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Research article

Bi-clean and clean Hopf modules

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Abstract: Let R be a commutative ring with multiplicative identity, C a coassociative and counital R-coalgebra, B an R-bialgebra. A clean comodule is a generalization and dualization of a clean module. An R-module M is called a clean module if the endomorphism ring of M over R (denoted by $End_R(M)$) is clean. Thus, any element of $End_R(M)$ can be expressed as a sum of a unit and an idempotent element of $End_R(M)$. Moreover, for a right C-comodule M, the endomorphism set of C-comodule M denoted by $End^C(M)$ is a subring of $End_R(M)$. A C-comodule M is a clean comodule if the $End^C(M)$ is a clean ring. A Hopf module M over B is a B-module and a B-comodule that satisfies the compatible conditions. This paper considers the notions of a clean ring, clean module, clean coalgebra, and clean comodule in relation to the Hopf Module. We divide our discussion into two parts, i.e., clean and bi-clean Hopf modules. A B-Hopf module M is said to be clean if the endomorphism ring of M is clean, and M is a bi-clean Hopf module if M is clean as a module over B and also clean as a comodule over B. Moreover, we give sufficient conditions of (bi)-clean bialgebras and Hopf modules related to the cleanness concept of modules and comodules.

Keywords: clean ring; clean module, clean comodule; clean coalgebra; clean Hopf module; bi-clean Hopf module
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1. Introduction

Throughout, R is a commutative ring with a multiplicative identity. In 1977, W. K. Nicholson introduced the notion of a clean ring. The ring R is clean if every element of R can be expressed as the sum of a unit and an idempotent element [1]. Based on the isomorphism properties of R, R is clean if and only if the ring $End_R(R)$ is clean. Some authors study clean modules by using the cleanness of their endomorphisms. An R-module M is said to be clean if the endomorphism ring $End_R(M)$ is a clean ring (see [2, 3]).

Sweedler introduces coalgebras over a field as the dualization of algebras over a field [4]. Furthermore, the ground field has been generalized to a commutative ring with a multiplicative identity. The readers are suggested to refer to [5] for more detailed basic notions of coalgebra and comodule over a commutative ring. For any comodule M over a coalgebra C, the endomorphism ring of C-comodule M is denoted by $End^C(M)$ is a subring of $End_R(M)$ over addition and composition functions. It is interesting since a subring of a clean ring is not need to be clean. Based on this fact and some results in module theory, Puspita et al. [6, 7] introduced the notion of clean comodule over a coalgebra.

Let (M, ϱ^M) be a right comodule over a coassociative and counital R-coalgebra (C, Δ, ε). The right C-comodule M is clean if the endomorphism ring of C-comodule M is clean. By taking $M = C$, the clean R-coalgebra (C, Δ, ε) is defined by the fact that C is a comodule over itself with left and right coaction Δ. The R-coalgebra C is said to be clean if the endomorphism ring of $End^C(C)$ is clean.

Now, we consider coalgebras with an algebraic structure. The Algebras with compatible coalgebras are known as bialgebra. A Hopf module over a bialgebra is a module over an algebra structure, and also, it is a comodule over a coalgebra structure. Both of these structures are compatible. The study of bialgebras and Hopf modules from the point of view of ring and module theory, for example, can be referred to [5, 8].

This paper divides the notion of clean in Hopf modules into two parts, i.e., clean Hopf modules and bi-clean Hopf modules. A B-Hopf module M is said to be bi-clean if M is clean as a module and it is also clean as a comodule over B. It means both of the endomorphisms of B-module M (or $End_B(M)$) and the endomorphisms of B-comodule (or $End^B(M)$) are clean rings. On the other hand, in [5], the endomorphisms of B-Hopf module is the intersection of $End_B(M)$ and $End^B(M)$ i.e., $End_B^B(M) = End_B(M) \cap End^B(M)$. A B-Hopf module M is a clean B-Hopf module if $End_B^B(M)$ is clean.

As we have already known from ring theory, if $End_B(M)$ and $End^B(M)$ are clean, it does not imply $End_B^B(M)$ is clean. If $End_B^B(M)$ is clean, we also can not conclude that $End_B(M)$ and $End^B(M)$ are clean. So the relationship of cleanness on $End_B(M)$, $End^B(M)$ and $End_B^B(M)$ are interesting to be observed.

The relationship for some clean algebra structures is clear for the trivial case. Any ring R with multiplication μ is an R-coalgebra by trivial comultiplication $\Delta_T : R \otimes_R R \to R, r \mapsto r \otimes 1$ and counit $\varepsilon_T = I_R$. Here, the R-algebra (R, μ, ι) and R-coalgebra $(R, \Delta_T, \varepsilon_T)$ satisfy the compatible properties on [5] such that R is a bialgebra over itself. Throughout, $(R, \mu, \iota, \Delta_T, \varepsilon_T)$ is said to be the trivial R-bialgebra R.

Moreover, for any R-module M and the trivial R-bialgebra R, we can define an R-coaction $\varrho_T^M : M \to M \otimes_R R, m \mapsto m \otimes 1$ such that for all $m \in M$ and $r \in R, \varrho_T^M(mr) = \varrho^M(m)\Delta_T(r)$. Consequently, any R-module M is a trivial R-Hopf module. Here, M is a clean R-module if and only if M is a clean R-comodule. Moreover, the trivial R-comodule M is clean if and only if M is a (bi)-clean Hopf module over R.

In general conditions, the perfect relationships on the trivial R-Hopf module $(R, \mu, \iota, \Delta_T, \varepsilon_T)$ does not imply that every ring R is clean if and only if R is a clean Hopf module. For example, although $\mathbb{Z}_4$ is a clean ring (clean module over itself), by changing the trivial comultiplication with $n \mapsto n \otimes 1 + 1 \otimes n$ for any $n \in \mathbb{Z}_4$, it shows that $\mathbb{Z}_4$ is not a clean coalgebra (comodule over itself). Thus, $\mathbb{Z}_4$ is not automatically a bi-clean bialgebra or clean Hopf module over itself. The transfer of clean properties among rings, modules, coalgebras, and comodules is not obvious. It motivates us to investigate clean Hopf modules and bialgebras and their relation to clean modules and clean comodules. In the category of modules, we have some examples of the clean module, which are also clean as a comodule and satisfy the compatible axioms of a Hopf module. This work connects clean modules and comodules, which brings us to some properties on clean Hopf modules and clean bialgebras.

2. Some basic notions

In this section, we give some basic notions of Hopf modules before we study the properties of clean Hopf modules. To understand the R-bialgebra and a Hopf module, we need to study coalgebra, and comodule structures [5].

Definition 2.1. An R-module B that is an algebra (B, μ, ι) and a coalgebra (B, Δ, ε) is called a bialgebra if Δ and ε are algebra morphisms or, equivalently, μ and ι are coalgebra morphisms.

An R-linear map $f : B \to B_0$ of bialgebras is called a bialgebra morphism if f is both an algebra and a coalgebra morphism. An R-submodule $I \subseteq B$ is a sub-bialgebra if it is a subalgebra as well as a subcoalgebra. Moreover, I is a bi-ideal if it is both an ideal and a coideal.

Since an R-bialgebra B has both a coalgebra and an algebra structure, R-coalgebra B must be compatible with the algebra structure of B. One can require compatibility conditions for corresponding modules and comodules. Throughout, B is an R-bialgebra with product μ, coproduct Δ, unit map ι and counit ε.

Definition 2.2. An R-module M is called a right B-Hopf module if M is

(1) a right B-module with an action $\varrho_M : M \otimes_R B \to M$;
(2) a right B-comodule with a coaction $\varrho^M : M \to M \otimes_R B$;
(3) for all $m \in M, b \in B$, $\varrho^M(mb) = \varrho^M(m)\Delta(b)$.

In Definition 2.2, it is essential to see that when B is a Hopf algebra over a field, then any Hopf B-modules are a free module over B [4,9]. It is importantly related to the property of cleanness on free modules.

Let B be an R-bialgebra, M and N B-Hopf modules. An R-linear map $f : M \to N$ is a B-Hopf module morphism if it is both a right B-module and a right B-comodule morphism. We denote the set of B-module homomorphisms, B-comodule morphisms, and B-Hopf module homomorphisms from M to N as $Hom_B(M, N)$, $Hom^B(M, N)$ and $Hom_B^B(M, N)$, respectively where $Hom_B^B(M, N) = Hom_B(M, N) \cap Hom^B(M, N)$. In case $M = N$ we have $End_B^B(M) = End_B(M) \cap End^B(M)$.

For an R-bialgebra B, the dual $B^* = Hom_R(B, R)$ also has a natural (B^*, B^*)-bimodule structure with the following right and left scalar multiplication [5]:

$$\begin{aligned} \rightharpoonup: B \otimes_R B^* &\to B^*, \quad b \otimes f \mapsto [c \mapsto f(cb)], \\ \leftharpoonup: B^* \otimes_R B &\to B^*, \quad f \otimes b \mapsto [c \mapsto f(bc)]. \end{aligned}$$

For $a \in B$ and $f, g \in B^*$

$$a \rightharpoonup (f * g) = \sum (a_{\underline{1}} \rightharpoonup f) * (a_{\underline{2}} \rightharpoonup g).$$

Thus, a right B-Hopf module M is a left B^*-module. The clean Hopf module and bi-clean Hopf module come from the module's and comodule's clean properties. In the main result, we will focus our investigations on the compatible condition of the cleanness of modules and comodules and the cleanness of algebras and bialgebras.

3. Results

We start our result by defining clean and bi-clean Hopf modules.

Definition 3.1. Let B be an R-bialgebra. A B-Hopf module M is bi-clean if both $End_B(M)$ and $End^B(M)$ are clean.

In [5], the set of endomorphism of B-Hopf module M are denoted by $End_B^B(M) = End_B(M) \cap End^B(M)$. A clean B-Hopf module is defined as follows.

Definition 3.2. Let B be a bialgebra. A B-Hopf module M is clean if $End_B^B(M)$ is clean.

Since $End_B^B(M) = End_B(M) \cap End^B(M)$ and for any clean rings does not imply its subring being clean, we have that the bi-cleanness and cleanness of Hopf modules are independent. A clean and a bi-clean R-bialgebra is a special case of a clean and bi-clean Hopf module in Definition 3.1 and Definition 3.2, when $B = M$ (or B consider as a Hopf module over itself).

In [5], for an R-module K and R-bialgebra B, we can construct a right B-Hopf module $K \otimes_R B$ with the canonical structure, $I_K \otimes \Delta : K \otimes_R B \to (K \otimes_R B) \otimes_R B$ and $I_K \otimes \mu : (K \otimes_R B) \otimes_R B \to (K \otimes_R B)$. Here, we give a relationship of the clean property of the R-module K, bialgebra B, and B-Hopf module $K \otimes_R B$.

Lemma 3.3. *Let B be an R-bialgebra and K a clean R-module. If $f \in End_R(K)$, then the endomorphism of B-Hopf module $f \otimes_R I_B : K \otimes_R B \to K \otimes_R B$ is a clean element of the $End_B^B(K \otimes_R B)$.*

Proof. Suppose that K is a clean R-module, $f \in End_R(K)$, and B is an R-bialgebra. By [5], clearly that $K \otimes_R B$ is a right B-Hopf module and for an R-module endomorphism $f : K \to K$, $f \otimes I_B : K \otimes_R B \to K \otimes_R B$ is a B-Hopf module endomorphism or $f \otimes I_B \in End_B^B(K \otimes_R B)$. Here, we prove that $f \otimes I_B$ is a clean B-Hopf module morphism.

(1) Since K is a clean R-module, $f = u + e$ for an idempotent e and a unit u in $End_R(K)$. Thus,

$$\begin{aligned} f \otimes I_B &= (u + e) \otimes I_B \\ &= u \otimes I_B + e \otimes I_B \in End_R(K \otimes_R B). \end{aligned}$$

Clearly, the identity map I_B is a unit and also it is an idempotent of $End_R(B)$. Therefore, $u \otimes I_B$ is a unit of $End_R(K \otimes_R B)$ and $e \otimes I_B$ is an idempotent of $End_R(K \otimes_R B)$. Consequently, $f \otimes I_B$ can be expressed as a sum of an idempotent and a unit element of $End_R(K \otimes_R B)$. Thus, $f \in End_R(K \otimes_R B)$ is a clean element in $\mathbf{M}_R$.

(2) We need prove f is a clean endomorphism of B Hopf module. Here, we need to make sure that $u \otimes I_B$ is a unit of $End_B^B(K \otimes_R B)$ and $e \otimes I_B$ is an idempotent of $End_B^B(K \otimes_R B)$. Based on [5] (page 135), since $u, e \in End_R(K)$ and B is an R-bialgebra, we have $u \otimes I_B$ and $e \otimes I_B$ are B-Hopf module morphisms. Analogue, we obtain $f \otimes I_B = u \otimes I_B + e \otimes I_B \in End^B(K \otimes_R B)$.

It means, if K is a clean R-module, then for any $f \in End_R(K)$, $f \otimes I_B$ is a clean element of $End_B^B(K \otimes_R B)$. □

As a corollary, if $K = R$ is a clean ring (or R-module R), then the identity map of R-bialgebra B is a clean element of the endomorphism of R-bialgebra B. The simple result above brings us on some special cases related to the clean modules and clean comodules.

In ring theory, if R is a clean ring, then the ring of $n \times n$-matrices over R (denoted by $M_n(R)$) is also a clean ring [10]. In [5], The ring $M_n(R)$ is an R-coalgebra with a comultiplication and counit as below:

$$\Delta : M_n(R) \to M_n(R) \otimes_R M_n(R), e_{ij} \mapsto \Sigma_{i,j} e_{i,k} \otimes e_{kj}, \tag{3.1}$$

and

$$\varepsilon : M_n(R) \to R, e_{ij} \mapsto \delta_{ij}, \tag{3.2}$$

where $\delta_{ij} = 1$ if $i = j$ and equal to zero if $i \neq j$. Throughout, the R-coalgebra $M_n(R)$ with the comultiplication (3.1) and the counit (3.2) is denoted by $M_n^C(R)$. In [7], we also have if R is a clean ring, then R-coalgebra $M_n^C(R)$ is clean. The following proposition gives a bi-clean R-bialgebra from the set of all $n \times n$-matrices over R.

Proposition 3.4. *If R is a clean ring, then $M_n^C(R)$ is a bi-clean R-bialgebra.*

Proof. It is clear that if R is a clean ring, then $M_n^C(R)$ is a clean ring and also clean as an R-coalgebra [7, 10]. For this proposition, we only need to check that $M_n^C(R)$ is an R-bialgebra. Let $\{e_{ij}\}_{1 \le i,j \le n}$ be the set of canonical basis of $n \times n$-matrices over R. Consider the R-algebra $(M_n^C(R), \mu, \iota)$ and the R-coalgebra $(M_n^C(R), \Delta, \varepsilon)$. See the commutative diagram on [5] page 130 and put $e_{ij}, e_{kl} \in M_n^C(R)$ where $i, j, k, l = 1, 2, \ldots, n$.

(1) For $j = k$, we have

$$\begin{aligned} \Delta \circ \mu(e_{ij} \otimes e_{kl}) &= \Delta(e_{il}) \\ &= \sum_a e_{ia} \otimes e_{al}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &(\mu \otimes \mu) \circ (I_{M_n^C(R)} \otimes tw \otimes I_{M_n^C(R)}) \circ (\Delta \otimes \Delta)(e_{ij} \otimes e_{kl}) \\ &= (\mu \otimes \mu) \circ (I_{M_n^C(R)} \otimes tw \otimes I_{M_n^C(R)})(\Delta(e_{ij}) \otimes \Delta(e_{kl})) \\ &= (\mu \otimes \mu) \circ (I_{M_n^C(R)} \otimes tw \otimes I_{M_n^C(R)})((\sum_a e_{ia} \otimes e_{aj}) \otimes (\sum_b e_{kb} \otimes e_{bl})) \\ &= (\mu \otimes \mu)(I_{M_n^C(R)} \otimes tw \otimes I_{M_n^C(R)})(e_{i1} \otimes e_{1j} + e_{i2} \otimes e_{2j} + \ldots + e_{in} \otimes e_{nj}) \\ &\quad \otimes (e_{l1} \otimes e_{1k} + e_{l2} \otimes e_{2k} + \ldots + e_{ln} \otimes e_{nk}) \\ &= (\mu \otimes \mu)(I_{M_n^C(R)} \otimes tw \otimes I_{M_n^C(R)})(e_{i1} \otimes e_{1j} \otimes e_{l1} \otimes e_{1k}) + \ldots \\ &\quad (e_{in} \otimes e_{nj} \otimes e_{ln} \otimes e_{nk}) \\ &= (\mu \otimes \mu)(e_{i1} \otimes e_{l1} \otimes e_{1j} \otimes e_{1k}) + \ldots (e_{in} \otimes e_{ln} \otimes e_{nj} \otimes e_{nk}) \\ &= \mu(e_{i1} \otimes e_{l1}) \otimes \mu(e_{1j} \otimes e_{1k}) + \ldots + \mu(e_{in} \otimes e_{ln}) \otimes \mu(e_{nj} \otimes e_{nk}) \\ &= e_{i1}e_{l1} \otimes e_{1j}e_{1k} + \ldots + e_{in}e_{ln} \otimes e_{nj}e_{nk} \end{aligned}$$

$$= \sum_{a} e_{ia}e_{la} \otimes e_{aj}e_{ak}.$$

In general case, for any k and j, the equation $\sum_a e_{ia}e_{la} \otimes e_{aj}e_{ak}$ means:

$$\sum_{a} e_{ia}e_{la} \otimes e_{aj}e_{ak} = \begin{cases} \sum_a e_{ia} \otimes e_{ak}, & \text{a=l=j;} \\ 0, & a \neq l, j \neq a. \end{cases}$$
$$= \begin{cases} \sum_k e_{ik} \otimes e_{ka}, & \text{k=a=l=j;} \\ 0, & a \neq l, j \neq a. \end{cases}$$
$$= \begin{cases} \sum_k e_{ik} \otimes e_{kl}, & \text{k=j;} \\ 0, & j \neq k. \end{cases}$$

(2) For $j \neq k$, we have $\Delta \circ \mu(e_{ij} \otimes e_{kl}) = \Delta(0) = 0$ and

$$\begin{aligned}
&(\mu \otimes \mu) \circ (I_{M_n^C(R)} \otimes tw \otimes I_{M_n^C(R)}) \circ (\Delta \otimes \Delta)(e_{ij} \otimes e_{kl}) \\
&= (\mu \otimes \mu) \circ (I_{M_n^C(R)} \otimes tw \otimes I_{M_n^C(R)})(\Delta(e_{ij}) \otimes \Delta(e_{kl})) \\
&= (\mu \otimes \mu)(I_{M_n^C(R)} \otimes tw \otimes I_{M_n^C(R)})((\sum_n e_{in} \otimes e_{nj}) \otimes (\sum_m e_{km} \otimes e_{ml})) \\
&= 0, \text{ since } k \neq j.
\end{aligned}$$

From Points 1 and 2, we have $\Delta \circ \mu = (\mu \otimes \mu) \circ (I_{M_n^C(R)} \otimes tw \otimes I_{M_n^C(R)}) \circ (\Delta \otimes \Delta)$.

For the second step, prove that $\Delta \circ \iota = (\iota \otimes \iota) \circ (\text{“} \simeq \text{”})$. Here, the unit of $M_n^C(R)$ is $\iota : R \to M_n^C(R), 1 \mapsto I_n$ such that for any $r \in R, \iota(r) = rI_n$ and the counit of $M_n^C(R)$ is $\varepsilon : M_n^C(R) \to R$ where $e_{i,j} \mapsto \begin{cases} 1, & \text{i= j;} \\ 0, & i \neq j. \end{cases}$ Thus, for any $r \in R$ we have:

$$\begin{aligned}
\Delta \circ \iota(r) &= \Delta(rI_n) \\
&= r(\Delta(I_n)) \\
&= r(\Delta(\sum_{i=1}^{n} e_{ii})) \\
&= r(\sum_k e_{1k} \otimes e_{k1} + \sum_k e_{2k} \otimes e_{k2} + \ldots + \sum_k e_{nk} \otimes e_{kn}) \\
&\simeq r(e_{11} + e_{22} + \ldots + e_{nn}) \\
&= r(I_n) \\
&= rI_n.
\end{aligned}$$

Moreover,

$$\begin{aligned}
(\iota \otimes \iota) \circ (\text{“} \simeq \text{”})(r) &= (\iota \otimes \iota)(r \otimes 1) \\
&= \iota(r) \otimes \iota(1) \\
&= rI_n \otimes I_n \\
&= r(I_n \otimes I_n)
\end{aligned}$$

$$= rI_n.$$

Since $\Delta \circ \iota = (\iota \otimes \iota) \circ (\text{“} \simeq \text{”})$, ι is an R-coalgebra morphism. Thus, $(M_n^C(R), \mu, \iota, \Delta, \varepsilon)$ is a bi-clean R-bialgebra. □

Let G be an abelian group. For the next result, we take the R-coalgebra $R[G]$ with comultiplication $\Delta : R[G] \to R[G] \otimes R[G], g \mapsto g \otimes g$. Hence, $R[G]$ is an R-bialgebra (see on [5] page 130). We have the following result.

Proposition 3.5. *Let G be a torsion group. If R is an Artinian principal ideal and a Boolean ring, then* $R[G]$ *is a bi-clean R-bialgebra.*

Proof. It is clear that $R[G] = \{\sum_{g \in G} a_g g | a_g \in R\}$ [11] is an R-bialgebra and also $R[G]$ is also an R-Hopf algebra. In [12], $R[G]$ is a clean ring if R is a Boolean ring and G is torsion where $End_{R[G]}(R[G]) \simeq R[G]$ is a clean ring. For its dual, in [6], we have any G-graded module M over R is clean if and only if M is a clean $R[G]$-comodule. Take $M = R[G]$ and consider $R[G]$ as a G-graded module over R. Since R Artinian principal ideal, $R[G]$ is a clean G-graded module over R, then $R[G]$ is a clean comodule over itself or $R[G]$ a clean. Therefore, if R is an Artinian principal ideal and Boolean ring, then $R[G]$ is a bi-clean R-bialgebra. □

In Proposition 3.5, since R is Artinian and Boolean, then every element of R is idempotent. It implies R is semi-simple with no nonzero nilpotent elements. Furthermore, R is a finite direct of some division rings, and the only Boolean division ring is the field with two elements. Thus, R is a finite direct sum of copies of $\mathbb{Z}_2$.

Functor F from the category of G-graded R-module to the category of $R[G]$-comodule is equivalence. Thus, a G-graded R-module M is clean if and only if M is a clean $R[G]$-comodule [6]. We will observe the cleanness for $R[G]$-Hopf module M.

Proposition 3.6. *Let* $R[G]$ *be an Artinian principal ideal and M a G-graded module over R. If M is a Hopf* $R[G]$*-modules, then M is a bi-clean Hopf* $R[G]$*-module.*

Proof. Let M be a G-graded module over R. Since the category of G-graded module isomorphic to the category of $R[G]$-comodule, M is a $R[G]$-comodule with the following conditions:

(1) From [6], $End_R(M) \simeq End^{R[G]}(M)$ is a clean ring since M is clean as a G-graded module over R.
(2) As a module over $R[G]$, $End_{R[G]}(M)$ is clean since $R[G]$ is an Artinian principal ideal [2].

Points 1 and 2 imply M is a clean $R[G]$-Hopf module. □

We have already known from some previous results that the direct product $\prod_{\lambda \in \Lambda} R_\lambda$ is clean if and only if R_λ is clean for any λ. Analog to the family of R-coalgebra $\{C_\lambda\}_{\lambda \in \Lambda}$ we have a similar property. We are going to bring these concepts for the direct sum of bialgebra.

Proposition 3.7. *Let* $\{(B_\lambda, \mu_\lambda, \iota_\lambda, \Delta_\lambda, \varepsilon_\lambda)\}_{\lambda \in \Lambda}$ *be a family of R-bialgebras and* $B = \oplus_{\lambda \in \Lambda} B_\lambda$ *the direct sums of the family of R-bialgebra* $\{B_\lambda\}_{\lambda \in \Lambda}$*. Then, B is clean if and only if* B_λ *is clean for every* $\lambda \in \Lambda$*.*

Proof. Let $B = \oplus_{\lambda \in \Lambda} B_\lambda$ be an R-bialgebra. Thus B is an R-algebra and B is an R-colagebra.

(1) From [13], we have $End^B(B)$ is clean if and only if $End^{B^\lambda}(B_\lambda)$ (as a comodule over itself) is a clean ring for any $\lambda \in \Lambda$.

(2) Futhermore, [14] proved that B is a clean algebra if and only if $B_\lambda \simeq End_{B_\lambda}(B_\lambda)$ is a clean ring for any $\lambda \in \Lambda$. Since $B_\lambda \simeq End_{B_\lambda}(B_\lambda)$ implies that $B = \oplus_{\lambda\in\Lambda} B_\lambda$ is clean.

Consequently, B is a clean R-bialgebra if and only if B_λ is clean R-algebra for all $\lambda \in \Lambda$. □

4. Conclusions

Bi-cleanness and cleanness on the Hopf modules category can be considered as generalizations of the clean module. This concept is motivated based on the fact that not any comodules are clean as a module, and not every clean module is a clean comodule. Therefore, we divided our result into bi-clean Hopf modules and clean Hopf modules using the property of their endomorphism as a module over a ring and as a comodule over coalgebra.

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Conflict of interest

The authors declare that they have no competing interests.

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