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Research article

On approximate vector variational inequalities and vector optimization problem using convexificator

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Abstract: In the present article, we study a vector optimization problem involving convexificator-based locally Lipschitz approximately convex functions and give some ideas for approximate efficient solutions. In terms of the convexificator, we approximate Stampacchia-Minty type vector variational inequalities and use them to describe an approximately efficient solution to the nonsmooth vector optimization problem. Moreover, we give a numerical example that attests to the credibility of our results.

Keywords: approximate efficient solutions; convexificator; nonsmooth vector optimization problem; vector variational inequality
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1. Introduction

Nonsmooth phenomena happen often in optimization theory, leading to the advancement of numerous concepts of subdifferentials and generalized directional derivatives. Convexificators generalize various well-known subdifferentials, such as Mordukhovich, Michel-Penot, and Clarke subdifferentials. In 1994, Demyanov [11] introduced and studied the concept of convexificators as an overview of the lower concave and upper convex approximations. Convexificators for positively homogeneous and locally Lipschitz functions were studied by Demyanov and Jeyakumar [12]. Recently, Golestani and Nobakhtian [14], Li and Zhang [22], Long and Huang [23] and Luu [24] used convexificators to get the best possible conditions for nonsmooth optimization problems. For more information on convexificators, see [10, 21, 25] and the references therein.

Approximation methods are very crucial in optimization theory because finding an exact solution is sometimes unattainable or computationally very expensive. As a result, approximate efficient solutions (AESs) can permeate the challenges given by computational flaws and modeling constraints. Mishra and Laha [27] using locally Lipschitz approximately convex functions gave the idea of AESs for a vector optimization problem (VOP) and by using approximate vector variational inequalities (VVIs) of the Minty and Stampacchia form about the Clarke subdifferentials describe these approximate efficient solutions. We refer the reader to [1, 6, 4, 3, 8, 7, 2, 33, 5, 15, 16, 19, 32] and the references therein for additional literature on approximation and its applications.

Laha and Mishra [20] define Stampacchia and Minty VVIs in terms of a convexificator and use them to identify necessary and sufficient criteria for a point to be a vector minimum point of the VOP. Mishra and Upadhyay [28] and Upadhyay et al. [30] identified connections between a nonsmooth vector optimization problems (NVOPs) and VVIs. Gupta and Mishra [17] introduced generalized approximate convex functions and related vector variational inequalities to vector optimization problems using Clarke's subdifferentials. Joshi [19] considered a VOP using locally Lipschitz convex maps and formulated a Minty and Stampacchia inequality. Motivated and inspired by the recent work in this direction, we show some relationships between nonsmooth VVI problems and NVOPs using a convexificator.

In Section 2, we review some notions and definitions that will be used in this paper. In Section 3, we formulate approximate Stampacchia-Minty type VVIs in terms of a convexificator and utilize them to describe an AES to the NVOP. A numerical example has also been shown to check the credibility of the main results.

2. Preliminaries

In this section, we review some concepts of nonsmooth analysis; for more details one may refer to [9]. Suppose $\mathbb{R}^n$ is the Euclidean space, $\mathbb{R}^n_+$ is its nonnegative orthant and $int\mathbb{R}^n_+$ is the positive orthant of $\mathbb{R}^n$. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and $\langle ., . \rangle$ be the notations for the extended real line and Euclidean inner product, respectively. Let (μ, ν) and $[\mu, \nu]$ be the notations for the open and closed line segments joining μ and ν, respectively.

For $\mu, \nu \in \mathbb{R}^n$, the conventions for the inequalities and equality are as follows:
$\mu \geqq \nu \;\Leftrightarrow\; \mu_i \geq \nu_i,\; i = 1, 2, 3, ..., n \;\Leftrightarrow\; \mu - \nu \in \mathbb{R}^n_+$;
$\mu > \nu \;\Leftrightarrow\; \mu_i > \nu_i,\; i = 1, 2, 3, ..., n \;\Leftrightarrow\; \mu - \nu \in int\mathbb{R}^n_+$;
$\mu \geq \nu \;\Leftrightarrow\; \mu_i \geq \nu_i,\; i = 1, 2, 3, ..., n$, but $\mu \neq \nu \;\Leftrightarrow\; \mu - \nu \in \mathbb{R}^n_+ \setminus \{0\}$.

In the sequel, let E be a nonempty subset of $\mathbb{R}^n$. First of all, we need the following definitions.

Definition 2.1. *[29] A function $\theta : E \to \mathbb{R}$ is called Lipschitz near $\mu \in E$ if*

$$\|\theta(\nu) - \theta(\omega)\| \leq k \,\|\nu - \omega\|$$

for some $k > 0$ and for all $\nu, \omega \in \mathcal{B}_\delta(\mu)$.

Definition 2.2. *Let $\theta : E \to \overline{\mathbb{R}}$ be a function, $\mu \in E$ and $\theta(\mu)$ be finite. The upper and lower Dini derivatives of θ at $\mu \in E$ in the direction $r \in \mathbb{R}^n$ are denoted by $\theta^+(\mu, r)$ and $\theta^-(\mu, r)$, respectively and are defined as follows:*

$$\theta^+(\mu, r) = \limsup_{\lambda \to 0} \frac{\theta(\mu + \lambda r) - \theta(\mu)}{\lambda},$$

$$\theta^{-}(\mu, r) = \liminf_{\lambda \to 0} \frac{\theta(\mu + \lambda r) - \theta(\mu)}{\lambda}.$$

Definition 2.3. *[18] Let $\theta : E \to \overline{\mathbb{R}}$ be a function, $\mu \in E$ and $\theta(\mu)$ be finite. Then θ is said to be*

(i) an upper convexificator $\partial^{*}\theta(\mu) \subseteq \mathbb{R}^{n}$ at $\mu \in E$ iff $\partial^{*}\theta(\mu)$ is closed and for every $r \in \mathbb{R}^{n}$ we have

$$\theta^{-}(\mu, r) \leq \sup_{\xi \in \partial^{*}\theta(\mu)} \langle \xi, r \rangle,$$

(ii) a lower convexificator $\partial_{*}\theta(\mu) \subseteq \mathbb{R}^{n}$ at $\mu \in E$ iff $\partial_{*}\theta(\mu)$ is closed and for every $r \in \mathbb{R}^{n}$ we have

$$\theta^{+}(\mu, r) \geq \inf_{\xi \in \partial_{*}\theta(\mu)} \langle \xi, r \rangle,$$

(iii) a convexificator $\partial^{*}_{*}\theta(\mu) \subseteq \mathbb{R}^{n}$ at $\mu \in E$ iff $\partial^{*}_{*}\theta(\mu)$ is both an upper and a lower convexificator of θ at μ.
Consequently, for every $r \in \mathbb{R}^{n}$, we have

$$\theta^{-}(\mu, r) \leq \sup_{\xi \in \partial^{*}_{*}\theta(\mu)} \langle \xi, r \rangle, \quad \theta^{+}(\mu, r) \geq \inf_{\xi \in \partial^{*}_{*}\theta(\mu)} \langle \xi, r \rangle.$$

We can extend the above definitions and properties to a locally Lipschitz vector-valued function $\theta : E \to \mathbb{R}^{p}$. We denote the components of θ by θ_i, $i \in N = \{1, 2, 3, ..., p\}$. The set $\partial^{*}_{*}\theta(\mu) = \partial^{*}_{*}\theta_1(\mu) \times \partial^{*}_{*}\theta_2(\mu) \times \partial^{*}_{*}\theta_3(\mu) \times ... \times \partial^{*}_{*}\theta_p(\mu)$ is the convexificator of θ at $\mu \in E$.

Now onwards, we assume that E is a nonempty convex set unless otherwise specified.

Definition 2.4. *Suppose $\theta : E \to \mathbb{R}^{p}$ is a locally Lipschitz map which permits a bounded convexificator $\partial^{*}_{*}\theta(\nu)$ at $\nu \in E$. Then θ is called ∂^{*}_{*}-approximate convex at $\mu_0 \in E$ if for every $\epsilon > 0$, $\exists\, \delta > 0$ satisfying*

$$\theta(\nu) - \theta(\mu) \geqq \langle \xi, \nu - \mu \rangle - e\|\nu - \mu\|, \ \forall\, \xi \in \partial^{*}_{*}\theta(\mu), \ \forall\, \mu, \nu \in \mathcal{B}_{\delta}(\mu_0),$$

where $e = \underbrace{(\epsilon, \epsilon, \ldots, \epsilon)}_{p} \in int\mathbb{R}^{p}_{+}$.

3. Approximate Minty and Stampacchia vector variational inequalities

We study the NVOP as follows:

$$\text{Minimize } \left\{\theta(\mu) = (\theta_1(\mu), \theta_2(\mu), ..., \theta_p(\mu))\right\} \text{ such that } \mu \in E,$$

where $\theta_i : E \to \mathbb{R},\ i = 1, 2, 3, ..., p$ are nondifferentiable locally Lipschitz functions on E and ∂^{*}_{*}-approximately convex at $\nu \in E$.

Definition 3.1. *[28] Let $\theta : E \to \mathbb{R}^{p}$ be a function. Then a vector $\nu \in E$ is as follows:*

(i) an efficient solution to the NVOP if $\exists$ no $\mu \in E$ such that $\theta(\mu) \leq \theta(\nu)$,

(ii) a local weak efficient solution to the NVOP if $\exists$ a $\delta > 0$ such that the following inequality does not hold

$$\theta(\mu) < \theta(\nu), \ \forall\, \mu \in E \cap \mathcal{B}_{\delta}(\nu).$$

The following notions of AESs were presented by Mishra and Laha [26]. These concepts are helpful when the existence of an efficient solution cannot be shown.

Definition 3.2. *Let $\theta : E \to \mathbb{R}^p$ be a function. A vector $\nu \in E$ is called an*

(i) AES of kind I of the NVOP, represented by $(AES)_1$, if and only if for some $\epsilon > 0$ however small, $\exists$ no $\delta > 0$ satisfying
$$\theta(\mu) - \theta(\nu) \leq e\|\mu - \nu\|, \ \forall \ \mu \in \mathcal{B}_\delta(\nu) \setminus \{\nu\},$$
where $e = \underbrace{(\epsilon, \epsilon, \ldots, \epsilon)}_{p} \in int\mathbb{R}^p_+$.

(ii) AES of kind II of the NVOP, represented by $(AES)_2$, if and only if for some $\epsilon > 0$ however small, $\exists$ a $\delta > 0$ satisfying
$$\theta(\mu) - \theta(\nu) \nleq e\|\mu - \nu\|, \ \forall \ \mu \in \mathcal{B}_\delta(\nu),$$

(iii) AES of kind III of the NVOP, represented by $(AES)_3$, if and only if for some $\epsilon > 0$, $\exists$ a $\delta > 0$ satisfying
$$\theta(\mu) - \theta(\nu) \nleq -e\|\mu - \nu\|, \ \forall \ \mu \in \mathcal{B}_\delta(\nu).$$

It is evident that $(AES)_2$ of the NVOP is equivalent to both $(AES)_1$ and $(AES)_3$ of the NVOP. A concept of local efficiency is that $(AES)_3$ is considered to be in a weaker state than one that is $(AES)_2$, which is considered to be in a stronger state. Additionally, $(AES)_1$ has the potential to be considered a quasi-efficient solution to the NVOP.

Example 3.3. *Suppose the optimization problem in the sense of a multiobjective (MOP) is given as*
$$\textit{minimize } \theta(\mu) = (\theta_1(\mu), \theta_2(\mu)), \ \textit{such that } \mu \in \mathbb{R},$$
where $\theta_1(\mu) = |sin\mu|$ and $\theta_2(\mu) = |\mu| - \mu^2$. Suppose $\nu = 0$; for any $\epsilon > 0$, we have
$$\theta_1(\mu) - \theta_1(\nu) + \epsilon\|\mu - \nu\| \geq |sin\mu| \ \geq 0, \ \forall \ \mu \in [-\pi, \pi],$$
$$\theta_2(\mu) - \theta_2(\nu) + \epsilon\|\mu - \nu\| \geq |\mu| - \mu^2 \ \geq 0, \ \forall \ \mu \in [-1, 1].$$
Choose $0 < \delta < 1$; we have
$$\theta(\mu) - \theta(\nu) + e\|\mu - \nu\| \notin -\mathbb{R}^2_+ \setminus \{0\}, \ \forall \ \mu \in \ \mathcal{B}_\delta(0).$$
Thus $\nu = 0$ is an $(AES)_3$ of the MOP.

Example 3.4. *Let $\theta_1(\mu) = |\mu| - \mu^4$ and $\theta_2(\mu) = |\mu|^3 - \mu^2$. Suppose $\nu = 0$; for any $0 < \epsilon < 1$, we have*
$$\theta_1(\mu) - \theta_1(\nu) - \epsilon\|\mu - \nu\| = |\mu| - \mu^4 - \epsilon|\mu| \ \geq 0, \ \forall \ \mu \in [-(1 - \epsilon), (1 - \epsilon)],$$
$$\theta_2(\mu) - \theta_2(\nu) - \epsilon\|\mu - \nu\| = |\mu|^3 - \mu^2 - \epsilon|\mu| \ \geq 0, \ \forall \ \mu \geq \frac{1 + \sqrt{1 + 4\epsilon}}{2}.$$
Choose $0 < \delta < (1 - \epsilon)$; we have
$$\theta(\mu) - \theta(\nu) - e\|\mu - \nu\| \notin -\mathbb{R}^2_+ \setminus \{0\}, \ \forall \ \mu \in \ \mathcal{B}_\delta(0).$$
Thus $\nu = 0$ is an $(AES)_2$ of the MOP. In the same way, we can find that $\mu = 1$ is also an $(AES)_2$.

Example 3.5. *By taking $\nu = 0$, in Example 3.3, and following the process of Example 3.4, it can be proved that, for $0 < \epsilon < 1$ and for some $\delta > 0$, $\exists\, \mu \in \mathcal{B}_\delta(\nu) \setminus \{\nu\}$ satisfying*

$$\theta(\mu) - \theta(\nu) - e\|\mu - \nu\| \notin -\mathbb{R}^2_+ \setminus \{0\}.$$

Hence $\nu = 0$ is an $(AES)_1$ of the MOP.

Now, we present VVI problems of the Minty type in terms of the convexificator, which has been utilized to describe an AES of the NVOP in the next section.

$(AMVVI)_1$: Find $\nu \in E$ such that, for some $\epsilon > 0$ however small, $\exists$ no $\delta > 0$ satisfying

$$\langle \xi, \mu - \nu \rangle \leq e\|\mu - \nu\|, \ \forall\, \mu \in \mathcal{B}_\delta(\nu) \setminus \{\nu\} \text{ and } \xi \in \partial^*_*\theta(\mu).$$

$(AMVVI)_2$: Find $\nu \in E$ such that, for some $\epsilon > 0$ however small, $\exists$ a $\delta > 0$ satisfying

$$\langle \xi, \mu - \nu \rangle \nleq e\|\mu - \nu\|, \ \forall\, \mu \in \mathcal{B}_\delta(\nu) \text{ and } \xi \in \partial^*_*\theta(\mu).$$

$(AMVVI)_3$: Find $\nu \in E$ such that, for some $\epsilon > 0$, $\exists$ a $\delta > 0$ satisfying

$$\langle \xi, \mu - \nu \rangle \nleq -e\|\mu - \nu\|, \ \forall\, \mu \in \mathcal{B}_\delta(\nu) \text{ and } \xi \in \partial^*_*\theta(\mu).$$

The next theorem illustrates the relationship between solutions of VVIs of the Stampacchia kind and an AES of the NVOP.

Theorem 3.6. *Consider $\theta : E \to \mathbb{R}^p$ to be a locally Lipschitz function on E, which permits a bounded convexificator $\partial^*_*\theta(\nu)$ at $\nu \in E$. Suppose θ is ∂^*_*-approximately convex function at $\nu \in E$; then,*

(i) if $\nu \in E$ is an $(AES)_1$ of the NVOP, then ν is also a solution of the $(AMVVI)_1$,
(ii) if $\nu \in E$ is a solution of the $(AMVVI)_2$, then ν is also an $(AES)_2$ of the NVOP,
(iii) if $\nu \in E$ is an $(AES)_3$ of the NVOP, then ν is also a solution of the $(AMVVI)_3$.

Proof. (i) Assume that ν is not a solution of the $(AMVVI)_1$. It follows that, for any $\overline{\epsilon} > 0$ however small, $\exists$ a $\tilde{\delta} > 0$ satisfying

$$\langle \xi, \mu - \nu \rangle \leq \frac{\overline{e}}{2}\, \|\mu - \nu\|, \ \forall\, \mu \in \mathcal{B}_{\tilde{\delta}}(\nu) \text{ and } \xi \in \partial^*_*\theta(\mu),$$

where $\overline{e} = \underbrace{(\overline{\epsilon}, \overline{\epsilon}, \ldots, \overline{\epsilon})}_{p} \in \text{int}\mathbb{R}^p_+$.

Since θ is ∂^*_*-approximately convex at $\nu \in E$ and a locally Lipschitz function on E, then, for every $\epsilon > 0$, $\exists$ a $\check{\delta} > 0$ satisfying

$$\theta(\nu) - \theta(\mu) \geqq \langle \xi, \nu - \mu \rangle - \frac{e}{2}\, \|\nu - \mu\|, \ \forall\, \mu \in \mathcal{B}_{\check{\delta}}(\nu), \text{ and } \xi \in \partial^*_*\theta(\mu).$$

Taking $\hat{\delta} = \inf\{\tilde{\delta}, \check{\delta}\}$, we have, for any $\overline{\epsilon} > 0$ however small, $\exists$ a $\hat{\delta} > 0$ satisfying

$$\theta(\mu) - \theta(\nu) \leq \overline{e}\, \|\mu - \nu\|, \ \forall\, \mu \in \mathcal{B}_{\hat{\delta}}(\nu),$$

which is a contradiction that $\nu \in E$ is an $(AES)_1$ of the NVOP.

□

Proof. (ii) Assume that $\nu \in E$ is not an $(AES)_2$ of the NVOP. Then, for some $\overline{\epsilon} > 0$ however small and for every $\delta > 0$, we have

$$\theta(\mu) - \theta(\nu) \leq \frac{\overline{e}}{2} \|\mu - \nu\|, \text{ for some } \mu \in \mathcal{B}_\delta(\nu) \setminus \{\nu\}.$$

Take $\hat{\delta} > 0$ such that the condition of $(AMVVI)_2$ holds. Thus, for this $\hat{\delta}$, suppose $\mu_0 \in \mathcal{B}_{\hat{\delta}}(\nu)$ is such that

$$\theta(\mu_0) - \theta(\nu) \leq \frac{\overline{e}}{2} \|\mu_0 - \nu\|. \tag{3.1}$$

Applying the mean value theorem, there exists $\xi_0 \in co(\partial^*_*\theta([\nu, \mu_0]))$ such that

$$\theta(\mu_0) - \theta(\nu) = \langle \xi_0, \mu_0 - \nu \rangle . \tag{3.2}$$

Accordingly there exist $\mu_1, \mu_2, ..., \mu_k$ from the open segment (ν, μ_0), $\xi_1 \in \partial^*_*\theta(\mu_1)$, $\xi_2 \in \partial^*_*\theta(\mu_2), ..., \xi_k \in \partial^*_*\theta(\mu_k)$, $0 < \beta_1 < \beta_2 < ... < \beta_k < 1$, $\alpha_1, \alpha_2, ..., \alpha_k > 0$ such that

$$\alpha_1 + \alpha_2 + ... + \alpha_k = 1,$$

$$\xi_0 = \alpha_1\xi_1 + \alpha_2\xi_2 + ... + \alpha_k\xi_k = \xi_1 + \sum_{i=2}^{k} \alpha_i(\xi_i - \xi_1),$$

$$\mu_i - \nu = \beta_i(\mu_0 - \nu), \ i = 1, 2, ..., k. \tag{3.3}$$

Using ∂^*_*-approximate convexity of θ at $\nu \in E$, it follows that

$$\langle \xi_i - \xi_1, \mu_i - \mu_1 \rangle \geqq -\frac{e}{2}\|\mu_i - \mu_1\|, \ \forall \ i = 1, 2, ..., k. \tag{3.4}$$

Owing to (3.1)–(3.3), we obtain

$$\begin{aligned} \langle \xi_0, \mu_0 - \nu \rangle &= \langle \xi_1, \mu_0 - \nu \rangle + \sum_{i=2}^{k} \alpha_i \ \langle \xi_i - \xi_1, \mu_0 - \nu \rangle \\ &= \frac{1}{\beta_1} \langle \xi_1, \mu_1 - \nu \rangle + \sum_{i=2}^{k} \frac{\alpha_i}{\beta_i - \beta_1} \ \langle \xi_i - \xi_1, \mu_i - \mu_1 \rangle \\ &\leq \frac{\overline{e}}{2} \|\mu_0 - \nu\|. \end{aligned}$$

From (3.4), it follows that

$$\frac{1}{\beta_1} \langle \xi_1, \mu_1 - \nu \rangle \leq \frac{\overline{e}}{2} \|\mu_0 - \nu\| + \frac{\overline{e}}{2} \sum_{i=2}^{k} \frac{\alpha_i}{\beta_i - \beta_1} \|\mu_i - \mu_1\|,$$

which in lieu of (3.3) becomes

$$\langle \xi_1, \mu_1 - \nu \rangle \leq \frac{\overline{e}}{2} \left(1 + \sum_{i=2}^{k} \alpha_i\right) \|\mu_1 - \nu\| \leqq \overline{e}\|\mu_1 - \nu\|,$$

with $\mu_1 \in [\nu, \mu_0] \subset \mathcal{B}_{\hat{\delta}}(\nu)$ and $\xi_1 \in \partial^*_*\theta(\mu_1)$, which is a contradiction that $\nu \in E$ is a solution of the $(AMVVI)_2$. □

Proof. (iii) Suppose ν is an $(AES)_3$ of the NVOP. Then, for every $\epsilon > 0, \exists\, \tilde{\delta} > 0$ satisfying

$$\theta(\mu) - \theta(\nu) \nleq -\frac{e}{2}\, \|\mu - \nu\|,\ \forall\, \mu \in \mathcal{B}_{\tilde{\delta}}(\nu).$$

Since θ is ∂^*_*-approximately convex at $\nu \in E$ and a locally Lipschitz function on E, we have, for a given $\epsilon > 0,\ \exists\, \check{\delta} > 0$ satisfying

$$\theta(\nu) - \theta(\mu) \geqq \langle \xi, \nu - \mu \rangle - \frac{e}{2}\, \|\nu - \mu\|,\ \forall\, \mu \in \mathcal{B}_{\check{\delta}}(\nu),\ \text{and } \xi \in \partial^*_*\theta(\mu).$$

Taking $\hat{\delta} = \inf\{\tilde{\delta}, \check{\delta}\}$, we have that, for every $\epsilon > 0,\ \exists\, \hat{\delta} > 0$ satisfying

$$\langle \xi, \mu - \nu \rangle \nleq -e\, \|\mu - \nu\|,\ \forall\, \mu \in \mathcal{B}_{\hat{\delta}}(\nu) \text{ and } \xi \in \partial^*_*\theta(\mu).$$

Hence $\nu \in E$ is a solution of the $(AMVVI)_3$. □

Now, we define an approximation of the Stampacchia VVI problems by expressing them about the convexificator.

$(ASVVI)_1$: Find $\nu \in E$ so that, for any $\epsilon > 0$ however small, there are some $\mu \in E \setminus \{\nu\}$ and $\xi_0 \in \partial^*_*\theta(\nu)$ satisfying

$$\langle \xi_0, \mu - \nu \rangle \nleq e\|\mu - \nu\|.$$

$(ASVVI)_2$: Find $\nu \in E$ so that, for any $\epsilon > 0$ however small, for every $\mu \in E$ and $\xi_0 \in \partial^*_*\theta(\nu)$, we have

$$\langle \xi_0, \mu - \nu \rangle \nleq e\|\mu - \nu\|.$$

$(ASVVI)_3$: Find $\nu \in E$ so that, for some $\epsilon > 0,\ \exists\, \delta > 0$ satisfying

$$\langle \xi_0, \mu - \nu \rangle \nleq -e\|\mu - \nu\|,\ \forall\, \mu \in \mathcal{B}_{\delta}(\nu) \text{ and } \xi_0 \in \partial^*_*\theta(\nu).$$

The following theorem gives the conditions under which a solution of the ASVVI is an AES of the NVOP.

Theorem 3.7. *Let $f : E \to \mathbb{R}^p$ be a locally Lipschitz function on E which permits a bounded convexificator $\partial^*_*\theta(\nu)$ at $\nu \in E$. Suppose θ is a ∂^*_*-approximately convex function at $\nu \in E$. Then the following hold:*

(i) If $\nu \in E$ is a solution of the $(ASVVI)_1$, then ν is also an $(AES)_1$ of the NVOP,
(ii) If $\nu \in E$ is a solution of the $(ASVVI)_2$, then ν is also an $(AES)_2$ of the NVOP,
(iii) If $\nu \in E$ is a solution of the $(ASVVI)_3$, then ν is also an $(AES)_3$ of the NVOP.

Proof. (i) Let $\nu \in E$ is a solution of the $(ASVVI)_1$ and suppose that ν is not an $(AES)_1$ of the NVOP. Then, there exist $\epsilon > 0$ and $\delta > 0$ satisfying

$$\theta(\mu) - \theta(\nu) \leq e\|\mu - \nu\|,\ \forall\, \mu \in \mathcal{B}_{\delta}(\nu),\ \mu \neq \nu.$$

Since θ is ∂^*_*-approximately convex, it follows that there is $\check{\delta} < \delta$ such that

$$\langle \xi_0, \mu - \nu \rangle \leqq \theta(\mu) - \theta(\nu)$$

for all $\mu \in \mathcal{B}_{\check{\delta}}(\nu)$ and $\xi_0 \in \partial^*_*\theta(\nu)$. Then

$$\langle \xi_0, \mu - \nu \rangle \leq e\| \mu - \nu\|, \ \forall \ \mu \in \mathcal{B}_\delta(\nu), \ \mu \neq \nu \text{ and hence for all } \mu \in E \setminus \{\nu\},$$

which contradicts the hypothesis that $\nu \in E$ is a solution of $(ASVVI)_1$.
(ii) Suppose $\nu \in E$ is a solution of the $(ASVVI)_2$; then, ν is a solution of the $(AMVVI)_2$. By Theorem 3.6, $\nu \in E$ is an $(AES)_2$ of the NVOP. Indeed, for every $\epsilon > 0$, for all $\mu \in E$ and $\xi_0 \in \partial^*_*\theta(\nu)$, we have

$$\langle \xi_0, \mu - \nu \rangle \nleq \frac{e}{2} \ \|\mu - \nu\|.$$

Since θ is ∂^*_*-approximately convex, for μ however close to ν, we have

$$\langle \xi - \xi_0, \mu - \nu \rangle \geq -\frac{e}{2} \ \|\mu - \nu\|, \ \forall \ \xi \in \partial^*_*\theta(\mu).$$

Consequently,

$$\langle \xi, \mu - \nu \rangle \nleq e\|\mu - \nu\|$$

for all μ in a small neighborhood of ν and for all $\xi \in \partial^*_*\theta(\mu)$. Hence $\nu \in E$ is an $(AES)_2$ of the NVOP.
(iii) Suppose ν is a solution of the $(ASVVI)_3$. Therefore, for every $\epsilon > 0$, $\exists \ \tilde{\delta} > 0$ satisfying

$$\langle \xi_0, \mu - \nu \rangle \nleq -\frac{e}{2} \ \|\mu - \nu\|, \ \forall \ \mu \in \mathcal{B}_{\tilde{\delta}}(\nu) \text{ and } \xi_0 \in \partial^*_*\theta(\nu).$$

Since θ is ∂^*_*-approximately convex at $\nu \in E$ and a locally Lipschitz function on E, we have, for every $\epsilon > 0$, there exists $\check{\delta} > 0$ satisfying

$$\theta(\mu) - \theta(\nu) \geqq \langle \xi_0, \mu - \nu \rangle - \frac{e}{2} \ \|\mu - \nu\|, \ \forall \ \mu \in \mathcal{B}_{\check{\delta}}(\nu), \text{ and } \xi_0 \in \partial^*_*\theta(\nu).$$

Taking $\hat{\delta} = \inf\{\tilde{\delta}, \check{\delta}\}$, we have, for every $\epsilon > 0$, there exists $\hat{\delta} > 0$ satisfying

$$\theta(\mu) - \theta(\nu) \nleq -e \ \|\mu - \nu\|, \ \forall \ \mu \in \mathcal{B}_{\hat{\delta}}(\nu).$$

Hence $\nu \in E$ is an $(AES)_3$ of NVOP. □

The authenticity of the main results is shown in the following example:

Example 3.8. *Consider the NVOP as follows:*

$$\min \theta(\mu) = (\theta_1(\mu), \theta_2(\mu)), \ \textit{subject to } \mu \in \ \mathbb{R},$$

where

$$\theta_1(\mu) = \begin{cases} 2\mu + 1, & \textit{if } \ \mu \geq 0, \\ 2\mu + e^\mu, & \textit{if } \ \mu < 0, \end{cases}$$

and

$$\theta_2(\mu) = \begin{cases} 4\mu - \mu^2, & \textit{if } \ \mu \geq 0, \\ 2\mu, & \textit{if } \ x < 0. \end{cases}$$

The convexificator of θ_1 and θ_2 at μ are defined as follows:

$$\partial_*^*\theta_1(\mu) = \begin{cases} 2, & if\ \mu > 0, \\ [2,3], & if\ \mu = 0, \\ 2 + e^{\mu}, & if\ \mu < 0, \end{cases}$$

and

$$\partial_*^*\theta_2(\mu) = \begin{cases} 4 - 2\mu, & if\ \mu > 0, \\ [2,4], & if\ \mu = 0, \\ 2, & if\ \mu < 0. \end{cases}$$

Suppose $e = (\epsilon, \epsilon)$ for $\epsilon > 0$ and take $\delta = min(1, \frac{\epsilon}{2})$ such that for any $\mu, \nu \in \mathcal{B}_\delta(0)$, $\xi_1 \in \partial_*^*\theta_1(\mu)$ and $\xi_2 \in \partial_*^*\theta_2(\mu)$, we have

$$\theta_1(\nu) - \theta_1(\mu) = \begin{cases} 2(\nu - \mu), & if\, \mu > 0, \nu > 0, \nu - \mu > 0, \\ 2\nu + 1 - 2\mu - e^{\mu}, & if\, \mu < 0, \nu > 0, \\ 2(\nu - \mu) + e^{\nu} - e^{\mu}, & if\, \mu < 0, \mu < 0, \nu - \mu > 0, \\ 2\nu, & if\, \mu = 0, \nu > 0, \end{cases}$$

and

$$\theta_2(\nu) - \theta_2(\mu) = \begin{cases} (\nu - \mu)(4 - \nu - \mu), & if\, \mu > 0, \nu > 0, \nu - \mu > 0, \\ 4\nu - \nu^2 - 2\mu, & if\, \mu < 0, \nu > 0, \\ 2(\nu - \mu), & if\, \mu < 0, \nu < 0, \nu - \mu > 0, \\ 4\nu - \nu^2, & if\, \mu = 0, \nu > 0. \end{cases}$$

Also,

$$\langle \xi_1, \nu - \mu \rangle - \epsilon \|\, \nu - \mu \| = \begin{cases} (4 - 2\mu)(\nu - \mu) - \epsilon\|\nu - \mu\| > 0, & if\, \mu > 0, \nu > 0, \nu - \mu > 0, \\ (4 - 2\mu)(\nu - \mu) - \epsilon\|\nu - \mu\| < 0, & if\, \mu > 0, \nu > 0, \nu - \mu < 0, \\ (4 - 2\mu)(\nu - \mu) - \epsilon\|\nu - \mu\| < 0, & if\, \mu > 0, \nu \le 0, \\ 2(\nu - \mu) - \epsilon\|\nu - \mu\| > 0, & if\, \mu > 0, \nu > 0, \nu - \mu > 0, \\ 2(\nu - \mu) - \epsilon\|\nu - \mu\| < 0, & if\, \mu > 0, \nu > 0, \nu - \mu < 0, \\ 2(\nu - \mu) - \epsilon\|\nu - \mu\| > 0, & if\, \mu > 0, \nu \le 0, \\ r_1(\nu - \mu) - \epsilon\|\nu - \mu\| > 0, & if\, \mu = 0, \nu > 0,\ r_1 \in [2,3], \\ r_2(\nu - \mu) - \epsilon\|\nu - \mu\| < 0, & if\, \mu = 0, \nu < 0,\ r_2 \in [2,3], \end{cases}$$

and

$$\langle \xi_2, \nu - \mu \rangle - \epsilon \|\, \nu - \mu \| = \begin{cases} 2(\nu - \mu) - \epsilon\|\nu - \mu\| > 0, & if\, \mu > 0, \nu > 0, \nu - \mu > 0, \\ 2(\nu - \mu) - \epsilon\|\nu - \mu\| < 0, & if\, \mu > 0, \nu > 0, \nu - \mu < 0, \\ 2(\nu - \mu) - \epsilon\|\nu - \mu\| < 0, & if\, \mu > 0, \nu \le 0, \\ (2 + e^{\mu})(\nu - \mu) - \epsilon\|\nu - \mu\| > 0, & if\, \mu > 0, \nu > 0, \nu - \mu > 0, \\ (2 + e^{\mu})(\nu - \mu) - \epsilon\|\nu - \mu\| < 0, & if\, \mu > 0, \nu > 0, \nu - \mu < 0, \\ (2 + e^{\mu})(\nu - \mu) - \epsilon\|\nu - \mu\| > 0, & if\, \mu > 0, \nu \le 0, \\ r_1(\nu - \mu) - \epsilon\|\nu - \mu\| > 0, & if\, \mu = 0, \nu > 0,\ r_1 \in [2,4], \\ r_2(\nu - \mu) - \epsilon\|\nu - \mu\| < 0, & if\, \mu = 0, \nu < 0,\ r_2 \in [2,4]. \end{cases}$$

We can easily verify that $\theta(\nu) - \theta(\mu) \geqq \langle \xi, \nu - \mu \rangle - e\| \nu - \mu\|$. *Hence* θ *is* ∂^*_**-approximate convex at 0.*
Since, for any $\mu > 0$, $\mu \in \mathcal{B}_\delta(\mu_0)$, *then*

$$\langle \xi_{0_1}, \mu - \mu_0 \rangle + \epsilon\|\mu - \mu_0)\| = r_1\mu + \epsilon\|\mu\| > 0,\ r_1 \in [2,3]\,,$$

and

$$\langle \xi_{0_2}, \mu - \mu_0 \rangle + \epsilon\|\mu - \mu_0)\| = r_2\mu + \epsilon\|\mu\| > 0,\ r_2 \in [2,4]\,.$$

That is, $\langle \xi_0, \mu - \mu_0 \rangle + e\|\mu - \mu_0)\| \nleq 0$.
Hence $\mu_0 = 0$ *is a solution of* $(ASVVI)_3$.
Since, for every $\epsilon > 0$, *there exists* $\delta > 0$ *such that for every* $\mu \in \mathcal{B}_\delta(\mu_0)$ *and* $\mu > 0$, *we have*

$$\theta_1(\mu) - \theta_1(\mu_0) + \epsilon\|\mu - \mu_0\| = 2\mu + 1 + \epsilon\|\mu\| > 0,$$

and

$$\theta_2(\mu) - \theta_2(\mu_0) + \epsilon\|\mu - \mu_0\| = 4\mu - \mu^2 + \epsilon\|\mu\| > 0.$$

That is, $\theta(\mu) - \theta(\mu_0) + \epsilon\|\mu - \mu_0\| \nleq 0$.
Hence $\mu_0 = 0$ *is an* $(AES)_3$ *of the NVOP.*
Thus, Theorem 3.7 is verified.
Since, for any $\mu > 0$, $\mu \in \mathcal{B}_\delta(\mu_0)$, *then*

$$\langle \xi_1, \mu - \mu_0 \rangle + \epsilon\|\mu - \mu_0)\| = 2\mu + \epsilon\|\mu\| > 0,$$

and

$$\langle \xi_2, \mu - \mu_0 \rangle + \epsilon\|\mu - \mu_0)\| = 4\mu - 2\mu^2 + \epsilon\|\mu\| > 0.$$

That is, $\langle \xi, \mu - \mu_0 \rangle + e\|\mu - \mu_0)\| \nleq 0$.
Hence $\mu_0 = 0$ *is a solution of* $(AMVVI)_3$. *Thus, Theorem 3.6 is verified.*

Remark 3.9. *It is clear that in the above example, the convexificators* $\partial^*_*\theta_1$ *and* $\partial^*_*\theta_2$ *of* θ_1 *and* θ_2*, respectively, are strictly contained in the corresponding Clarke or Michel-Penot subdifferentials. Convexity in terms of convexificators, as opposed to other subdifferentials, is hence simpler to verify. Because of this, the results of our paper are easier to use.*

4. Conclusions

In this manuscript, we have constructed approximation Stampacchia-Minty type VVIs in terms of the convexificator and used them to describe an AES to the NVOP. It will also be fascinating to think about an MOP with an objective function between Hilbert spaces. This will be an interesting problem for the future research point of view. In addition, the associated equilibrium problem and its various iterations may be investigated by making slight adjustments to the methodologies described in this study. Thus, the results of the current manuscript are very useful and interesting.

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Conflicts of interest

The authors declare no conflicts of interest.

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