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*Research article*

# Further irreducibility criteria for polynomials associated with the complete residue systems in any imaginary quadratic field

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**Abstract:** Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $O_K$ its ring of integers. Let $\pi$ and $\beta$ be an irreducible element and a nonzero element, respectively, in $O_K$. In the authors' earlier work, it was proved for the cases, $m \not\equiv 1 \pmod 4$ and $m \equiv 1 \pmod 4$ that if $\pi = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0 =: f(\beta)$, where $n \geq 1$, $\alpha_n \in O_K \setminus \{0\}$, $\alpha_0, \ldots, \alpha_{n-1}$ belong to a complete residue system modulo $\beta$, and the digits $\alpha_{n-1}$ and $\alpha_n$ satisfy certain restrictions, then the polynomial $f(x)$ is irreducible in $O_K[x]$. In this paper, we extend these results by establishing further irreducibility criteria for polynomials in $O_K[x]$. In addition, we provide elements of $\beta$ that can be applied to the new criteria but not to the previous ones.

**Keywords:** imaginary quadratic field; ring of integers; complete residue system; irreducible element; irreducible polynomial
**Mathematics Subject Classification:** 11R04, 11R09, 11R11

## 1. Introduction

Determining the irreducibility of a polynomial has been one of the most intensively studied problems in mathematics. Among many irreducibility criteria for polynomials in $\mathbb{Z}[x]$, a classical result of A. Cohn [1] states that if we express a prime $p$ in the decimal representation as

$$p = a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0,$$

then the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is irreducible in $\mathbb{Z}[x]$. This result was subsequently generalized to any base $b$ by Brillhart et al. [2]. In 2002, Murty [3] gave another proof of this fact that was conceptually simpler than the one in [2].

In the present work, we are interested in studying the result of A. Cohn in any imaginary quadratic field. Let $K = \mathbb{Q}(\sqrt{m})$ with a unique squarefree integer $m \neq 1$, be a quadratic field. We have seen that the quadratic field $K$ is said to be *real* if $m > 0$ and *imaginary* if $m < 0$. The set of algebraic integers that lie in $K$ is denoted by $O_K$. Indeed,

$$O_K = \{a + b\sigma_m \mid a, b \in \mathbb{Z}\},$$

where

$$\sigma_m := \begin{cases} \sqrt{m} & \text{if } m \not\equiv 1 \pmod 4, \\ \dfrac{1+\sqrt{m}}{2} & \text{if } m \equiv 1 \pmod 4 \end{cases}$$

[4]. Clearly, $O_{\mathbb{Q}(i)} = \mathbb{Z}[i]$, the ring of Gaussian integers, where $i = \sqrt{-1}$. It is well known that $O_K$ is an integral domain and $K$ is its quotient field. Then the set of units in $O_K[x]$ is $U(O_K)$, the group of units in $O_K$.

In general, we know that a prime element in $O_K$ is an irreducible element and the converse holds if $O_K$ is a unique factorization domain. A nonzero polynomial $p(x) \in O_K[x]$ is said to be *irreducible* in $O_K[x]$ if $p(x)$ is not a unit and if $p(x) = f(x)g(x)$ in $O_K[x]$, then either $f(x)$ or $g(x)$ is a unit in $O_K$. Polynomials that are not irreducible are called *reducible*. For $\beta = a + b\sigma_m \in O_K$, we denote the *norm* of $\beta$ by

$$N(\beta) = \begin{cases} a^2 - mb^2 & \text{if } m \not\equiv 1 \pmod 4, \\ a^2 + ab + b^2\left(\dfrac{1-m}{4}\right) & \text{if } m \equiv 1 \pmod 4. \end{cases}$$

Clearly, $N(\beta) \in \mathbb{Z}$ for all $\beta \in O_K$. To determine whether $\alpha \in O_K$ is an irreducible element, we often use the fact that if $N(\alpha) = \pm p$, where $p$ is a rational prime, then $\alpha$ is an irreducible element [4].

For $\alpha, \beta \in O_K$ with $\alpha \neq 0$, we say that $\alpha$ *divides* $\beta$, denoted by $\alpha \mid \beta$, if there exists $\delta \in O_K$ such that $\beta = \alpha\delta$. For $\alpha, \beta, \gamma \in O_K$ with $\gamma \neq 0$, we say that $\alpha$ is *congruent* to $\beta$ modulo $\gamma$ and we write $\alpha \equiv \beta \pmod{\gamma}$, if $\gamma \mid (\alpha - \beta)$. By a *complete residue system* modulo $\beta$ in $O_K$, abbreviated by $CRS(\beta)$ [5], we mean a set of $|N(\beta)|$ elements $C = \{\alpha_1, \alpha_2, \ldots, \alpha_{|N(\beta)|}\}$ in $O_K$, which satisfies the following.

(i) For each $\alpha \in O_K$, there exists $\alpha_i \in C$ such that $\alpha \equiv \alpha_i \pmod{\beta}$.
(ii) For all $i, j \in \{1, 2, \ldots, |N(\beta)|\}$ with $i \neq j$, we have $\alpha_i \not\equiv \alpha_j \pmod{\beta}$.

We have seen from [6] that

$$C = \left\{x + yi \mid x = 0, 1, \ldots, \frac{a^2+b^2}{d} - 1 \text{ and } y = 0, 1, \ldots, d-1\right\} \tag{1.1}$$

is a $CRS(\beta)$, where $\beta = a + bi \in \mathbb{Z}[i]$ and $d = \gcd(a, b)$. It is clear that

$$C' := \{x + yi \mid x = 0, 1, \ldots, \max\{|a|, |b|\} - 1 \text{ and } y = 0, 1, \ldots, d-1\} \subseteq C.$$

In 2017, Singthongla et al. [7] established the result of A. Cohn in $O_K[x]$, where $K$ is an imaginary quadratic field such that $O_K$ is a Euclidean domain, namely $m = -1, -2, -3, -7$, and $-11$ [4]. Regarding the complete residue system (1.1), they established irreducibility criteria for polynomials in $\mathbb{Z}[i][x]$ as the following results.

**Theorem A.** *[7] Let $\beta \in \{2 \pm 2i, 1 \pm 3i, 3 \pm i\}$ or $\beta = a + bi \in \mathbb{Z}[i]$ be such that $|\beta| \geq 2 + \sqrt{2}$ and $a \geq 1$. For a Gaussian prime $\pi$, if*

$$\pi = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0 =: f(\beta),$$

*with $n \geq 1$, $\mathrm{Re}(\alpha_n) \geq 1$, and $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in C'$ satisfying $\mathrm{Re}(\alpha_{n-1})\,\mathrm{Im}(\alpha_n) \geq \mathrm{Re}(\alpha_n)\,\mathrm{Im}(\alpha_{n-1})$, then $f(x)$ is irreducible in $\mathbb{Z}[i][x]$.*

In the proof of Theorem A in [7], the inequality

$$|\beta| \geq \frac{3+\sqrt{1+4M}}{2}, \tag{1.2}$$

where $M = \sqrt{(\max\{a,|b|\}-1)^2+(d-1)^2}$ is necessary. It can be verified that for $\beta = a+bi \in \mathbb{Z}[i]$, if $|\beta| < 2+\sqrt{2}$ and $a \geq 1$, then $\beta \in \{3\pm i, 2\pm 2i, 2\pm i, 1\pm 3i, 1\pm 2i, 1\pm i, 3, 2, 1\}$. It is clear that the Gaussian integers $2\pm 2i$, $1\pm 3i$, and $3\pm i$ satisfy (1.2), while $2\pm i, 1\pm 2i, 1\pm i, 3, 2, 1$ do not. Consequently, we cannot apply Theorem A for these numbers. However, there is an irreducibility criterion for polynomials in $\mathbb{Z}[i][x]$ using $\beta = 3$ in [7].

**Theorem B.** *[7] If $\pi$ is a Gaussian prime such that*

$$\pi = \alpha_n 3^n + \alpha_{n-1}3^{n-1} + \cdots + \alpha_1 3 + \alpha_0,$$

*where $n \geq 3$, $\mathrm{Re}(\alpha_n) \geq 1$, and $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in C'$ satisfying the conditions*

$$\mathrm{Re}(\alpha_{n-1})\,\mathrm{Im}(\alpha_n) \geq \mathrm{Re}(\alpha_n)\,\mathrm{Im}(\alpha_{n-1}),$$

$$\mathrm{Re}(\alpha_{n-2})\,\mathrm{Im}(\alpha_n) \geq \mathrm{Re}(\alpha_n)\,\mathrm{Im}(\alpha_{n-2}),$$

$$\mathrm{Re}(\alpha_{n-2})\,\mathrm{Im}(\alpha_{n-1}) \geq \mathrm{Re}(\alpha_{n-1})\,\mathrm{Im}(\alpha_{n-2}),$$

*then the polynomial $f(x) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x + \alpha_0$ is irreducible in $\mathbb{Z}[i][x]$.*

In 2017, Tadee et al. [8] derived three explicit representations for a complete residue system in a general quadratic field $K = \mathbb{Q}(\sqrt{m})$. We are interested in the first one and only the case $m \not\equiv 1 \pmod 4$ because the complete residue system in another case, $m \equiv 1 \pmod 4$ is inapplicable for our work. The $CRS(\beta)$ for $m \not\equiv 1 \pmod 4$ in [8] is the set

$$C := \left\{ x + y\sqrt{m} \mid x = 0, 1, \ldots, \frac{|N(\beta)|}{d} - 1 \text{ and } y = 0, 1, \ldots, d-1 \right\}, \tag{1.3}$$

where $\beta = a + b\sqrt{m}$ and $d = \gcd(a, b)$.

Recently, Phetnun et al. [9] constructed a complete residue system in a general quadratic field $K = \mathbb{Q}(\sqrt{m})$ for the case $m \equiv 1 \pmod 4$, which is similar to that in (1.3). They then determined the so-called base-$\beta(C)$ representation in $O_K$ and generalized Theorem A for any imaginary quadratic field by using such representation. These results are as the following.

**Theorem C.** *[9] Let $K = \mathbb{Q}(\sqrt{m})$ be a quadratic field with $m \equiv 1 \pmod 4$. If $\beta = a + b\sigma_m \in O_K \setminus \{0\}$ with $d = \gcd(a, b)$, then the set*

$$C = \left\{ x + y\sigma_m \mid x = 0, 1, \ldots, \frac{|N(\beta)|}{d} - 1 \textit{ and } y = 0, 1, \ldots, d-1 \right\} \tag{1.4}$$

*is a $CRS(\beta)$.*

From (1.3) and (1.4), we have shown in [9] for any $m < 0$, that the set

$$C' := \{x + y\sigma_m \mid x = 0, 1, \ldots, \max\{|a|, |b|\} - 1 \text{ and } y = 0, 1, \ldots, d-1\} \subseteq C. \tag{1.5}$$

Moreover, if $d = 1$, then $C' = \{0, 1, \ldots, \max\{|a|, |b|\} - 1\}$, while $b = 0$ implies $C' = \{x + y\sigma_m \mid x, y = 0, 1, \ldots, |a| - 1\} = C$.

**Definition A.** [9] Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field. Let $\beta \in O_K \setminus \{0\}$ and $C$ be a $CRS(\beta)$. We say that $\eta \in O_K \setminus \{0\}$ has a *base-$\beta(C)$ representation* if

$$\eta = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0, \tag{1.6}$$

where $n \geq 1$, $\alpha_n \in O_K \setminus \{0\}$, and $\alpha_i \in C$ $(i = 0, 1, \ldots, n-1)$. If $\alpha_i \in C'$ $(i = 0, 1, \ldots, n-1)$, then (1.6) is called a *base-$\beta(C')$ representation* of $\eta$.

**Theorem D.** *[9] Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $m \not\equiv 1 \pmod 4$. Let $\beta = a + b\sqrt{m} \in O_K$ be such that $|\beta| \geq 2 + \sqrt{1-m}$ and $a \geq 1 + \sqrt{1-m}$. For an irreducible element $\pi$ in $O_K$ with $|\pi| \geq |\beta|$, if*

$$\pi = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0 =: f(\beta)$$

*is a base-$\beta(C')$ representation with $\mathrm{Re}(\alpha_n) \geq 1$ satisfying $\mathrm{Re}(\alpha_{n-1})\,\mathrm{Im}(\alpha_n) \geq \mathrm{Re}(\alpha_n)\,\mathrm{Im}(\alpha_{n-1})$, then $f(x)$ is irreducible in $O_K[x]$.*

**Theorem E.** *[9] Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $m \equiv 1 \pmod 4$. Let $\beta = a + b\sigma_m \in O_K$ be such that $|\beta| \geq 2 + \sqrt{(9-m)/4}$, $a \geq 1$, and $a + (b/2) \geq 1$. For an irreducible element $\pi$ in $O_K$ with $|\pi| > \sqrt{(9-m)/4}\,(|\beta| - 1)$, if*

$$\pi = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0 =: f(\beta)$$

*is a base-$\beta(C')$ representation with $\mathrm{Re}(\alpha_n) \geq 1$ satisfying $\mathrm{Re}(\alpha_{n-1})\,\mathrm{Im}(\alpha_n) \geq \mathrm{Re}(\alpha_n)\,\mathrm{Im}(\alpha_{n-1})$, then $f(x)$ is irreducible in $O_K[x]$.*

In this work, we first establish further irreducibility criteria for polynomials in $O_K[x]$, where $K = \mathbb{Q}(\sqrt{m})$ is an imaginary quadratic field, which extend Theorem D and Theorem E. We observe that the result for the case $m \not\equiv 1 \pmod 4$ is a generalization of Theorem B. Furthermore, we provide elements of $\beta$ that can be applied to the new criteria but not to the previous ones.

## 2. Further irreducibility criteria

In this section, we establish irreducibility criteria for polynomials in $O_K[x]$, where $K$ is an imaginary quadratic field. To prove this, we first recall the essential lemmas in [7, 10] as the following.

**Lemma 1.** *[10] Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field. Then $|\beta| \geq 1$ for all $\beta \in O_K \setminus \{0\}$.*

We note for an imaginary quadratic field $K$ that $|\alpha| = 1$ for all $\alpha \in U(O_K)$.

**Lemma 2.** *[7] Let $f(x) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x + \alpha_0 \in \mathbb{C}[x]$ be such that $n \geq 3$ and $|\alpha_i| \leq M$ $(0 \leq i \leq n-2)$ for some real number $M \geq 1$. If $f(x)$ satisfies the following:*

*(i)* $\mathrm{Re}(\alpha_n) \geq 1$, $\mathrm{Re}(\alpha_{n-1}) \geq 0$, $\mathrm{Im}(\alpha_{n-1}) \geq 0$, $\mathrm{Re}(\alpha_{n-2}) \geq 0$, *and* $\mathrm{Im}(\alpha_{n-2}) \geq 0$,
*(ii)* $\mathrm{Re}(\alpha_{n-1})\,\mathrm{Im}(\alpha_n) \geq \mathrm{Re}(\alpha_n)\,\mathrm{Im}(\alpha_{n-1})$,
*(iii)* $\mathrm{Re}(\alpha_{n-2})\,\mathrm{Im}(\alpha_n) \geq \mathrm{Re}(\alpha_n)\,\mathrm{Im}(\alpha_{n-2})$, *and*
*(iv)* $\mathrm{Re}(\alpha_{n-2})\,\mathrm{Im}(\alpha_{n-1}) \geq \mathrm{Re}(\alpha_{n-1})\,\mathrm{Im}(\alpha_{n-2})$,

*then any complex zero $\alpha$ of $f(x)$ satisfies $|\alpha| < M^{1/3} + 0.465572$ if $\left|\arg\alpha\right| \leq \pi/6$; otherwise*

$$\mathrm{Re}(\alpha) < \frac{\sqrt{3}}{2}\left(\frac{1 + \sqrt{1+4M}}{2}\right).$$

We note that the inequality $|\alpha| < M^{1/3} + 0.465572$ appears in Lemma 2 follows from the proof of the lemma in [7] as follows: It was shown in [7] that

$$0 = \left|\frac{f(\alpha)}{\alpha^n}\right| > \frac{|\alpha|^3 - |\alpha|^2 - M}{|\alpha|^2(|\alpha| - 1)} =: \frac{h(|\alpha|)}{|\alpha|^2(|\alpha| - 1)}, \tag{2.1}$$

where $h(x) = x^3 - x^2 - M$. To obtain such inequality, the authors suppose to the contrary that $|\alpha| \geq M^{1/3} + 0.465572$. One can show that $h(x)$ is increasing on $(-\infty, 0) \cup (2/3, \infty)$. Since $M^{1/3} + 0.465572 > 2/3$, it follows that

$$\begin{aligned} h(|\alpha|) &\geq h(M^{1/3} + 0.465572) \\ &= 0.396716M^{2/3} - 0.280872138448M^{1/3} - 0.115841163475170752 \\ &> 0.396716M^{2/3} - 0.280873M^{1/3} - 0.115842 \\ &= M^{1/3}(0.396716M^{1/3} - 0.280873) - 0.115842 \\ &\geq 0.000001, \text{ since } M \geq 1 \\ &> 0, \end{aligned}$$

which contradicts to (2.1).

Now, we proceed to our first main results. To obtain an irreducibility criterion for the case $m \not\equiv 1 \pmod 4$, we begin with the following lemma.

**Lemma 3.** *Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $m \not\equiv 1 \pmod 4$. Let $\beta = a + b\sqrt{m} \in O_K$ be such that $a > 1$ and*

$$M := \sqrt{(\max\{a, |b|\} - 1)^2 - m(d-1)^2}, \tag{2.2}$$

*where $d = \gcd(a, b)$. Then $M \geq 1$.*

*Proof.* If $b = 0$, then $M = \sqrt{(a-1)^2 - m(a-1)^2} = \sqrt{1-m}(a-1) > 1$. Now, assume that $b \neq 0$ and we treat two separate cases.
**Case 1:** $|b| \geq a$. Then $M = \sqrt{(|b|-1)^2 - m(d-1)^2} \geq \sqrt{(|b|-1)^2} = |b| - 1 \geq 1$.
**Case 2:** $|b| < a$. Then $M = \sqrt{(a-1)^2 - m(d-1)^2} \geq \sqrt{(a-1)^2} = a - 1 \geq 1$.
From every case, we conclude that $M \geq 1$. □

By applying Lemmas 1–3, we have the following.

**Theorem 1.** *Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $m \not\equiv 1 \pmod 4$. Let $\beta = a + b\sqrt{m} \in O_K$ be such that $|\beta| \geq M^{1/3} + 1.465572$ and $a \geq 1 + (\sqrt{3}/2)\left((1 + \sqrt{1+4M})/2\right)$, where $M$ is defined as in (2.2). For an irreducible element $\pi$ in $O_K$, if*

$$\pi = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0 =: f(\beta)$$

*is a base-$\beta(C')$ representation with $n \geq 3$ and $\mathrm{Re}(\alpha_n) \geq 1$ satisfying conditions (ii)–(iv) of Lemma 2, then $f(x)$ is irreducible in $O_K[x]$.*

*Proof.* Suppose to the contrary that $f(x)$ is reducible in $O_K[x]$. Then $f(x) = g(x)h(x)$ with $g(x)$ and $h(x)$ in $O_K[x] \setminus U(O_K)$. We first show that either $\deg g(x) \geq 1$ and $|g(\beta)| = 1$ or $\deg h(x) \geq 1$ and $|h(\beta)| = 1$. It follows from $\deg f(x) \geq 3$ that $g(x)$ or $h(x)$ is a positive degree polynomial. If either $\deg g(x) = 0$

or $\deg h(x) = 0$, we may assume that $h(x) = \alpha \in O_K$. Then $\deg g(x) = \deg f(x)$ and $f(x) = \alpha g(x)$ so that $\pi = \alpha g(\beta)$. Since $\pi$ is an irreducible element and $\alpha \notin U(O_K)$, we obtain $g(\beta) \in U(O_K)$ and thus, $|g(\beta)| = 1$. Otherwise, both $\deg g(x) \geq 1$ and $\deg h(x) \geq 1$, we have that $\pi = g(\beta)h(\beta)$. Using the irreducibility of $\pi$ again, we deduce that either $g(\beta)$ or $h(\beta)$ is a unit and hence, either $|g(\beta)| = 1$ or $|h(\beta)| = 1$, as desired.

We now assume without loss of generality that $\deg g(x) \geq 1$ and $|g(\beta)| = 1$. We will show that this cannot happen. Note that $M \geq 1$ by Lemma 3. Moreover, since $\alpha_i \in C'$ for all $i \in \{0, 1, \dots, n-1\}$, where $C'$ is defined as in (1.5), we have

$$|\alpha_i| \leq \left|(\max\{a, |b|\} - 1) + (d-1)\sqrt{m}\right| = \sqrt{(\max\{a, |b|\} - 1)^2 - m(d-1)^2} = M$$

for all $i \in \{0, 1, \dots, n-1\}$. Since $\deg g(x) \geq 1$, $g(x)$ can be expressed in the form

$$g(x) = \varepsilon \prod_i (x - \gamma_i),$$

where $\varepsilon \in O_K$ is the leading coefficient of $g(x)$ and the product is over the set of complex zeros of $g(x)$. It follows from Lemma 2 that any complex zero $\gamma$ of $g(x)$ satisfies either

$$|\gamma| < M^{1/3} + 0.465572 \text{ or } \operatorname{Re}(\gamma) < \frac{\sqrt{3}}{2}\left(\frac{1 + \sqrt{1+4M}}{2}\right).$$

In the first case, it follows from $|\beta| \geq M^{1/3} + 1.465572$ that

$$|\beta - \gamma| \geq |\beta| - |\gamma| > |\beta| - \left(M^{1/3} + 0.465572\right) \geq 1.$$

In the latter case, it follows from $a \geq 1 + (\sqrt{3}/2)\left((1 + \sqrt{1+4M})/2\right)$ that

$$|\beta - \gamma| \geq \operatorname{Re}(\beta - \gamma) = \operatorname{Re}(\beta) - \operatorname{Re}(\gamma) = a - \operatorname{Re}(\gamma) > a - \frac{\sqrt{3}}{2}\left(\frac{1 + \sqrt{1+4M}}{2}\right) \geq 1.$$

From both cases, by using Lemma 1, we obtain

$$1 = |g(\beta)| = |\varepsilon| \prod_i |\beta - \gamma_i| \geq \prod_i |\beta - \gamma_i| > 1,$$

which is a contradiction. This completes the proof. □

By taking $\beta = 3$ together with $m = -1$ in Theorem 1, we obtain Theorem B. This shows that Theorem 1 is a generalization of Theorem B. We will show in the next section that if $\beta = a + bi \in \mathbb{Z}[i] \setminus \{0\}$ with $b = 0$, then $\beta = 3$ is the only element that can be applied to Theorem 1.

Next, we illustrate the use of Theorem 1 by the following example.

**Example 1.** Let $K = \mathbb{Q}(\sqrt{-5})$, $\beta = 3 + \sqrt{-5} \in O_K$, and $\pi = -9069 - 5968\sqrt{-5}$. Then $d = 1$ and so $C' = \{0, 1, 2\}$. Note that $M = \sqrt{(3-1)^2 + 5(1-1)^2} = 2$, $|\beta| = \sqrt{14} > M^{1/3} + 1.465572$, $a = 3 > 1 + (\sqrt{3}/2)\left((1 + \sqrt{1+4M})/2\right)$, and $\pi$ is an irreducible element because $N(\pi) = (-9069)^2 + 5(-5968)^2 = 260331881$ is a rational prime. Now, we have

$$\pi = (13 + 8\sqrt{-5})\beta^5 + 2\beta^4 + 2\beta^3 + \beta^2 + 2\beta + 1$$

is its base-$\beta(C')$ representation with $n = 5$ and $\mathrm{Re}(\alpha_n) = 13$ satisfying conditions (ii)–(iv) of Lemma 2. By using Theorem 1, we obtain that

$$f(x) = (13 + 8\sqrt{-5})x^5 + 2x^4 + 2x^3 + x^2 + 2x + 1$$

is irreducible in $O_K[x]$.

Note from Example 1 that we cannot apply Theorem D to conclude the irreducibility of the polynomial $f(x)$ because $|\beta| = |3 + \sqrt{-5}| < 2 + \sqrt{6} = 2 + \sqrt{1-m}$. Moreover, we see that $a = 3 < 1 + \sqrt{6} = 1 + \sqrt{1-m}$.

For the case $m \equiv 1 \pmod 4$, we start with the following lemma.

**Lemma 4.** *Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $m \equiv 1 \pmod 4$. Let $\beta = a + b\sigma_m \in O_K$ be such that $a + (b/2) \geq 1 + (\sqrt{3}/2)\left((1 + \sqrt{1+4M})/2\right)$ and*

$$M := \sqrt{(\max\{|a|, |b|\} - 1)^2 + (\max\{|a|, |b|\} - 1)(d-1) + (d-1)^2\left(\frac{1-m}{4}\right)}, \tag{2.3}$$

*where $d = \gcd(a, b)$. Then $M \geq 1$.*

*Proof.* If $b = 0$, then $a \geq 1 + (\sqrt{3}/2)\left((1 + \sqrt{1+4M})/2\right) > 1$. It follows that

$$M = \sqrt{(a-1)^2 + (a-1)^2 + (a-1)^2\left(\frac{1-m}{4}\right)} > \sqrt{(a-1)^2} = a - 1 \geq 1.$$

If $a = 0$, then $b/2 \geq 1 + (\sqrt{3}/2)\left((1 + \sqrt{1+4M})/2\right) > 1$. Thus, $b > 2$ and so

$$M = \sqrt{(b-1)^2 + (b-1)^2 + (b-1)^2\left(\frac{1-m}{4}\right)} > \sqrt{(b-1)^2} = b - 1 > 1.$$

Now, assume that $|a| \geq 1$ and $|b| \geq 1$. If $|a| = 1$ and $|b| = 1$, then $M = 0$, yielding a contradiction because $a + (b/2) \geq 1 + (\sqrt{3}/2)\left((1 + \sqrt{1+4M})/2\right)$. Then $|a| > 1$ or $|b| > 1$. It follows from $d \geq 1$ that

$$M \geq \sqrt{(2-1)^2 + (2-1)(d-1) + (d-1)^2\left(\frac{1-m}{4}\right)} \geq \sqrt{(2-1)^2} = 1.$$

□

By applying Lemmas 1, 2 and 4, we obtain an irreducibility criterion for the case $m \equiv 1 \pmod 4$ as the following theorem.

**Theorem 2.** *Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $m \equiv 1 \pmod 4$. Let $\beta = a + b\sigma_m \in O_K$ be such that $|\beta| \geq M^{1/3} + 1.465572$ and $a + (b/2) \geq 1 + (\sqrt{3}/2)\left((1 + \sqrt{1+4M})/2\right)$, where $M$ is defined as in (2.3). For an irreducible element $\pi$ in $O_K$, if*

$$\pi = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0 =: f(\beta)$$

*is a base-$\beta(C')$ representation with $n \geq 3$ and $\mathrm{Re}(\alpha_n) \geq 1$ satisfying conditions (ii)–(iv) of Lemma 2, then $f(x)$ is irreducible in $O_K[x]$.*

*Proof.* Suppose to the contrary that $f(x)$ is reducible in $O_K[x]$. Then $f(x) = g(x)h(x)$ with $g(x)$ and $h(x)$ in $O_K[x] \setminus U(O_K)$. It can be proved similarly to the proof of Theorem 1 that either $\deg g(x) \geq 1$ and $|g(\beta)| = 1$ or $\deg h(x) \geq 1$ and $|h(\beta)| = 1$. We may assume without loss of generality that $\deg g(x) \geq 1$ and $|g(\beta)| = 1$. We will show that this cannot happen. By Lemma 4, we have $M \geq 1$. For $i \in \{0, 1, \ldots, n-1\}$, since $\alpha_i \in C'$, it follows from the definition of $C'$ in (1.5) that

$$\begin{aligned}
|\alpha_i| &\leq \left|(\max\{|a|,|b|\}-1) + (d-1)\left(\frac{1+\sqrt{m}}{2}\right)\right| \\
&= \left|\left((\max\{|a|,|b|\}-1) + \frac{d-1}{2}\right) + \left(\frac{d-1}{2}\right)\sqrt{m}\right| \\
&= \sqrt{(\max\{|a|,|b|\}-1)^2 + (\max\{|a|,|b|\}-1)(d-1) + (d-1)^2\left(\frac{1-m}{4}\right)} \\
&= M.
\end{aligned}$$

The remaining proof is again similar to that of Theorem 1 by using Lemmas 1, 2 and $\mathrm{Re}(\beta) = a + (b/2)$. $\square$

We illustrate the use of Theorem 2 by the following example.

**Example 2.** Let $K = \mathbb{Q}(\sqrt{-3})$, $\beta = 4 - \sigma_{-3}$, and $\pi = 359 - 278\sigma_{-3}$. Then $d = 1$ and so $C' = \{0, 1, 2, 3\}$. Note that $M = \sqrt{(4-1)^2 + (4-1)(1-1) + (1-1)^2} = 3$, $|\beta| = \sqrt{13} > M^{1/3} + 1.465572$, $a + (b/2) = 3.5 > 1 + (\sqrt{3}/2)\left((1 + \sqrt{1+4M})/2\right)$, and $\pi$ is an irreducible element because $N(\pi) = 359^2 - 359 \cdot 278 + (-278)^2 = 106363$ is a rational prime. Now, we have

$$\pi = \beta^4 + 3\beta^3 + \beta^2 + 2\beta + 1$$

is its base-$\beta(C')$ representation with $n = 4$ and $\mathrm{Re}(\alpha_n) = 1$ satisfying conditions (ii)–(iv) of Lemma 2. By using Theorem 2, we obtain that

$$f(x) = x^4 + 3x^3 + x^2 + 2x + 1$$

is irreducible in $O_K[x]$.

From Example 2, we emphasize that we cannot apply Theorem E to conclude the irreducibility of the polynomial $f(x)$ because $|\beta| = |4 - \sigma_{-3}| < 2 + \sqrt{3} = 2 + \sqrt{(9-m)/4}$, although $a = 4 > 1$ and $a + (b/2) = 4 - (1/2) > 1$.

## 3. Comparison of the criteria

Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field. In this section, we will try to find elements of $\beta = a + b\sigma_m \in O_K \setminus \{0\}$ that can be applied to Theorem 1, respectively, Theorem 2 but not to Theorem D, respectively, Theorem E. We are only interested in two cases, namely $b = 0$ and $b \neq 0$ with $d = \gcd(a, b) = 1$ because the remaining case, $b \neq 0$ with $d > 1$ requires us to solve a multi-variable system of inequalities, which is more complicated. To proceed with this objective, we begin with the following remarks.

**Remark 1.** *Let a and m be integers with $m < 0$. Then the following statements hold.*

*(i)* $a \geq 1 + \frac{\sqrt{3}}{2}\left(\frac{1+\sqrt{1+4(a-1)}}{2}\right)$ *if and only if* $a \geq 3$.

*(ii)* $a \geq 1 + \frac{\sqrt{3}}{2}\left(\frac{1+\sqrt{1+4\sqrt{1-m}(a-1)}}{2}\right)$ *if and only if* $a \geq \frac{4+2\sqrt{3}+3\sqrt{1-m}}{4}$.

*(iii)* $a \geq 1 + \frac{\sqrt{3}}{2}\left(\frac{1+\sqrt{1+4\sqrt{(9-m)/4}(a-1)}}{2}\right)$ *if and only if* $a \geq \frac{4+2\sqrt{3}+3\sqrt{(9-m)/4}}{4}$.

*Proof.* For convenience, we let $A = a - 1$. We have for any real number $x > 0$ that

$$\begin{aligned}
a \geq 1 + \frac{\sqrt{3}}{2}\left(\frac{1+\sqrt{1+4x(a-1)}}{2}\right) &\text{ if and only if } A \geq \frac{\sqrt{3}}{4}\left(1+\sqrt{1+4xA}\right), \\
&\text{ if and only if } \left(\frac{4\sqrt{3}A}{3} - 1\right)^2 \geq 1 + 4xA, \\
&\text{ if and only if } \frac{16A^2}{3} - \frac{(8\sqrt{3}+12x)A}{3} \geq 0, \\
&\text{ if and only if } A[4A - (2\sqrt{3}+3x)] \geq 0, \\
&\text{ if and only if } 4A - (2\sqrt{3}+3x) \geq 0, \\
&\text{ if and only if } A \geq \frac{2\sqrt{3}+3x}{4}, \\
&\text{ if and only if } a \geq \frac{4+2\sqrt{3}+3x}{4}.
\end{aligned} \tag{3.1}$$

Substituting $x = 1$, $x = \sqrt{1-m}$, and $x = \sqrt{(9-m)/4}$ in (3.1) lead to (i)–(iii), respectively, as desired. □

To compare Theorem 1 with Theorem D and to compare Theorem 2 with Theorem E, we require the following remark.

**Remark 2.** *For any real number x, the following statements hold.*

*(i)* $\frac{4+2\sqrt{3}+3\sqrt{x}}{4} \geq \left(x+\sqrt{x}\right)^{1/3} + 1.465572$ *for all* $x \in [3, \infty)$.

*(ii)* $\sqrt{x^2+5} \geq (x-1)^{1/3} + 1.465572$ *for all* $x \in [1, \infty)$.

*(iii)* $\sqrt{3x+1} \geq (x-1)^{1/3} + 1.465572$ *for all* $x \in [1, \infty)$.

*(iv)* $\sqrt{\frac{x^2}{2}+1} \geq (x-1)^{1/3} + 1.465572$ *for all* $x \in [4, \infty)$.

*(v)* $x \geq \left(\sqrt{2}(x-1)\right)^{1/3} + 1.465572$ *for all* $x \in [2.85, \infty)$.

*(vi)* $\sqrt{x^2+1} \geq (x-1)^{1/3} + 1.465572$ *for all* $x \in [3, \infty)$.

*(vii)* $\sqrt{-73-121x} > 4 + \sqrt{9-x}$ *for all* $x \in (-\infty, -2]$.

*(viii)* $\sqrt{29-9x} > 4 + \sqrt{9-x}$ *for all* $x \in (-\infty, -3]$.

*Proof of Remark 2.* By using the WolframAlpha computational intelligence (www.wolframalpha.com), it can be verified by considering the graphs of both left and right functions of each inequality. □

### 3.1. Comparison of the criteria for $m \not\equiv 1 \pmod 4$

Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $m \not\equiv 1 \pmod 4$. In this subsection, we will find elements of $\beta \in O_K \setminus \{0\}$ that can be applied to Theorem 1 but not to Theorem D. Now, let $\beta = a + b\sqrt{m}$ be a nonzero element in $O_K$ that can be applied to Theorem 1 but not to Theorem D. Then $|\beta| \geq M^{1/3} + 1.465572$ and $a \geq 1 + (\sqrt{3}/2)\left((1 + \sqrt{1+4M})/2\right)$, where $M$ is defined as in (2.2). Since $\beta$ cannot be applied to Theorem D, one can consider two possible cases, namely, $|\beta| < 2 + \sqrt{1-m}$ or $|\beta| \geq 2 + \sqrt{1-m}$ as follows:

**Case A:** $|\beta| < 2 + \sqrt{1-m}$. Then, we now try to find elements of $\beta$ that satisfy the following inequality system:

$$\begin{aligned} |\beta| &< 2 + \sqrt{1-m} \\ |\beta| &\geq M^{1/3} + 1.465572 \\ a &\geq 1 + \frac{\sqrt{3}}{2}\left(\frac{1 + \sqrt{1+4M}}{2}\right). \end{aligned} \tag{3.2}$$

We consider two cases as follows:

**Case 1:** $b = 0$. Then $\beta = a$ and $M = \sqrt{(a-1)^2 - m(a-1)^2} = \sqrt{1-m}(a-1)$. Thus, the system (3.2) becomes

$$a < 2 + \sqrt{1-m} \tag{3.3}$$

$$a \geq \left(\sqrt{1-m}(a-1)\right)^{1/3} + 1.465572 \tag{3.4}$$

$$a \geq 1 + \frac{\sqrt{3}}{2}\left(\frac{1 + \sqrt{1 + 4\sqrt{1-m}(a-1)}}{2}\right). \tag{3.5}$$

By (3.5) and Remark 1(ii), we have $a \geq (4 + 2\sqrt{3} + 3\sqrt{1-m})/4$, which together with (3.3) yield

$$\frac{4 + 2\sqrt{3} + 3\sqrt{1-m}}{4} \leq a < 2 + \sqrt{1-m}. \tag{3.6}$$

To show that the integers $\beta = a$ satisfying (3.6) are solutions of the system above, we must show that they also satisfy (3.4). If $m = -1$, then $a \geq (4 + 2\sqrt{3} + 3\sqrt{2})/4 \approx 2.93$. It follows from Remark 2(v) with $x = a$ that $a \geq \left(\sqrt{2}(a-1)\right)^{1/3} + 1.465572 = \left(\sqrt{1-m}(a-1)\right)^{1/3} + 1.465572$. Assume that $m \leq -2$. By taking $x = 1 - m$ in Remark 2(i), we obtain that

$$\begin{aligned} \frac{4 + 2\sqrt{3} + 3\sqrt{1-m}}{4} &\geq \left(1 - m + \sqrt{1-m}\right)^{1/3} + 1.465572 \\ &= \left(\sqrt{1-m}(2 + \sqrt{1-m} - 1)\right)^{1/3} + 1.465572 \\ &> \left(\sqrt{1-m}(a-1)\right)^{1/3} + 1.465572, \text{ by (3.3)}, \end{aligned}$$

implying (3.4).

We note for $m = -1$ that the inequality (3.6) implies $a = 3$. Hence, $\beta = 3 \in \mathbb{Z}[i]$ is the only element that can be applied to Theorem 1 but not to Theorem D.

**Case 2:** $b \neq 0$ and $d = 1$. There are two further subcases:

*Subcase 2.1*: $|b| \geq a$. Then $|\beta| = \sqrt{a^2 - mb^2}$ and $M = \sqrt{(|b|-1)^2} = |b| - 1$. Thus, the system (3.2) becomes

$$\sqrt{a^2 - mb^2} < 2 + \sqrt{1-m} \tag{3.7}$$
$$\sqrt{a^2 - mb^2} \geq (|b|-1)^{1/3} + 1.465572$$
$$a \geq 1 + \frac{\sqrt{3}}{2}\left(\frac{1 + \sqrt{1 + 4(|b|-1)}}{2}\right). \tag{3.8}$$

Since $|b| \geq a$, we obtain from (3.8) that $a \geq 1 + (\sqrt{3}/2)\left(\left(1 + \sqrt{1+4(a-1)}\right)/2\right)$. Using Remark 1(i), we have that $a \geq 3$. It follows from $|b| \geq a$, $a \geq 3$, and $m \leq -1$ that

$$\sqrt{a^2 - mb^2} \geq \sqrt{a^2 - ma^2} = \sqrt{a^2(1-m)} \geq \sqrt{9(1-m)} = 3\sqrt{1-m} > 2 + \sqrt{1-m},$$

which is contrary to (3.7). Thus, the system above has no integer solution $(a, b)$. This means that the assumptions in the system generate no pairs $(a, b)$ that are solutions to Theorem 1 and that are also not solutions to Theorem D.

*Subcase 2.2*: $|b| < a$. Then $|\beta| = \sqrt{a^2 - mb^2}$ and $M = \sqrt{(a-1)^2} = a - 1$. Thus, the system (3.2) becomes

$$\sqrt{a^2 - mb^2} < 2 + \sqrt{1-m} \tag{3.9}$$
$$\sqrt{a^2 - mb^2} \geq (a-1)^{1/3} + 1.465572 \tag{3.10}$$
$$a \geq 1 + \frac{\sqrt{3}}{2}\left(\frac{1 + \sqrt{1 + 4(a-1)}}{2}\right). \tag{3.11}$$

Using Remark 1(i) and (3.11), we have $a \geq 3$. Since $m \leq -1$, we obtain $(6-5m)^2 = 25m^2 - 60m + 36 > 16m^2 - 52m + 36 = 4(9-4m)(1-m)$, yielding $6 - 5m > 2\sqrt{(9-4m)(1-m)}$. It follows that

$$\left(\sqrt{9-4m} - \sqrt{1-m}\right)^2 = 10 - 5m - 2\sqrt{(9-4m)(1-m)} > 4$$

and so $\sqrt{9-4m} - \sqrt{1-m} > 2$. If $|b| \geq 2$, then $\sqrt{a^2 - mb^2} \geq \sqrt{9-4m} > 2 + \sqrt{1-m}$, which is contrary to (3.9). Thus, $|b| = 1$. Using (3.9) and $a \geq 3$, we have $\sqrt{9-m} \leq \sqrt{a^2 - m} < 2 + \sqrt{1-m}$ and so $9 \leq a^2 < 5 + 4\sqrt{1-m}$, i.e., $3 \leq a < \sqrt{5 + 4\sqrt{1-m}}$. We next show that the pairs $(a, b)$ with

$$3 \leq a < \sqrt{5 + 4\sqrt{1-m}} \text{ and } b = \pm 1 \tag{3.12}$$

also satisfy (3.10). Since $|b| = 1$, $a \geq 3$, and Remark 2(vi) with $x = a$, we have

$$\sqrt{a^2 - mb^2} = \sqrt{a^2 - m} \geq \sqrt{a^2 + 1} \geq (a-1)^{1/3} + 1.465572,$$

yielding (3.10). Thus, we conclude that the pairs $(a, b)$ satisfying (3.12) are solutions of the system above.
**Case B:** $|\beta| \geq 2+\sqrt{1-m}$. Since we cannot apply the element $\beta$ to Theorem D, we have $a < 1+\sqrt{1-m}$. Now, we try again to find elements of $\beta$ that satisfy the following inequality system:

$$\begin{aligned} |\beta| &\geq 2+\sqrt{1-m} \\ a &< 1+\sqrt{1-m} \\ |\beta| &\geq M^{1/3}+1.465572 \\ a &\geq 1+\frac{\sqrt{3}}{2}\left(\frac{1+\sqrt{1+4M}}{2}\right). \end{aligned} \tag{3.13}$$

We consider two cases as follows:
**Case 1:** $b = 0$. Then $a < 1+\sqrt{1-m} < 2+\sqrt{1-m} \leq |\beta| = a$, which is a contradiction. Hence, the system (3.13) has no integer solution $\beta = a$. In other words, the assumptions in the system generate no pairs $(a, b)$ that are solutions to Theorem 1 and that are also not solutions to Theorem D.
**Case 2:** $b \neq 0$ and $d = 1$. There are two further subcases:
*Subcase 2.1*: $|b| \geq a$. Then $|\beta| = \sqrt{a^2-mb^2}$ and $M = \sqrt{(|b|-1)^2} = |b|-1$. Thus, the system (3.13) becomes

$$\sqrt{a^2-mb^2} \geq 2+\sqrt{1-m} \tag{3.14}$$

$$a < 1+\sqrt{1-m} \tag{3.15}$$

$$\sqrt{a^2-mb^2} \geq (|b|-1)^{1/3}+1.465572 \tag{3.16}$$

$$a \geq 1+\frac{\sqrt{3}}{2}\left(\frac{1+\sqrt{1+4(|b|-1)}}{2}\right). \tag{3.17}$$

Since $|b| \geq a$, we obtain from (3.17) that $a \geq 1+(\sqrt{3}/2)\left(\left(1+\sqrt{1+4(a-1)}\right)/2\right)$. It follows from Remark 1(i) that $a \geq 3$. Since $d = 1$, we have $|b| > a$. By using (3.15) together with $a \geq 3$, we have $3 \leq a < 1+\sqrt{1-m}$, implying $m \leq -5$. It can be verified by using (3.17) that $|b| \leq \left(\left(4\sqrt{3}(a-1)-3\right)^2+27\right)/36$. Now, we have that

$$3 \leq a < 1+\sqrt{1-m} \text{ and } a < |b| \leq \frac{\left(4\sqrt{3}(a-1)-3\right)^2+27}{36}. \tag{3.18}$$

To show that the pairs $(a, b)$ satisfying (3.18) are solutions of the system, it remains to show that they also satisfy (3.14) and (3.16). Since $|b| > a \geq 3$ and $m < 0$, we obtain

$$\sqrt{a^2-mb^2} > \sqrt{a^2-ma^2} = a\sqrt{1-m} \geq 3\sqrt{1-m} > 2+\sqrt{1-m},$$

yielding (3.14). From Remark 2(ii) with $x = |b|$, we have

$$\sqrt{a^2-mb^2} > \sqrt{5+b^2} \geq (|b|-1)^{1/3}+1.465572,$$

showing (3.16).
*Subcase 2.2*: $|b| < a$. Then $|\beta| = \sqrt{a^2-mb^2}$ and $M = \sqrt{(a-1)^2} = a-1$. Thus, the system (3.13) becomes

$$\sqrt{a^2-mb^2} \geq 2+\sqrt{1-m} \tag{3.19}$$

$$a < 1 + \sqrt{1-m} \tag{3.20}$$

$$\sqrt{a^2 - mb^2} \geq (a-1)^{1/3} + 1.465572 \tag{3.21}$$

$$a \geq 1 + \frac{\sqrt{3}}{2}\left(\frac{1 + \sqrt{1+4(a-1)}}{2}\right). \tag{3.22}$$

Again, using Remark 1(i) and (3.22), we obtain $a \geq 3$. By using (3.20) together with $a \geq 3$, we have $3 \leq a < 1 + \sqrt{1-m}$, implying $m \leq -5$. Using (3.19), we can verify that $|b| \geq \sqrt{(5 - m + 4\sqrt{1-m} - a^2)/(-m)}$. Now, we have that

$$3 \leq a < 1 + \sqrt{1-m} \text{ and } \sqrt{\frac{5 - m + 4\sqrt{1-m} - a^2}{-m}} \leq |b| < a. \tag{3.23}$$

To show that the pairs $(a, b)$ satisfying (3.23) are solutions of the system, it remains to show that they also satisfy (3.21). It follows from $b^2 \geq 1$, $m \leq -5$, and Remark 2(ii) with $x = a$ that

$$\sqrt{a^2 - mb^2} \geq \sqrt{a^2+5} \geq (a-1)^{1/3} + 1.465572,$$

yielding (3.21).

From every case, we conclude that elements of $\beta = a + b\sqrt{m} \in O_K \setminus \{0\}$ with $m \not\equiv 1 \pmod 4$ that can be applied to Theorem 1 but not to Theorem D are shown in the following tables.

We note from Subcase 2.2 in Table 1 that the number of $a$ roughly grows as $2\sqrt[4]{1-m}$. To see this, since $8\sqrt[4]{1-m} > 1$, we have

$$5 + 4\sqrt{1-m} < 4\sqrt{1-m} + 8\sqrt[4]{1-m} + 4 = \left(2\sqrt[4]{1-m} + 2\right)^2$$

and so $3 \leq a < \sqrt{5 + 4\sqrt{1-m}} < 2\sqrt[4]{1-m} + 2$. This means that the number of such $a$ is approximately $2\sqrt[4]{1-m}$.

**Table 1.** Case A: $|\beta| < 2 + \sqrt{1-m}$.

| $\beta = a + b\sqrt{m}$, $d = \gcd(a, b)$ | Integer solutions $(a, b)$ |
|---|---|
| Case 1: $b = 0$ | $\frac{4 + 2\sqrt{3} + 3\sqrt{1-m}}{4} \leq a < 2 + \sqrt{1-m}$ and $b = 0$ |
| Case 2: $b \neq 0$ and $d = 1$ | |
| Subcase 2.1: $\lvert b\rvert \geq a$ | none |
| Subcase 2.2: $\lvert b\rvert < a$ | $3 \leq a < \sqrt{5 + 4\sqrt{1-m}}$ and $b = \pm 1$ |

We note from Table 2 that the complicated lower bound in Subcase 2.2 is actually very close to 1. Indeed, we show that

$$\sqrt{\frac{5 - m + 4\sqrt{1 - m} - a^2}{-m}} < 2.$$

Since $m \le -1$, it follows that

$$(4 - 3m)^2 - 16(1 - m) = (9m^2 - 24m + 16) - 16 + 16m = 9m^2 - 8m = m(9m - 8) > 0,$$

showing $(4 - 3m)^2 > 16(1 - m)$ and so $4 - 3m > 4\sqrt{1 - m}$. Using $3 \le a < 1 + \sqrt{1 - m}$, we have that $-2 + m - 2\sqrt{1 - m} < -a^2 \le -9$. It follows that

$$\begin{aligned} 0 < \frac{3 + 2\sqrt{1 - m}}{-m} &= \frac{\left(5 - m + 4\sqrt{1 - m}\right) + \left(-2 + m - 2\sqrt{1 - m}\right)}{-m} \\ &< \frac{\left(5 - m + 4\sqrt{1 - m}\right) - a^2}{-m} \\ &\le \frac{\left(5 - m + 4\sqrt{1 - m}\right) - 9}{-m} \\ &= \frac{-4 - m + 4\sqrt{1 - m}}{-m} \\ &< \frac{-4 - m + (4 - 3m)}{-m} \\ &= 4. \end{aligned}$$

This shows that $\sqrt{(5 - m + 4\sqrt{1 - m} - a^2)/(-m)} < \sqrt{4} = 2$, as desired.

**Table 2.** Case B: $|\beta| \ge 2 + \sqrt{1 - m}$.

| $\beta = a + b\sqrt{m}$, $d = \gcd(a, b)$ | Integer solutions $(a, b)$ |
| --- | --- |
| Case 1: $b = 0$ | none |
| Case 2: $b \neq 0$ and $d = 1$ | |
| Subcase 2.1: $\lvert b\rvert \ge a$ | $3 \le a < 1 + \sqrt{1 - m}$ and $a < \lvert b\rvert \le \dfrac{\left(4\sqrt{3}(a - 1) - 3\right)^2 + 27}{36}$ |
| Subcase 2.2: $\lvert b\rvert < a$ | $3 \le a < 1 + \sqrt{1 - m}$ and $\sqrt{\dfrac{5 - m + 4\sqrt{1 - m} - a^2}{-m}} \le \lvert b\rvert < a$ |

### 3.2. *Comparison of the criteria for* $m \equiv 1 \pmod 4$

Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $m \equiv 1 \pmod 4$. In this subsection, we find elements of $\beta \in O_K \setminus \{0\}$ that can be applied to Theorem 2 but not to Theorem E. Now, let $\beta = a + b\sigma_m$ be a nonzero element in $O_K$ that can be applied to Theorem 2 but not to Theorem E. Then $|\beta| \geq M^{1/3} + 1.465572$ and $a + (b/2) \geq 1 + (\sqrt{3}/2)\left((1 + \sqrt{1+4M})/2\right)$, where $M$ is defined as in (2.3). Since $\beta$ cannot be applied to Theorem E, one can consider two possible cases, namely, $|\beta| < 2 + \sqrt{(9-m)/4}$ or $|\beta| \geq 2 + \sqrt{(9-m)/4}$ as follows:

**Case A:** $|\beta| < 2 + \sqrt{(9-m)/4}$. Then we will find elements of $\beta$ that satisfy the inequality system:

$$\begin{aligned} |\beta| &< 2 + \sqrt{\frac{9-m}{4}} \\ |\beta| &\geq M^{1/3} + 1.465572 \\ a + \frac{b}{2} &\geq 1 + \frac{\sqrt{3}}{2}\left(\frac{1+\sqrt{1+4M}}{2}\right). \end{aligned} \tag{3.24}$$

We consider two cases as follows:

**Case 1:** $b = 0$. Then $\beta = a$ and $M = \sqrt{(a-1)^2 + (a-1)(a-1) + (a-1)^2(1-m)/4} = \sqrt{(9-m)/4}(a-1)$. Thus, the system (3.24) becomes

$$a < 2 + \sqrt{\frac{9-m}{4}} \tag{3.25}$$

$$a \geq \left(\sqrt{\frac{9-m}{4}}(a-1)\right)^{1/3} + 1.465572 \tag{3.26}$$

$$a \geq 1 + \frac{\sqrt{3}}{2}\left(\frac{1+\sqrt{1+4\sqrt{(9-m)/4}(a-1)}}{2}\right). \tag{3.27}$$

By (3.27) and Remark 1(iii), we have that $a \geq (4 + 2\sqrt{3} + 3\sqrt{(9-m)/4})/4$, which together with (3.25) yield

$$\frac{4 + 2\sqrt{3} + 3\sqrt{(9-m)/4}}{4} \leq a < 2 + \sqrt{\frac{9-m}{4}}. \tag{3.28}$$

To show that the integers $\beta = a$ satisfying (3.28) are solutions of the system above, we must show that they also satisfy (3.26). By taking $x = (9-m)/4$ in Remark 2(i) and using (3.25), we obtain that

$$\begin{aligned} \frac{4 + 2\sqrt{3} + 3\sqrt{(9-m)/4}}{4} &\geq \left(\frac{9-m}{4} + \sqrt{\frac{9-m}{4}}\right)^{1/3} + 1.465572 \\ &= \left[\sqrt{\frac{9-m}{4}}\left(2 + \sqrt{\frac{9-m}{4}} - 1\right)\right]^{1/3} + 1.465572 \\ &> \left(\sqrt{\frac{9-m}{4}}(a-1)\right)^{1/3} + 1.465572. \end{aligned} \tag{3.29}$$

It follows from (3.28) and (3.29) that $a > \left(\sqrt{(9-m)/4}(a-1)\right)^{1/3} + 1.465572$, yielding (3.26).
**Case 2:** $b \neq 0$ and $d = 1$. There are two further subcases:
*Subcase 2.1*: $|b| \geq |a|$. Then $|\beta| = \sqrt{a^2 + ab + b^2(1-m)/4}$ and $M = \sqrt{(|b|-1)^2} = |b| - 1$. Thus, the system (3.24) becomes

$$\sqrt{a^2 + ab + b^2\left(\frac{1-m}{4}\right)} < 2 + \sqrt{\frac{9-m}{4}} \tag{3.30}$$

$$\sqrt{a^2 + ab + b^2\left(\frac{1-m}{4}\right)} \geq (|b|-1)^{1/3} + 1.465572$$

$$a + \frac{b}{2} \geq 1 + \frac{\sqrt{3}}{2}\left(\frac{1 + \sqrt{1 + 4(|b|-1)}}{2}\right). \tag{3.31}$$

In this subcase, we now show that the system has no integer solution $(a, b)$. If $a < 0$, then it follows from (3.31) that $b > 0$ and so $(b/2) - 1 \geq 1 + (\sqrt{3}/2)\left(\left(1 + \sqrt{1 + 4(b-1)}\right)/2\right)$. Then $b^2 - (11 + \sqrt{3})b + (19 + 4\sqrt{3}) \geq 0$, implying $b \geq 11$. It follows from $a^2 \geq 1$, $a > 1 - (b/2)$, $b \geq 11$, and Remark 2(vii) with $x = m$ that

$$\begin{aligned}
\sqrt{a^2 + ab + b^2\left(\frac{1-m}{4}\right)} &> \sqrt{1 + \left(1 - \frac{b}{2}\right)b + b^2\left(\frac{1-m}{4}\right)} \\
&= \sqrt{b^2\left(\frac{-1-m}{4}\right) + b + 1} \\
&\geq \sqrt{\frac{-121 - 121m + 48}{4}} \\
&= \frac{1}{2}\sqrt{-73 - 121m} \\
&> \frac{1}{2}\left(4 + \sqrt{9-m}\right) \\
&= 2 + \sqrt{\frac{9-m}{4}},
\end{aligned} \tag{3.32}$$

which is contrary to (3.30). Thus, $a \geq 0$. If $a = 0$, then $|b| = 1$ because $d = 1$. This contradicts to (3.31), so $a \geq 1$. If $|b| = 1$, then $a = 1$ and so (3.31) is false. Thus, $|b| \geq 2$ and so $|b| > a$ because $d = 1$. It follows from (3.31) and $|b| \geq 2$ that $a + (b/2) > 2.4$ and so $|b| + (b/2) > 2.4$. This implies that $b \geq 2$ or $b \leq -5$. If $b = 2$, then we obtain that $2 = |b| > a \geq (\sqrt{3}/2)\left(\left(1 + \sqrt{1 + 4(2-1)}\right)/2\right) > 1.4$, which is a contradiction. If $b = 3$, then we obtain that $3 = |b| > a \geq (\sqrt{3}/2)\left(\left(1 + \sqrt{1 + 4(3-1)}\right)/2\right) - (1/2) > 1.2$, which implies that $a = 2$. It follows that

$$\begin{aligned}
\sqrt{a^2 + ab + b^2\left(\frac{1-m}{4}\right)} &= \sqrt{2^2 + 2\cdot 3 + 3^2\left(\frac{1-m}{4}\right)} \\
&= \frac{1}{4}\left(\sqrt{49 - 9m} + \sqrt{49 - 9m}\right)
\end{aligned}$$

$$> \frac{1}{4}\left(8+\sqrt{36-4m}\right)$$
$$= 2+\sqrt{\frac{9-m}{4}},$$

which is contrary to (3.30). If $b \geq 4$, then

$$\begin{aligned}\sqrt{a^2+ab+b^2\left(\frac{1-m}{4}\right)} &\geq \sqrt{1+4+16\left(\frac{1-m}{4}\right)}\\ &= \frac{1}{2}\left(\sqrt{9-4m}+\sqrt{9-4m}\right)\\ &> \frac{1}{2}\left(4+\sqrt{9-m}\right)\\ &= 2+\sqrt{\frac{9-m}{4}},\end{aligned}$$

which is contrary to (3.30). If $b \leq -5$, then

$$a-\frac{5}{2} \geq a+\frac{b}{2} \geq 1+\frac{\sqrt{3}}{2}\left(\frac{1+\sqrt{1+4(|b|-1)}}{2}\right) \geq 1+\frac{\sqrt{3}}{2}\left(\frac{1+\sqrt{1+4(5-1)}}{2}\right) > 3.22,$$

showing $a \geq 6$. Since $b \leq -5$ and $a \geq 6$, it follows from $-b = |b| > a$ that

$$\begin{aligned}\sqrt{a^2+ab+b^2\left(\frac{1-m}{4}\right)} &> \sqrt{a^2-b^2+b^2\left(\frac{1-m}{4}\right)}\\ &= \sqrt{b^2\left(\frac{1-m}{4}-1\right)+a^2}\\ &\geq \sqrt{25\left(\frac{1-m}{4}-1\right)+36}\\ &= \frac{1}{4}\left(\sqrt{69-25m}+\sqrt{69-25m}\right)\\ &> \frac{1}{4}\left(8+\sqrt{36-4m}\right)\\ &= 2+\sqrt{\frac{9-m}{4}},\end{aligned}$$

which is contrary to (3.30).

Thus, in this subcase, we conclude that the assumptions in the system generate no pairs $(a, b)$ that are solutions to Theorem 2 and that are also not solutions to E.

*Subcase 2.2*: $|b| < |a|$. Then $|\beta| = \sqrt{a^2+ab+b^2(1-m)/4}$ and $M = \sqrt{(|a|-1)^2} = |a|-1$. Thus, the system (3.24) becomes

$$\sqrt{a^2+ab+b^2\left(\frac{1-m}{4}\right)} < 2+\sqrt{\frac{9-m}{4}} \tag{3.33}$$

$$\sqrt{a^2 + ab + b^2\left(\frac{1-m}{4}\right)} \geq (|a| - 1)^{1/3} + 1.465572 \tag{3.34}$$

$$a + \frac{b}{2} \geq 1 + \frac{\sqrt{3}}{2}\left(\frac{1 + \sqrt{1 + 4(|a| - 1)}}{2}\right). \tag{3.35}$$

If $a < 0$, then it follows from $a + (b/2) > 1$ that $b > 0$. Since $|a| > |b| = b$ and (3.35), we obtain $(b/2) - 1 \geq a + (b/2) > 1 + (\sqrt{3}/2)\left(\left(1 + \sqrt{1 + 4(b-1)}\right)/2\right)$, implying $b \geq 11$. Now, we have that $a^2 > 1$, $a > 1 - (b/2)$, and $b \geq 11$. It can be proved similarly to (3.32) that

$$\sqrt{a^2 + ab + b^2\left(\frac{1-m}{4}\right)} > 2 + \sqrt{\frac{9-m}{4}},$$

which is contrary to (3.33). Thus, $a \geq 0$. If $a = 0$ or $a = 1$, then $0 < |b| < |a| \leq 1$, which is impossible so that $a \geq 2$. If $b = -1$, then it follows from (3.35) that $a - (1/2) \geq 1 + (\sqrt{3}/2)\left(\left(1 + \sqrt{1 + 4(a-1)}\right)/2\right)$, implying $a \geq 4$. By taking $x = a$ in Remark 2(iii), we have

$$\sqrt{a^2 - a + \frac{1-m}{4}} = \sqrt{a(a-1) + \frac{1-m}{4}} \geq \sqrt{3a+1} \geq (a-1)^{1/3} + 1.465572,$$

yielding (3.34). It can be verified by (3.33) with $b = -1$ that $a < \left(\sqrt{8\sqrt{9-m} + 25} + 1\right)/2$. This shows that

$$4 \leq a < \frac{\sqrt{8\sqrt{9-m} + 25} + 1}{2}, \text{ when } b = -1. \tag{3.36}$$

If $b = 1$, then it follows from (3.35) that $a + (1/2) \geq 1 + (\sqrt{3}/2)\left(\left(1 + \sqrt{1 + 4(a-1)}\right)/2\right)$, implying $a \geq 2$. By taking $x = a$ in Remark 2(iii), we have that

$$\sqrt{a^2 + a + \frac{1-m}{4}} = \sqrt{a(a+1) + \frac{1-m}{4}} \geq \sqrt{3a+1} \geq (a-1)^{1/3} + 1.465572,$$

yielding (3.34). It can be verified by (3.33) with $b = 1$ that $a < \left(\sqrt{8\sqrt{9-m} + 25} - 1\right)/2$ and thus

$$2 \leq a < \frac{\sqrt{8\sqrt{9-m} + 25} - 1}{2}, \text{ when } b = 1. \tag{3.37}$$

We next show for $b \geq 2$ or $b \leq -2$ that the system above has no integer solution $(a, b)$. If $b \geq 2$, then $a = |a| > |b| = b \geq 2$ and so $a \geq 3$. It follows that

$$\begin{aligned}\sqrt{a^2 + ab + b^2\left(\frac{1-m}{4}\right)} &\geq \sqrt{3^2 + 3 \cdot 2 + 2^2\left(\frac{1-m}{4}\right)} \\ &= \frac{1}{2}\left(\sqrt{16-m} + \sqrt{16-m}\right)\end{aligned}$$

$$
\begin{aligned}
&> \frac{1}{2}\left(4+\sqrt{9-m}\right)\\
&= 2+\sqrt{\frac{9-m}{4}},
\end{aligned}
$$

which is contrary to (3.33). If $b = -2$, then we obtain from (3.35) that $a - 1 \geq 1 + (\sqrt{3}/2)\left(\left(1+\sqrt{1+4(a-1)}\right)/2\right)$, implying $a \geq 4$. Since $d = 1$ and $b = -2$, we have that $a \geq 5$. Hence,

$$
\begin{aligned}
\sqrt{a^2-2a+1-m} &= \sqrt{a(a-2)+1-m}\\
&\geq \sqrt{5(3)+1-m}\\
&= \frac{1}{2}\left(\sqrt{16-m}+\sqrt{16-m}\right)\\
&> \frac{1}{2}\left(4+\sqrt{9-m}\right)\\
&= 2+\sqrt{\frac{9-m}{4}},
\end{aligned}
$$

which is contrary to (3.33). If $b \leq -3$, then we have $a - (3/2) \geq a + (b/2) \geq 1 + (\sqrt{3}/2)\left(\left(1+\sqrt{1+4(a-1)}\right)/2\right)$. This implies that $a \geq 5$. Since $a > |b| = -b$, we obtain that $-b \leq a-1$ and so $ab \geq -a^2 + a$. It follows from $b \leq -3$, $a \geq 5$, $ab \geq -a^2 + a$, and Remark 2(viii) with $x = m$ that

$$
\begin{aligned}
\sqrt{a^2+ab+b^2\left(\frac{1-m}{4}\right)} &\geq \sqrt{a^2-a^2+a+b^2\left(\frac{1-m}{4}\right)}\\
&\geq \sqrt{9\left(\frac{1-m}{4}\right)+5}\\
&= \frac{1}{2}\sqrt{29-9m}\\
&> \frac{1}{2}\left(4+\sqrt{9-m}\right)\\
&= 2+\sqrt{\frac{9-m}{4}},
\end{aligned}
$$

which is contrary to (3.33).

Thus, in this subcase, we obtain that the pairs $(a, b)$ with $b \neq 0$ and $d = 1$ satisfying (3.36) or (3.37) are integer solutions of the system (3.24).

**Case B:** $|\beta| \geq 2 + \sqrt{(9-m)/4}$. Since $a + (b/2) > 1$ and we cannot apply $\beta$ to Theorem E, it follows that $a < 1$. Thus, we have to find elements of $\beta$ that satisfy the following inequality system:

$$
\begin{aligned}
|\beta| &\geq 2+\sqrt{\frac{9-m}{4}},\ a<1\\
|\beta| &\geq M^{1/3}+1.465572\\
a+\frac{b}{2} &\geq 1+\frac{\sqrt{3}}{2}\left(\frac{1+\sqrt{1+4M}}{2}\right).
\end{aligned}
\tag{3.38}
$$

Note that $M \geq 1$ by Lemma 4. Then $b/2 \geq 1 + (\sqrt{3}/2)\left((1+\sqrt{5})/2\right) > 2.4$ and so $b \geq 5$. If $b < |a|$, then $a \leq -6$ and so $a + (b/2) < a + b < a + |a| = 0$, which is a contradiction. Thus, $b \geq |a| = -a$ and so $M = \sqrt{(b-1)^2} = b - 1$. Hence, the system (3.38) becomes

$$\sqrt{a^2 + ab + b^2\left(\frac{1-m}{4}\right)} \geq 2 + \sqrt{\frac{9-m}{4}},\ a < 1 \tag{3.39}$$

$$\sqrt{a^2 + ab + b^2\left(\frac{1-m}{4}\right)} \geq (b-1)^{1/3} + 1.465572 \tag{3.40}$$

$$a + \frac{b}{2} \geq 1 + \frac{\sqrt{3}}{2}\left(\frac{1+\sqrt{1+4(b-1)}}{2}\right). \tag{3.41}$$

Since $b \geq 5$ and $d = 1$, we have $a \leq -1$. It follows by (3.41) that $(b/2) - 1 \geq 1 + (\sqrt{3}/2)\left(\left(1+\sqrt{1+4(b-1)}\right)/2\right)$, implying $b \geq 11$. Note that $b \geq -a$, $b \geq 11$, and $d = 1$ imply $b > -a$. That is, $-b < a \leq -1$. Now, we have that

$$b \geq 11 \text{ and } 1 + \frac{\sqrt{3}}{2}\left(\frac{1+\sqrt{1+4(b-1)}}{2}\right) - \frac{b}{2} \leq a \leq -1 \tag{3.42}$$

To show that the pairs $(a, b)$ satisfying (3.42) are solutions of the system, it remains to show that they also satisfy (3.39) and (3.40). Since $a^2 \geq 1$, $a > 1 - (b/2)$, and $b \geq 11$, we obtain by Remark 2(vii) with $x = m$ that

$$\begin{aligned}
\sqrt{a^2 + ab + b^2\left(\frac{1-m}{4}\right)} &> \sqrt{1 + \left(1 - \frac{b}{2}\right)b + b^2\left(\frac{1-m}{4}\right)} \\
&= \sqrt{b^2\left(\frac{-1-m}{4}\right) + b + 1} \\
&\geq \sqrt{121\left(\frac{-1-m}{4}\right) + 12} \\
&= \frac{1}{2}\sqrt{-73 - 121m} \\
&> \frac{1}{2}\left(4 + \sqrt{9-m}\right) \\
&= 2 + \sqrt{\frac{9-m}{4}},
\end{aligned}$$

showing (3.39). It follows from $a^2 \geq 1$, $a > 1 - (b/2)$, $m \leq -3$, and Remark 2(iv) with $x = b$ that

$$\sqrt{a^2 + ab + b^2\left(\frac{1-m}{4}\right)} > \sqrt{1 + \left(1 - \frac{b}{2}\right)b + b^2} > \sqrt{\frac{b^2}{2} + 1} \geq (b-1)^{1/3} + 1.465572,$$

yielding (3.40), as desired.

From every case, we conclude that elements of $\beta = a + b\sigma_m \in O_K \setminus \{0\}$ with $m \equiv 1 \pmod 4$ that can be applied to Theorem 2 but not to Theorem E are shown in the following tables.

We note from Subcase 2.2 in Table 3 that when $b = -1$, the number of $a$ roughly grows as $\sqrt[4]{4(9-m)}$. Otherwise, $b = 1$ implies that the number of $a$ roughly grows as $\sqrt[4]{4(9-m)} + 1$. To see these, one can see that

$$8\sqrt{9-m} + 25 < 8\sqrt{9-m} + 20\sqrt[4]{4(9-m)} + 25 = \left(2\sqrt[4]{4(9-m)} + 5\right)^2$$

and so $\sqrt{8\sqrt{9-m} + 25} < 2\sqrt[4]{4(9-m)} + 5$. If $b = -1$, then

$$4 \le a < \frac{\sqrt{8\sqrt{9-m} + 25} + 1}{2} < \frac{2\sqrt[4]{4(9-m)} + 6}{2} = \sqrt[4]{4(9-m)} + 3,$$

showing that the number of such $a$ is approximately $\sqrt[4]{4(9-m)}$. If $b = 1$, we obtain

$$2 \le a < \frac{\sqrt{8\sqrt{9-m} + 25} - 1}{2} < \frac{2\sqrt[4]{4(9-m)} + 4}{2} = \sqrt[4]{4(9-m)} + 2,$$

showing that the number of such $a$ is approximately $\sqrt[4]{4(9-m)} + 1$.

**Table 3.** Case A: $|\beta| < 2 + \sqrt{\dfrac{9-m}{4}}$.

| $\beta = a + b\sigma_m$, $d = \gcd(a, b)$ | Integer solutions $(a, b)$ |
|---|---|
| Case 1: $b = 0$ | $\dfrac{4 + 2\sqrt{3} + 3\sqrt{(9-m)/4}}{4} \le a < 2 + \sqrt{\dfrac{9-m}{4}}$ and $b = 0$ |
| Case 2: $b \neq 0$ and $d = 1$ | |
| Subcase 2.1: $\lvert b\rvert \ge \lvert a\rvert$ | none |
| Subcase 2.2: $\lvert b\rvert < \lvert a\rvert$ | $4 \le a < \dfrac{\sqrt{8\sqrt{9-m} + 25} + 1}{2}$, when $b = -1$, <br> $2 \le a < \dfrac{\sqrt{8\sqrt{9-m} + 25} - 1}{2}$, when $b = 1$ |

From Table 4, one can verify that if $b \ge |a|$ and $d = 1$, then $b \ge 11$ and

$$4.2 - \frac{b}{2} \approx 1 + \frac{\sqrt{3}}{2}\left(\frac{1 + \sqrt{1 + 4(11-1)}}{2}\right) - \frac{b}{2} \le 1 + \frac{\sqrt{3}}{2}\left(\frac{1 + \sqrt{1 + 4(b-1)}}{2}\right) - \frac{b}{2} \le a \le -1.$$

This implies that the number of possible values of $a$ is at most $\lfloor (b/2) - 4.2 \rfloor$, the greatest integer less than or equal to $(b/2) - 4.2$.

**Table 4.** Case B: $|\beta| \geq 2 + \sqrt{\dfrac{9-m}{4}}$.

| $\beta = a + b\sigma_m$, $d = \gcd(a, b)$ | Integer solutions $(a, b)$ |
|---|---|
| $b < \lvert a \rvert$ | none |
| $b \geq \lvert a \rvert$ and $d = 1$ | $b \geq 11$ and $1 + \dfrac{\sqrt{3}}{2}\left(\dfrac{1 + \sqrt{1 + 4(b-1)}}{2}\right) - \dfrac{b}{2} \leq a \leq -1$ |

## 4. Conclusions

Let $K = \mathbb{Q}(\sqrt{m})$ be an imaginary quadratic field with $O_K$ its ring of integers. In this paper, further irreducibility criteria for polynomials in $O_K[x]$ are established which extend the authors' earlier works (Theorems D and E). Moreover, elements of $\beta \in O_K$ that can be applied to the new criteria but not to the previous ones are also provided.

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## Conflicts of interest

All authors declare no conflicts of interest in this paper.

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