\} \nsupseteq \{u_9\}= R_{2\bigstar}(A)$.

(b) $R_1{}^{\bigstar}(A) = \{u_2, u_4\} \nsubseteq \{u_2\} = R_2{}^{\bigstar}(A)$.

(c) $B^{\bigstar}_{R_1}(A) = \{u_4\} \nsubseteq \{u_2\} = B^{\bigstar}_{R_2}(A)$.

(d) $Acc^{\bigstar}_{R_1}(A) = \frac{1}{2} < 1 = Acc^{\bigstar}_{R_2}(A)$.

Consequently, the computations show that Kandil et al.'s methods [23] are not monotonic, so it cannot be used to evaluate the uncertainty in the data. Meantime, the current methods are monotonic which overcome the defect of Kandil et al.'s methods [23] and can be used to evaluate the uncertainty of the given data. On the other hand, in the lower, upper approximations, boundary and the accuracy measure of A generated from our first approach with respect to R_1 are $\underline{LOW}^{\mathcal{I}}_{R_1}(A) = \phi, \overline{UPP}^{\mathcal{I}}_{R_1}(A) = \{u_2, u_4, u_5, u_8\}, BND^{\mathcal{I}}_{R_1}(A) = \{u_2, u_4, u_5, u_8\}, ACC^{\mathcal{I}}_{R_1}(A) = 0$ while these computations using R_2 are $\underline{LOW}^{\mathcal{I}}_{R_2}(A) = \phi, \overline{UPP}^{\mathcal{I}}_{R_2}(A) = \{u_1, u_2, u_3, u_4, u_5, u_8, u_9\}, BND^{\mathcal{I}}_{R_2}(A) = \{u_1, u_2, u_3, u_4, u_5, u_8, u_9\}, ACC^{\mathcal{I}}_{R_2}(A) = 0$. Therefore, it is elucidated that the corresponding lower and upper approximations, boundary regions and accuracy measures are monotonic.

(ii) if $\mathcal{I} = \{\phi, \{u_1\}, \{u_2\}, \{u_3\}, \{u_9\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_1, u_9\}, \{u_2, u_3\}, \{u_2, u_9\}, \{u_3, u_9\}, \{u_1, u_2, u_3\}, \{u_1, u_2, u_9\}, \{u_1, u_3, u_9\}, \{u_2, u_3, u_9\}, \{u_1, u_2, u_3, u_9\}\}$ and $A = \{u_2, u_3, u_4, u_5, u_8\}$. Thus, we computed the lower, upper approximations, boundary and the accuracy measure of A using R_1 in the first kind of Dai et al.'s approach [14] are $\underline{LOW}_{R_1}(A) = \{u_2, u_4, u_5\}$, $\overline{UPP}_{R_1}(A) = \{u_1, u_2, u_3, u_4, u_5, u_8, u_9\}$, $Boundary_{R_1}(A) = \{u_1, u_3, u_8, u_9\}$, $Accuracy_{R_1}(A) = \frac{3}{7}$. Whereas the lower, upper approximations, boundary and the accuracy measure of A generated from our first approach with respect to R_1 are $\underline{LOW}^{\mathcal{I}}_{R_1}(A) = \{u_1, u_2, u_3, u_4, u_5, u_8, u_9\}$, $\overline{UPP}^{\mathcal{I}}_{R_1}(A) = \{u_2, u_3, u_4, u_5, u_8\}$, $BND^{\mathcal{I}}_{R_1}(A) = \{u_1\}$, $ACC^{\mathcal{I}}_{R_1}$ $(A) = 1$. So, the current techniques are successful and powerful techniques to reduce the boundary region and improve the accuracy measure. Therefore, this helps the medical staff to make an accurate decision about the diagnosis of patients.

6. Conclusions

In recent years, many researchers interested in the rough set theory. As, it is a new efficacious tool to get rid of vagueness; it pursue to minimize the boundary region for the sake of maximizing the accuracy measure. To this end, many approximation spaces (or rough set models) inspired by different types of neighborhoods have been proposed and discussed.

In line with this trend, we have dedicated this manuscript to form new approximation spaces with a view to remove the uncertainty in data and improve the accuracy measure. These approximation spaces have been constructed utilizing the concepts of "intersection of maximal right and left neighborhoods" and "ideals". The main properties and characterizations of these approximation spaces have been investigated. With the help of counterexamples, it was elucidated the relationships between the present approximation spaces and the previous ones introduced in [8, 11, 14, 20, 23]. According to the given comparisons, the present techniques " maximal right and left neighborhoods with ideal structure" were the largest achievement and contribution that made the accuracy measures are greater than the prior studies. As evidence of the promising domain of the followed approaches, a medical application has been presented to elucidate the significance of applying the new methods in decision-making problems.

In upcoming papers, the following goals will be achieved.

(i) New types of approximation spaces induced from neighborhoods and bi-ideals which helps to solve more real-life applications in decision-making problems.

(ii) Extensions of the current approximation spaces in rough multisets via multisets ideals.

(iii) Investigation the current approximation spaces with respect to the other types of maximal neighborhoods.

Acknowledgments

The first author extends her appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through a research groups program under grant (R. G. P. 2/144/43).

Conflicts of interest:

The authors declare no conflicts of interest.

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