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Research article

A new self-adaptive inertial algorithm with W-mapping for solving split feasibility problem in Banach spaces

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Abstract: In this paper, the split feasibility problem is studied in real Banach spaces. Through the W-mapping, a new iterative algorithm with the inertial technique for solving the split feasibility problem is proposed, which the step size is self-adaptive and no prior estimation of operator norm is required. We prove that the proposed algorithm converges weakly to a solution of the split feasibility problem under some mild conditions. Finally, the effectiveness of the proposed algorithm is indicated by numerical experiments. Our results are innovative and can enrich recently announced related results in the literature.

Keywords: split feasibility problem; inertial technique; W-mapping; weak convergence; Banach spaces
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1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. The split feasibility problem (SFP) was first proposed by Censor and Elfving [4] for modeling inverse problems in finite dimensional Hilbert spaces. The SFP for mapping $A : H_1 \to H_2$ is described as follows:

$$\text{find } \ u \in C \ \text{ such that } \ Au \in Q, \tag{1.1}$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 , respectively.

The SFP has been widely studied and promoted by many scholars due to its crucial role in practical problems such as medical image reconstruction [1], signal processing [2] and intensity-modulated radiation therapy [5]. Therefore, some iterative algorithms have been proposed to approximate the solution of SFP (see [10, 12, 15, 18, 21, 22, 27]).

The algorithm to solve SFP in finite dimensional real Hilbert spaces was first proposed by Censor and Elfving [4] in 1994. This iterative algorithm is based on projections and involves computing the inverse of a matrix. For the sake of overcoming this drawback, a popular CQ algorithm was proposed by Byrne [1] as follows:

$$u_{n+1} = P_C(u_n - \gamma A^*(I - P_Q)Au_n), \tag{1.2}$$

where $0 < \gamma < 2/\|A\|^2$, P_C and P_Q are metric projections onto C and Q, respectively. The authors proved that the sequence $\{u_n\}$ generated by this algorithm converges weakly to a solution of SFP.

To obtain strongly convergent results, Berinde et al. [3] proposed the following algorithm:

$$u_{n+1} = (1 - \lambda)u_n + \lambda P_C(I - \gamma A^*(I - P_Q)A)u_n, \tag{1.3}$$

where $0 < \lambda \leq 1$ and $0 < \gamma < 2/\|A\|^2$. The authors proved that the sequence $\{u_n\}$ generated by this algorithm converges strongly to a unique solution u^* of the SFP.

In paper [27], Wang proposed the following new algorithm for solving the SFP by applying Polyak's gradient method and the calculation of the projections is easily implemented:

$$u_{n+1} = u_n - \gamma_n[(u_n - P_{C_n}u_n) + A^*(I - P_{Q_n})Au_n], \tag{1.4}$$

where C_n, Q_n and γ_n are given by

$$C_n = \{z \in H_1 : c(u_n) \leq \langle \eta_n, u_n - z\rangle\}, \quad \eta_n \in \partial c(u_n),$$

$$Q_n = \{w \in H_2 : q(Au_n) \leq \langle \theta_n, Au_n - w\rangle\}, \quad \theta_n \in \partial q(Au_n),$$

and

$$\gamma_n = \lambda_n \frac{\|u_n - P_{C_n}u_n\|^2 + \|(I - P_{Q_n})Au_n\|^2}{2\|(I - P_{C_n})u_n + A^*(I - P_{Q_n})Au_n\|^2},$$

where $\lambda_n \in (0, 4)$. The author proved that the sequence $\{u_n\}$ generated by (1.4) converges weakly to a solution of the SFP. To obtain the strong convergence theorem, an improved algorithm was proposed by Wang [27] as follows:

$$\begin{cases} w_n = u_n - \gamma_n[(u_n - P_{C_n}u_n) + A^*(I - P_{Q_n})Au_n], \\ u_{n+1} = \nu_n u + (1 - \nu_n)w_n, \end{cases} \tag{1.5}$$

where C_n, Q_n and γ_n are the same as those given in algorithm (1.4), $\{\nu_n\}$ is a sequence in $[0, 1]$ and satisfies

$$\lim_{n\to\infty} \nu_n = 0, \ \sum_{n=0}^{\infty} \nu_n = \infty,$$

$$\text{either } \sum_{n=0}^{\infty} |\nu_{n+1} - \nu_n| < \infty \text{ or } \lim_{n\to\infty} \frac{\nu_{n+1}}{\nu_n} = 1.$$

Since the setting of Banach spaces sometimes allows for more realistic modeling of problems arising in industrial and natural science applications, solving SFP in Banach space is interesting not only from a theoretical point of view, but also for solving related application problems in the real world.

In [19], Schöpfer et al. first proposed the following algorithm for solving the SFP in Banach spaces:

$$u_{n+1} = \Pi_C J_q^*[J_p(u_n) - t_n A^* J(Au_n - P_Q(Au_n))], \tag{1.6}$$

where Π_C denotes the Bregman projection, J_p, J_q^*, and J are duality mappings, and P_Q denotes the metric projection. The authors proved the result of weak convergence of algorithm (1.6).

In [23], Tang et al. introduced an iteration algorithm in Banach spaces which does not involve the projection operator to approximate a solution of the split common fixed point problem. The split common fixed point problem (SCFP) for two mappings $K : C \to C$ and $L : Q \to Q$ was introduced in 2009 by Censor et al. [6] as finding a point $u^* \in C$ such that

$$u^* \in F(K) \text{ and } Au^* \in F(L), \tag{1.7}$$

where $F(K)$ and $F(L)$ represent the sets of fixed points of K and L, respectively. It is worth underlining that the SCFP can be considered as a promotion of the SFP. The iterative algorithm was proposed by Tang et al. [23] as follows: For each $u_1 \in E_1$,

$$\begin{cases} w_n = u_n + \rho J_{E_1}^{-1} A^* J_{E_2}(T - I)Au_n, \\ u_{n+1} = (1 - \nu_n)w_n + \nu_n S^n w_n, \quad \forall n \geq 1, \end{cases} \tag{1.8}$$

where $J_{E_1}^{-1} : E_1^* \to 2^{E_1}$, $J_{E_2} : E_2 \to 2^{E_2^*}$ are the normalized duality mappings, $\{\nu_n\}$ is a sequence in $(0, 1)$ and ρ is a positive constant satisfying $0 < \rho < \min\{\frac{1-2k^2}{\|A\|^2}, \frac{1-\tau}{\|A\|^2}\}$. They proved that the sequence $\{u_n\}$ converges weakly to a point of the SCFP.

To accelerate the convergence, Polyak [17] firstly proposed the inertial extrapolation method for solving the smooth convex minimization problem. The inertial algorithm is a two-step iterative method, using the first two iterations to define the next iteration. Nesterov [16] introduced a modified method to improve the convergence rate as follows:

$$\begin{cases} w_n = u_n + \beta_n(w_n - w_{n-1}), \\ u_{n+1} = w_n - \lambda_n A^*(I - P_Q A w_n), \quad \forall n \geq 1, \end{cases} \tag{1.9}$$

where $\beta_n \in [0, 1)$ is an extrapolation factor, and $\{\lambda_n\}$ is a positive sequence. The inertia is denoted by the term $\beta_n(w_n - w_{n-1})$. It is worth noting that the inertial term greatly improves the performance of the algorithm and has a good convergence property [16]. Encouraged by the inertial term, many authors have proposed different algorithms with inertial techniques to solve a number of different problems(see [7–9, 14, 18, 30]).

Inspired and motivated by previous works, this paper aims to construct a new iterative method with the inertial technique for solving the SFP, which is self-adaptive in step size and does not require the prior estimate of operator norm. Through the new W-mapping, the algorithm is given in 2-uniformly convex and uniformly smooth Banach spaces, and then we show that the algorithm weakly converges to a solution of SFP.

The rest of this paper is organized as follows: In section 2, some basic facts and helpful lemmas are given for use in subsequent proofs. In section 3, a new iterative method with the inertial technique based on W-mapping is presented to find the solution of the SFP and the weak convergence theorem is proved under some mild conditions. Finally, in section 4, we give the numerical experiment to verify the effectiveness of the proposed iterative method and compare it with other methods.

2. Preliminaries

In this section, we first recall some notations and results which are needed in the sequel. We assume that E is a real Banach space and E^* represents its dual, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E and E^*. C is a nonempty closed convex subset of E. We denote the identity operator by I and the set of fixed points of T by $F(T)$. The notation "$\rightarrow$" stands for strong convergence while "$\rightharpoonup$" stands for weak convergence.

The normalized duality mapping $J_E : E \rightarrow 2^{E^*}$ is defined by

$$J_E(u) = \{u^* \in E^* : \langle u^*, u\rangle = \|u\|^2 = \|u^*\|^2\}, \quad \forall u \in E. \tag{2.1}$$

Let $1 < q \leq 2 \leq p < \infty$. The modulus of convexity $\delta_E(\varepsilon) : [0,2] \rightarrow [0,1]$ is defined as

$$\delta_E(\varepsilon) = \inf\{1 - \frac{\|u+v\|}{2} : \|u\|, \|v\| \leq 1, \|u-v\| \geq \varepsilon\},$$

E is called uniformly convex if $\delta_E(\varepsilon) > 0$ for any $\varepsilon \in (0,2]$, strictly convex if $\delta_E(2) = 1$. If there is a $c_p > 0$ such that $\delta_E(\varepsilon) \geq c_p\varepsilon^p$ for any $\varepsilon \in (0,2]$, then E is called p-uniformly convex. The modulus of smoothness $\rho_E(\tau) : [0,\infty) \rightarrow [0,\infty)$ is defined by

$$\rho_E(\tau) = \sup\{\frac{\|u+\tau v\| + \|u-\tau v\|}{2} - 1 : \|u\| = 1, \|v\| \leq \tau\},$$

E is called uniformly smooth if $\lim\limits_{\tau\rightarrow\infty} \frac{\rho_E(\tau)}{\tau} = 0$, q-uniformly smooth if there is a $c_q > 0$ so that $\rho_E(\tau) \leq c_q\tau^q$ for any $\tau > 0$. It is known that E is p-uniformly convex if and only if its dual E^* is q-uniformly smooth [13].

Definition 2.1. [29] Let $T : C \rightarrow C$ be a nonlinear mapping. Then, T is
(1) Nonexpansive if

$$\|Tu - Tv\| \leq \|u - v\|, \quad \forall u, v \in C.$$

(2) Firmly nonexpansive if

$$\|Tu - Tv\|^2 + \|(I-T)u - (I-T)v\|^2 \leq \|u - v\|^2, \quad \forall u, v \in C.$$

Definition 2.2. The metric projection $P_C : E \rightarrow C$ is defined by

$$P_C u := \arg\min_{v\in C} \|u - v\|, \quad u \in E.$$

Lemma 2.3. *[19] Let $\{u_n\}$ be a sequence in E. Then, for $u \in E$ the following holds:*
(1) $\langle w - P_C u, J_E(I - P_C)u\rangle \leq 0$, $\forall w \in C$.
(2) $\|u - P_C u\|^2 \leq \langle u - w, J_E(I - P_C)u\rangle$, $\forall w \in C$.

The uniformly convex or 2-uniformly smooth Banach space has the following estimate.

Lemma 2.4. *[28] Let E be a uniformly convex Banach space. Then, for any $r > 0$ and $s \in [0,1]$, there exists a continuous strictly increasing function $g : [0,\infty) \rightarrow [0,\infty)$, $g(0) = 0$ such that for all $u, v \in E$ the following inequality exists:*

$$\|su + (1-s)v\|^2 \leq s\|u\|^2 + (1-s)\|v\|^2 - s(1-s)g(\|u - v\|),$$

where $\|u\| \leq r$ and $\|v\| \leq r$.

Lemma 2.5. *[28] If E is a 2-uniformly smooth Banach space with the optimal smoothness constant $a > 0$, then the following inequality holds:*

$$\|u + v\|^2 \le \|u\|^2 + 2\langle v, J_E u\rangle + 2\|av\|^2, \quad \forall u, v \in E.$$

Lemma 2.6. *[25] In a strictly convex Banach space E, if*

$$\|u\| = \|v\| = \|\lambda u + (1 - \lambda)v\|,$$

for all $u, v \in E$ and $\lambda \in (0, 1)$, then $u = v$.

Definition 2.7. [26] Let $T_t(1 \le t \le n)$ be a finite family of nonexpansive mapping on C and λ_t be real numbers with $0 < \lambda_t \le \lambda < 1$. The definition of mapping W_n on C is as follows:

$$\begin{aligned}
H_{n,n+1} &= I,\\
H_{n,n} &= (1 - \lambda_n)I + \lambda_n T_n H_{n,n+1},\\
H_{n,n-1} &= (1 - \lambda_{n-1})I + \lambda_{n-1} T_{n-1} H_{n,n},\\
&\vdots\\
H_{n,m} &= (1 - \lambda_m)I + \lambda_m T_m H_{n,m+1},\\
H_{n,m-1} &= (1 - \lambda_{m-1})I + \lambda_{m-1} T_{m-1} H_{n,m},\\
&\vdots\\
H_{n,2} &= (1 - \lambda_2)I + \lambda_2 T_2 H_{n,3},\\
H_{n,1} &= (1 - \lambda_1)I + \lambda_1 T_1 H_{n,2},\\
W_n &= H_{n,1},
\end{aligned}$$

then the mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

The following tools will be used for analysis in the sequel.

Lemma 2.8. *[20] Let $W_n : E \to E$ be a W-mapping, then W_n is a nonexpansive mapping.*

Definition 2.9. Let $T : C \to C$ be an operator. Then we say that $I - T$ is demiclosed at zero, if for any $\{u_n\}$ in E, the following meaning holds:

$$u_n \rightharpoonup u \text{ and } (I - T)u_n \to 0 \quad \Rightarrow \quad u \in F(T).$$

Lemma 2.10. *[11] If $T : C \to C$ is a nonexpansive operator, then $I - T$ is demiclosed at zero.*

Definition 2.11. Let E be a Banach space, E is said to have the Opial property if for any sequence $\{u_n\}$ in E with $u_n \rightharpoonup u^*$, for any $v \in E$ with $v \neq u^*$, we have

$$\liminf_{n\to\infty} \|u_n - u^*\| < \liminf_{n\to\infty} \|u_n - v\|.$$

Lemma 2.12. *[14] Let $\{\phi_n\} \subset [0, \infty)$ and $\{\eta_n\} \subset [0, \infty)$ be two nonnegative real sequences satisfying the following conditions:*
(1) $\phi_{n+1} - \phi_n \le \alpha_n(\phi_n - \phi_{n-1})$,
(2) $\sum_{n=1}^{\infty} \eta_n < \infty$,
(3) $\{\alpha_n\} \subset [0, \alpha]$, where $\alpha \in [0, 1)$.

Then, $\{\phi_n\}$ is a converging sequence and $\sum_{n=1}^{\infty}[\phi_{n+1} - \phi_n]_+ < \infty$, where $[t]_+ = \max\{t, 0\}$ for any $t \in R$.

Lemma 2.13. *[24] Assume that $\{r_n\}$ and $\{t_n\}$ are two sequences of nonnegative numbers satisfying*

$$r_{n+1} \leq r_n + t_n, \quad \forall n \in \mathbb{N}.$$

If $\sum_{i=1}^{\infty} t_n \leq \infty$, then $\lim_{n\to\infty} r_n$ exists.

3. Main results

In this section, we propose our self-adaptive iterative algorithm to solve the split feasibility problem in Banach spaces. Subsequently, the weak convergence of the proposed algorithm is analyzed and established. In order to introduce our algorithm, we first define a special W-mapping W_n as follows:

$$\begin{aligned}
H_{n,n+1} &= I,\\
H_{n,n} &= (1-\lambda_n)I + \lambda_n P_C H_{n,n+1},\\
H_{n,n-1} &= (1-\lambda_{n-1})I + \lambda_{n-1} P_C H_{n,n},\\
&\vdots\\
H_{n,m} &= (1-\lambda_m)I + \lambda_m P_C H_{n,m+1},\\
H_{n,m-1} &= (1-\lambda_{m-1})I + \lambda_{m-1} P_C H_{n,m},\\
&\vdots\\
H_{n,2} &= (1-\lambda_2)I + \lambda_2 P_C H_{n,3},\\
H_{n,1} &= (1-\lambda_1)I + \lambda_1 P_C H_{n,2},\\
W_n &= H_{n,1},
\end{aligned}$$

where $\lambda_t = \frac{1}{n-t+1}$ and $1 \leq t \leq n$.

Assumption 3.1. (i) E_1 is a real 2-uniformly smooth and uniformly convex Banach space and E_2 is a real Banach space.
(ii) E_1 has the Opial property and the optimal smoothness constant a satisfies $0 < a < \frac{1}{\sqrt{2}}$.
(iii) $A : E_1 \to E_2$ is a bounded linear operator and the adjoint of A is represented by A^*.
(iv) The solution set Ω of SFP is nonempty:

$$\Omega = \{u \in C : \ Au \in Q\} \neq \emptyset.$$

Remark 3.2. Let $W_n : E_1 \to E_1$ be a W-mapping. It's obvious from Lemma 2.8 that the W_n is a nonexpansive mapping.

Lemma 3.3. *Let C be a nonempty closed convex subset of a strictly convex Banach space E. P_C is the metric projection, and it is a nonexpansive mapping. $F(P_C) \neq \emptyset$ and let $\lambda_1, \lambda_2, \cdots$ be real numbers such that $0 < \lambda_i \leq b < 1$ for every $i = 1, 2, \cdots$. Then*
(i) $\lim_{n\to\infty} H_{n,m}u$ exists, $\forall u \in C, m \in \mathbb{N}$ and $Wu = \lim_{n\to\infty} H_{n,m}u$;
(ii) $F(W) = F(P_C)$.

Proof. (i) Let $u \in E_1$. Without loss of generality, suppose $u \neq P_C u$, then $\forall n \in \mathbb{N},\ n \geq m$, we have

$$\begin{aligned}
&\|H_{n+1,m}u - H_{n,m}u\| \\
&= \|\lambda_m P_C H_{n+1,m+1}u + (1-\lambda_m)u - \lambda_m P_C H_{n,m+1}u - (1-\lambda_m)u\| \\
&= \lambda_m \|P_C H_{n+1,m+1}u - P_C H_{n,m+1}u\| \\
&\leq \lambda_m \|H_{n+1,m+1}u - H_{n,m+1}u\| \\
&= \lambda_m \|\lambda_{m+1} P_C H_{n+1,m+2}u + (1-\lambda_{m+1})u - \lambda_{m+1} P_C H_{n,m+2}u - (1-\lambda_{m+1})u\| \\
&\leq \lambda_m \lambda_{m+1} \|H_{n+1,m+2}u - H_{n,m+2}u\| \\
&\ \ \vdots \\
&\leq (\prod_{i=m}^{n} \lambda_i) \|H_{n+1,n+1}u - H_{n,n+1}u\| \\
&= (\prod_{i=m}^{n} \lambda_i) \|\lambda_{n+1} P_C H_{n+1,n+2}u + (1-\lambda_{n+1})u - u\| \\
&= (\prod_{i=m}^{n+1} \lambda_i) \|P_C u - u\|.
\end{aligned}$$

Let $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ with $n_0 \geq m$, such that

$$b^{n_0-m+2} < \frac{\epsilon(1-b)}{\|P_C u - u\|}.$$

Hence for every s, n when $s > n > n_0$, we have

$$\begin{aligned}
\|H_{s,m}u - H_{n,m}u\| &\leq \sum_{j=n}^{s-1} \|H_{j+1,m}u - H_{j,m}u\| \\
&\leq \sum_{j=n}^{s-1} \{(\prod_{i=m}^{j+1} \lambda_i) \|P_C u - u\|\} \\
&\leq \|P_C u - u\| \sum_{j=n}^{s-1} b^{j-m+2} \\
&\leq \frac{b^{n-m+2}\|P_C u - u\|}{1-b} \\
&< \epsilon.
\end{aligned}$$

So, $\{U_{n,m}u\}$ is a Cauchy sequence, and $\lim_{n\to\infty} H_{n,m}u$ exists. Let

$$H_{\infty,m}u := \lim_{n\to\infty} H_{n,m}u,$$

and

$$Wu := \lim_{n\to\infty} W_n u = \lim_{n\to\infty} H_{n,1}u.$$

(ii) Let $w \in F(P_C)$, clearly, $H_{n,m}w = w$ for all $n, m \in \mathbb{N},\ n \geq m$. Therefore, $H_{\infty,1}w = w$. In particular, $Ww = H_{\infty,1}w = w$. So $F(P_C) \subset F(W)$.

Next we prove $F(W) \subset F(P_C)$.

Let $u \in F(W),\ v \in F(P_C)$, then

$$\begin{aligned}
&\|W_n u - W_n v\| \\
&= \|H_{n,1}u - H_{n,1}v\| \\
&= \|(1-\lambda_1)u + \lambda_1 P_C H_{n,2}u - (1-\lambda_1)v - \lambda_1 P_C H_{n,2}v\| \\
&= (1-\lambda_1)\|u - v\| + \lambda_1\|P_C H_{n,2}u - P_C H_{n,2}v\| \\
&\leq (1-\lambda_1)\|u - v\| + \lambda_1\|H_{n,2}u - H_{n,2}v\| \\
&\quad\vdots \\
&\leq (1 - \prod_{i=1}^{m-1}\lambda_i)\|u - v\| + (\prod_{i=1}^{m-1}\lambda_i)\|H_{n,m}u - H_{n,m}v\| \\
&\leq (1 - \prod_{i=1}^{m-1}\lambda_i)\|u - v\| + (\prod_{i=1}^{m-1}\lambda_i)\|(1-\lambda_m)u + \lambda_m P_C H_{n,m+1}u - (1-\lambda_m)v - \lambda_m P_C H_{n,m+1}v\| \\
&= (1 - \prod_{i=1}^{m-1}\lambda_i)\|u - v\| + (\prod_{i=1}^{m-1}\lambda_i)\|(1-\lambda_m)(u - v) + \lambda_m(P_C H_{n,m+1}u - P_C H_{n,m+1}v)\| \\
&\leq (1 - \prod_{i=1}^{m}\lambda_i)\|u - v\| + (\prod_{i=1}^{m}\lambda_i)\|P_C H_{n,m+1}u - P_C H_{n,m+1}v\| \\
&\leq (1 - \prod_{i=1}^{m}\lambda_i)\|u - v\| + (\prod_{i=1}^{m}\lambda_i)\|H_{n,m+1}u - H_{n,m+1}v\| \\
&\quad\vdots \\
&\leq \|u - v\|.
\end{aligned}$$

So as $n \to \infty$ we have

$$\begin{aligned}
\|Wu - Wv\| &\leq (\prod_{i=1}^{m-1}\lambda_i)\|\lambda_m(P_C H_{\infty,m+1}u - P_C H_{\infty,m+1}v) + (1-\lambda_m)(u - v)\| \\
&\quad + (1 - \prod_{i=1}^{m-1}\lambda_i)\|u - v\| \\
&\leq (\prod_{i=1}^{m}\lambda_i)\|P_C H_{\infty,m+1}u - P_C H_{\infty,m+1}v)\| + (1 - \prod_{i=1}^{m}\lambda_i)\|u - v\| \\
&\leq \|u - v\|.
\end{aligned}$$

Since

$$\|Wu - Wv\| = \|u - v\|,$$

and $0 < \lambda_i < 1$ for all $m \in \mathbb{N}$,

$$\begin{aligned} & \|\lambda_m(P_C H_{\infty,m+1}u - P_C H_{\infty,m+1}v) + (1-\lambda_m)(u-v)\| \\ = & \|P_C H_{\infty,m+1}u - P_C H_{\infty,m+1}v\| \\ = & \|u-v\|. \end{aligned}$$

Since $y \in F(P_C)$ and from Lemma 2.6, we have

$$\|u-v\| = \|P_C H_{\infty,m+1}u - P_C H_{\infty,m+1}v\| = \|P_C H_{\infty,m+1}u - v\|,$$

and

$$u = P_C H_{\infty,m+1}u.$$

On the other hand, due to

$$H_{n,m+1}u = \lambda_{m+1}P_C H_{n,m+2}u + (1-\lambda_{m+1})u,$$

we have

$$H_{\infty,m+1}u = \lim_{n\to\infty} H_{n,m+1}u = \lambda_{m+1}P_C H_{\infty,m+2}u + (1-\lambda_{m+1})u = \lambda_{m+1}u + (1-\lambda_{m+1})u = u.$$

So $u = P_C H_{\infty,m+1}u = P_C u$, which means that $x \in F(P_C)$ and $F(W) \subset F(P_C)$. Therefore, we have $F(W) = F(P_C)$. □

We now introduce our self-adaptive inertial algorithm for solving SFP as follows.

Algorithm 3.4 The inertial algorithm for SFP.

Initialization: Given $\tau_0 > 0$, $\epsilon > 0$. Choose initial point $u_0,\ u_1 \in E_1$, and set $n = 1$.

Iterative Steps: Calculate u_{n+1} as follows:

Step 1: Given the iterates u_{n-1} and u_n. Set

$$z_n = u_n + \alpha_n(u_n - u_{n-1}),$$

where $0 \le \alpha_n < \overline{\alpha_n}$, $\{\epsilon_n\}$ is a positive sequence satisfying $\sum_{n=0}^{\infty}\epsilon_n < \infty$, and

$$\overline{\alpha_n} = \begin{cases} \min\{\frac{\epsilon_n}{\max\{\|u_n-u_{n-1}\|,\ g(\|u_n-u_{n-1}\|)\}},\ \alpha\}, & if\ u_n \neq u_{n-1}, \\ \alpha, & otherwise, \end{cases}$$

and calculate the step size as shown below:

$$\tau_n \in (\epsilon,\ \frac{2\|(I-P_Q)Az_n\|^2}{\|A^*J_{E_2}(I-P_Q)Az_n\|^2} - \epsilon), \quad n \in \Gamma$$

where $\Gamma := \{n \in \mathbb{N} : Az_n - P_Q Az_n \neq 0\}$, otherwise $\tau_n = \tau$, τ is nonnegative real number.

Step 2: Compute

$$w_n = z_n - \tau_n J_{E_1}^{-1}[A^*J_{E_2}(I-P_Q)Az_n].$$

Step 3: If $w_n = z_n$ stop. Otherwise, calculate u_{n+1} via

$$u_{n+1} = \kappa_n z_n + (1-\kappa_n)W_n w_n, \quad \forall n \ge 1.$$

where κ_n satisfies $\lim_{n\to\infty}\kappa_n = 0$ and $\sum_{n=0}^{\infty}\kappa_n = \infty$.

Step 4: Set $n = n+1$ and go to ***Step 1***.

Remark 3.5. It is easy to see from the definition of $\overline{\alpha_n}$ that

$$\sum_{n=1}^{\infty} \alpha_n \|u_n - u_{n-1}\| < \infty, \quad \sum_{n=1}^{\infty} \alpha_n g(\|u_n - u_{n-1}\|) < \infty,$$

which means

$$\lim_{n\to\infty} \alpha_n \|u_n - u_{n-1}\| = 0, \quad \lim_{n\to\infty} \alpha_n g(\|u_n - u_{n-1}\|) = 0.$$

Theorem 3.6. *The sequence $\{u_n\}$ is generated by Algorithm 3.4, then $\{u_n\}$ converges weakly to a solution $\tilde{u}$ of the SFP.*

Proof. Let $w \in \Omega$. According to the definition of W_n, we obtain $w = W_n w$. In the following, the proof will be divided into three steps.

To begin with, we show that the sequence $\{u_n\}$ is bounded. From Lemmas 2.3 and 2.5, we get

$$\begin{aligned}
\|w_n - w\|^2 &= \|z_n - w - \tau_n J_{E_1}^{-1}[A^* J_{E_2}(I - P_Q)Az_n]\|^2 \\
&\le \tau_n^2 \|J_{E_1}^{-1}[A^* J_{E_2}(I - P_Q)Az_n]\|^2 - 2\tau_n \langle z_n - w,\ A^* J_{E_2}(I - P_Q)Az_n\rangle + 2a^2\|z_n - w\|^2 \\
&= \tau_n^2 \|A^* J_{E_2}(I - P_Q)Az_n\|^2 + 2a^2\|z_n - w\|^2 + 2\tau_n \langle Aw - Az_n,\ J_{E_2}(I - P_Q)Az_n\rangle \\
&= \tau_n^2 \|A^* J_{E_2}(I - P_Q)Az_n\|^2 + 2a^2\|z_n - w\|^2 + 2\tau_n \langle Aw - P_Q Az_n,\ J_{E_2}(I - P_Q)Az_n\rangle \\
&+ 2\tau_n \langle P_Q Az_n - Az_n,\ J_{E_2}(I - P_Q)Az_n\rangle \\
&\le \tau_n^2 \|A^* J_{E_2}(I - P_Q)Az_n\|^2 + 2a^2\|z_n - w\|^2 - 2\tau_n \langle Az_n - P_Q Az_n,\ J_{E_2}(I - P_Q)Az_n\rangle \\
&= \tau_n^2 \|A^* J_{E_2}(I - P_Q)Az_n\|^2 + 2a^2\|z_n - w\|^2 - 2\tau_n \|(I - P_Q)Az_n\|^2 \\
&= 2a^2\|z_n - w\|^2 - \tau_n(2\|(I - P_Q)Az_n\|^2 - \tau_n \|A^* J_{E_2}(I - P_Q)Az_n\|^2) \\
&\le \|z_n - w\|^2 - \tau_n(2\|(I - P_Q)Az_n\|^2 - \tau_n \|A^* J_{E_2}(I - P_Q)Az_n\|^2).
\end{aligned} \tag{3.1}$$

And then, based on the construction of z_n and Lemma 2.4, we have

$$\begin{aligned}
\|z_n - w\|^2 &= \|(1 + \alpha_n)(u_n - w) - \alpha_n(u_{n-1} - w)\|^2 \\
&\le (1 + \alpha_n)\|u_n - w\|^2 - \alpha_n\|u_{n-1} - w\|^2 + (1 + \alpha_n)\alpha_n g(\|u_n - u_{n-1}\|).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|u_{n+1} - w\|^2 &= \|\kappa_n z_n + (1 - \kappa_n)W_n w_n - w\|^2 \\
&= \|\kappa_n(z_n - w) + (1 - \kappa_n)(W_n w_n - w)\|^2 \\
&\le \kappa_n\|z_n - w\|^2 + (1 - \kappa_n)\|w_n - w\|^2 - \kappa_n(1 - \kappa_n)g(\|z_n - W_n w_n\|) \\
&\le \|z_n - w\|^2 - (1 - \kappa_n)\tau_n(2\|(I - P_Q)Az_n\|^2 - \tau_n\|A^* J_{E_2}(I - P_Q)Az_n\|^2) \\
&- \kappa_n(1 - \kappa_n)g(\|z_n - W_n w_n\|) \\
&\le (1 + \alpha_n)\|u_n - w\|^2 - \alpha_n\|u_{n-1} - w\|^2 + (1 + \alpha_n)\alpha_n g(\|u_n - u_{n-1}\|) \\
&- (1 - \kappa_n)\tau_n(2\|(I - P_Q)Az_n\|^2 - \tau_n\|A^* J_{E_2}(I - P_Q)Az_n\|^2) \\
&- \kappa_n(1 - \kappa_n)g(\|z_n - W_n w_n\|) \\
&\le (1 + \alpha_n)\|u_n - w\|^2 - \alpha_n\|u_{n-1} - w\|^2 + 2\alpha_n g(\|u_n - u_{n-1}\|).
\end{aligned} \tag{3.2}$$

Let $\phi_n = \|u_n - w\|^2$, from (3.2), we can obtain

$$\phi_{n+1} - \phi_n \leq \alpha_n(\phi_n - \phi_{n-1}) + 2\alpha_n g(\|u_n - u_{n-1}\|),$$

where $\sum_{n=1}^{\infty} \alpha_n g(\|u_n - u_{n-1}\|) < \infty$.

Therefore, it follows from Lemma 2.12 that the limit of ϕ_n exists, and

$$\sum_{n=1}^{\infty}(\|u_{n+1} - w\|^2 - \|u_n - w\|^2)_+ < \infty,$$

which means that

$$\sum_{n=1}^{\infty}(\|u_{n+1} - w\|^2 - \|u_n - w\|^2) < \infty.$$

It also means that $\{u_n\}$ is bounded, so $\{z_n\}$ is bounded.

Secondly, we show that $\lim_{n\to\infty} \|Az_n - P_Q Az_n\| = 0$ and $\lim_{n\to\infty} \|z_n - W_n z_n\| = 0$. From (3.2) we have

$$\begin{aligned}
&2(1-\kappa_n)\tau_n(\|(I-P_Q)Az_n\|^2 - \frac{\tau_n}{2}\|A^* J_{E_2}(I-P_Q)Az_n\|^2) + \kappa_n(1-\kappa_n)g(\|z_n - W_n w_n\|) \\
&\leq \|u_n - w\|^2 - \|u_{n+1} - w\|^2 + \alpha_n(\|u_n - w\|^2 - \|u_{n-1} - w\|^2) + 2\alpha_n g(\|u_n - u_{n-1}\|).
\end{aligned}$$

Taking the limit of the above inequality, we get

$$\|(I-P_Q)Az_n\|^2 - \frac{\tau_n}{2}\|A^* J_{E_2}(I-P_Q)Az_n\|^2 \to 0, \quad n \to \infty, \tag{3.3}$$

and

$$g(\|z_n - W_n w_n\|) \to 0, \quad n \to \infty.$$

Since $\tau_n < \frac{2\|(I-P_Q)Az_n\|^2}{\|A^* J_{E_2}(I-P_Q)Az_n\|^2} - \epsilon$, we obtain

$$\frac{\epsilon}{2}\|A^* J_{E_2}(I-P_Q)Az_n\|^2 \leq \|(I-P_Q)Az_n\|^2 - \frac{\tau_n}{2}\|A^* J_{E_2}(I-P_Q)Az_n\|^2.$$

Thus,

$$\lim_{n\to\infty} \|A^* J_{E_2}(I-P_Q)Az_n\| = 0.$$

Then, according to (3.3), we have

$$\lim_{n\to\infty} \|(I-P_Q)Az_n\| = 0.$$

From the properties of g in Lemma 2.4 and $\lim_{n\to\infty} g(\|z_n - W_n w_n\|) = 0$, we know that

$$\lim_{n\to\infty} \|z_n - W_n w_n\| = 0.$$

In addition, since

$$\|z_n - w_n\| = \|J_{E_1}(z_n - w_n)\| = \tau_n\|A^* J_{E_2}(I-P_Q)Az_n\|,$$

then, we have

$$\lim_{n\to\infty} \|z_n - w_n\| = 0.$$

Therefore, according to W_n is a nonexpansive mapping, we obtain

$$\begin{aligned}\|z_n - W_n z_n\| &= \|z_n - W_n w_n + W_n w_n - W_n z_n\| \\ &\leq \|z_n - W_n w_n\| + \|W_n w_n - W_n z_n\| \\ &\leq \|z_n - W_n w_n\| + \|w_n - z_n\| \to 0, \quad n \to \infty. \end{aligned} \tag{3.4}$$

Finally, $\{u_n\}$ converges weakly to $\tilde{u} \in \Omega$ will be proved.

Due to the assumption that E_1 is a uniformly convex reflexive Banach space, as well as the fact that $\{u_n\}$ is a bounded sequence from the first step, so there exits a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \rightharpoonup \tilde{u}$. On the other hand, $\lim\limits_{n\to\infty} \|u_n - z_n\| = \lim\limits_{n\to\infty} \alpha_n \|u_n - u_{n-1}\| = 0$, so we obtain that $z_{n_j} \rightharpoonup \tilde{u}$. Then, since A is a bounded linear operator, we can get $Az_{n_j} \rightharpoonup A\tilde{u}$. By the previous step, we get

$$\|Az_{n_j} - P_Q A z_{n_j}\| \to 0, \ \|z_{n_j} - W_{n_j} z_{n_j}\| \to 0, \ j \to \infty.$$

Since W_n and P_Q are nonexpansive mapping, then it follows from Lemma 2.10 that $I - P_Q$ is demiclosed at zero, and by Lemma 3.3 we know that $F(W) = F(P_C)$, so we can obtain

$$\tilde{u} = P_C \tilde{u} \text{ and } A\tilde{u} = P_Q A\tilde{u},$$

that is $\tilde{u} \in C, \ A\tilde{u} \in Q$, hence $\tilde{u} \in \Omega$.

Now, we prove that $\{u_n\}$ converges weakly to $\tilde{u} \in \Omega$. As a matter of fact, we can assume that there exists another subsequence $\{u_{n_k}\}$ of $\{u_n\}$, such that $\{u_{n_k}\}$ converges weakly to $\check{u} \in \Omega$. Since E_1 has the Opial property, then we get as follows:

$$\begin{aligned}\liminf_{j\to\infty} \|u_{n_j} - \tilde{u}\| &< \liminf_{j\to\infty} \|u_{n_j} - \check{u}\| = \lim_{n\to\infty} \|u_n - \check{u}\| \\ &= \liminf_{k\to\infty} \|u_{n_k} - \check{u}\| < \lim\inf_{k\to\infty} \|u_{n_k} - \tilde{u}\| \\ &= \lim_{n\to\infty} \|u_n - \tilde{u}\| = \liminf_{j\to\infty} \|u_{n_j} - \tilde{u}\|.\end{aligned}$$

It's a contradiction. Hence $\{u_n\}$ converges weakly to $\tilde{u} \in \Omega$. As this point, the proof is completed. □

Remark 3.7. One of the significant advantages of the results in this paper is that the proposed inertial algorithm for solving the split feasibility problem in Banach space based on W-mapping is new, and the methods in this paper can be applied to solve SFP in Banach spaces, which are more general than Hilbert spaces ([1, 27]). On the other hand, the inertial technique can improve the performance of the algorithm. At last, the step size of our proposed algorithm is self-adaptive and does not require prior knowledge of the operator norm. This makes the calculation of our algorithm easier to implement than [1, 19, 23, 27].

It is well known that P_C is a nonexpansive mapping. Next, we consider replacing nonexpansive mapping W_n with P_C, and then we have the following corollary.

Corollary 3.8. *Let E_1 and E_2 be 2-uniformly smooth and uniformly convex real Banach spaces. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2, respectively, $A : E_1 \to E_2$ be a bounded* *linear operator and A^* is the adjoint of A. Let $\{\kappa_n\}$ be a sequence in $(0,1)$. For fixed point $u_0,\ u_1 \in E_1$, let $\{u_n\}$ be the sequence generated by the following iteration:*

$$\begin{cases} z_n = u_n + \alpha_n(u_n - u_{n-1}), \\ w_n = z_n - \tau_n J_{E_1}^{-1}[A^* J_{E_2}(I - P_Q)Az_n], \\ u_{n+1} = \kappa_n z_n + (1-\kappa_n)P_C w_n. \end{cases} \tag{3.5}$$

Suppose α_n, κ_n and τ_n satisfy the following conditions:
(i) $0 \le \alpha_n < \overline{\alpha_n}$, and

$$\overline{\alpha_n} = \begin{cases} \min\{\frac{\epsilon_n}{\max\{\|u_n - u_{n-1}\|,\ g(\|u_n - u_{n-1}\|)\}},\ \alpha\}, & if\ u_n \neq u_{n-1}, \\ \alpha, & otherwise, \end{cases}$$

(i) $\tau_n \in (\epsilon,\ \frac{2\|(I-P_Q)Az_n\|^2}{\|A^* J_{E_2}(I-P_Q)Az_n\|^2} - \epsilon)$,
(ii) $\lim\limits_{n\to\infty} \kappa_n = 0,\ \sum_0^\infty \kappa_n = \infty$.

Then the sequence $\{u_n\}$ converges weakly to a solution $\bar{u}$ of the SFP.

4. Numerical example

In this section, we give some preliminary numerical results and compare Algorithm 3.4 with (1.2), (1.3) and (3.5) to demonstrate the effectiveness of our proposed algorithm. All codes were written in MATLAB2015B. The numerical results were carried out on a personal Lenovo computer with Intel Core(TM) i5-7200 CPU @ 3.1GHz.

We give the numerical example in $(\mathbb{R}^3, \|\cdot\|_2)$ of the problem considered in Theorem 3.6.

Let

$$C := \{u = (u_1, u_2, u_3) \in \mathbb{R}^3 : \|u\| \le 1\},$$

and

$$Q := \{v = (v_1, v_2, v_3) \in \mathbb{R}^3 : \|v\| \le 1\}.$$

For Algorithm 3.4 and Eq (3.5), we take $\alpha_n = \alpha = 0.01$, $\kappa_n = \frac{1}{n+1}$ and $\tau_n = \tau = 0.12$, then the Algorithm 3.4 becomes

$$\begin{cases} z_n = u_n + \alpha_n(u_n - u_{n-1}), \\ w_n = z_n - \tau_n[A^*(I - P_Q)Az_n], \\ u_{n+1} = \frac{1}{n+1}z_n + (1 - \frac{1}{n+1})W_n w_n. \end{cases}$$

For algorithm (1.2), we take $\gamma = 0.12$, for algorithm (1.3), we take $\gamma = 0.12$ and $\lambda = 1/4$. The iterations stop with

$$\text{error} = \frac{\|u_{n+1} - u_n\|}{\|u_2 - u_1\|} < 10^{-5}.$$

We assume $u_0 = (0,0,0)$ and make different choices of u_1 and A:
Case I: Take $u_1 = (4, -6, -1)$ and

$$A = \begin{pmatrix} 3 & -2 & 1 \\ 2 & -1 & -2 \\ 1 & 0 & -1 \end{pmatrix}.$$

In this case, the numerical values are shown in Figure 1.

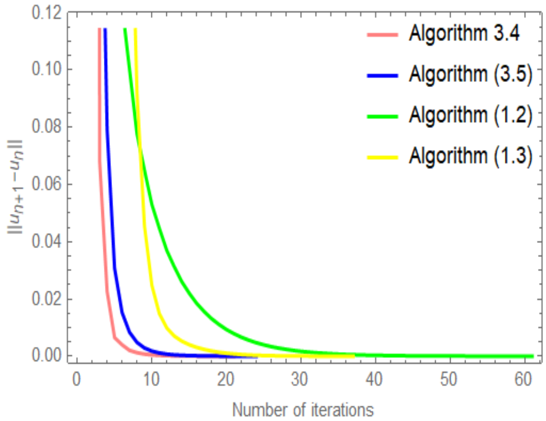

Figure 1. Case I: $u_1 = (4, -6, -1)$.

Case II: Take $u_1 = (-1, 3, 8)$ and

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{pmatrix}.$$

In this case, the numerical values are shown in Figure 2.

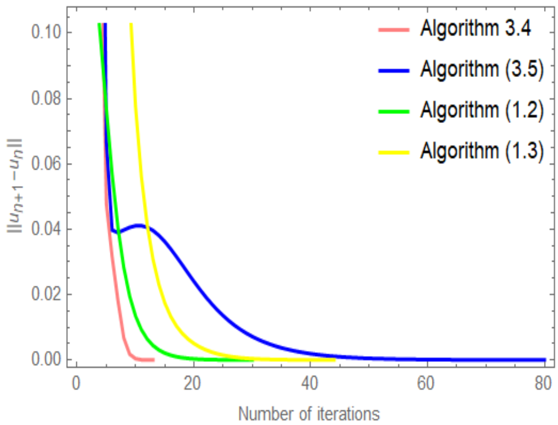

Figure 2. Case II: $u_1 = (-1, 3, 8)$.

From the above Figures and Table 1, we can observe that the error value decreases with the increase of the number of iterative steps, which means that all the algorithms for solving the SFP are effective. In addition, Algorithm 3.4 shows a faster decrease in error values, fewer iteration steps and shorter CPU time than (3.5), (1.2) and (1.3), which reflects the better effect of Algorithm 3.4.

Table 1. Comparison of Algorithm 3.4, (3.5), (1.2) and (1.3).

	Case	Number of iterations	Time
Algorithm 3.4	I	21	0.0057619
Algorithm (3.5)	I	24	0.0062141
Algorithm (1.2)	I	61	0.015625
Algorithm (1.3)	I	37	0.0068346
Algorithm 3.4	II	13	0.0034033
Algorithm (3.5)	II	80	0.015625
Algorithm (1.2)	II	30	0.0042064
Algorithm (1.3)	II	44	0.015625

5. Conclusions

A new self-adaptive algorithm with the inertial technique for solving the SFP in Banach spaces has been proposed in this work. The inertial term greatly improves the performance of the algorithm and has a good convergence property. Furthermore, the choice of step size is self-adaptive, which means that τ_n does not depend on a prior estimate of the operator norm A. This allows our algorithm to be computed more simply. Under some mild conditions, the weak convergence theorem of the algorithm for solving SFP is obtained. Through numerical experiments, the effectiveness of the algorithm was verified by comparing it with existing results.

In the future work, we shall develop other algorithms for solving SFP that are designed to converge faster and in less time. Furthermore, we shall also focus our attention on the applications of the theoretical results obtained on SFP to solve problems arising from medical sciences, economics and engineering.

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Conflict of interest

The author declares that there are no conflicts of interest.

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