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Research article

Extended Prudnikov sum

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Abstract: A Prudnikov sum is extended to derive the finite sum of the Hurwitz-Lerch Zeta function in terms of the Hurwitz-Lerch Zeta function. This formula is then used to evaluate a number trigonometric sums and products in terms of other trigonometric functions. These sums and products are taken over positive integers which can be simplified in certain circumstances. The results obtained include generalizations of linear combinations of the Hurwitz-Lerch Zeta functions and involving powers of 2 evaluated in terms of sums of Hurwitz-Lerch Zeta functions. Some of these derivations are in the form of a new recurrence identity and finite products of trigonometric functions.

Keywords: finite sum; finite product; trigonometric function; Catalan's constant; Hurwitz-Lerch Zeta function; Cauchy integral
Mathematics Subject Classification: 30E20, 33-01, 33-03, 33-04

1. Introduction

Mathematical functions can be represented in various forms. These forms can be a Fourier series, orthogonal polynomials, finite series, and finite products. Finite sums and products will be considered as the background for the derivation and evaluation of trigonometric functions in this work. The theory of finite series and products are used in a many areas of mathematics. These series are used in differential equations where this topic is treated in Sections (15.10) and (15.11) of [1]. In the field of Conformal Mappings, Section (15.17) of [1] these series are used in finding the quotient of two solutions of which map the closed upper half-plane conformally onto a curvilinear triangle. In the area of Group Representations, these series are used in harmonic analysis where it is more natural to represent hypergeometric functions as a Jacobi function Section (15.9(ii)) in [1]. In Combinatorics finite series are used wth respect to hypergeometric identities to classify single sums of products of binomial coefficients, Section (15.17(iv)) in [1]. These series are also used in Monodromy Groups, Section (15.17(v)) in [1] where the three singular points in Riemann's differential equation lead to an interesting Riemann sheet structure.

Finite sums of special functions was studied in the work of Apostol [2] where the analytic continuation of the Zeta and Dirichlet functions proved. In the work of Nakamura [3] consideration of the universality for linear combinations of Lerch zeta functions was studied.

In the book of Prudnikov et al. [4], one will find an elaborate list of indefinite and definite integrals, finite and infinite sums and products of elementary and special functions. Multidimensional forms of the latter are also listed in this volume which is used in almost all areas of mathematics. In the work of Khan et al. [5] two types of splitting algorithms were proposed for approximation of Cauchy type singular integrals having high frequency Fourier kernel.

In this present work we look to expand upon previous work featuring the finite sum of Special functions. We proceed by using the contour integral method [6] applied to Eq (4.4.6.18) in [4] to yield the contour integral representation given by:

$$\begin{aligned}\frac{1}{2\pi i}\int_C\sum_{p=0}^{n-1}\Big(&2^{-p-1}a^w w^{-k-1}\csc\left(2^{-p}(m+w)\right)\\&-2^{-p-1}a^w w^{-k-1}\cos\left(2^{-p-1}(m+w)\right)\csc\left(2^{-p}(m+w)\right)\Big)dw\\&=\frac{1}{2\pi i}\int_C\left(\frac{1}{2}a^w w^{-k-1}\csc(m+w)-2^{-n-1}a^w w^{-k-1}\csc\left(2^{-n}(m+w)\right)\right)dw\end{aligned} \tag{1.1}$$

where $a,m,k\in\mathbb{C},Re(m+w)>0,n\in\mathbb{Z}^+$. Using Eq (1.1) the main Theorem to be derived and evaluated is given by

$$\begin{aligned}\sum_{p=0}^{n-1}2^{-1}\left(i2^{-p}\right)^{k+1}e^{im2^{-p-1}}\Bigg(&\Phi\left(e^{i2^{1-p}m},-k,\frac{1}{4}-i2^{p-1}\log(a)\right)\\&+e^{im2^{-p}}\Phi\left(e^{i2^{1-p}m},-k,\frac{3}{4}-i2^{p-1}\log(a)\right)\\&-2e^{im2^{-p-1}}\Phi\left(e^{i2^{1-p}m},-k,\frac{1}{2}\left(1-i2^{p}\log(a)\right)\right)\Bigg)\\&=\left(i2^{-n}\right)^{k+1}e^{im2^{-n}}\Phi\left(e^{i2^{1-n}m},-k,\frac{1}{2}\left(1-i2^{n}\log(a)\right)\right)\\&\quad-ie^{\frac{1}{2}i(\pi k+2m)}\Phi\left(e^{2im},-k,\frac{1}{2}-\frac{1}{2}i\log(a)\right)\end{aligned} \tag{1.2}$$

where the variables k,a,m are general complex numbers and n is any positive integer. This new expression is then used to derive special cases in terms of trigonometric functions. The derivations follow the method used by us in [6]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)}=\frac{1}{2\pi i}\int_C\frac{e^{wy}}{w^{k+1}}dw, \tag{1.3}$$

where $y,w\in\mathbb{C}$ and C is in general an open contour in the complex plane where the bilinear concomitant [6] has the same value at the end points of the contour. This method involves using a form of Eq (1.3) then multiplies both sides by a function, then takes the definite integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Eq (1.3) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

2. The Hurwitz-Lerch Zeta function

We use Eq (1.11.3) in [7] where $\Phi(z, s, v)$ is the Lerch function which is a generalization of the Hurwitz zeta $\zeta(s, v)$ and Polylogarithm function $\mathrm{Li}_n(z)$. The Lerch function has a series representation given by

$$\Phi(z, s, v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n \tag{2.1}$$

where $|z| < 1, v \neq 0, -1, -2, -3, ..,$ and is continued analytically by its integral representation given by

$$\Phi(z, s, v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt \tag{2.2}$$

where $Re(v) > 0$, and either $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

3. Contour integral representation for the finite sum of the Hurwitz-Lerch Zeta functions

3.1. Derivation of the first finite sum of the contour integral

We use the method in [6]. The cut and contour are in the first quadrant of the complex w-plane with $0 < Re(w+m)$. The cut approaches the origin from the interior of the first quadrant and goes to infinity vertically and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula (1.3) we first replace y by $ix+y$ then multiply both sides by e^{imx} then form a second equation by replacing x by $-x$ and adding both equations to get

$$\frac{e^{-imx}\left(e^{2imx}(y+ix)^k + (y-ix)^k\right)}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C 2w^{-k-1}e^{wy}\cos(x(m+w))dw \tag{3.1}$$

Next we replace y by $2^{-p}i(2y+1)+\log(a)$, x by 2^{-p} and multiply both sides by $e^{im2^{-p}(2y+1)}$ and take the infinite and finite sums over $y \in [0, \infty)$ and $p \in [0, n-1]$, respectively and simplify in terms of the Hurwitz-Lerch zeta function to get

$$\begin{aligned}
&\sum_{p=0}^{n-1}\frac{1}{\Gamma(k+1)}i2^{k-p-1}\left(i2^{-p}\right)^k e^{im\left(2^{-p}-2^{-p-1}\right)}\left(\Phi\left(e^{i2^{1-p}m}, -k, 2^{p-1}\left(-i\log(a) - 2^{-p-1} + 2^{-p}\right)\right)\right.\\
&\qquad\left.+e^{im2^{-p}}\Phi\left(e^{i2^{1-p}m}, -k, 2^{p-1}\left(-i\log(a) + 2^{-p-1} + 2^{-p}\right)\right)\right)\\
&= \frac{1}{2\pi i}\sum_{y=0}^{\infty}\sum_{p=0}^{n-1}\int_C 2a^w w^{-k-1}e^{i2^{-p}(2y+1)(m+w)}\cos\left(2^{-p-1}(m+w)\right)dw
\end{aligned}$$

$$= \frac{1}{2\pi i} \int_C \sum_{p=0}^{n-1} \sum_{y=0}^{\infty} 2a^w w^{-k-1} e^{i2^{-p}(2y+1)(m+w)} \cos\left(2^{-p-1}(m+w)\right) dw$$

$$= -\frac{1}{2\pi i} \int_C \sum_{p=0}^{n-1} 2^{-p-1} a^w w^{-k-1} \cos\left(2^{-p-1}(m+w)\right) \csc\left(2^{-p}(m+w)\right) dw \quad (3.2)$$

from Eq (1.232.3) in [8] where $Re(w+m) > 0$ and $Im(m+w) > 0$ in order for the sums to converge. We apply Tonelli's theorem for multiple sums, see page 177 in [9] as the summands are of bounded measure over the space $\mathbb{C} \times [0, n-1] \times [0, \infty)$.

3.2. Derivation of the second finite sum of the contour integral

We use the method in [6]. Using Eq (1.3) we first replace $\log(a) + i2^{-p}(2y+1)$ and multiply both sides by $i2^{-n} e^{im2^{-n}(2y+1)}$ then take the finite and infinite sums over $p \in [0, n-1]$ and $y \in [0, \infty)$ and simplify in terms of the Hurwitz-Lerch Zeta function to get

$$-\sum_{p=0}^{n-1} \frac{2^k \left(i2^{-p}\right)^{k+1} e^{im2^{-p}} \Phi\left(e^{i2^{1-p}m}, -k, \frac{1}{2}\left(1 - i2^p \log(a)\right)\right)}{\Gamma(k+1)}$$

$$= -\frac{1}{2\pi i} \sum_{y=0}^{\infty} \sum_{p=0}^{n-1} \int_C i2^{-p} a^w w^{-k-1} e^{i2^{-p}(2y+1)(m+w)} dw$$

$$= -\frac{1}{2\pi i} \int_C \sum_{p=0}^{n-1} \sum_{y=0}^{\infty} i2^{-p} a^w w^{-k-1} e^{i2^{-p}(2y+1)(m+w)} dw$$

$$= \frac{1}{2\pi i} \int_C \sum_{p=0}^{n-1} 2^{-p-1} a^w w^{-k-1} \csc\left(2^{-p}(m+w)\right) dw \quad (3.3)$$

from Eq (1.232.3) in [8] where $Re(w+m) > 0$ and $Im(m+w) > 0$ in order for the sums to converge. We apply Tonelli's theorem for multiple sums, see page 177 in [9] as the summands are of bounded measure over the space $\mathbb{C} \times [0, n-1] \times [0, \infty)$.

3.3. Derivation of the finite sum of the contour integral

Here we formulate the finite sum of the Hurwitz-Lerch Zeta function in terms of the contour integral.

$$\sum_{p=0}^{n-1} \frac{1}{\Gamma(k+1)} 2^{k-1} \left(i2^{-p}\right)^{k+1} e^{im2^{-p-1}} \left(\Phi\left(e^{i2^{1-p}m}, -k, \frac{1}{4} - i2^{p-1}\log(a)\right)\right.$$
$$\left. + e^{im2^{-p}} \Phi\left(e^{i2^{1-p}m}, -k, \frac{3}{4} - i2^{p-1}\log(a)\right) - 2e^{im2^{-p-1}} \Phi\left(e^{i2^{1-p}m}, -k, \frac{1}{2}\left(1 - i2^p \log(a)\right)\right)\right)$$

$$= -\frac{1}{2\pi i}\int_C \sum_{p=0}^{n-1} 2^{-p-1} a^w w^{-k-1} \left(\cos\left(2^{-p-1}(m+w)\right) - 1\right) \csc\left(2^{-p}(m+w)\right) dw \quad (3.4)$$

from the addition of Eqs (3.3) and (3.2) and Eq (4.4.6.18) in [4] where $Re(m+w) > 0, Im(m+w) > 0$.

4. Contour integral representations for the Hurwitz-Lerch Zeta function

4.1. Derivation of the first contour integral

We use the method in [6]. Using Eq (1.3) we first replace $\log(a) + i(2y+1)$ and multiply both sides by $-ie^{im(2y+1)}$ then take the infinite sum over $y \in [0, \infty)$ and simplify in terms of the Hurwitz-Lerch Zeta function to get

$$-\frac{i2^k e^{\frac{1}{2}i(\pi k+2m)} \Phi\left(e^{2im}, -k, \frac{1}{2} - \frac{1}{2} i \log(a)\right)}{\Gamma(k+1)}$$

$$= -\frac{1}{2\pi i} \sum_{y=0}^{\infty} \int_C i a^w w^{-k-1} e^{i(2y+1)(m+w)} dw$$

$$= -\frac{1}{2\pi i} \int_C \sum_{y=0}^{\infty} i a^w w^{-k-1} e^{i(2y+1)(m+w)} dw$$

$$= \frac{1}{2\pi i} \int_C \frac{1}{2} a^w w^{-k-1} \csc(m+w) dw \quad (4.1)$$

from Eq (1.232.3) in [8] where $Im(w+m) > 0$ in order for the sum to converge. We apply Fubini's theorem for integrals and sums, see page 178 in [9] as the summand is of bounded measure over the space $\mathbb{C} \times [0, \infty)$.

4.2. Derivation of the second contour integral

We use the method in [6]. Using Eq (1.3) we first replace $\log(a) + i2^{-n}(2y+1)$ and multiply both sides by $i2^{-n} e^{im2^{-n}(2y+1)}$ then take the infinite sum over $y \in [0, \infty)$ and simplify in terms of the Hurwitz-Lerch Zeta function to get

$$\frac{2^k \left(i2^{-n}\right)^{k+1} e^{im2^{-n}} \Phi\left(e^{i2^{1-n}m}, -k, \frac{1}{2}\left(1 - i2^n \log(a)\right)\right)}{\Gamma(k+1)}$$

$$= \frac{1}{2\pi i} \sum_{y=0}^{\infty} \int_C i2^{-n} a^w w^{-k-1} e^{i2^{-n}(2y+1)(m+w)} dw$$

$$= \frac{1}{2\pi i}\int_C \sum_{y=0}^{\infty} i2^{-n}a^w w^{-k-1}e^{i2^{-n}(2y+1)(m+w)}dw$$

$$= -\frac{1}{2\pi i}\int_C 2^{-n-1}a^w w^{-k-1}\csc\left(2^{-n}(m+w)\right)dw \quad (4.2)$$

from Eq (1.232.3) in [8] where $Im(w+m) > 0$ in order for the sum to converge. We apply Fubini's theorem for integrals and sums, see page 178 in [9] as the summand is of bounded measure over the space $\mathbb{C}\times[0,\infty)$.

5. The finite sum of Hurwitz-Lerch Zeta functions in terms of Hurwitz-Lerch Zeta functions

In this section we will derive the finite sum of Hurwitz-Lerch Zeta functions in terms of the Hurwitz-Lerch Zeta function.

Theorem 5.1. *For all $k, a, m \in \mathbb{C}, n \in \mathbb{Z}^+$ then,*

$$\begin{aligned}
\sum_{p=0}^{n-1}\left(i2^{-p}\right)^{k+1}e^{im2^{-p-1}}\Bigg(\Phi\left(e^{i2^{1-p}m}, -k, \frac{1}{4} - i2^{p-1}\log(a)\right)& \\
+e^{im2^{-p}}\Phi\left(e^{i2^{1-p}m}, -k, \frac{3}{4} - i2^{p-1}\log(a)\right)& \\
-2e^{im2^{-p-1}}\Phi\left(e^{i2^{1-p}m}, -k, \frac{1}{2}\left(1 - i2^{p}\log(a)\right)\right)\Bigg)& \\
= 2\left(i2^{-n}\right)^{k+1}e^{im2^{-n}}\Phi\left(e^{i2^{1-n}m}, -k, \frac{1}{2}\left(1 - i2^{n}\log(a)\right)\right)& \\
- i2e^{\frac{1}{2}i(\pi k+2m)}\Phi\left(e^{2im}, -k, \frac{1}{2} - \frac{1}{2}i\log(a)\right)& \quad (5.1)
\end{aligned}$$

Proof. With respect to Eq (1.1) and observing the addition of the right-hand sides of relations (3.2) and (3.3), and the addition of relations (4.1) and (4.2) are identical; hence, the left-hand sides of the same are identical too. Simplifying with the Gamma function yields the desired conclusion. □

6. Special cases and finite sums and products involving trigonometric functions

In this section we will evaluate Eq (5.1) for various values of the parameters involved to derive special cases in terms of mathematical constants, trigonometric and special functions. We will also look at plots of finite sums and products functions involving mathematical constants.

Example 6.1. The degenerate case.

$$\sum_{p=0}^{n-1} 2^{-p}\sin^2\left(m2^{-p-2}\right)\csc\left(m2^{-p}\right) = \frac{1}{2}\left(\csc(m) - 2^{-n}\csc\left(m2^{-n}\right)\right) \quad (6.1)$$

Proof. Use Eq (5.1) and set $k = 0$ and simplify using entry (2) in Table below (64:12:7) in [10]. □

Example 6.2. A finite product involving quotient of cosine functions.

$$\prod_{p=0}^{n-1} \cos^4\left(2^{-p-4}x\right)\cos^5\left(2^{-p-2}x\right)\sec^8\left(2^{-p-3}x\right)\sec\left(2^{-p-1}x\right)$$

$$= \cos^4\left(\frac{x}{4}\right)\sec^4\left(\frac{x}{8}\right)\sec\left(\frac{x}{2}\right)\cos^4\left(2^{-n-3}x\right)\cos\left(2^{-n-1}x\right)\sec^4\left(2^{-n-2}x\right) \quad (6.2)$$

Proof. Use Eq (5.1) and set $k = 1, a = 1, m = x$ and simplify using the method in Section (8.1) in [11]. □

Example 6.3. A finite product involving the exponential of trigonometric functions.

$$\prod_{p=0}^{n-1} \cos^3\left(2^{-p-2}x\right)\sec^2\left(2^{-p-3}x\right)\sec\left(2^{-p-1}x\right)$$
$$\exp\left(-\frac{i2^{-p}\left(4\sin^2\left(2^{-p-2}x\right)\csc\left(2^{-p}x\right) - \tan\left(2^{-p-3}x\right)\sec\left(2^{-p-2}x\right)\right)}{\pi}\right)$$

$$= \tan\left(\frac{x}{2}\right)\cot\left(\frac{x}{4}\right)\tan\left(2^{-n-2}x\right)\cot\left(2^{-n-1}x\right)$$
$$\exp\left(\frac{i2^{1-n}\left(2^n\left(\csc\left(\frac{x}{2}\right) - \csc(x)\right) - \csc\left(2^{-n-1}x\right) + \csc\left(2^{-n}x\right)\right)}{\pi}\right) \quad (6.3)$$

Proof. Use Eq (5.1) and set $k = 1, a = i, m = x$ and simplify using the method in Section (8.1) in [11]. □

Example 6.4. The finite sum of the difference of the secant function squared.

$$\sum_{p=0}^{n-1} 4^{-p}\left(\sec^2\left(m2^{-p-2}\right) - 2\sec^2\left(m2^{-p-1}\right)\right) = 8\left(\cot(m)\csc(m) - 4^{-n}\cot\left(m2^{-n}\right)\csc\left(m2^{-n}\right)\right) \quad (6.4)$$

Proof. Use Eq (5.1) and set $k = 1, a = 1$ and simplify using the method in Section (8.1) in [11]. □

Example 6.5. The finite product of quotient tangent functions.

$$\sum_{p=0}^{n-1} \frac{\cos\left(2^{-p}m\right)\cos^2\left(2^{-1-p}r\right)}{\cos^2\left(2^{-1-p}m\right)\cos\left(2^{-p}r\right)} = \frac{\tan\left(2^{-n}m\right)\tan(r)}{\tan(m)\tan\left(2^{-n}r\right)} \quad (6.5)$$

Proof. Use Eq (5.1) and form a second equation by replacing $m \to r$ take the difference of both these equations then set $k = -1, a = 1$ and simplify using entry (3) of Section (64:12) in [10]. □

Example 6.6. Recurrence identity with consecutive neighbours.

$$\Phi(z,s,a)=\frac{1}{2z^{1/4}}\left(2^{1-s}z^{1/4}\left(\Phi\left(z^2,s,\frac{a}{2}\right)+z\Phi\left(z^2,s,\frac{a+1}{2}\right)\right)\right.$$
$$\left.-2^s\Phi\left(z^{1/2},s,2a-\frac{1}{2}\right)+\Phi\left(z,s,a-\frac{1}{4}\right)+z^{1/2}\Phi\left(z,s,a+\frac{1}{4}\right)\right) \quad (6.6)$$

Proof. Use Eq (5.1) and set $n=2, m=\log(z)/i, k=-s, a=e^{(a-1/2)i}$ and simplify. □

Example 6.7. The derivative of the Hurwitz-Lerch Zeta function.

$$\Phi'(i,0,u)=\log\left(\frac{\Gamma\left(\frac{u}{4}\right)}{2\Gamma\left(\frac{u+2}{4}\right)}\right)+i\log\left(\frac{\Gamma\left(\frac{u+1}{4}\right)}{2\Gamma\left(\frac{u+3}{4}\right)}\right) \quad (6.7)$$

Proof. Use Eq (6.6) and set $z=-1, a=u$ and simplify in terms of the Hurwitz Zeta function using entry (4) in Table below (64:12:7) in [10]. Next take the first partial derivative with respect to s and set $s=0$ and simplify using Eq (64:10:2) in [10]. □

Example 6.8. The derivative of the Hurwitz-Lerch Zeta function in terms of the Stieltjes constant γ_1.

$$\Phi'\left(i,1,\frac{3}{2}\right)=\frac{1}{4}\left(\gamma_1\left(\frac{7}{8}\right)+i\left(\gamma_1\left(\frac{1}{8}\right)+i\gamma_1\left(\frac{3}{8}\right)-\gamma_1\left(\frac{5}{8}\right)+\log(256)\right)\right.$$
$$\left.+4\left(\frac{1+i}{\sqrt{2}}\right)\log(2)\left(2\coth^{-1}\left(\sqrt{2}\right)+i\pi\right)\right) \quad (6.8)$$

Proof. Use Eq (6.6) and set $a=e^i, m=\pi/2$ and simplify in terms of the Zeta and Hurwitz Zeta functions using entry (4) of Section (64:12) and entry (2) of Section (64:7) in [10]. Next take the first partial derivative with respect to s and apply l'Hopital's rule as $s\to 0$ and simplify using Eq (3:6:8) in [10]. □

Example 6.9. A finite sum involving the cosecant function.

$$\sum_{p=0}^{n-1}\frac{2^{-p}\sin\left(m2^{-p}\right)}{\left(\cos\left(m2^{-p-2}\right)+\cos\left(3m2^{-p-2}\right)\right)^2}=2\left(\csc(m)-2^{-n}\csc\left(m2^{-n}\right)\right) \quad (6.9)$$

Proof. Use Eq (5.1) and set $a=e, k=1$ and simplify using entry (3) of Section (64:12) in [10]. Next we form a second equation by replacing $m\to -m$ and taking their difference. In this example $m\in\mathbb{C}$. □

Example 6.10. A finite sum involving Catalan's constant K.

$$\sum_{p=0}^{n-1}2\pi\log\left(\sec\left(\pi 2^{-p-2}\right)+1\right)-i2^{p+2}\left(4\mathrm{Li}_2\left(-e^{i2^{-p-2}\pi}\right)-\mathrm{Li}_2\left(-e^{i2^{-p-1}\pi}\right)\right)$$

$$=8K+i2^n\left(-4\mathrm{Li}_2\left(e^{-i2^{-n-1}\pi}\right)+4\mathrm{Li}_2\left(e^{i2^{-n-1}\pi}\right)+\mathrm{Li}_2\left(e^{-i2^{-n}\pi}\right)-\mathrm{Li}_2\left(e^{i2^{-n}\pi}\right)+\pi^2\right)$$
$$+\pi\log\left(\cot^2\left(\pi 2^{-n-2}\right)\right)-i\pi^2 \quad (6.10)$$

Proof. Use Eq (6.9) and multiply both sides by m and take the definite integral over $m \in [-\pi/2, \pi/2]$ and simplify using Eq (3.521.2) in [8]. □

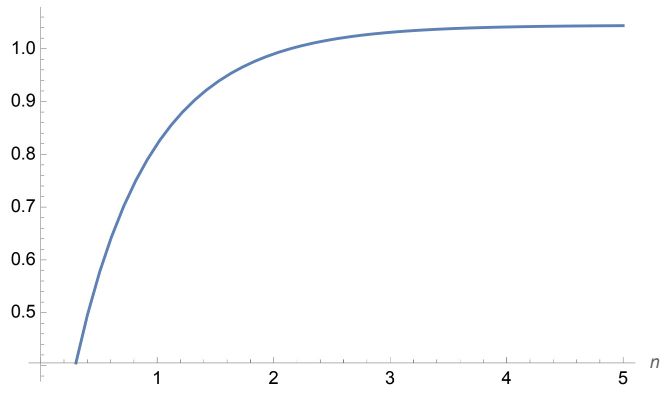

Figure 1. Real part of rhs of Eq (6.10).

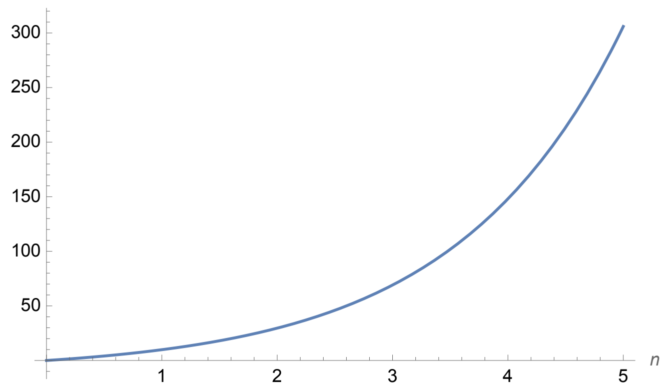

Figure 2. Imaginary part of rhs of Eq (6.10).

Example 6.11. A finite product involving Catalan's constant K.

$$\prod_{p=0}^{n-1} \exp\left(-i2^{2+p}\left(4\mathrm{Li}_2\left(-e^{i2^{-2-p}\pi}\right) - \mathrm{Li}_2\left(-e^{i2^{-1-p}\pi}\right)\right)\right)\left(1 + \sec\left(2^{-2-p}\pi\right)\right)^{2\pi}$$

$$= \exp\left(8K - i\pi^2 + i2^n\left(\pi^2 - 4\mathrm{Li}_2\left(e^{-i2^{-1-n}\pi}\right)\right)\right)$$
$$\exp\left(i2^n\left(4\mathrm{Li}_2\left(e^{i2^{-1-n}\pi}\right) + \mathrm{Li}_2\left(e^{-i2^{-n}\pi}\right) - \mathrm{Li}_2\left(e^{i2^{-n}\pi}\right)\right)\right)\cot^2\left(2^{-2-n}\pi\right)^{\pi} \quad (6.11)$$

Proof. Use Eq (6.10) and take the exponential function of both sides and simplify using Theorem 1 on page 133 in [12]. □

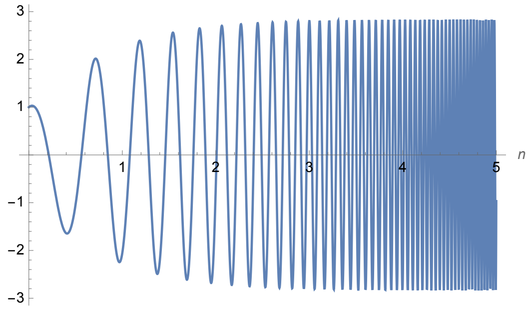

Figure 3. Real part of rhs of Eq (6.11) for n=5.

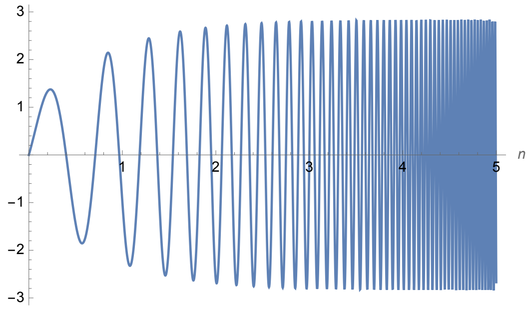

Figure 4. Imaginary part of rhs of Eq (6.11) for n=5.

Example 6.12. A finite sum involving quotient trigonometric functions.

$$\begin{aligned}\sum_{p=0}^{n-1}\Bigg(&\frac{2^{3+2p}a^2 8^{-p}\tan\left(2^{-2-p}m\right)\left(\cos\left(2^{-2-p}m\right)+\cos\left(3\,2^{-2-p}m\right)\right)^2}{\cos^2\left(2^{-2-p}m\right)\cos^3\left(2^{-1-p}m\right)}\\&+\frac{8^{-p}\tan\left(2^{-2-p}m\right)\left(\cos\left(3\,2^{-1-p}m\right)-13\cos\left(2^{-1-p}m\right)\right)}{\cos^2\left(2^{-2-p}m\right)\cos^3\left(2^{-1-p}m\right)}\\&-\frac{4\,8^{-p}\tan\left(2^{-2-p}m\right)\left(3+\cos\left(2^{-p}m\right)\right)}{\cos^2\left(2^{-2-p}m\right)\cos^3\left(2^{-1-p}m\right)}\Bigg)\\&=32\left(2\left(1+a^2\right)\csc(m)-4\csc^3(m)-2^{1-3n}\csc\left(2^{-n}m\right)\left(1+4^n a^2-2\csc^2\left(2^{-n}m\right)\right)\right)\end{aligned}\tag{6.12}$$

Proof. Use Eq (5.1) and set $a = e^{ai}, k = 2$ and simplify using entry (4) of Section (64:12) in [10]. Next we form a second equation by replacing $m \to -m$ and taking their difference. □

7. Conclusions

In this paper, we used a contour integration method to derive a new finite summation formula involving the Hurwitz-Lerch Zeta function along with some interesting special cases in terms of mathematical constants and plots, see Figures 1–4. A new derivative of the Hurwitz-Lerch Zeta function involving an imaginary parameter expressed in terms of the Log-gamma function was produced. We will be applying this contour integral method to other trigonometric functions to derive other finite and infinite sums and products for future work.

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Conflict of interest

The authors declare no conflict of interest.

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