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Research article

On n-ary ring congruences of n-ary semirings

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Abstract: In universal algebra, it is well-known that if S is an algebraic structure, then the kind of algebraic structure of S/ρ is similar to S where ρ is a congruence relation on S. In this work, we study the notion of a full k-ideal A of an n-ary semiring S and construct a congruence relation ρ on S with respect to the full k-ideal A in order to make the quotient n-ary semiring S/ρ to be an n-ary ring. Moreover, the notion of an h-ideal of an n-ary semiring was studied and connections between an h-ideal and a k-ideal of an n-ary semiring were investigated.

Keywords: n-ary ring; n-ary semiring; ring congruence; k-ideal; h-ideal
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1. Introduction

A semiring which is a common generalization of rings and distributive lattices was introduced first by Vandiver [26] in 1934. This algebraic structure appears in a natural manner in some applications to the theory of automata, formal languages, optimization theory and other branches of applied mathematics (for example, see [4, 5, 11–14, 18]). In algebraic structure point of view, we are able to study the concept of an n-ary semiring as a generalization of a semiring because a semiring is a special kind of an n-ary semiring where $n = 2$ and so every results on an n-ary semiring is also true on a semiring but not conversely.

In modern algebra, it is well-known that the kernel of a ring homomorphism is an ideal [2] and also true for a semiring homomorphism. Conversely, each ideal of a ring can be considered as the kernel of a ring homomorphism. Notwithstanding, this condition is not generally true in case of an ideal of a semiring [1]. However, this condition can be true if we replace the ideal by a special ideal which is called a k-ideal defined by Henriksen [15]. A more restrict class of ideals of a semiring which is called an h-ideal was introduced by Iizuka [16].

To generalize the algebraic system of an algebra from a binary operation to an n-ary operation, Dörnte [8] first defined the notion of an n-ary group in 1928. Later, Timm [25] studied an n-ary group with commutative property. As a generalization of a semigroup and an n-ary group, the notion of an n-ary semigroup was introduced by Siosson [22, 23]. Some properties of idempotent elements of an n-ary semigroup were studied by Dudek [10]. The concepts of homomorphism, quotient structures, and some ideal theoretic were studied by Crombez and Timm [6, 7]. Later, in 1981, Dudek [9] studied the divisibility property of an (m, n)-ring. As a generalization of a semiring and an (m, n)-ring, Alam, Rao and Davvaz [3, 19] introduced the notion of an (m, n)-semiring.

In 1992, Sen and Adhikari [20] studied the notion of a full k-ideal which is a k-ideal containing the set of all additively idempotent elements of a semiring and use it to construct a congruence relation in order to make the quotient semiring to be a ring. More results of full k-ideals of a semiring were also investigated by Sen and Maity [21] in 2021. As a similar way of Sen and Adhikari [20], Sunitha, Nagi Reddy, and Shobhalatha [24] studied full k-ideals of ternary semirings.

It is well-known that if we have a congruence relation ρ on an n-ary semiring S, then we can immediately obtain that S/ρ is also an n-ary semiring. It is interesting that what is the kind of a congruence relation ρ affecting S/ρ to be an n-ary ring. In this work, we study the notions of k-ideals and h-ideals of n-ary semirings and also investigate their connections. Finally, we use a full k-ideal to construct a congruence relation in order to make the quotient n-ary semiring to be an n-ary ring.

2. Preliminaries

Let $\mathbb{N}$ be the set of all natural numbers and $i, j, n \in \mathbb{N}$. An algebra $\langle S; f\rangle$ consisting of a nonempty set S together with an n-ary operation $f : S^n \to S$ is called an n-ary groupoid [8]. For $1 \leq i < j \leq n$, the sequence $y_i, y_{i+1}, y_{i+2}, \ldots, y_j$ of elements of S is denoted by y_i^j. If $j < i$, then we denote it to be the empty symbol. If $x_1 = x_2 = \cdots = x_{i-1} = x$ where $x_1, x_2, \ldots, x_{i-1}, x \in S$, we write $\overset{(i-1)}{x}$ instead of x_1^{i-1}. So, the term

$$f(\underbrace{x, x, \ldots, x}_{i-1 \text{ terms}}, y_i, y_{i+1}, \ldots, y_j, z_{j+1}, z_{j+2}, \ldots, z_n)$$

where $z_{j+1}, z_{j+2}, \ldots, z_n \in S$ can be simply represented by

$$f(\overset{(i-1)}{x}, y_i^j, z_{j+1}^n).$$

Similarly, for $1 \leq i < j \leq n$, we also denote the sequence $A_i, A_{i+1}, A_{i+2}, \ldots, A_j$ of nonempty subsets of S by A_i^j. If $A_1 = A_2 = \cdots = A_k = A$, where $1 < k \leq n$ and $A_1, A_2, \ldots, A_k$, A are nonempty subsets of S, then we write $\overset{(k)}{A}$ instead of A_1^k.

Let $x_1^{2n-1} \in S$. The associative law [10] for the n-ary operation f on S is defined by for all $1 \leq i < j \leq n$,

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}).$$

If this law holds for all elements $x_1^{2n-1} \in S$, an n-ary groupoid $\langle S; f\rangle$ is called an n-ary semigroup.

An n-ary semiring is an algebra $\langle S; +, f\rangle$ type $(2, n)$ for which $\langle S; +\rangle$ is a semigroup, $\langle S; f\rangle$ is an n-ary semigroup and for all $x_1^n, a, b \in S$, $1 \leq i \leq n$,

$$f(x_1^{i-1}, a + b, x_{i+1}^n) = f(x_1^{i-1}, a, x_{i+1}^n) + f(x_1^{i-1}, b, x_{i+1}^n).$$

Indeed, an n-ary semiring is a $(2, n)$-semiring [3]. An n-ary semiring $\langle S; +, f\rangle$ is said to be additively commutative if $a + b = b + a$ for all $a, b \in S$.

In this work, we simply write S instead of an additively commutative n-ary semiring $\langle S;+,f\rangle$.

For any nonempty subsets A, B, A_1^n of an n-ary semiring S, we denote

$$A+B=\{a+b\in S\mid a\in A,b\in B\}$$

and

$$f(A_1^n)=\{f(a_1^n)\in S\mid a_i\in A_i,1\leq i\leq n\}.$$

A nonempty subset T of an n-ary semiring S is called a subalgebra of S if $T+T\subseteq T$ and $f(\overset{(n)}{T})\subseteq T$.

Definition 1. *[3] Let $1\leq i\leq n$. A nonempty subset A of an n-ary semiring S is called an i-ideal of S if $A+A\subseteq A$ and $f(\overset{(i-1)}{S},A,\overset{(n-i)}{S})\subseteq A$. If A is an i-ideal of S for all $1\leq i\leq n$, then A is called an ideal of S.*

An element a of an n-ary semiring S is called additively regular if $a=a+b+a$ for some $b\in S$. If in addition, the element b is unique and satisfies $b=b+a+b$, then b is called the additively inverse of a in S and will be denoted by the notation a'. Particularly, if every element of S is additively regular, then S is called an additively regular n-ary semiring. Furthermore, if every additively regular element of S has the unique additively inverse, then S is called an additively inverse n-ary semiring.

Let S be an additively inverse n-ary semiring. It is obvious that $x=(x')'$ and $(x+y)'=x'+y'$ for all $x,y\in S$.

Lemma 1. *Let S be an additively inverse n-ary semiring. Then for any $x_1^n\in S$, $(f(x_1^n))'=f(x_1^{i-1},x_i',x_{i+1}^n)$, for all $1\leq i\leq n$.*

Proof. Let $x_1^n\in S$ and $1\leq i\leq n$. Since

$$f(x_1^n)+f(x_1^{i-1},x_i',x_{i+1}^n)+f(x_1^n)=f(x_1^{i-1},x_i+x_i'+x_i,x_{i+1}^n)=f(x_1^{i-1},x_i,x_{i+1}^n)=f(x_1^n)$$

and

$$f(x_1^{i-1},x_i',x_{i+1}^n)+f(x_1^n)+f(x_1^{i-1},x_i',x_{i+1}^n)=f(x_1^{i-1},x_i'+x_i+x_i',x_{i+1}^n)=f(x_1^{i-1},x_i',x_{i+1}^n),$$

we obtain that

$$(f(x_1^n))'=f(x_1^{i-1},x_i',x_{i+1}^n).$$

□

An element x of an n-ary semiring S is called additively idempotent if $x+x=x$. We define the set of all additively idempotent elements of S by $E^+=\{x\in S\mid x+x=x\}$. It is not difficult to verify that E^+ is an ideal of S.

A partially ordered set $(L,\prec)$ is said to be a lattice if every pair of elements a,b of L has both greatest lower bound and least upper bound. If every subset A of a lattice L has both greatest lower bound and least upper bound, then L is called a complete lattice. It is not difficult to show that a partially ordered set $(L,\prec)$ has the greatest element and every subset of L has the greatest lower bound if and only if L is a complete lattice.

A lattice L is called modular [17] if L satisfies the following law; for all $a,b\in L$, $a\leq b$ implies $a\vee(x\wedge b)=(a\vee x)\wedge b$, for every $x\in L$, where $x\vee y$ and $x\wedge y$ is the least upper bound and the greatest lower bound of $x,y\in L$, respectively.

Lemma 2. *[17] A lattice L is modular if and only if for any $a, b, c \in L$, $a \wedge b = a \wedge c$, $a \vee b = a \vee c$, and $b \leq c$ implies $b = c$.*

3. *k*-ideals and *h*-ideals of *n*-ary semirings

In this section, we introduce the notions of k-ideals and h-ideals of n-ary semirings and study some of their properties.

Definition 2. *A nonempty subset A of an n-ary semiring S is called a k-ideal of S if $A + A \subseteq A$, $f(\overset{(i-1)}{S}, A, \overset{(n-i)}{S}) \subseteq A$ for all $1 \leq i \leq n$ and the following condition is satisfied: for any $x \in S$, $x + a = b$ for some $a, b \in A$ implies $x \in A$. If A is a k-ideal of S and $E^+ \subseteq A$, then A is said to be a full k-ideal.*

According to Definition 2, it is clear that every k-ideal of an n-ary semiring is an ideal. However, the converse is not generally true as the following example shows.

Example 1. Define an n-ary operation f on $\mathbb{N}$ by $f(a_1^n) = a_1 \cdot a_2 \cdot a_3 \cdots a_n$ for any $a_1^n \in \mathbb{N}$. Then $\langle \mathbb{N}; \max, f \rangle$ is an n-ary semiring. We have that $2\mathbb{N}$ is an ideal of $\langle \mathbb{N}; \max, f \rangle$ but not a k-ideal because $\max\{1, 2\} = 2$ but $1 \notin 2\mathbb{N}$.

The following example is an example of a k-ideal of an n-ary semiring which is not a full k-ideal.

Example 2. Define an n-ary operation f on $\mathbb{N}$ by $f(a_1^n) = \min\{a_1, a_2, a_3 \ldots, a_n\}$ for any $a_1^n \in \mathbb{N}$. Then $\langle \mathbb{N}; \max, f \rangle$ is an n-ary semiring and $E^+ = \mathbb{N}$. It is easy to obtain that the set $\mathbb{I}_m = \{1, 2, 3, \ldots, m\}$ is a k-ideal of $\langle \mathbb{N}; \max, f \rangle$ but not a full k-ideal because $E^+ \nsubseteq \mathbb{I}_m$.

The following example is an example of a k-ideal of a finite n-ary semiring which is not a full k-ideal.

Example 3. Let $S = \{a, b\}$. Then $\langle P(S); \cup, f \rangle$ is an n-ary semiring where $P(S)$ is the power set of S and f is the n-ary operation on $P(S)$ defined by $f(A_1^n) = \bigcap_{i=1}^n A_i$ for any $A_i \in P(S)$. It is easy to show that $\{\emptyset, \{a\}\}$ is a k-ideal of $\langle P(S); \cup, f \rangle$ but not full because $E^+ = P(S) \nsubseteq \{\emptyset, \{a\}\}$.

We give an example of a proper full k-ideal of an n-ary semiring as follows.

Example 4. Consider the n-ary semiring $\langle \mathbb{N} \cup \{0\}; +, f \rangle$ where $+$ is the usual addition and f is the n-ary operation defined in Example 1. We have that the set of all additively idempotent elements of $\langle \mathbb{N} \cup \{0\}; +, f \rangle$ is $\{0\}$ and $2\mathbb{N} \cup \{0\}$ is a full k-ideal.

Remark 1. Let $\{A\}_{i \in I}$ be a family of full k-ideals of an n-ary semiring S. Then $\bigcap_{i \in I} A_i$ is a full k-ideal as well if it is not empty.

Remark 2. Every k-ideal of an additively inverse n-ary semiring S is an additively inverse subalgebra of S.

Proof. Let K be a k-ideal of S. Clearly, K is a subalgebra of S. Let $a \in K$. Then $(a + a') + a = a \in K$ and so $a + a' \in K$. This implies that $a' \in K$. Hence, K is additively inverse. □

The k-closure of a nonempty subset A of an n-ary semiring S is defined by

$$[A]_k = \{x \in S \mid x + a = b \text{ for some } a, b \in A\}.$$

It is easy to prove that for any $\emptyset \neq A \subseteq S$, $A \subseteq [A]_k$ if $A + A \subseteq A$. Furthermore, if A is closed under the addition, then $[A]_k$ is also closed. Now, we give some necessary properties of k-closure of nonempty subsets of an n-ary semiring as follows.

Lemma 3. *Let A, B, and A_1^n be nonempty subsets of an n-ary semiring S. Then the following statements hold:*

(1) if $A + A \subseteq A$, then $[A]_k = [[A]_k]_k$;
(2) if $A \subseteq B$, then $[A]_k \subseteq [B]_k$;
(3) $[A]_k + [B]_k \subseteq [A + B]_k$;
(4) if A_1^n are closed under the addition, then $f(A_i^{i-1}, [A_i]_k, A_{i+1}^n) \subseteq [f(A_1^n)]_k$ for all $1 \leq i \leq n$.

Proof. (1) Let $\emptyset \neq A \subseteq S$ be such that $A + A \subseteq A$. Obviously, $[A]_k \subseteq [[A]_k]_k$. If $x \in [[A]_k]_k$, then $x + y = z$ for some $y, z \in [A]_k$ such that $y + a_1 = b_1$ and $z + a_2 = b_2$ for some $a_1, a_2, b_1, b_2 \in A$. Then

$$x + y + a_1 + a_2 = z + a_1 + a_2 = z + a_2 + a_1 = b_2 + a_1. \tag{3.1}$$

We have $y + a_1 + a_2 = b_1 + a_2 \in A + A \subseteq A$ and $b_2 + a_1 \in A + A \subseteq A$. Using (3.1), we get $x \in [A]_k$ and so $[[A]_k]_k \subseteq [A]_k$.

(2)–(4) are straightforward. □

Lemma 4. *If A is an ideal of an n-ary semiring S, then $[A]_k$ is a k-ideal of S.*

Proof. Let A be an ideal of S. It is clear that $[A]_k$ is closed under the addition. Using A being an ideal of S and Lemma 3(2) and (4), we obtain that $f(\overset{(i-1)}{S}, [A]_k, \overset{(n-i)}{S}) \subseteq [f(\overset{(i-1)}{S}, A, \overset{(n-i)}{S})]_k \subseteq [A]_k$. If $x \in S$ is such that $x + a = b$ for some $a, b \in [A]_k$, then by Lemma 3(1), we get $x \in [[A]_k]_k = [A]_k$. Therefore, $[A]_k$ is a k-ideal of S. □

The following corollary is directly obtained by Lemma 4.

Corollary 1. *Let S be an n-ary semiring. The following statements hold:*

(1) an ideal A of S is a k-ideal if and only if $A = [A]_k$;
(2) $[E^+]_k$ is a full k-ideal of S.

Lemma 5. *Let A and B be two full k-ideals of an additively inverse n-ary semiring S. Then $[A + B]_k$ is a full k-ideal of S such that $A \subseteq [A + B]_k$ and $B \subseteq [A + B]_k$.*

Proof. Clearly, $A + B$ is closed under the addition. It holds that

$$f(\overset{(i-1)}{S}, A + B, \overset{(n-i)}{S}) \subseteq f(\overset{(i-1)}{S}, A, \overset{(n-i)}{S}) + f(\overset{(i-1)}{S}, B, \overset{(n-i)}{S}) \subseteq A + B$$

for all $1 \leq i \leq n$. Now, $A + B$ is an ideal of S. Using Lemma 4, we immediately get that $[A + B]_k$ is a k-ideal. Since $E^+ \subseteq A$ and $E^+ \subseteq B$, $E^+ = E^+ + E^+ \subseteq A + B \subseteq [A + B]_k$. Hence, $[A + B]_k$ is a full k-ideal of S.

Let $a \in A$. Then

$$a = a + a' + a = a + (a' + a) \in A + E^+ \subseteq A + B \subseteq [A + B]_k.$$

Hence, $A \subseteq [A + B]_k$. Similarly, we are able to get that $B \subseteq [A + B]_k$. □

Theorem 1. *Let $K(S)$ be the set of all full k-ideals of an additively inverse n-ary semiring S. Then $K(S)$ is a complete lattice which is also modular.*

Proof. We have that $K(S)$ is a partially ordered set with respect to usual set inclusion. Let $A, B \in K(S)$. By Remark 1 and Lemma 5, we obtain that $A \cap B \in K(S)$ and $[A + B]_k \in K(S)$, respectively. Define $A \wedge B = A \cap B$ and $A \vee B = [A + B]_k$. Obviously, $A \cap B$ is the greatest lower bound of A and B. Let $C \in K(S)$ such that $A \subseteq C$ and $B \subseteq C$. Then $A + B \subseteq C + C \subseteq C$. By Remark 3(2) and Corollary 1(1), we get $[A + B]_k \subseteq [C]_k = C$. Hence, $[A + B]_k$ is the least upper bound of A and B. Now, $K(S)$ is a lattice. Clearly, S is the greatest element of $K(S)$. Let $\{C_i\}_{i\in I}$ be a family of elements in $K(S)$. By Remark 1, we get that $\bigcap\{C_i\}_{i\in I} \in K(S)$. These imply that $K(S)$ is a complete lattice.

Finally, let $A, B, C \in K(S)$ such that

$$A \wedge B = A \wedge C \text{ and } A \vee B = A \vee C \text{ and } B \subseteq C.$$

Let $x \in C$. Then $x \in C \subseteq A \vee C = A \vee B = [A + B]_k$. It follows that there exist $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $x + a_1 + b_1 = a_2 + b_2$. Then

$$x + a_1 + a_1' + b_1 = x + a_1 + b_1 + a_1' = a_2 + b_2 + a_1' = a_2 + a_1' + b_2. \tag{3.2}$$

Now, $x \in C$, $a_1 + a_1' \in E^+ \subseteq C$ and $b_1, b_2 \in B \subseteq C$. Using (3.2), $a_2 + a_1' \in [C]_k = C$. At this point, $a_1 + a_1', a_2 + a_1' \in A \cap C = A \wedge C = A \wedge B = A \cap B \subseteq B$. It follows that $a_1 + a_1' + b_1 \in B$ and $a_2 + a_1' + b_2 \in B$. Using (3.2) again, we obtain that $x \in [B]_k = B$ and so $C \subseteq B$. Hence, $B = C$. By Lemma 2, $K(S)$ is a modular lattice. □

Now, we introduce a more restricted class of k-ideals of an n-ary semiring as follows.

Definition 3. *A nonempty subset A of an n-ary semiring S is called an h-ideal of S if $A + A \subseteq A$, $f(\overset{(i-1)}{S}, A, \overset{(n-i)}{S}) \subseteq A$ for all $1 \leq i \leq n$ and the following condition is satisfied: for any $x \in S$, $x + a + s = b + s$ for some $a, b \in A$ and $s \in S$ implies $x \in A$.*

It is unnecessary to define a full h-ideal of an n-ary semiring because every h-ideal is immediately full, i.e., if A is an h-ideal of S and $x \in E^+$, then for any $a \in A$, $x + a + x = a + x$ implies $x \in A$.

It is obvious that every h-ideal of an n-ary semiring is a k-ideal. In general, the converse is not true as it is shown by the following example.

Example 5. Let $S = \{a, b, c\}$.
Define an n-ary operation f on the power set $P(S)$ of S by $f(A_1^n) = \bigcap_{i=1}^n A_i$ for any $A_1^n \in P(S)$. Then $\langle P(S); \cup, f\rangle$ is an n-ary semiring. We have that $T = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ is a k-ideal of $\langle P(S); \cup, f\rangle$. However, T is not an h-ideal because $\{c\} \cup \{a, b\} \cup \{a, c\} = S = \{b\} \cup \{a, c\}$ where $\{a, b\}, \{b\} \in T$ but $\{c\} \notin T$.

Remark 3. Let $\{A\}_{i\in I}$ be a family of h-ideals of an n-ary semiring S. Then $\bigcap_{i\in I} A_i$ is an h-ideal as well if it is not empty.

Remark 4. Every h-ideal of an additively inverse n-ary semiring S is an additively inverse subalgebra of S.

Proof. Let H be an h-ideal of S. Clearly, H is a subalgebra of S. Let $a \in H$. Then $(a+a')+a+s = a+s$ for all $s \in S$. So, $a + a' \in H$. This means that $a' + a = b$ for some $b \in H$ and thus $a' + a + t = b + t$ for any $t \in S$. This implies that $a' \in H$. Hence, H is additively inverse. □

The h-closure of a nonempty subset A of an n-ary semiring S is defined by

$$[A]_h = \{x \in S \mid x + a + s = b + s \text{ for some } a, b \in A \text{ and } s \in S\}.$$

It is obvious that $[A]_k \subseteq [A]_h$ for any $\emptyset \neq A \subseteq S$. Moreover, it is not difficult to verify that for any $\emptyset \neq A \subseteq S$, $A \subseteq [A]_h$ if $A + A \subseteq A$. Furthermore, if A is closed under the addition, then $[A]_h$ is also closed. Now, we give some necessary properties of h-closure of nonempty subsets on an n-ary semiring as follows.

Lemma 6. *Let A, B and A_1^n be nonempty subsets of an n-ary semiring S. Then the following statements hold:*

(1) if $A + A \subseteq A$, then $[A]_h = [[A]_h]_h$;
(2) if $A \subseteq B$, then $[A]_h \subseteq [B]_h$;
(3) $[A]_h + [B]_h \subseteq [A + B]_h$;
(4) if A_1^n are closed under the addition, then $f(A_i^{i-1}, [A_i]_h, A_{i+1}^n) \subseteq [f(A_1^n)]_h$ for all $1 \leq i \leq n$.

Proof. (1) Let $\emptyset \neq A \subseteq S$ be such that $A + A \subseteq A$. Obviously, $[A]_h \subseteq [[A]_h]_h$. If $x \in [[A]_h]_h$, then $x + y + s = z + s$ for some $y, z \in [A]_h$ and $s \in S$ where $y + a_1 + u = b_1 + u$ and $z + a_2 + v = b_2 + v$ for some $a_1, a_2, b_1, b_2 \in A$ and $u, v \in S$. Then

$$\begin{aligned} x + y + s + a_1 + u + a_2 + v &= x + (y + a_1 + u) + a_2 + s + v \\ &= x + b_1 + u + a_2 + s + v \\ &= x + b_1 + a_2 + u + s + v \qquad (3.3) \\ x + y + s + a_1 + u + a_2 + v &= z + s + a_1 + u + a_2 + v \\ &= a_1 + (z + a_2 + v) + s + u \\ &= a_1 + b_2 + v + s + u. \qquad (3.4) \end{aligned}$$

Using (3.3) and (3.4), we get that $x + (b_1 + a_2) + u + s + v = (a_1 + b_2) + u + s + v$ where $b_1 + a_2, a_1 + b_2 \in A + A \subseteq A$ and $u + s + v \in S$ implies $x \in [A]_h$ and so $[[A]_h]_h \subseteq [A]_h$.

(2)–(4) are straightforward. □

Lemma 7. *If A is an ideal of an n-ary semiring S, then $[A]_h$ is an h-ideal of S.*

Proof. Let A be an ideal of S. Clearly, $[A]_h$ is closed under the addition. Using A being an ideal of S and Lemma 6(2) and (4), we obtain that $f(\overset{(i-1)}{S}, [A]_h, \overset{(n-i)}{S}) \subseteq [f(\overset{(i-1)}{S}, A, \overset{(n-i)}{S})]_h \subseteq [A]_h$. If $x \in S$ is such that $x + a + s = b + s$ for some $a, b \in [A]_h$ and $s \in S$, then by Lemma 6(1), we get $x \in [[A]_h]_h = [A]_h$. Therefore, $[A]_h$ is an h-ideal of S. □

The following corollary is directly obtained by Lemma 7.

Corollary 2. *Let S be an n-ary semiring. The following statements hold:*

(1) an ideal A of S is an h-ideal if and only if $A = [A]_h$;

(2) $[E^+]_h$ *is an h-ideal of S.*

Lemma 8. *Let A and B be two h-ideals of an additively inverse n-ary semiring S. Then* $[A+B]_h$ *is an h-ideal of S such that* $A \subseteq [A+B]_h$ *and* $B \subseteq [A+B]_h$.

Proof. Since $f(\overset{(i-1)}{S}, A+B, \overset{(n-i)}{S}) \subseteq f(\overset{(i-1)}{S}, A, \overset{(n-i)}{S}) + f(\overset{(i-1)}{S}, B, \overset{(n-i)}{S}) \subseteq A+B$ for all $1 \le i \le n$ and $A+B$ is closed under the addition, we get $A+B$ is an ideal of S. Using Lemma 7, we obtain that $[A+B]_h$ is an h-ideal. Let $a \in A$. Then $a = a + a' + a = a + (a' + a) \in A + E^+ \subseteq A + B \subseteq [A+B]_h$. Hence, $A \subseteq [A+B]_h$. Similarly, we are able to get that $B \subseteq [A+B]_h$. □

Theorem 2. *Let H(S) be the set of all h-ideals of an additively inverse n-ary semiring S. Then H(S) is a complete lattice which is also modular.*

Proof. We have that $H(S)$ is a partially ordered set with respect to the usual set inclusion. Let $A, B \in H(S)$. By Remark 3 and Lemma 8, we obtain that $A \cap B \in H(S)$ and $[A+B]_h \in H(S)$, respectively. Define $A \wedge B = A \cap B$ and $A \vee B = [A+B]_h$. Obviously, $A \cap B$ is the greatest lower bound of A and B. Let $C \in H(S)$ such that $A \subseteq C$ and $B \subseteq C$. Then $A + B \subseteq C + C \subseteq C$. By Remark 6(2) and Corollary 2(1), we get $[A+B]_h \subseteq [C]_h = C$. Hence, $[A+B]_h$ is the least upper bound of A and B. Now, $H(S)$ is a lattice. Clearly, S is the greatest element of $H(S)$. Let $\{C_i\}_{i \in I}$ be a family of elements of $H(S)$. By Remark 3, we obtain that $\bigcap\{C_i\} \in H(S)$. These imply that $H(S)$ is a complete lattice.

Finally, let $A, B, C \in H(S)$ such that

$$A \wedge B = A \wedge C \text{ and } A \vee B = A \vee C \text{ and } B \subseteq C.$$

Let $x \in C$. Then $x \in C \subseteq A \vee C = A \vee B = [A+B]_h$. It follows that there exist $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $s \in S$ such that $x + a_1 + b_1 + s = a_2 + b_2 + s$. Then

$$\begin{aligned} x + a_1 + a_1' + b_1 + s &= x + a_1 + b_1 + s + a_1' \\ &= a_2 + b_2 + s + a_1' \\ &= a_2 + a_1' + b_2 + s. \end{aligned} \tag{3.5}$$

Since, $x \in C$, $a_1 + a_1' \in E^+ \subseteq C$ and $b_1 \in B \subseteq C$, we have $x + a_1 + a_1' + b_1 \in C$. Using (3.5) and $b_2 \in B \subseteq C$, we get $a_2 + a_1' \in [C]_h = C$. At this point, $a_1 + a_1', a_2 + a_1' \in A \cap C = A \wedge C = A \wedge B = A \cap B \subseteq B$. It follows that $a_1 + a_1' + b_1 \in B$ and $a_2 + a_1' + b_2 \in B$. Using (3.5) again, we obtain that $x \in [B]_h = B$ and so $C \subseteq B$. Hence, $B = C$. By Lemma 2, $H(S)$ is a modular lattice. □

4. *n*-ary ring congruences

In this section, we characterize an n-ary ring congruence with respect to a full k-ideal of an additively inverse n-ary semiring.

Definition 4. *A binary relation ρ on an n-ary semigroup* $\langle S; f \rangle$ *is said to be a congruence if ρ is an equivalence relation and satisfies the following property; for any* $a_1^n, b_1^n \in S$,

$$(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n) \in \rho \text{ implies } (f(a_1^n), f(b_1^n)) \in \rho.$$

Lemma 9. *An equivalence relation ρ on an n-ary semigroup $\langle S\,;f\rangle$ is a congruence if and only if for any $a, b, x_1^n \in S$,*

$$(a,b) \in \rho \text{ implies } (f(x_1^{i-1}, a, x_{i+1}^n), f(x_1^{i-1}, b, x_{i+1}^n)) \in \rho$$

for each $1 \leq i \leq n$.

An equivalence relation ρ on an n-ary semiring $\langle S\,;+,f\rangle$ is a congruence if ρ is a congruence on $\langle S\,;+\rangle$ and $\langle S\,;f\rangle$.

Definition 5. *An n-ary semiring $\langle S\,;+,f\rangle$ is called an n-ary ring if $\langle S\,;+\rangle$ is a group. In other words, the following conditions are satisfied;*

(1) there exists $0 \in S$ such that $x + 0 = x = 0 + x$ for all $x \in S$;
(2) for each $x \in S$, there is $y \in S$ such that $x + y = 0 = y + x$.

If $\langle S\,;+,f\rangle$ is an n-ary ring, then the element y in the condition 2 is usually denoted by $-x$.

Definition 6. *A congruence ρ on an n-ary semiring S is called an n-ary ring congruence if the quotient n-ary semiring $S/\rho := \{[a]_\rho \mid a \in S\}$ is an n-ary ring.*

Theorem 3. *Let A be a full k-ideal of an additively inverse n-ary semiring S. Then the relation $\rho_A = \{(a,b) \in S \times S \mid a + b' \in A\}$ is an n-ary ring congruence such that $-[a]_{\rho_A} = [a']_{\rho_A}$.*

Proof. Let A be a full k-ideal of S.

Firstly, we show that ρ is an equivalence relation on S. Let $a, b, c \in S$. Since $a + a' \in E^+ \subseteq A$, ρ_A is reflexive. If $a + b' \in A$, then by Remark 2, we get $b + a' = (b')' + a' = (a + b')' \in A$ and so ρ_A is symmetric. Assume that $a + b' \in A$ and $b + c' \in A$. Then $a + c' + b + b' \in A$. Since $b + b' \in E^+ \subseteq A$, $a + c' \in [A]_k = A$. So, ρ_A is transitive. Now, ρ_A is an equivalence relation.

Secondly, let $a, b, c, x_1^n \in S$. Assume that $(a,b) \in \rho_A$. Then $a + b' \in A$ and so

$$\begin{aligned}(a+c) + (b+c)' &= a + c + c' + b' \\ &= (a + b') + (c + c') \\ &\in A + E^+ \subseteq A + A \subseteq A.\end{aligned}$$

Hence, $(a + c, b + c) \in \rho_A$. Using Lemma 1, we obtain that for each $1 \leq i \leq n$,

$$\begin{aligned}f(x_1^{i-1}, a, x_{i+1}^n) + (f(x_1^{i-1}, b, x_{i+1}^n))' &= f(x_1^{i-1}, a, x_{i+1}^n) + f(x_1^{i-1}, b', x_{i+1}^n) \\ &= f(x_1^{i-1}, a + b', x_{i+1}^n) \\ &\in f(\overset{(i-1)}{S}, A, \overset{(n-i)}{S}) \subseteq A.\end{aligned}$$

Hence, $(f(x_1^{i-1}, a, x_{i+1}^n), f(x_1^{i-1}, b, x_{i+1}^n)) \in \rho_A$. Therefore, we obtain that ρ_A is a congruence on S.

Finally, we show that S/ρ_A is an n-ary ring together with the operations $\oplus$ and F on S/ρ_A defined by

$$[a]_{\rho_A} \oplus [b]_{\rho_A} = [a + b]_{\rho_A} \text{ and } F([a_1]_{\rho_A}, [a_2]_{\rho_A}, \ldots, [a_n]_{\rho_A}) = [f(a_1^n)]_{\rho_A}$$

for any $a, b, a_1^n \in S$. It is immediately to obtain that $\langle S/\rho_A; \oplus, F\rangle$ is a quotient n-ary semiring of $\langle S\,;+,f\rangle$. Let $e \in E^+$ and $x \in S$. Then $(e + x) + x' = e + (x + x') \in E^+ + E^+ = E^+ \subseteq A$ and so $(e + x, x) \in \rho_A$. It follows that

$$[e]_{\rho_A} \oplus [x]_{\rho_A} = [e + x]_{\rho_A} = [x]_{\rho_A}.$$

Since $e + (x + x')' = e + x' + x \in A$, $(e, x + x') \in \rho_A$. It turns out that

$$[x]_{\rho_A} \oplus [x']_{\rho_A} = [x + x']_{\rho_A} = [e]_{\rho_A}.$$

Therefore, S/ρ_A is an n-ary ring. □

Theorem 4. *Let ρ be a congruence on an additively inverse n-ary semiring S such that S/ρ is an n-ary ring and $-[a]_\rho = [a']_\rho$. Then there exists a full k-ideal A of S such that $\rho_A = \rho$.*

Proof. Let $A = \{a \in S \mid (a, e) \in \rho \text{ for some } e \in E^+\}$. Using the reflexivity of ρ, we get $E^+ \subseteq A \neq \emptyset$. Let $a, b \in A$. Then there exist $e, f \in E^+$ such that $(a, e) \in \rho$ and $(b, f) \in \rho$. Then $(a + b, e + f) \in \rho$ and $e + f \in E^+$. Hence, $a + b \in A$ and thus $A + A \subseteq A$. Let $x \in f(\overset{(i-1)}{S}, A, \overset{(n-i)}{S})$ for each $1 \leq i \leq n$. Then $x = f(x_1^{i-1}, c, x_{i+1}^n)$ for some $x_1^n \in S$ and $c \in A$ such that $(c, g) \in \rho$ for some $g \in E^+$. It follows that $(x, f(x_1^{i-1}, g, x_{i+1}^n)) = (f(x_1^{i-1}, c, x_{i+1}^n), f(x_1^{i-1}, g, x_{i+1}^n)) \in \rho$. Since E^+ is an ideal of S, $f(x_1^{i-1}, g, x_{i+1}^n) + f(x_1^{i-1}, g, x_{i+1}^n) = f(x_1^{i-1}, g + g, x_{i+1}^n) \in f(\overset{(i-1)}{S}, E^+, \overset{(n-i)}{S}) \subseteq E^+$. So, $x \in A$ leads to $f(\overset{(i-1)}{S}, A, \overset{(n-i)}{S}) \subseteq A$. Now, A is an ideal of S.

Let $x \in [A]_k$. Then $x + a = b$ for some $a, b \in A$ where $(a, e) \in \rho$ and $(b, f) \in \rho$ for some $e, f \in E^+$. However, $[f]_\rho$ and $[e]_\rho$ are additively idempotent in the ring S/ρ. This obtains that $[e]_\rho = [f]_\rho$ is the zero element of S/ρ. It follows that $[f]_\rho = [b]_\rho = [x + a]_\rho = [x]_\rho \oplus [a]_\rho = [x]_\rho \oplus [e]_\rho = [x]_\rho$. Thus, $x \in A$ and so $[A]_k = A$. By Corollary 1(1), A is a full k-ideal of S.

Finally, we show that $\rho = \rho_A$. Let $(a, b) \in \rho$. Then $(a + b', b + b') \in \rho$. Since $b + b' \in E^+$, $a + b' \in A$ and thus $(a, b) \in \rho_A$. Hence, $\rho \subseteq \rho_A$. If $(a, b) \in \rho_A$, then $a + b' \in A$. Thus, $(a + b', e) \in \rho$ for some $e \in E^+$. We have that $[b]_\rho = [e]_\rho \oplus [b]_\rho = [a + b']_\rho \oplus [b]_\rho = [a]_\rho \oplus [b']_\rho \oplus [b]_\rho = [a]_\rho \oplus [b + b']_\rho = [a]_\rho$, since $b + b' \in E^+$. This shows that $(a, b) \in \rho$ and so $\rho_A \subseteq \rho$. Therefore, $\rho = \rho_A$. □

The concepts of full k-ideals and h-ideals of an additively inverse n-ary semiring are coincidence as the following remark.

Remark 5. If S is an additively inverse n-ary semiring, then full k-ideals and h-ideals coincide in S.

Proof. Since every h-ideal is a full k-ideal, we show that every full k-ideal is an h-ideal. Let A be a full k-ideal. Then S/ρ_A is an n-ary ring and A is its zero element by Theorem 3. Let $x \in S$ and $x + a + s = b + s$ for some $a, b \in A$ and $s \in S$. Then $[x]_\rho + [a]_\rho + [s]_\rho = [b]_\rho + [s]_\rho$. It follows that $[x]_\rho + 0 + [s]_\rho = 0 + [s]_\rho$ where 0 is the zero of S/ρ_A. Hence, $[x]_\rho + [s]_\rho = [s]_\rho$ and so $[x]_\rho = 0$. It turns out that $x \in A$. Therefore, A is an h-ideal. □

5. Discussion and conclusions

In algebraic structure point of view, we are able to say that an n-ary semiring is a generalization of a semiring and a ternary semiring, and any results on an n-ary semiring are also true on a semiring and a ternary semiring because a semiring is an n-ary semiring where $n = 2$ and a ternary semiring is an n-ary semiring where $n = 3$.

The notions of a k-ideal and a full k-ideal of an n-ary semiring were defined in Section 3. There is a k-ideal which is not full as it is shown by Example 2. However, every h-ideal of an n-ary semiring is immediately full. Moreover, h-ideals and full k-ideals are coincidence in an additively inverse n-ary semiring and the set of all of them forms a complete lattice and also a modular lattice.

A group (ring) congruence is such a congruence relation on a semigroup (semiring) that the quotient semigroup (semiring) is a group (ring). Similarly, an n-ary ring congruence is such a congruence relation on an n-ary semiring that the quotient n-ary semiring is an n-ary ring. Constructing a relation with respect to a full k-ideal of an additively inverse n-ary semiring is a way to obtain an n-ary ring congruence. Indeed, if A is a full k-ideal of an additively inverse n-ary semiring S, then the relation $\rho_A = \{(a, b) \in S \times S \mid a + b' \in A\}$ is an n-ary ring congruence where b' is the additively inverse of b. Conversely, if ρ is an n-ary ring congruence on an additively inverse n-ary semiring S, then there exists a such full k-ideal A of S that $\rho = \rho_A$.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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