Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(10): 18793–18808.
DOI: 10.3934/math.20221034
Received: 02 June 2022
Revised: 21 July 2022
Accepted: 02 August 2022
Published: 23 August 2022

*Research article*

# Stability property of the boundary equilibria of a symbiotic model of commensalism and parasitism with harvesting in commensal populations

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**Abstract:** This article demonstrates the stability property of two boundary equilibria of a symbiotic model of commensalism and parasitism with harvesting in the commensal population. The model was proposed by Nurmaini Puspitasari, Wuryansari Muharini Kusumawinahyu, Trisilowati (2021). We first give two numeric examples to show that the corresponding results of the mentioned paper may be incorrect. Then, by analysis of the characteristic roots of the characteristic equations, we obtain sufficient conditions that ensure the locally asymptotic stability of the equilibria. After that, by applying the standard comparison theorem, some novel results on the global attractivity of these two equilibria are obtained respectively. Our results complement and supplement some known results.

**Keywords:** commensalism; parasitism; comparison theorem; global attractivity
**Mathematics Subject Classification:** 34C25, 92D25, 34D20, 34D40

## 1. Introduction

The aim of this paper is to present an investigation of the dynamic behaviors of the following symbiotic model of commensalism and parasitism with harvesting in the commensal population:

$$\begin{aligned}
\frac{dx}{dt} &= r_1x\Big(1-\frac{x}{k_1}+a\frac{y}{k_1}\Big)-\frac{qEx}{m_1E+m_2x},\\
\frac{dy}{dt} &= r_2y\Big(1-\frac{y}{k_2}-b\frac{z}{k_2}\Big),\\
\frac{dz}{dt} &= r_3z\Big(1-\frac{z}{k_3}+c\frac{y}{k_3}\Big),
\end{aligned} \tag{1.1}$$

where $x(t), y(t)$ and $z(t)$ denote the commensal population, host population and parasite species, respectively. All parameters used in this model are positive. The definitions of these parameters are as follows: $r_i, i = 1, 2, 3$ represents the intrinsic growth of $x, y$ and $z$; $k_i, i = 1, 2, 3$ show the carrying capacities of $x$, $y$ and $z$. The parameter $a$ is the interaction parameter for $x$ and $y$ and $b$ and $c$ the interaction parameters between $y$ and $z$. The parameter $E$ is the fishing effort parameter used to harvest, $q$ is the catching power coefficient and $m_1$ and $m_2$ are the suitable constants.

During the lase few decades, many scholars have investigated the dynamic behaviors of the commensalism model [1–38]. Biologists have studied the phenomenon of commensalism through direct observation of wild species [7–9]. Mathematicians have studied the dynamic behavior of biased populations through hypothesis and theoretical derivation, with relevant topics including the influence of the Allee effect [1–6, 26–28], the influence of functional response [10, 13, 21, 22, 25, 28, 29], the influence of feedback controls [28–30], the influence of linear harvesting or Michaelis-Menten type harvesting [15, 16, 23, 31–38], the existence of a positive periodic solution or an almost periodic solution [11, 12, 14, 20, 22, 31, 36], the influence of stage structure [17], the stability of the system [10, 18, 32–35], the influence of nonlinear birth rates [19], the influence of noise [24], the influence of delays [25]; These topics have been extensively investigated and many important results were obtained. On the other hand, to meet human needs, the exploitation of biological resources and harvesting of populations are commonly practiced in fishery, forestry and wildlife management. Ecological modeling incorporates linear harvesting or Michaelis-Menten type harvesting, and it has become one of the main study topics in the area of population dynamics [15, 16, 23], [31–48]. Chen [38] was first to incorporate the Michaelis-Menten-type harvesting into the two species Lotka-Volterra commensalism model, and he investigated the local and global stability of the equilibria. Since then, many scholars have conducted works on commensalism systems with Michaelis-Menten type harvesting [31–38]. However, the commensalism models were not well studied in the sense that to this day, most of the works have focused on the two species case. Only recently did Puspitasari, Kusumawinahyu and Trisilowati [32, 33] begin to study three species and four species cases. In [32], Puspitasari, Kusumawinahyu and Trisilowati proposed System (1.1). The system has eight equilibria, which take the following forms:

$$T_0(0,0,0),\ \ T_1(0,0,k_3),\ \ T_2(0,k_2,0),\ \ T_3(x_3^*,0,0),$$
$$T_4\Big(0,\frac{k_2-bk_3}{1+bc},\frac{k_3+ck_2}{1+bc}\Big),\ \ T_5(x_5^*,0,k_3),\ \ T_6(x_6^*,k_2,0),\ \ T_7(x_7^*,y_7^*,z_7^*).$$

Concerned with the local stability property of those equilibria, the authors declared that “of the eight points, only two points are asymptotically stable if they meet certain conditions.” Indeed, they showed that $T_4$ and $T_7$ are locally asymptotically stable while the other six equilibria are all unstable. Note that from the third equation of System (1.1), we have that

$$\begin{aligned}\frac{dz}{dt} &= r_3z\Big(1-\frac{z}{k_3}+c\frac{y}{k_3}\Big)\\ &\geq r_3z\Big(1-\frac{z}{k_3}\Big).\end{aligned}$$

Hence,

$$\liminf_{t\to+\infty} z(t)\geq k_3.$$

That is, for any positive initial conditions, $z(t)$ can not approach 0 as $t \to +\infty$, which means that the equilibria $T_0, T_2, T_3$ and $T_6$ are always unstable. Hence, it is only necessary to investigate the stability property of the remaining four equilibria.

Now let us consider the following two examples.

**Example 1.1.** Consider the following system

$$\begin{aligned} \frac{dx}{dt} &= x\Big(1 - x + y\Big) - \frac{3x}{1+x}, \\ \frac{dy}{dt} &= y\Big(1 - y - 2z\Big), \\ \frac{dz}{dt} &= z\Big(1 - z + y\Big). \end{aligned} \tag{1.2}$$

Here, for System (1.1), we chose $r_i = k_i = E = c = a = 1, i = 1,2,3, b = 2, q = 3$ and $m_1 = m_2 = 1$. The results of numerical simulations (Figures 1–3) show that in this case, $T_1(0,0,1)$ is locally asymptotically stable.

**Example 1.2.** Consider the following system

$$\begin{aligned} \frac{dx}{dt} &= x\Big(1 - x + y\Big) - \frac{0.1x}{1+x}, \\ \frac{dy}{dt} &= y\Big(1 - y - 2z\Big), \\ \frac{dz}{dt} &= z\Big(1 - z + y\Big). \end{aligned} \tag{1.3}$$

Here, all the coefficients are the same as that of System (1.2), only with $q$ changing from 3 to 0.1. The results of numerical simulations (Figures 4–6) show that in this case, $T_5(x_5^*, 0, k_3) = (0.949, 0, 1)$ is locally asymptotically stable.

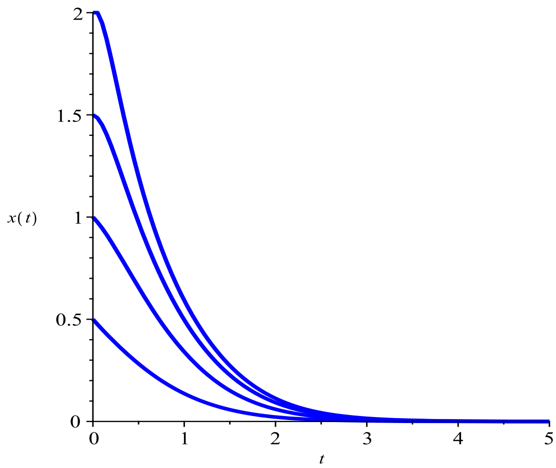


**Figure 1.** Dynamic behaviors of the first component $x$ in System (1.2) with the initial conditions $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$, $(1, 1, 1)$, $(1.5, 1.5, 1.5)$ and $(2, 2, 2)$, respectively.

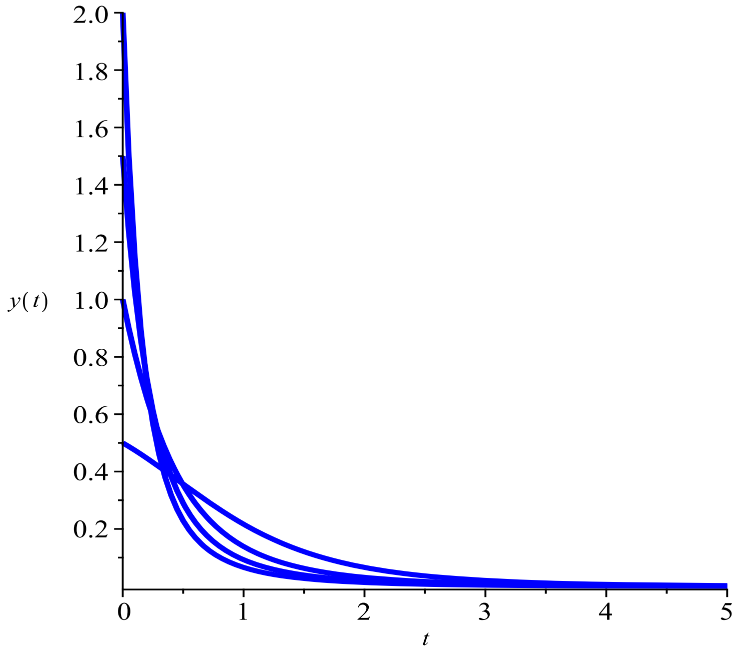


**Figure 2.** Dynamic behaviors of the second component $y$ in System (1.2) with the initial conditions $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$, $(1, 1, 1)$, $(1.5, 1.5, 1.5)$ and $(2, 2, 2)$, respectively.

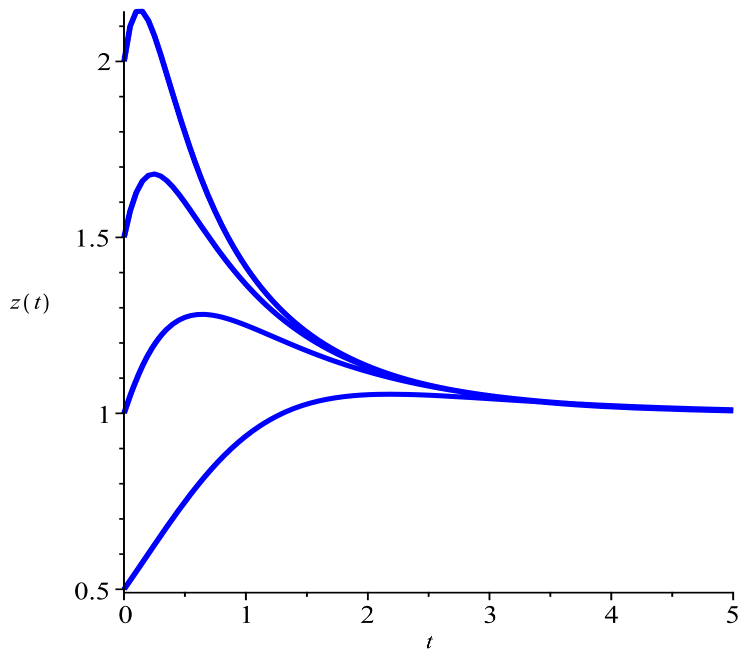


**Figure 3.** Dynamic behaviors of the third component $z$ in System (1.2) with the initial conditions $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$, $(1, 1, 1)$, $(1.5, 1.5, 1.5)$ and $(2, 2, 2)$, respectively.

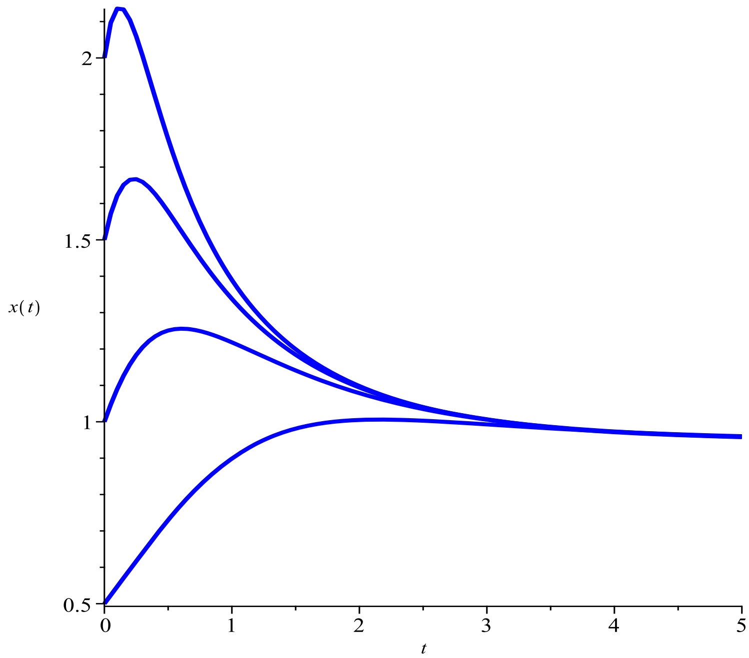


**Figure 4.** Dynamic behaviors of the first component $x$ in System (1.3) with the initial conditions $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$, $(1, 1, 1)$, $(1.5, 1.5, 1.5)$ and $(2, 2, 2)$, respectively.

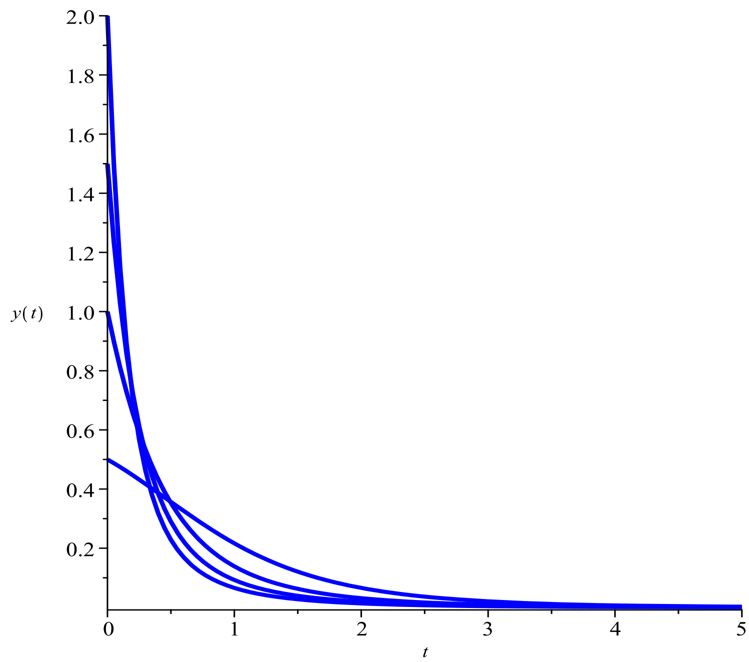


**Figure 5.** Dynamic behaviors of the second component $y$ in System (1.3) with the initial conditions $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$, $(1, 1, 1)$, $(1.5, 1.5, 1.5)$ and $(2, 2, 2)$, respectively.

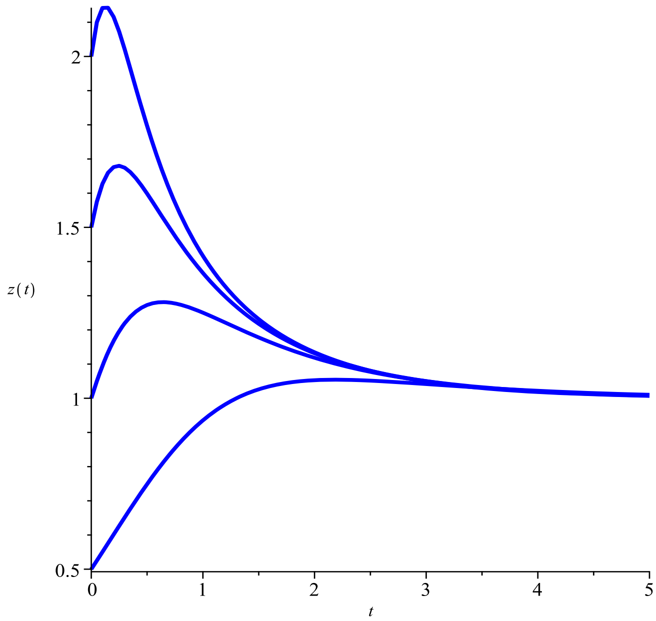


**Figure 6.** Dynamic behaviors of the third component $z$ in System (1.3) with the initial conditions $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$, $(1, 1, 1)$, $(1.5, 1.5, 1.5)$ and $(2, 2, 2)$, respectively.

The above two examples show that in System (1.1), $T_1(0, 0, k_3)$ and $T_5(x_5^*, 0, k_3)$ are all possibly asymptotically stable. The conclusion of Puspitasari, Kusumawinahyu and Trisilowati [32] may be incorrect.

Now, one natural problem is to obtain sufficient conditions to ensure the local asymptotic stability of the equilibria $T_1(0, 0, k_3)$ and $T_5(x_5^*, 0, k_3)$. Another interesting thing is that the conclusions of Puspitasari, Kusumawinahyu and Trisilowati [32] are all local ones; thus, we wanted to to whether we could obtain some sufficient conditions to ensure the globally stability property of the equilibrium.

The aim of this study was to give definitive answers to the above two problems.

## 2. Local stability of $T_1$ and $T_5$

We will investigate the local stability property of $T_1(0, 0, k_3)$ and $T_5(x_5^*, 0, k_3)$ in this section.

The variational matrix of System (1.1) is

$$J(x_1, x_2, y) = \begin{pmatrix} F_{11} & \frac{r_1 x a}{k_1} & 0 \\ 0 & F_{22} & -\frac{r_2 y b}{k_2} \\ 0 & \frac{r_3 z c}{k_3} & F_{33} \end{pmatrix}, \tag{2.1}$$

where

$$\begin{aligned} F_{11} &= r_1\left(1 - \frac{x}{k_1} + \frac{ay}{k_1}\right) - \frac{r_1 x}{k_1} - \frac{qE}{m_1E + m_2x} + \frac{qExm_2}{(m_1E + m_2x)^2}, \\ F_{22} &= r_2\left(1 - \frac{y}{k_2} - \frac{bz}{k_2}\right) - \frac{r_2 y}{k_2}, \\ F_{33} &= r_3\left(1 - \frac{z}{k_3} + \frac{cy}{k_3}\right) - \frac{zr_3}{k_3}. \end{aligned}$$

**Theorem 2.1.** *Assume that*

$$r_1 < \frac{q}{m_1} \tag{2.2}$$

*and*

$$1 < \frac{bk_3}{k_2} \tag{2.3}$$

*hold, then,* $T_1(0, 0, k_3)$ *is locally asymptotically stable.*

*Proof.* The Jacobian matrix of the equilibrium point $T_1(0, 0, k_3)$ is given by

$$\begin{pmatrix} r_1 - \frac{q}{m_1} & 0 & 0 \\ 0 & r_2\left(1 - \frac{bk_3}{k_2}\right) & 0 \\ 0 & cr_3 & -r_3 \end{pmatrix}.$$

The characteristic equation of the above matrix is

$$\left(\lambda - (r_1 - \frac{q}{m_1})\right)\left(\lambda - r_2\left(1 - \frac{bk_3}{k_2}\right)\right)(\lambda + r_3) = 0.$$

Hence, the characteristic roots are

$$\lambda_1 = r_1 - \frac{q}{m_1} < 0, \lambda_2 = r_2\left(1 - \frac{bk_3}{k_2}\right) < 0, \lambda_3 = -r_3 < 0.$$

Consequently, $T_1(0, 0, k_3)$ is locally asymptotically stable. This ends the proof of Theorem 2.1.

**Remark 2.1.** In Example 1.1, one could easily check that

$$r_1 = 1 < \frac{q}{m_1} = 3$$

and

$$1 < \frac{bk_3}{k_2} = 2.$$

Hence, it follows from Theorem 2.1 that $T_1(0, 0, 1)$ is locally asymptotically stable, which is consistence with the numerical simulations (Figures 1–3).

**Theorem 2.2.** *Assume that*

$$q < r_1 m_1, \tag{2.4}$$

$$\frac{qm_2}{m_1^2 E} < \frac{r_1}{k_1} \tag{2.5}$$

*and*

$$1 < \frac{bk_3}{k_2} \tag{2.6}$$

*hold, then,* $T_5(x_5^*, 0, k_3)$ *is locally asymptotically stable.*

*Proof.* $x_5^*$ satisfies the equation

$$r_1\Big(1 - \frac{x}{k_1}\Big) - \frac{qE}{m_1E + m_2x} = 0. \tag{2.7}$$

It is easy to verify that under the assumption that Eq (2.4) holds, Eq (2.7) admits a unique positive equilibrium, indeed, $x_5^*$ could be expressed as follows:

$$x_5^* = \frac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1}, \tag{2.8}$$

where

$$B_1 = r_1m_2, \;\; B_2 = r_1m_1E - k_1m_2r_1, \;\; B_3 = Ek_1q - Ek_1m_1r_1.$$

The Jacobian matrix of the system about the equilibrium point $T_5(x_5^*, 0, k_3)$ is given by

$$\begin{pmatrix} G_{11} & \frac{r_1x_5^*a}{k_1} & 0 \\ 0 & r_2\Big(1 - \frac{bk_3}{k_2}\Big) & 0 \\ 0 & cr_3 & -r_3 \end{pmatrix}, \tag{2.9}$$

where

$$\begin{aligned} G_{11} &= r_1\left(1 - \frac{x_5^*}{k_1}\right) - \frac{r_1x_5^*}{k_1} - \frac{qE}{m_1E + m_2x_5^*} + \frac{qEx_5^*m_2}{\left(m_1E + m_2x_5^*\right)^2} \\ &= -\frac{r_1x_5^*}{k_1} + \frac{qEx_5^*m_2}{\left(m_1E + m_2x_5^*\right)^2} \quad \text{(by using Eq (2.6)).} \end{aligned}$$

The characteristic equation of the above matrix is

$$\Big(\lambda - G_{11}\Big)\Big(\lambda - r_2\Big(1 - \frac{bk_3}{k_2}\Big)\Big)(\lambda + r_3) = 0.$$

Hence, it follows from Eqs (2.5) and (2.6) that the characteristic roots are

$$\lambda_1 = -\frac{r_1x_5^*}{k_1} + \frac{qEx_5^*m_2}{\left(m_1E + m_2x_5^*\right)^2} < -\frac{r_1x_5^*}{k_1} + \frac{qm_2x_5^*}{m_1^2E} < 0,$$

$$\lambda_2 = r_2\Big(1 - \frac{bk_3}{k_2}\Big) < 0, \;\; \lambda_3 = -r_3 < 0.$$

Consequently, $T_5(x_5^*, 0, k_3)$ is locally asymptotically stable. This ends the proof of Theorem 2.2.

**Remark 2.2.** In Example 1.2, one could easily check that

$$q = 0.1 < r_1m_1 = 1,$$

$$\frac{qm_2}{m_1^2E} = 0.1 < 1 = \frac{r_1}{k_1}$$

and

$$1 < \frac{bk_3}{k_2} = 2.$$

Hence, it follows from Theorem 2.2, $T_5(x_5^*, 0, k_3)$ is locally asymptotically stable. This is consistent with the numerical simulations (Figures 4–6).

## 3. Global attractivity of $T_1$ and $T_5$

We showed in the previous section that $T_1(0, 0, k_3)$ and $T_5(x_5^*, 0, k_3)$ could be locally asymptotically stable under some suitable assumption. In this section we will further show that, under some suitable assumption, $T_1(0, 0, k_3)$ and $T_5(x_5^*, 0, k_3)$ could be globally attractive.

**Theorem 3.1.** *Assume that*

$$r_1 < \frac{qE}{m_1E + m_2k_1} \tag{3.1}$$

*and*

$$1 < \frac{bk_3}{k_2} \tag{3.2}$$

*hold, then, $T_1(0, 0, k_3)$ is globally attractive.*

*Proof.* For $\varepsilon > 0$ sufficiently small, Condition (3.1) implies that

$$r_1 + \frac{ar_1\varepsilon}{k_1} < \frac{qE}{m_1E + m_2(k_1 + \varepsilon)}. \tag{3.3}$$

Condition (3.2) implies that

$$1 < \frac{b(k_3 - \varepsilon)}{k_2}. \tag{3.4}$$

From the third equation of System (1.1), we have that

$$\frac{dz}{dt} \geq r_3z\Big(1 - \frac{z}{k_3}\Big), \tag{3.5}$$

hence

$$\liminf_{t\to+\infty} z(t) \geq k_3. \tag{3.6}$$

For aforementioned $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$z(t) > k_3 - \varepsilon \text{ for all } t > T_1.$$

From this and the second equation of System (1.1), for $t > T_1$, we have that

$$\begin{aligned} \frac{dy}{dt} &\leq r_2y\Big(1 - \frac{y}{k_2} - b\frac{k_3 - \varepsilon}{k_2}\Big) \\ &< r_2y\Big(1 - b\frac{k_3 - \varepsilon}{k_2}\Big). \end{aligned}$$

Hence, it follows from Eq (3.4) that

$$y(t) < y(T_1)\exp\Big\{r_2\Big(1 - b\frac{k_3-\varepsilon}{k_2}\Big)(t-T_1)\Big\} \to 0 \ \text{ as } \ t \to +\infty.$$

The nonnegative of $y(t)$ together with the above inequality leads to

$$\lim_{t\to+\infty} y(t) = 0. \tag{3.7}$$

From Eq (3.7), for aforementioned $\varepsilon > 0$, there exists a $T_2 > T_1$ such that

$$y(t) < \varepsilon \ \text{ as } \ t \geq T_2. \tag{3.8}$$

For $t \geq T_2$, from the above inequality and the third equation of System (1.1), we have that

$$\frac{dz}{dt} \leq r_3 z\Big(1 - \frac{z}{k_3} + c\frac{\varepsilon}{k_3}\Big);$$

so,

$$\limsup_{t\to+\infty} z(t) \leq k_3\Big(1 + c\frac{\varepsilon}{k_3}\Big). \tag{3.9}$$

Equation (3.6) together with Eq (3.9) leads to

$$k_3 \leq \liminf_{t\to+\infty} z(t) \leq \limsup_{t\to+\infty} z(t) \leq k_3\Big(1 + c\frac{\varepsilon}{k_3}\Big). \tag{3.10}$$

With $\varepsilon$ as sufficiently small positive constants, setting $\varepsilon \to 0$ in Eq (3.10) leads to

$$\lim_{t\to+\infty} z(t) = k_3. \tag{3.11}$$

For $t \geq T_2$, from Eq (3.8) and the first equation of System (1.1), we have that

$$\frac{dx}{dt} \leq r_1 x\Big(1 - \frac{x}{k_1} + a\frac{\varepsilon}{k_1}\Big).$$

Thus,

$$\limsup_{t\to+\infty} x(t) \leq k_1\Big(1 + a\frac{\varepsilon}{k_1}\Big).$$

Setting $\varepsilon \to 0$ in the above inequality leads to

$$\limsup_{t\to+\infty} x(t) \leq k_1.$$

Consequently, for $\varepsilon > 0$ sufficiently small, there exists a $T_3 > T_2$ such that

$$x(t) < k_1 + \varepsilon \ \text{ for all } \ t \geq T_3.$$

From the above inequality and the first equation of System (1.1), for $t \geq T_3$, we also have that

$$\begin{aligned}
\frac{dx}{dt} &\leq r_1 x\Big(1 - \frac{x}{k_1} + a\frac{\varepsilon}{k_1}\Big) - \frac{qEx}{m_1E + m_2(k_1+\varepsilon)} \\
&\leq x\Big(r_1 + a\frac{r_1\varepsilon}{k_1} - \frac{qE}{m_1E + m_2(k_1+\varepsilon)}\Big).
\end{aligned}$$

It immediately follows from Eq(3.3) and the above inequality that

$$x(t) \leq x(T_3)\exp\Big\{\Big(r_1 + a\frac{r_1\varepsilon}{k_1} - \frac{qE}{m_1E + m_2(k_1+\varepsilon)}\Big)(t-T_3)\Big\} \to 0 \ \text{ as } \ t \to +\infty,$$

that is

$$\lim_{t\to+\infty} x(t) = 0. \tag{3.12}$$

Equations (3.7), (3.11) and (3.12) show that $T_1(0,0,k_3)$ is globally attractive.

This completes the proof of Theorem 3.1.

**Remark 3.1.** In Example 1.1, one could easily check that

$$r_1 = 1 < \frac{qE}{m_1E + m_2k_1} = \frac{3}{2}$$

and

$$1 < \frac{bk_3}{k_2} = 2.$$

Hence, it follows from Theorem 3.1 that $T_1(0,0,1)$ is globally attractive, which is consistent with the numerical simulations (Figures 1–3).

**Theorem 3.2.** *Assume that*

$$q < r_1m_1, \tag{3.13}$$

$$\frac{qm_2}{m_1^2E} < \frac{r_1}{k_1} \tag{3.14}$$

*and*

$$1 < \frac{bk_3}{k_2} \tag{3.15}$$

*hold, then,* $T_5(x_5^*, 0, k_3)$ *is globally attractive.*

*Proof.* The idea for the proof of this theorem comes from the work of Chen [38].

By using Eq (3.15), similar to the analysis of Eqs (3.5)–(3.11) in the proof of Theorem 3.1, we have that

$$\lim_{t\to+\infty} y(t) = 0, \quad \lim_{t\to+\infty} z(t) = k_3. \tag{3.16}$$

Let $\varepsilon > 0$ be a sufficiently small positive constant; it follows from Eq (3.16) that there exists $T > 0$ such that

$$y(t) < \varepsilon \quad \text{for} \quad \text{all} \ \ t > T.$$

From the above inequality and the first equation of System (1.1), we have that

$$\frac{dx}{dt} \ \leq \ r_1x\Big(1 - \frac{x}{k_1} + \alpha\frac{\varepsilon}{k_1}\Big) - \frac{qEx}{m_1E + m_2x}. \tag{3.17}$$

Now, let us consider the equation

$$\frac{du}{dt} = r_1u\Big(1 - \frac{u}{k_1} + \alpha\frac{\varepsilon}{k_1}\Big) - \frac{qEu}{m_1E + m_2u}. \tag{3.18}$$

Condition (3.13) implies that

$$r_1\Big(1+\alpha\frac{\varepsilon}{k_1}\Big)>\frac{q}{m_1}$$

holds. Let

$$F(u)=r_1\Big(1-\frac{u}{k_1}+\alpha\frac{\varepsilon}{k_1}\Big)-\frac{qE}{m_1E+m_2u}.$$

Then, we have the following:
(1) There is a unique $u_\varepsilon^*$, such that $F(u_\varepsilon^*)=0$, where, by simple computation,

$$u_\varepsilon^*=\frac{-A_2+\sqrt{A_2^2-4A_1A_3}}{2A_1}. \tag{3.19}$$

$$\begin{aligned} A_1 &= m_2r_1,\ \ A_2=Em_1r_1-\varepsilon\alpha m_2r_1-k_1m_2r_1,\\ A_3 &= Ek_1q-Ek_1m_1r_1-E\alpha m_1r_1\varepsilon. \end{aligned}$$

(2) For all $u_\varepsilon^*>u>0, F(u)>0$.
(3) For all $u>u_\varepsilon^*>0, F(u)<0$.

Hence, it follows from Lemma 2.1 in [49] that the unique positive equilibrium $u_\varepsilon^*$ of System (3.18) is globally stable. By the comparison theorem of the differential equation, it immediately follows from Eqs (3.17) and (3.18) that

$$\limsup_{t\to+\infty}x(t)\le u_\varepsilon^*. \tag{3.20}$$

Since $\varepsilon>0$ is an arbitrary small positive constant, if we let $\varepsilon\to0$ in Eq (3.19), from the expression of $x_5^*$ (see Eq (2.8)), we have that

$$\limsup_{t\to+\infty}x(t)\le x_5^*. \tag{3.21}$$

From the first equation of Eq (1.1), we have that

$$\frac{dx}{dt}\ \ \ge\ \ r_1x\Big(1-\frac{x}{k_1}\Big)-\frac{qEx}{m_1E+m_2x}.$$

Now let us consider the equation

$$\frac{dw}{dt}=r_1w\Big(1-\frac{w}{k_1}\Big)-\frac{qEw}{m_1E+m_2w}. \tag{3.22}$$

Similar to the above analysis, we could show that (3.22) admits a unique positive equilibrium $w=x_5^*$ that is globally stable. By the comparison theorem, it immediately follows that

$$\liminf_{t\to+\infty}x(t)\ge x_5^*. \tag{3.23}$$

It follows from Eqs (3.21) and (3.23) that

$$\lim_{t\to+\infty}x(t)=x_5^*. \tag{3.24}$$

Equations (3.16) and (3.24) show that $T_5(x_5^*,0,k_3)$ is globally attractive.

This ends the proof of Theorem 3.2.

**Remark 3.2.** In Example 1.2, one could easily check that all the conditions of Theorem 3.2 are satisfied; hence, it follows from Theorem 3.2 that $T_5(x_5^*,0,k_3)$ is globally attractive. This assertion is consistent with the numerical simulations (Figures 4–6).

## 4. Conclusions

Puspitasari, Kusumawinahyu and Trisilowati [32] proposed System (1.1). The system has eight equilibria. By computation, they showed that $T_4$ and $T_7$ are locally asymptotically stable while the other six equilibria are all unstable. However, our numerical simulations (Examples 1.1 and 1.2) showed that $T_1$ and $T_5$ are also possibly locally stable.

By analyzing the Jacobian matrix's characteristic equation for the equilibria $T_1$ and $T_5$, some suitable conditions that ensure the local asymptotical stability of the equilibria $T_1$ and $T_5$ were obtained.

Next, by developing the analysis technique of Baoguo Chen [38], we also obtained sufficient conditions that ensure the global attractivity of the two equilibria.

At the end of the paper, we would like to mention that to this day, most of the works on commensalism models focused on the two species case, and seldom did scholars study the more complicated multispecies case; we will try to do more works on this topic.

## Acknowledgements

The research was supported by the Key Project of Humanities and Social Sciences in Anhui Province, (SK2020A0395), Key Research Project of College Level Humanities and Social Sciences of West Anhui University, (WXSK202037) and Key Program of Anhui Provincial Education Department (KJ2019A0618).

The authors would like to thank the two anonymous reviewers for their valuable comments, which have greatly improved the final expression of the paper.

## Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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