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Research article

Asymptotic behavior of plate equations with memory driven by colored noise on unbounded domains

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Abstract: The paper investigates mainly the asymptotic behavior of the non-autonomous random dynamical systems generated by the plate equations with memory driven by colored noise defined on $\mathbb{R}^n$. Firstly, we prove the well-posedness of the equation in the natural energy space. Secondly, we define a continuous cocycle associated with the solution operator. Finally, we establish the existence and uniqueness of random attractors of the equation by the uniform tail-ends estimates methods and the splitting technique.

Keywords: plate equation; colored noise; memory; asymptotic compactness; random attractor; unbounded domain
Mathematics Subject Classification: 35B40, 60H15

1. Introduction

The colored noise was first introduced in [23, 24] in order to obtain the information of velocity of randomly moving particles, which cannot be obtained from the white noise since the the Wiener process is nowhere differentiable. Moreover, for many physical systems, the stochastic fluctuations are correlated and should be modeled by the colored noise rather than the white noise, see [20].

This paper is concerned the asymptotic behavior of the plate equation driven by nonlinear colored noise in unbounded domains:

$$\begin{cases} u_{tt} + \alpha u_t + \Delta^2 u + \int_0^\infty \mu(s)\Delta^2(u(t) - u(t-s))ds + \nu u + f(x,u) \\ \qquad\qquad = g(x,t) + h(t,x,u)\zeta_\delta(\theta_t\omega),\ t > \tau,\ x \in \mathbb{R}^n, \\ u(x,\tau) = u_0(x),\ \ u_t(x,\tau) = u_{1,0}(x),\ x \in \mathbb{R}^n,\ \ t \le \tau, \end{cases} \tag{1.1}$$

where $\tau \in \mathbb{R}$, α, ν are positive constants, μ is the memory kernel, f and h are given nonlinearity, $g \in L^2_{loc}(\mathbb{R}, H^1(\mathbb{R}^n))$, and ζ_δ is a colored noise with correlation time $\delta > 0$.

It is clear that (1.1) becomes a deterministic plate equation as $\mu \equiv 0$ and $h \equiv 0$. In this case, we can characterize the long-time behavior of solutions by virtue of the concept of global attractors under the framework of semigroup. Some authors have extensively studied the existence of global attractors for the autonomous plate equation. For instance, the attractors of deterministic plate equations have been investigated in [2,8,12,14,30,32–35,44] in bounded domains. In [2,30,34,35], the authors considered global attractor for the plate equation with thermal memory; Khanmamedov investigated a global attractor for the plate equation with displacement-dependent damping in [8]; Liu and Ma obtained the existence of time-dependent strong pullback attractors for non-autonomous plate equations in [12,14]; Yang and Zhong studied the uniform attractor and global attractor for non-autonomous plate equations with nonlinear damping in [32, 33], respectively; In [44], the author obtained global existence and blow-up of solutions for a Kirchhoff type plate equation with damping. For the case of unbounded domains, see refereces [9, 10, 13, 31, 42].

The existence and uniqueness of pathwise random attractors of stochastic plate equations have been studied in [15, 16, 21, 22] in the case of bounded domains; and in [36–41] in the case of unbounded domains. In all these publications ([36–41]), only the additive white noise and linear multiplicative white noise were considered. Notice that the random equation (1.1) is driven by the colored noise rather than the white noise. In general, it is very hard to study the asymptotic dynamics of differential equations driven by nonlinear white noise, including the random attractors. Indeed, only when the white noise is linear, the stochastic equations can be transformed into a deterministic equations, then one can obtain the existence of random attractors of the plate equation (1.1). However, this transformation does not apply to stochastic equations driven by nonlinear white noise, and that is why we are currently unable to prove the existence of random attractors for systems with nonlinear white noise.

For the colored noise, even it is nonlinear, we are able to show system (1.1) has a random attractor in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_{\mu,2}$ (the definition of $\mathfrak{R}_{\mu,2}$ see Section 3), which is quite different from the nonlinear white noise. The reader is referred to [6, 7, 26, 27] for more details on random attractors of differential equations driven by colored noise. However, for the random plate equations driven by colored noise (1.1), we find that there is no results available to the existence of random attractors. In the present paper, we will prove that (1.1) is pathwise well-posed and generate a continuous cocycle, and the cocycle possesses a unique tempered random attractor. This is different from the corresponding stochastic system driven by white noise

$$u_{tt}+\alpha u_t+\Delta^2 u+\int_0^\infty \mu(s)\Delta^2(u(t)-u(t-s))ds+\nu u+f(x,u) = g(x,t)+h(t,x,u)\circ\frac{dW}{dt},\ t>\tau,\ x\in\mathbb{R}^n, \quad (1.2)$$

where the symbol $\circ$ indicates that the equation is understood in the sense of stratonovich integration. For (1.2), one can define a random dynamical system when $h(\cdot,\cdot,u)$ is a linear function, see [41]. But for a general nonlinear function h, random dynamical system associated with (1.2) can not be defined due to the absence of appropriate transformation, hence asymptotic behavior of such stochastic equations has not been investigated until now by the random dynamical system approach. This paper indicates that the colored noise is much easier to handle than the white noise for studying pathwise dynamics of such stochastic equations.

The main purpose of the paper is establish the existence and uniqueness of measurable tempered random attractors in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_{\mu,2}$ for the dynamical system associated with (1.1). The key for achieving our goal is to establish the tempered pullback asymptotic compactness of solutions of (1.1) in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \Re_{\mu,2}$. Involving to our problem (1.1), there are two essential difficulties in verifying the compactness. On the one hand, notice that system (1.1) is defined in the unbounded domain $\mathbb{R}^n$ where the noncompactness of Sobolev embeddings on unbounded domains gives rise to difficulty in showing the pullback asymptotic compactness of solutions, to get through of it, we use the tail-estimates method (as in [25]) and the splitting technique (see [3]) to obtain the pullback asymptotic compactness. On the other hand, there is no applicable compact embedding property in the "history" space. In this case, we solve it with the help of a useful result in [19]. For our purpose, we introduce a new variable and an extend Hilbert space.

The rest of this article consists of four sections. In the next section, we define some functions sets and recall some useful results. In Section 3, we first establish the existence, uniqueness and continuity of solutions in initial data of (1.1) in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \Re_{\mu,2}$, then define a non-autonomous random dynamical system based on the solution operator of problem (1.1). The last two section are devoted to derive necessary estimates of solutions of (1.1) and the existence of random attractors.

Throughout the paper, the inner product and the norm of $L^2(\mathbb{R}^n)$ will be denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. The letters c and c_i $(i = 1, 2, \ldots)$ are generic positive constants which may depend on some parameters in the contexts.

2. Asymptotic compactness of cocycles

In this section, we define some functions sets and recall some useful results, see [4,17,18,28,29,43]. These results will be used to establish the asymptotic compactness of the solutions and attractor for the random plate equation defined on the entire space $\mathbb{R}^n$.

From now on, we assume $(\Omega, \mathcal{F}, P)$ is the canonical probability space where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ with compact-open topology, $\mathcal{F}$ is the Borel σ-algebra of Ω, and P is the Wiener measure on $(\Omega, \mathcal{F})$. Recall the standard group of transformations $\{\theta_t\}_{t\in\mathbb{R}}$ on Ω:

$$\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t), \ \forall\, t \in \mathbb{R} \text{ and } \forall\, \omega \in \Omega.$$

Suppose $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is a continuous cocycle on X over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. Let $\mathcal{D}$ be a collection of some families of nonempty subset of X:

$$\mathcal{D} = \{D = \{D(\tau, \omega) \subseteq X : D(\tau, \omega) \neq \emptyset, \tau \in \mathbb{R}, \omega \in \Omega\}\}.$$

Suppose Φ has a $\mathcal{D}$-pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$; that is, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D \in \mathcal{D}$ there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega). \tag{2.1}$$

Assume that

$$\Phi(t, \tau, \omega, x) = \Phi_1(t, \tau, \omega, x) + \Phi_2(t, \tau, \omega, x), \ \forall\, t \in \mathbb{R}^+, \ \tau \in \mathbb{R}, \ \omega \in \Omega, \ x \in X, \tag{2.2}$$

where both Φ_1 and Φ_2 are mappings from $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times X$ to X.

Given $k \in \mathbb{N}$, denote by $\mathcal{O}_k = \{x \in \mathbb{R}^n : |x| < k\}$ and $\tilde{\mathcal{O}}_k = \{x \in \mathbb{R}^n : |x| > k\}$. Let X be a Banach space with norm $\|\cdot\|_X$ which consists of some functions defined on $\mathbb{R}^n$. Given a function $u : \mathbb{R}^n \to \mathbb{R}$, the restrictions of u to $\mathcal{O}_k$ and $\tilde{\mathcal{O}}_k$ are written as $u|_{\mathcal{O}_k}$ and $u|_{\tilde{\mathcal{O}}_k}$, respectively. Denote by

$$X_{\mathcal{O}_k} = \{u|_{\mathcal{O}_k} : u \in X\} \text{ and } X_{\tilde{\mathcal{O}}_k} = \{u|_{\tilde{\mathcal{O}}_k} : u \in X\}.$$

Suppose $X_{\mathcal{O}_k}$ and $X_{\tilde{\mathcal{O}}_k}$ are Banach spaces with norm $\|\cdot\|_{\mathcal{O}_k}$ and $\|\cdot\|_{\tilde{\mathcal{O}}_k}$, respectively, and

$$\|u\|_X \le \|u|_{\mathcal{O}_k}\|_{\mathcal{O}_k} + \|u|_{\tilde{\mathcal{O}}_k}\|_{\tilde{\mathcal{O}}_k}, \quad \forall\, u \in X. \tag{2.3}$$

We further assume that for every $\delta > 0$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$, there exists $t_0 = t_0(\delta, \tau, \omega, K) > 0$ and $k_0 = k_0(\delta, \tau, \omega) \ge 1$ such that

$$\|\Phi(t_0, \tau - t_0, \theta_{-t_0}\omega, x)|_{\tilde{\mathcal{O}}_{k_0}}\|_{\tilde{\mathcal{O}}_{k_0}} < \delta, \ \forall\, x \in K(\tau - t_0, \theta_{-t_0}\omega), \tag{2.4}$$

and

$$\Phi_1(t_0, \tau - t_0, \theta_{-t_0}\omega, K(\tau - t_0, \theta_{-t_0}\omega))|_{\mathcal{O}_{k_0}} \text{has a finite cover of balls of radius } \delta \text{ in } X|_{\mathcal{O}_{k_0}}. \tag{2.5}$$

In addition, we assume that for every $k \in \mathbb{N}$, $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$, the set

$$\Phi_2(t, \tau - t, \theta_{-t}\omega, K(\tau - t, \theta_{-t}\omega)) \text{ is precompact in } X|_{\mathcal{O}_k}. \tag{2.6}$$

Theorem 2.1 [29]. *If (2.1)-(2.6) hold, then the cocycle Φ is $\mathcal{D}$-pullback asymptotically compact in X; that is, the sequence $\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ is precompact in X for any $\tau \in \mathbb{R}, \omega \in \Omega, D \in \mathcal{D}, t_n \to \infty$ monotonically, and $x_n \in D(\tau - t_n, \theta_{-t_n}\omega)$.*

Theorem 2.2 [29]. *Let $\mathcal{D}$ be an inclusion closed collection of some families of nonempty subsets of X, and Φ be a continuous cocycle on X over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. Then Φ has a unique $\mathcal{D}$-pullback random attractor $\mathcal{A}$ in $\mathcal{D}$ if Φ is $\mathcal{D}$-pullback asymptotically compact in X and Φ has a closed measurable $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$.*

3. Cocycles of random plate equations

In this section, we first establish the existence of solution for problem (1.1), then define a non-autonomous cocycle of (1.1).

Given $\delta > 0$, let $\zeta_\delta(\theta_t\omega)$ be the unique stationary solution of the stochastic equation:

$$d\zeta_\delta + \frac{1}{\delta}\zeta_\delta dt = \frac{1}{\delta}dW, \tag{3.1}$$

where W is a two-sided real-valued Wiener process on $(\Omega, \mathcal{F}, P)$. The process $\zeta_\delta(\theta_t\omega)$ is called the one-dimensional colored noise. Recall that there exists a θ_t-invariant subset of full measure (see [1]), which is still denoted by Ω, such that for all $\omega \in \Omega$, $\zeta_\delta(\theta_t\omega)$ is continuous in $t \in \mathbb{R}$ and

$$\lim_{t\to\pm\infty} \frac{\zeta_\delta(\theta_t\omega)}{t} = 0.$$

Let $-\Delta$ denote the Laplace operator in $\mathbb{R}^n$, $A = \Delta^2$ with the domain $D(A) = H^4(\mathbb{R}^n)$. We can also define the powers A^ν of A for $\nu \in \mathbb{R}$. The space $V_\nu = D(A^{\frac{\nu}{4}})$ is a Hilbert space with the following inner product and norm

$$(u, v)_\nu = (A^{\frac{\nu}{4}} u, A^{\frac{\nu}{4}} v), \qquad \|\cdot\|_\nu = \|A^{\frac{\nu}{4}} \cdot\|.$$

Following Dafermos [5], we introduce a Hilbert "history" space $\mathfrak{R}_{\mu,2} = L^2_\mu(\mathbb{R}^+, V_2)$ with the inner product

$$(\eta_1, \eta_2)_{\mu,2} = \int_0^\infty \mu(s)(\Delta\eta_1(s), \Delta\eta_2(s))ds, \quad \forall\, \eta_1,\ \eta_2 \in \mathfrak{R}_{\mu,2},$$

and new variables

$$\eta = \eta^t(x, s) = u(x, t) - u(x, t - s),\ (x, s) \in \mathbb{R}^n \times \mathbb{R}^+,\ \ t \geq \tau.$$

By differentiation we have

$$\eta^t_t(x, s) = -\eta^t_s(x, s) + u_t(x, t),\ (x, s) \in \mathbb{R}^n \times \mathbb{R}^+,\ \ t \geq \tau.$$

Then (1.1) can be rewritten as the equivalent system

$$\begin{cases} u_{tt} + \alpha u_t + \Delta^2 u + \int_0^\infty \mu(s)\Delta^2\eta^t(s)ds + \nu u + f(x, u) \\ \qquad\qquad = g(x, t) + h(t, x, u)\zeta_\delta(\theta_t\omega),\ t > \tau,\ x \in \mathbb{R}^n, \\ \eta^t_t + \eta^t_s = u_t, \\ u(x, \tau) = u_0(x),\ \ u_t(x, \tau) = u_{1,0}(x),\ x \in \mathbb{R}^n,\ \ t \leq \tau, \\ \eta^\tau(x, s) = \eta^0(x, s) = u(x, \tau) - u(x, \tau - s),\ \ x \in \mathbb{R}^n,\ s \in \mathbb{R}^+. \end{cases} \tag{3.2}$$

We introduce the following hypotheses to complete the uniform estimates.

Assume that the memory kernel function $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, and satisfy the following conditions: $\forall\ s \in \mathbb{R}^+$ and some $\varrho > 0$.

$$\mu(s) \geq 0,\ \mu'(s) + \varrho\mu \leq 0, \tag{3.3}$$

note that (3.3) implies $\varpi \overset{\text{def}}{=} \|\mu\|_{L^1(\mathbb{R}^+)} = \int_0^\infty \mu(s)ds > 0$.

Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $F(x, r) = \int_0^r f(x, s)ds$ for all $x \in \mathbb{R}^n, r \in \mathbb{R}$ and $s, s_1, s_2 \in \mathbb{R}$,

$$\liminf_{|s|\to\infty} \inf_{x\in\mathbb{R}^n} (f(x, s)s) > 0, \tag{3.4}$$

$$f(x, 0) = 0,\ |f(x, s_1) - f(x, s_2)| \leq \alpha_1(\varphi(x) + |s_1|^p + |s_2|^p)|s_1 - s_2|, \tag{3.5}$$

$$F(x, s) + \varphi_1(x) \geq 0, \tag{3.6}$$

where $p > 0$ for $1 \leq n \leq 4$ and $0 < p \leq \frac{4}{n-4}$ for $n \geq 5$, α_1 is a positive constant, $\varphi_1 \in L^1(\mathbb{R}^n)$, and $\varphi \in L^\infty(\mathbb{R}^n)$.

Let $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \times\mathbb{R}$ be continuous such that for all $t, s, s_1, s_2 \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$|h(t, x, s)| \leq \alpha_2|s| + \varphi_2(t, x), \tag{3.7}$$

$$|h(t, x, s_1) - h(t, x, s_2)| \leq \alpha_3|s_1 - s_2|, \tag{3.8}$$

where α_2 and α_3 are positive constants, and $\varphi_2 \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$.

By (3.3), the space $\mathfrak{R}_{\mu,r} = L^2_\mu(\mathbb{R}^+, V_r)(r \in \mathbb{R})$ is a Hilbert space of V_r-valued functions on $\mathbb{R}^+$ with the inner product and norm

$$\begin{aligned}(\eta_1^t, \eta_2^t)_{\mu,r} &= \int_0^\infty \mu(s)(A^{\frac{r}{4}}\eta_1^t(s), A^{\frac{r}{4}}\eta_2^t(s))ds,\\ \|\eta^t\|^2_{\mu,r} &= \int_0^\infty \mu(s)(A^{\frac{r}{4}}\eta^t(s), A^{\frac{r}{4}}\eta^t(s))ds,\end{aligned} \qquad \forall\ \eta^t,\ \eta_1^t,\ \eta_2^t \in V_r,$$

and on $\mathfrak{R}_{\mu,r}$, the linear operator $-\partial_s$ has domain

$$D(-\partial_s) = \{\eta^t \in H^1_\mu(\mathbb{R}^+, V_r) : \eta^0 = 0\} \ \text{ where } \ H^1_\mu(\mathbb{R}^+, V_r) = \{\eta^t : \eta^t(s), \partial_s\eta^t \in L^2_\mu(\mathbb{R}^+, V_r)\}.$$

Definition 3.1. Given $\tau \in \mathbb{R}, \omega \in \Omega,\ T > 0, u_0 \in H^2(\mathbb{R}^n),\ u_{1,0} \in L^2(\mathbb{R}^n)$, and $\eta^0 \in \mathfrak{R}_{\mu,2}$, a function $z(t) = (u, u_t, \eta^t)$ is called a (weak) solution of (3.2) if the following conditions are fulfilled:

(i) $u(\cdot, \tau, \omega, u_0, u_{1,0}) \in L^\infty(\tau, \tau + T; H^2(\mathbb{R}^n)) \cap C([\tau, \tau + T], L^2(\mathbb{R}^n))$ with $u(\tau, \tau, \omega, u_0, u_{1,0}) = u_0, u_t(\cdot, \tau, \omega, u_0, u_{1,0}) \in L^\infty(\tau, \tau + T; L^2(\mathbb{R}^n)) \cap C([\tau, \tau + T], L^2(\mathbb{R}^n))$ with $u_t(\tau, \tau, \omega, u_0, u_{1,0}) = u_{1,0}$ and $\eta^t(\cdot, \tau, \omega, \eta^0, s) \in L^\infty(\tau, \tau + T; \mathfrak{R}_{\mu,2}) \cap C([\tau, \tau + T], L^2(\mathbb{R}^n))$ with $\eta^t(\tau, \tau, \omega, \eta^0, s) = \eta^0$.

(ii) $u(t, \tau, \cdot, u_0, u_{1,0}) : \Omega \to H^2(\mathbb{R}^n)$ is $(\mathcal{F}, \mathcal{B}(H^2(\mathbb{R}^n))$-measurable, $u_t(t, \tau, \cdot, u_0, u_{1,0}) : \Omega \to L^2(\mathbb{R}^n)$ is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^n))$-measurable, and $\eta^t(t, \tau, \cdot, \eta^0, s) : \Omega \to \mathfrak{R}_{\mu,2}$ is $(\mathcal{F}, \mathcal{B}(\mathfrak{R}_{\mu,2})$-measurable.

(iii) For all $\xi \in C_0^\infty((\tau, \tau + T) \times \mathbb{R}^n)$,

$$\begin{aligned}&-\int_\tau^{\tau+T}(u_t, \xi_t)dt + \alpha\int_\tau^{\tau+T}(u_t, \xi)dt + \int_\tau^{\tau+T}(\Delta u, \Delta\xi)dt\\ &\qquad + \int_0^\infty \mu(s)(\Delta^2\eta^t(s), \xi)ds + \nu\int_\tau^{\tau+T}(u, \xi)dt + \int_\tau^{\tau+T}\int_{\mathbb{R}^n} f(x, u(t, x))\xi(t, x)dxdt\\ &= \int_\tau^{\tau+T}(g(t, x), \xi)dt + \int_\tau^{\tau+T}\int_{\mathbb{R}^n} h(t, x, u(t, x))\zeta_\delta(\theta_t\omega)\xi(t, x)dxdt.\end{aligned}$$

In order to investigate the long-time dynamics, we are now ready to prove the existence and uniqueness of solutions of (3.2). We first recall the following well-known existence and uniqueness of solutions for the corresponding linear plate equations of (1.1)(see [34, 35]).

Lemma 3.1. *Let $u_0 \in H^2(\mathbb{R}^n), u_{1,0} \in L^2(\mathbb{R}^n)$ and $g \in L^1(\tau, \tau + T; L^2(\mathbb{R}^n))$ with $\tau \in \mathbb{R}$ and $T > 0$. Then the linear plate equation*

$$u_{tt} + \alpha u_t + \Delta^2 u + \int_0^\infty \mu(s)\Delta^2(u(t) - u(t - s))ds + \nu u = g(t), \quad \tau < t \le \tau + T,$$

with the initial conditions

$$u(\tau) = u_0, \quad \text{and} \quad u_t(\tau) = u_{1,0},$$

possesses a unique solution (u, u_t, η^t) in the sense of Definition 3.1. In addition,

$$u \in C([\tau, \tau + T], H^2(\mathbb{R}^n)), \ \ u_t \in C([\tau, \tau + T], L^2(\mathbb{R}^n)) \text{ and } \ \eta^t \in C([\tau, \tau + T], \mathfrak{R}_{\mu,2})$$

and there exists a positive number C depending only on ν (but independent of $\tau, T, u_0, u_{1,0}$ and g) such that for all $t \in [\tau, \tau + T]$,

$$\|u(t)\|_{H^2(\mathbb{R}^n)} + \|u_t(t)\| + \|\eta^t\|_{\mu,2} \le C(\|u_0\|_{H^2(\mathbb{R}^n)} + \|u_{1,0}\| + \int_\tau^{\tau+T}\|g(t)\|dt). \tag{3.9}$$

Furthermore, the solution (u, u_t, η^t) satisfies the energy equation

$$\frac{d}{dt}(\|u_t\|^2 + \|\Delta u\|^2 + \nu\|u\|^2 + \|\eta^t\|^2_{\mu,2}) = -2\alpha\|u_t\|^2 + \int_0^\infty \mu'(s)\|\Delta\eta^t\|^2 ds + 2(g(t), u_t), \tag{3.10}$$

and

$$\frac{d}{dt}(u(t), u_t(t)) + \alpha(u(t), u_t(t)) + \|\Delta u(t)\|^2 + (\eta^t(s), u(t))_{\mu,2} + \nu\|u(t)\|^2 = \|u_t(t)\|^2 + (g(t), u(t)), \tag{3.11}$$

for almost all $t \in [\tau, \tau + T]$.

Theorem 3.1. *Let $\tau \in \mathbb{R}, u_0 \in H^2(\mathbb{R}^n), u_{1,0} \in L^2(\mathbb{R}^n)$ and $\eta^0 \in \Re_{\mu,2}$. Suppose (3.3)-(3.8) hold, then:*
(a) Problem (3.2) possesses a solution $z(t) = (u, u_t, \eta^t)$ in the sense of Definition 3.1;
(b) The solution $z(t) = (u, u_t, \eta^t)$ to problem (3.2) is unique, continuous in initial data in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \Re_{\mu,2}$, and

$$u \in C([\tau, \tau + T], H^2(\mathbb{R}^n)), \;\; u_t \in C([\tau, \tau + T], L^2(\mathbb{R}^n)) \;\text{ and }\; \eta^t \in C([\tau, \tau + T], \Re_{\mu,2}). \tag{3.12}$$

Moreover, the solution $z(t) = (u, u_t, \eta^t)$ to problem (3.2) satisfies the energy equation:

$$\begin{aligned}&\frac{d}{dt}(\|u_t\|^2 + \nu\|u\|^2 + \|\Delta u\|^2 + \|\eta^t\|^2_{\mu,2} + 2\int_{\mathbb{R}^n} F(x, u(t, x))dx) + 2\alpha\|u_t\|^2\\ &= \int_0^\infty \mu'(s)\|\Delta\eta^t\|^2 ds + 2(g(t), u_t) + 2\zeta_\delta(\theta_t\omega)\int_{\mathbb{R}^n} h(t, x, u(t, x))u_t(t, x)dx\end{aligned} \tag{3.13}$$

for almost all $t \in [\tau, \tau + T]$.

Proof. The proof will be divided into four steps. We first construct a sequence of approximate solutions, and then derive uniform estimates, in the last two steps we take the limit of those approximate solutions to prove the uniqueness of solutions.

Step (i): Approximate solutions. Given $k \in \mathbb{N}$, define a function $\eta_k : \mathbb{R} \to \mathbb{R}$ by

$$\eta_k(s) = \begin{cases} s, & \text{if } -k \le s \le k,\\ k, & \text{if } s > k,\\ -k, & \text{if } s < -k.\end{cases} \tag{3.14}$$

Then for every fixed $k \in \mathbb{N}$, the function η_k as defined by (3.14) is bounded and Lipschitz continuous; more precisely, for all $s, s_1, s_2 \in \mathbb{R}$

$$\eta_k(0) = 0, |\eta_k(s)| \le |s| \;\text{ and }\; |\eta_k(s_1) - \eta_k(s_2)| \le |s_1 - s_2|. \tag{3.15}$$

For all $x \in \mathbb{R}^n$ and $t, s \in \mathbb{R}$, denote

$$f_k(x, s) = f(x, \eta_k(s)), \;\; F_k(x, s) = \int_0^s f_k(x, r)dr \;\text{ and }\; h_k(t, x, s) = h(t, x, \eta_k(s)). \tag{3.16}$$

By (3.4) we know that there exists $k_0 \in \mathbb{N}$ such that for all $|s| \ge k_0$ and $x \in \mathbb{R}^n$,

$$f(x, s)s > 0, \tag{3.17}$$

thus, for all $k \geq k_0$ and $x \in \mathbb{R}^n$,

$$f_k(x,k) > 0, \qquad f_k(x,-k) < 0. \tag{3.18}$$

By (3.5)-(3.6), (3.15)-(3.16) and (3.18) we know that for all $s, s_1, s_2 \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$|f_k(x,s_1) - f_k(x,s_2)| \leq \alpha_1(\varphi(x) + |s_1|^p + |s_2|^p)|s_1 - s_2|, \ \forall\, k \geq 1, \tag{3.19}$$

and

$$F_k(x,s) + \varphi_1(x) \geq 0, \ \forall\, k \geq k_0. \tag{3.20}$$

By (3.19) we get that for all $s \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$|F_k(x,s)| \leq \alpha_1(\varphi(x)|s|^2 + |s|^{p+2}), \ \ \forall\, k \geq 1. \tag{3.21}$$

By (3.7)-(3.8) and (3.15)-(3.16) we obtain that for all $k \geq 1, t, s, s_1, s_2 \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$|h_k(t,x,s)| \leq \alpha_2|s| + \varphi_2(t,x), \tag{3.22}$$

$$|h_k(t,x,s_1) - h_k(t,x,s_2)| \leq \alpha_3|s_1 - s_2|. \tag{3.23}$$

By (3.3) and (3.15)-(3.16), we find that for all $k \in \mathbb{N}, s, s_1, s_2 \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$|f_k(x,s)| \leq \alpha_1 k(\varphi(x) + k^p), \tag{3.24}$$

$$|f_k(x,s_1) - f_k(x,s_2)| \leq \alpha_1(\varphi(x) + 2k^p)|s_1 - s_2|. \tag{3.25}$$

For every $k \in \mathbb{N}$, consider the following approximate system for u_k, η_k^t:

$$\begin{cases} \frac{\partial^2}{\partial t^2}u_k + \alpha\frac{\partial}{\partial t}u_k + \Delta^2 u_k + \int_0^\infty \mu(s)\Delta^2\eta_k^t(s)ds + \nu u_k + f_k(\cdot,u_k) \\ \qquad = g(\cdot,t) + h_k(t,\cdot,u_k)\zeta_\delta(\theta_t\omega), \ t > \tau, \\ u_k(\tau) = u_0, \ \ \frac{\partial}{\partial t}u_k(\tau) = u_{1,0}, \ \ \eta_k^\tau(x,s) = \eta^0(x,s). \end{cases} \tag{3.26}$$

From (3.23)-(3.24), $\varphi \in L^\infty(\mathbb{R}^n)$ and the standard method (see, e.g., [11]), it follows that for each $\tau \in \mathbb{R}, \omega \in \Omega, u_0 \in H^2(\mathbb{R}^n), u_{1,0} \in L^2(\mathbb{R}^n)$ and $\eta^0 \in \mathfrak{R}_{\mu,2}$, problem (3.26) has a unique global solution $(u_k, \partial_t u_k, \eta_k^t)$ defined on $[\tau, \tau + T]$ for every $T > 0$ in the sense of Definition 3.1. In particular, $u_k(\cdot,\tau,\omega,u_0) \in C([\tau,\tau+T], H^2(\mathbb{R}^n))$ and $u_k(t,\tau,\omega,u_0)$ is measurable with respect to $\omega \in \Omega$ in $H^2(\mathbb{R}^n)$ for every $t \in [\tau,\tau+T]$; $\partial_t u_k(\cdot,\tau,\omega,u_0) \in C([\tau,\tau+T], L^2(\mathbb{R}^n))$ and $\partial_t u_k(t,\tau,\omega,u_0)$ is measurable with respect to $\omega \in \Omega$ in $L^2(\mathbb{R}^n)$ for every $t \in [\tau,\tau+T]$; $\eta_k^t(\cdot,\tau,\omega,\eta_0,s) \in C([\tau,\tau+T], \mathfrak{R}_{\mu,2})$ and $\eta_k^t(t,\tau,\omega,\eta_0,s)$ is measurable with respect to $\omega \in \Omega$ in $\mathfrak{R}_{\mu,2}$ for every $t \in [\tau,\tau+T]$ Furthermore, the solution u_k satisfies the energy equation:

$$\begin{aligned} &\frac{d}{dt}(\|\partial_t u_k\|^2 + \nu\|u_k\|^2 + \|\Delta u_k\|^2 + \|\eta_k^t\|^2_{\mu,2} + 2\int_{\mathbb{R}^n} F_k(x,u_k(t,x))dx) + 2\alpha\|\partial_t u_k\|^2 \\ &= \int_0^\infty \mu'(s)\|\Delta\eta_k^t\|^2 ds + 2(g(t),\partial_t u_k) + 2\zeta_\delta(\theta_t\omega)\int_{\mathbb{R}^n} h_k(t,x,u_k(t,x))\partial_t u_k(t,x)dx \end{aligned} \tag{3.27}$$

for almost all $t \in [\tau,\tau+T]$. Next, we use the energy equation (3.25) to derive uniform estimate on the sequence $\{u_k, \partial_t u_k, \eta_k^t\}_{k=1}^\infty$.

Step (ii): Uniform estimates.

For the last term on the right-hand side of (3.25), by (3.21) we have

$$\begin{aligned}&2\zeta_\delta(\theta_t\omega)\int_{\mathbb{R}^n} h_k(t,x,u_k(t,x))\partial_t u_k(t,x)dx\\ \leq&2|\zeta_\delta(\theta_t\omega)|\Big(\alpha_2\int_{\mathbb{R}^n}|u_k(t,x)|\cdot|\partial_t u_k(t,x)|dx+\int_{\mathbb{R}^n}|\varphi_2(t,x)|\cdot|\partial_t u_k(t,x)|dx\Big)\\ \leq&|\zeta_\delta(\theta_t\omega)|(\alpha_2\|u_k(t)\|^2+(1+\alpha_2)\|\partial_t u_k(t)\|^2+\|\varphi_2(t)\|^2).\end{aligned}\tag{3.28}$$

By Young's inequality, we get

$$2(g(t),\partial_t u_k)\leq\|\partial_t u_k(t)\|^2+\|g(t)\|^2.\tag{3.29}$$

By (3.27)–(3.29) together with (3.3), it follows that for almost all $t\in[\tau,\tau+T]$,

$$\begin{aligned}&\frac{d}{dt}(\|\partial_t u_k\|^2+\nu\|u_k\|^2+\|\Delta u_k\|^2+\|\eta_k^t\|_{\mu,2}^2+2\int_{\mathbb{R}^n}F_k(x,u_k(t,x))dx)+2\alpha\|\partial_t u_k\|^2\\ &\leq c_1(1+|\zeta_\delta(\theta_t\omega)|)(\|u_k(t)\|^2+\|\partial_t u_k(t)\|^2)+|\zeta_\delta(\theta_t\omega)|\cdot\|\varphi_2(t)\|^2+\|g(t)\|^2,\end{aligned}\tag{3.30}$$

where $c_1>0$ depends only on α_2, but independent of k.

By (3.20) and (3.30) we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\partial_t u_k\|^2+\nu\|u_k\|^2+\|\Delta u_k\|^2+\|\eta_k^t\|_{\mu,2}^2+2\int_{\mathbb{R}^n}F_k(x,u_k(t,x))dx)\\ \leq&c_2(1+|\zeta_\delta(\theta_t\omega)|)(\|\partial_t u_k(t)\|^2+\nu\|u_k(t)\|^2+\|\Delta u_k\|^2+\|\eta_k^t\|_{\mu,2}^2+2\int_{\mathbb{R}^n}F_k(x,u_k(t,x))dx)\\ &+|\zeta_\delta(\theta_t\omega)|\cdot\|\varphi_2(t)\|^2+2c_1(1+|\zeta_\delta(\theta_t\omega)|)\|\varphi_1\|_{L^1(\mathbb{R}^n)}+\|g(t)\|^2,\end{aligned}\tag{3.31}$$

where $c_2>0$ depends only on ν and α_2, but independent of k.

Multiplying (3.31) with $e^{-c_2\int_0^t(1+|\zeta_\delta(\theta_r\omega)|)dr}$, and then integrating the inequality on (τ,t), we have

$$\begin{aligned}&\|\partial_t u_k\|^2+\nu\|u_k\|^2+\|\Delta u_k\|^2+\|\eta_k^t\|_{\mu,2}^2+2\int_{\mathbb{R}^n}F_k(x,u_k(t,x))dx\\ \leq&e^{c_2\int_\tau^t(1+|\zeta_\delta(\theta_r\omega)|)dr}\Big(\|u_{1,0}\|^2+\nu\|u_0\|^2+\|\Delta u_0\|^2+\|\eta^0\|_{\mu,2}^2+2\int_{\mathbb{R}^n}F_k(x,u_0(x))dx\Big)\\ &+\int_\tau^t e^{c_2\int_s^t(1+|\zeta_\delta(\theta_r\omega)|)dr}(|\zeta_\delta(\theta_s\omega)|\cdot\|\varphi_2(s)\|^2+2c_1(1+|\zeta_\delta(\theta_s\omega)|)\|\varphi_1\|_{L^1(\mathbb{R}^n)}+\|g(s)\|^2)ds.\end{aligned}\tag{3.32}$$

By (3.21) we get, for all $k\geq1$,

$$\begin{aligned}2\int_{\mathbb{R}^n}|F_k(x,u_0(x))|dx&\leq2\alpha_1\Big(\|\varphi\|_{L^\infty(\mathbb{R}^n)}\|u_0\|^2+\|u_0\|_{L^{p+2}(\mathbb{R}^n)}^{p+2}\Big)\\ &\leq2\alpha_1\Big(\|\varphi\|_{L^\infty(\mathbb{R}^n)}\|u_0\|^2+\|u_0\|_{H^2(\mathbb{R}^n)}^{p+2}\Big).\end{aligned}\tag{3.33}$$

By (3.32)-(3.33) imply that there exists a positive constant $c_3=c_3(\tau,T,\varphi,\varphi_1,\varphi_2,g,\omega,\delta,\alpha_1,\nu)$ (but independent of $k,u_0,u_{1,0}$) such that for all $t\in[\tau,\tau+T]$ and $k\geq1$,

$$\|\partial_t u_k\|^2+\nu\|u_k\|^2+\|\Delta u_k\|^2+\|\eta_k^t\|_{\mu,2}^2+2\int_{\mathbb{R}^n}F_k(x,u_k(t,x))dx$$

$$\le c_3 + c_3(1 + \|u_{1,0}\|^2 + \|u_0\|^{p+2}_{H^2(\mathbb{R}^n)} + \|\eta^0\|^2_{\mu,2}),$$

which along with (3.20) show that for all $t \in [\tau, \tau + T]$ and $k \ge k_0$,

$$\|\partial_t u_k\|^2 + \nu\|u_k\|^2 + \|\Delta u_k\|^2 + \|\eta_k^t\|^2_{\mu,2} + 2\int_{\mathbb{R}^n} F_k(x, u_k(t, x))dx$$
$$\le c_3 + 2\|\varphi_1\|_{L^1(\mathbb{R}^n)} + c_3(1 + \|u_{1,0}\|^2 + \|u_0\|^{p+2}_{H^2(\mathbb{R}^n)} + \|\eta^0\|^2_{\mu,2}), \tag{3.34}$$

thus,

$$\{u_k\}_{k=1}^\infty \text{ is bounded in } L^\infty(\tau, \tau + T; H^2(\mathbb{R}^n)), \tag{3.35}$$
$$\{\partial_t u_k\}_{k=1}^\infty \text{ is bounded in } L^\infty(\tau, \tau + T; L^2(\mathbb{R}^n)). \tag{3.36}$$
$$\{\eta_k^t\}_{k=1}^\infty \text{ is bounded in } L^\infty(\tau, \tau + T; \mathfrak{R}_{\mu,2}), \tag{3.37}$$

By (3.19), there exists a positive constant $c_4 = c_4(p, n, \alpha_1)$ such that

$$\int_{\mathbb{R}^n} |f_k(x, u_k(t, x))|^2 dx \le c_4\Big(\int_{\mathbb{R}^n} |\varphi(x)|^2 dx + \int_{\mathbb{R}^n} |u_k(t, x)|^{2(p+1)} dx\Big),$$

which along with the embedding $H^2(\mathbb{R}^n) \hookrightarrow L^{2(p+1)}(\mathbb{R}^n)$ and the assumption $\varphi \in L^\infty(\mathbb{R}^n)$ implies that there exists $c_5 = c_5(p, n, \alpha_1, \varphi) > 0$ (independent of k) such that

$$\int_{\mathbb{R}^n} |f_k(x, u_k(t, x))|^2 dx \le c_5\Big(1 + \|u_k(t)\|^{2(p+1)}_{H^2(\mathbb{R}^n)}\Big). \tag{3.38}$$

By (3.35) and (3.38) we see that

$$\{f_k(\cdot, u_k)\}_{k=1}^\infty \text{ is bounded in } L^2(\tau, \tau + T; L^2(\mathbb{R}^n)). \tag{3.39}$$

By (3.22) we get

$$\int_{\mathbb{R}^n} |h_k(t, x, u_k(t, x))|^2 dx \le 2\alpha_2\|u_k\|^2 + 2\|\varphi_2(t)\|^2,$$

which together with (3.35) shows that

$$\{h_k(\cdot, \cdot, u_k)\}_{k=1}^\infty \text{ is bounded in } L^2(\tau, \tau + T; L^2(\mathbb{R}^n)). \tag{3.40}$$

By (3.35)–(3.37) and (3.39)-(3.40), it follows that there exists $u \in L^\infty(\tau, \tau + T; H^2(\mathbb{R}^n))$ with $\partial_t u \in L^\infty(\tau, \tau+T; L^2(\mathbb{R}^n))$, $\kappa_1 \in L^2(\tau, \tau+T; L^2(\mathbb{R}^n))$, $\kappa_2 \in L^2(\tau, \tau+T; L^2(\mathbb{R}^n))$, $v^{\tau+T} \in H^2(\mathbb{R}^n)$ and $v_1^{\tau+T} \in L^2(\mathbb{R}^n)$ such that

$$u_k \to u \text{ weak-star in } L^\infty(\tau, \tau + T; H^2(\mathbb{R}^n)), \tag{3.41}$$
$$\partial_t u_k \to \partial_t u \text{ weak-star in } L^\infty(\tau, \tau + T; L^2(\mathbb{R}^n)), \tag{3.42}$$
$$\eta_k^t \to \eta^t \text{ weak-star in } L^\infty(\tau, \tau + T; \mathfrak{R}_{\mu,2}), \tag{3.43}$$
$$f_k(\cdot, u_k) \to \kappa_1 \text{ weakly in } L^2(\tau, \tau + T; L^2(\mathbb{R}^n)), \tag{3.44}$$
$$h_k(\cdot, \cdot, u_k) \to \kappa_2 \text{ weakly in } L^2(\tau, \tau + T; L^2(\mathbb{R}^n)), \tag{3.45}$$
$$u_k(\tau + T) \to v^{\tau+T} \text{ weakly in } H^2(\mathbb{R}^n), \tag{3.46}$$

$$\partial_t u_k(\tau+T) \to v_1^{\tau+T} \text{ weakly in } L^2(\mathbb{R}^n). \tag{3.47}$$

It follows from (3.41)-(3.42) that there exists a subsequence which is still denoted u_k, such that

$$u_k(t,x) \to u(t,x) \text{ for almost all } (t,x) \in [\tau, \tau+T] \times \mathbb{R}^n. \tag{3.48}$$

By (3.15) and (3.48) we get that for almost all $(t,x) \in [\tau, \tau+T] \times \mathbb{R}^n$,

$$\begin{aligned} &|\eta_k(u_k(t,x)) - u(t,x)| \le |\eta_k(u_k(t,x)) - \eta_k(u(t,x))| + |\eta_k(u(t,x)) - u(t,x)| \\ \le &|u_k(t,x) - u(t,x)| + |\eta_k(u(t,x)) - u(t,x)| \to 0, \text{ as } k \to \infty. \end{aligned} \tag{3.49}$$

By (3.49), we have

$$f_k(x, u_k(t,x)) \to f(x, u(t,x)) \text{ for almost all } (t,x) \in [\tau, \tau+T] \times \mathbb{R}^n, \tag{3.50}$$

$$h_k(t, x, u_k(t,x)) \to h(t, x, u(t,x)) \text{ for almost all } (t,x) \in [\tau, \tau+T] \times \mathbb{R}^n. \tag{3.51}$$

It follows from (3.44)-(3.45), (3.50)-(3.51) that

$$f_k(\cdot, u_k) \to f(\cdot, u) \text{ weakly in } L^2(\tau, \tau+T; L^2(\mathbb{R}^n)), \tag{3.52}$$

$$h_k(\cdot, \cdot, u_k) \to h(\cdot, \cdot, u) \text{ weakly in } L^2(\tau, \tau+T; L^2(\mathbb{R}^n)). \tag{3.53}$$

Step (iii): Existence of solutions.

Choosing an arbitrary $\xi \in C_0^\infty((\tau, \tau+T) \times \mathbb{R}^n)$. By (3.26) we get

$$\begin{aligned} &-\int_\tau^{\tau+T} (\partial_t u_k, \xi_t) dt + \alpha \int_\tau^{\tau+T} (\partial_t u_k, \xi) dt + \int_\tau^{\tau+T} (\Delta u_k, \Delta \xi) dt + \nu \int_\tau^{\tau+T} (u_k, \xi) dt \\ &\quad + \int_\tau^{\tau+T} \int_0^\infty \mu(s) (\Delta^2 \eta_k^t(s), \xi) ds dt + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} f_k(x, u_k(t,x)) \xi(t,x) dx dt \\ = &\int_\tau^{\tau+T} (g(t), \xi) dt + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} h_k(t, x, u_k(t,x)) \zeta_\delta(\theta_t \omega) \xi(t,x) dx dt. \end{aligned} \tag{3.54}$$

Letting $k \to \infty$ in (3.54), it follows from (3.41)-(3.43) and (3.52)-(3.53) that for any $\xi \in C_0^\infty((\tau, \tau+T) \times \mathbb{R}^n)$,

$$\begin{aligned} &-\int_\tau^{\tau+T} (u_t, \xi_t) dt + \alpha \int_\tau^{\tau+T} (u_t, \xi) dt + \int_\tau^{\tau+T} (\Delta u, \Delta \xi) dt + \nu \int_\tau^{\tau+T} (u, \xi) dt \\ &\quad + \int_\tau^{\tau+T} \int_0^\infty \mu(s) (\Delta^2 \eta^t(s), \xi) ds dt + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} f(x, u(t,x)) \xi(t,x) dx dt \\ = &\int_\tau^{\tau+T} (g(t), \xi) dt + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} h(t, x, u(t,x)) \zeta_\delta(\theta_t \omega) \xi(t,x) dx dt. \end{aligned} \tag{3.55}$$

Notice that

$$u \in L^\infty(\tau, \tau+T; H^2(\mathbb{R}^n)) \text{ and } \partial_t u \in L^\infty(\tau, \tau+T; L^2(\mathbb{R}^n)). \tag{3.56}$$

By (3.56) we obtain

$$h(\cdot, \cdot, u) \in L^2(\tau, \tau+T; L^2(\mathbb{R}^n)). \tag{3.57}$$

We claim that

$$f(\cdot,u) \quad \text{belongs to} \quad L^\infty(\tau,\tau+T;L^2(\mathbb{R}^n)). \tag{3.58}$$

In fact, by (3.5) we obtain that there exists some $c_6 = c_6(p,n,\alpha_1,\varphi) > 0$ such that

$$\begin{aligned}\|f(\cdot,u(t))\|^2 &\le 2\alpha_1^2(\|\varphi\|^2_{L^\infty(\mathbb{R}^n)}\|u(t)\|^2 + \|u(t)\|^{2(p+1)}_{L^{2(p+1)}(\mathbb{R}^n)})\\ &\le c_6(\|u(t)\|^2 + \|u(t)\|^{2(p+1)}_{H^2(\mathbb{R}^n)}),\end{aligned}$$

which along with (3.56) to obtain (3.58).

By (3.54)–(3.58), we can get

$$u_{tt} \quad \text{belongs to} \quad L^2(\tau,\tau+T;H^{-2}(\mathbb{R}^n)), \tag{3.59}$$

where $H^{-2}(\mathbb{R}^n)$ is the dual space of $H^2(\mathbb{R}^n)$.

Next, we prove (u,u_t,η^t) satisfy the initial conditions $(3.2)_2$.

By (3.26), we get that for any $v \in C_0^\infty(\mathbb{R}^n)$ and $\psi \in C^2([\tau,\tau+T])$,

$$\begin{aligned}&\int_\tau^{\tau+T}(u_k(t),v)\psi''(t)dt + (\partial_t u_k(\tau+T),v)\psi(\tau+T) - (u_k(\tau+T),v)\psi'(\tau+T)\\ &+ (u_0,v)\psi'(\tau) - (u_{1,0},v)\psi(\tau) + \alpha\int_\tau^{\tau+T}(\partial_t u_k(t),v)\psi(t)dt + \int_\tau^{\tau+T}(\Delta u_k(t),\Delta v)\psi(t)dt\\ &+ \int_\tau^{\tau+T}\int_0^\infty \mu(s)(\Delta^2\eta_k^t(s),v)\psi(t)dsdt + \nu\int_\tau^{\tau+T}(u_k(t),v)\psi(t)dt\\ &+ \int_\tau^{\tau+T}\int_{\mathbb{R}^n} f_k(x,u_k(t,x))v(x)\psi(t)dxdt\\ =&\int_\tau^{\tau+T}(g(t),v)\psi(t)dt + \int_\tau^{\tau+T}\int_{\mathbb{R}^n} h_k(t,x,u_k(t,x))\zeta_\delta(\theta_t\omega)v(x)\psi(t)dxdt.\end{aligned} \tag{3.60}$$

Letting $k \to \infty$ in (3.60), by (3.41)-(3.43), (3.46)-(3.47) and (3.52)-(3.53) we obtain, for any $v \in C_0^\infty(\mathbb{R}^n)$ and $\psi \in C^2([\tau,\tau+T])$,

$$\begin{aligned}&\int_\tau^{\tau+T}(u(t),v)\psi''(t)dt + (v_1^{\tau+T},v)\psi(\tau+T) - (v^{\tau+T},v)\psi'(\tau+T)\\ &+ (u_0,v)\psi'(\tau) - (u_{1,0},v)\psi(\tau) + \alpha\int_\tau^{\tau+T}(\partial_t u(t),v)\psi(t)dt + \int_\tau^{\tau+T}(\Delta u(t),\Delta v)\psi(t)dt\\ &+ \int_\tau^{\tau+T}\int_0^\infty \mu(s)(\Delta^2\eta^t(s),v)\psi(t)dsdt + \nu\int_\tau^{\tau+T}(u(t),v)\psi(t)dt\\ &+ \int_\tau^{\tau+T}\int_{\mathbb{R}^n} f(x,u(t,x))v(x)\psi(t)dxdt\\ =&\int_\tau^{\tau+T}(g(t),v)\psi(t)dt + \int_\tau^{\tau+T}\int_{\mathbb{R}^n} h(t,x,u(t,x))\zeta_\delta(\theta_t\omega)v(x)\psi(t)dxdt.\end{aligned} \tag{3.61}$$

By (3.55) we get that for any $v \in C_0^\infty(\mathbb{R}^n)$,

$$\frac{d}{dt}(u_t,v) + \alpha(u_t,v) + (\Delta u,\Delta v) + \int_0^\infty \mu(s)(\Delta^2\eta^t(s),v)ds + \nu(u,v) + \int_{\mathbb{R}^n} f(x,u(t,x))v(x)dx$$

$$=(g(t),v)+\int_{\mathbb{R}^n} h(t,x,u(t,x))\zeta_\delta(\theta_t\omega)v(x)dx. \tag{3.62}$$

By (3.62) we find that for any $v\in C_0^\infty(\mathbb{R}^n)$ and $\psi\in C^2([\tau,\tau+T])$,

$$\begin{aligned}
&\int_\tau^{\tau+T}(u(t),v)\psi''(t)dt+(\partial_t u(\tau+T),v)\psi(\tau+T)-(u(\tau+T),v)\psi'(\tau+T)\\
&+(u(\tau),v)\psi'(\tau)-(\partial_t u(\tau),v)\psi(\tau)+\alpha\int_\tau^{\tau+T}(\partial_t u(t),v)\psi(t)dt+\int_\tau^{\tau+T}(\Delta u(t),\Delta v)\psi(t)dt\\
&+\int_\tau^{\tau+T}\int_0^\infty\mu(s)(\Delta^2\eta^t(s),v)\psi(t)dsdt+\nu\int_\tau^{\tau+T}(u(t),v)\psi(t)dt\\
&+\int_\tau^{\tau+T}\int_{\mathbb{R}^n} f(x,u(t,x))v(x)\psi(t)dxdt\\
=&\int_\tau^{\tau+T}(g(t,\cdot),v)\psi(t)dt+\int_\tau^{\tau+T}\int_{\mathbb{R}^n} h(t,x,u(t,x))\zeta_\delta(\theta_t\omega)v(x)\psi(t)dxdt,
\end{aligned}\tag{3.63}$$

together with (3.61) to obtain, for $v\in C_0^\infty(\mathbb{R}^n)$ and $\psi\in C^2([\tau,\tau+T])$,

$$\begin{aligned}
&(v_1^{\tau+T},v)\psi(\tau+T)-(v^{\tau+T},v)\psi'(\tau+T)+(u_0,v)\psi'(\tau)-(u_{1,0},v)\psi(\tau)\\
=&(\partial_t u(\tau+T),v)\psi(\tau+T)-(u(\tau+T),v)\psi'(\tau+T)+(u(\tau),v)\psi'(\tau)-(\partial_t u(\tau),v)\psi(\tau).
\end{aligned}\tag{3.64}$$

Let $\psi\in C^2([\tau,\tau+T])$ such that $\psi(\tau+T)=\psi'(\tau+T)=\psi'(\tau)=0$ and $\psi(\tau)=1$, by (3.64), we have

$$(\partial_t u(\tau),v)=(u_{1,0},v),\quad \forall\, v\in C_0^\infty(\mathbb{R}^n). \tag{3.65}$$

Let $\psi\in C^2([\tau,\tau+T])$ such that $\psi(\tau+T)=\psi'(\tau+T)=\psi(\tau)=0$ and $\psi'(\tau)=1$, by (3.64), we have

$$(u(\tau),v)=(u_0,v),\quad \forall\, v\in C_0^\infty(\mathbb{R}^n), \tag{3.66}$$

which together with (3.65) that (u,u_t,η^t) satisfies the initial conditions $(3.2)_2$.

Through choosing proper $\psi\in C^2([\tau,\tau+T])$, we can also obtain from (3.64) that

$$u(\tau+T)=v^{\tau+T},\quad \text{and}\quad \partial_t u(\tau+T)=v_1^{\tau+T},$$

which along with (3.46)-(3.47) implies that

$$u_k(\tau+T)\to u(\tau+T)\ \text{ weakly in }\ H^2(\mathbb{R}^n), \tag{3.67}$$

$$\partial_t u_k(\tau+T)\to \partial_t u(\tau+T)\ \text{ weakly in }\ L^2(\mathbb{R}^n), \tag{3.68}$$

thereby,

$$\eta_k^t(\tau+T)\to\eta^t(\tau+T)\ \text{ weakly in }\ \Re_{\mu,2}. \tag{3.69}$$

Similar to (3.67)-(3.69), one can verify that for any $t\in[\tau,\tau+T]$,

$$u_k(t)\to u(t)\ \text{ weakly in }\ H^2(\mathbb{R}^n), \tag{3.70}$$

$$\partial_t u_k(t)\to\partial_t u(t)\ \text{ weakly in }\ L^2(\mathbb{R}^n), \tag{3.71}$$

$$\eta_k^t \to \eta^t \ \text{ weakly in } \ \Re_{\mu,2}. \tag{3.72}$$

By (3.70)–(3.72), we get the that (u, u_t, η^t) is a solution of (3.2) in the sense of Definition 3.1.

Step (iv): Uniqueness of solutions.

Let $(u_1, (u_1)_t, \eta_1^t)$ and $(u_2, (u_2)_t, \eta_2^t)$ be solutions to (3.2), denote $v = u_1 - u_2, \bar{\eta}^t = \eta_1^t - \eta_2^t$. Then we have

$$\begin{cases} v_{tt} + \alpha v_t + \Delta^2 v + \int_0^\infty \mu(s)\Delta^2 \bar{\eta}^t(s)ds + \nu v \\ \qquad = f(\cdot, u_2) - f(\cdot, u_1) + (h(t, \cdot, u_1) - h(t, \cdot, u_2))\zeta_\delta(\theta_t\omega), \\ v(\tau) = 0, \ \ v_t(\tau) = 0. \end{cases} \tag{3.73}$$

by (3.10), we get

$$\begin{aligned} &\frac{d}{dt}(\|v_t\|^2 + \|\Delta v\|^2 + \|\bar{\eta}^t(s)\|_{\mu,2}^2 + \nu\|v\|^2) \\ =& -2\alpha\|v_t\|^2 + 2(f(\cdot, u_2) - f(\cdot, u_1), v_t) + 2(h(t, \cdot, u_1) - h(t, \cdot, u_2), v_t)\zeta_\delta(\theta_t\omega). \end{aligned} \tag{3.74}$$

Since $H^2(\mathbb{R}^n) \hookrightarrow L^{2(p+1)}(\mathbb{R}^n)$ for $0 < p \leq \frac{4}{n-4}$, by (3.5), we get

$$\|f(\cdot, u_2) - f(\cdot, u_1)\| \leq \alpha_1\|\varphi\|_{L^\infty(\mathbb{R}^n)}\|v\| + \alpha_1(\|u_1\|_{H^2(\mathbb{R}^n)}^p + \|u_2\|_{H^2(\mathbb{R}^n)}^p)\|v\|_{H^2(\mathbb{R}^n)}$$

and hence

$$\begin{aligned} &2(f(\cdot, u_2) - f(\cdot, u_1), v_t) \\ \leq& 2\|f(\cdot, u_2) - f(\cdot, u_1)\|\|v_t\| \\ \leq& \alpha_1(\|\varphi\|_{L^\infty(\mathbb{R}^n)} + \|u_1\|_{H^2(\mathbb{R}^n)}^p + \|u_2\|_{H^2(\mathbb{R}^n)}^p)(\|v\|_{H^2(\mathbb{R}^n)}^2 + \|v_t\|^2). \end{aligned} \tag{3.75}$$

By (3.8) we get

$$\begin{aligned} &2(h(t, \cdot, u_1) - h(t, \cdot, u_2), v_t)\zeta_\delta(\theta_t\omega) \\ \leq& \|h(t, \cdot, u_1) - h(t, \cdot, u_2)\|\|v_t\||\zeta_\delta(\theta_t\omega)| \\ \leq& 2\alpha_3\|v\|\|v_t\||\zeta_\delta(\theta_t\omega)| \\ \leq& \alpha_3(\|v\|^2 + \|v_t\|^2)|\zeta_\delta(\theta_t\omega)|. \end{aligned} \tag{3.76}$$

It follows from (3.74)–(3.76) that

$$\begin{aligned} &\frac{d}{dt}(\|v_t\|^2 + \|\Delta v\|^2 + \|\bar{\eta}^t(s)\|_{\mu,2}^2 + \nu\|v\|^2) \\ \leq& c_7(1 + \|u_1\|_{H^2(\mathbb{R}^n)}^p + \|u_2\|_{H^2(\mathbb{R}^n)}^p)(\|v_t\|^2 + \|\Delta v\|^2 + \|\bar{\eta}(s)\|_{\mu,2}^2 + \nu\|v\|^2), \end{aligned} \tag{3.77}$$

where $c_7 > 0$ depends on τ and T. Since $u_1, u_2 \in L^\infty(\tau, \tau + T; H^2(\mathbb{R}^n))$, then applying the Gronwall's lemma on $[\tau, \tau+T]$, we can obtain that the uniqueness of solution as well as the continuous dependence property of solution with initial data. □

We now define a mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \Re_{\mu,2} \to H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \Re_{\mu,2}$ such that for all $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$ and $(u_0, u_{1,0}, \eta^0) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \Re_{\mu,2}$,

$$\Phi(t, \tau, \omega, (u_0, u_{1,0}, \eta^0)) = (u(t + \tau, \tau, \theta_{-\tau}\omega, u_0), u_t(t + \tau, \tau, \theta_{-\tau}\omega, u_{1,0}), \eta^t(t + \tau, \tau, \theta_{-\tau}\omega, \eta^0, s)), \tag{3.78}$$

where (u, u_t, η^t) is the solution of (3.2). Then Φ is a continuous cocycle on $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \Re_{\mu,2}$ over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$.

4. Uniform estimates of solutions

In this section, we derive necessary estimates of solutions of (3.2) under stronger conditions than (3.4)-(3.8) on the nonlinear functions f and h. These estimates are useful for proving the asymptotic compactness of the solutions and the existence of pullback random attractors.

From now on, we assume f satisfies: for all $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$,

$$f(x,s)s - \gamma F(x,s) \geq \varphi_3(x), \tag{4.1}$$

$$F(x,s) + \varphi_1(x) \geq \alpha_4 |s|^{p+2}, \tag{4.2}$$

$$|\partial_s f(x,s)| \leq \iota |s|^p + \varsigma, \quad |\partial_x f(x,s)| \leq \varphi_4(x), \tag{4.3}$$

where $p > 0$ for $1 \leq n \leq 4$ and $0 < p \leq \frac{4}{n-4}$ for $n \geq 5$, $\gamma \in (0,1]$, α_4, ς are positive constants, $\varphi_3 \in L^1(\mathbb{R}^n)$, and $\varphi_4 \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\iota > 0$ will be denoted later.

By (3.5) and (4.1) we get that for all $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$,

$$\gamma F(x,s) \leq \alpha_1 s^2 \varphi(x) + \alpha_1 |s|^{p+2} - \varphi_3(x). \tag{4.4}$$

Assume the nonlinearity h satisfies: for all $x \in \mathbb{R}^n$ and $t, s \in \mathbb{R}$,

$$|h(t,x,s)| \leq \varphi_5(x)|s| + \varphi_6(x), \tag{4.5}$$

$$|\partial_x h(t,x,s)| + |\partial_s h(t,x,s)| \leq \varphi_7(x), \tag{4.6}$$

where $\varphi_5 \in L^\infty(\mathbb{R}^n) \cap L^{2+\frac{4}{p}}(\mathbb{R}^n)$, $\varphi_6 \in L^2(\mathbb{R}^n)$, and $\varphi_7 \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Let $\mathcal{D}$ be the set of all tempered families of nonempty bounded subsets of $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \Re_{\mu,2}$. $D = \{D(\tau,\Omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called tempered if for any $c > 0$,

$$\lim_{t \to +\infty} e^{-ct} \|D(\tau - t, \theta_{-t}\omega)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \Re_{\mu,2}} = 0,$$

where $\|D\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \Re_{\mu,2}} = \sup\limits_{\xi \in D} \|\xi\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \Re_{\mu,2}}$.

Under $\alpha > 0, \nu > 0, \varrho > 0, \varpi > 0$ and $\gamma \in (0,1]$, we can choose a sufficiently small positive constant ε such that

$$\varepsilon < \min\{1, \nu, \frac{2\alpha}{5}, \frac{3}{2}\varrho, \frac{3\varrho}{\gamma}\}, \ \frac{1}{2}\alpha - 2\varepsilon - \frac{1}{8}\varepsilon\gamma > 0, \ \nu - \frac{1}{2}\nu\gamma - \varepsilon\alpha + \frac{1}{8}\varepsilon^2\gamma > 0,$$
$$\nu - \varepsilon - \varepsilon\alpha + \frac{1}{2}\varepsilon^2 > 0, \qquad 1 - \frac{\gamma}{2} - \frac{2\varpi\varepsilon}{\varrho} > 0. \tag{4.7}$$

We also assume

$$\int_{-\infty}^{\tau} e^{\frac{1}{4}\varepsilon\gamma s} \|g(s)\|_1^2 ds < \infty, \quad \forall\, \tau \in \mathbb{R}, \tag{4.8}$$

$$\lim_{t \to +\infty} e^{-ct} \int_{-\infty}^{0} e^{\frac{1}{4}\varepsilon\gamma s} \|g(s-t)\|_1^2 ds = 0, \quad \text{for } \forall\, c > 0. \tag{4.9}$$

Lemma 4.1. *Let (3.3)–(3.5), (3.8), (4.1)-(4.2) and (4.5)–(4.8) hold. Then for any $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$, the solution of (3.2) satisfies*

$$\|u_t(\tau, \tau - t, \theta_{-\tau}\omega, u_{1,0})\|^2 + \|u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)\|^2_{H^2(\mathbb{R}^n)} + \|\eta^t(\tau, \tau - t, \theta_{-\tau}\omega, \eta^0, s)\|^2_{\mu,2}$$

$$+\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}(\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2+\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2_{H^2(\mathbb{R}^n)}+\|\eta^t(s,\tau-t,\theta_{-\tau}\omega,\eta^0,s)\|^2_{\mu,2})ds$$
$$\le M_1+M_1\int_{-\infty}^{0} e^{\frac{1}{4}\varepsilon\gamma s}(1+\|g(s+\tau)\|^2+|\zeta_\delta(\theta_s\omega)|^{2+\frac{4}{p}})ds,$$

where $(u_0,u_{1,0},\eta^0)\in D(\tau-t,\theta_{-t}\omega)$ and M_1 is a positive constant independent of τ,ω and D.

Proof. By (3.11), (3.13), (4.1) and (4.10) we obtain, for almost all $t\in[\tau,\tau+T]$,

$$\begin{aligned}&\frac{d}{dt}\Big(\|u_t\|^2+\nu\|u\|^2+\|\Delta u\|^2+\|\eta^t\|^2_{\mu,2}+2\int_{\mathbb{R}^n}F(x,u(t,x))dx+\varepsilon(u,u_t)\Big)\\&+(2\alpha-\varepsilon)\|u_t\|^2+\varepsilon\alpha(u,u_t)+\varepsilon\|\Delta u\|^2+\varepsilon(\eta^t(s),u(t))_{\mu,2}\\&-\int_0^\infty\mu'(s)\|\Delta\eta^t\|^2ds+\varepsilon\nu\|u\|^2+\varepsilon\gamma\int_{\mathbb{R}}F(x,u(t,x))dx\\\le&\varepsilon\|\varphi_3\|_{L^1(\mathbb{R}^n)}+(g(t)+h(t,\cdot,u(t))\zeta_\delta(\theta_t\omega),\varepsilon u+2u_t).\end{aligned}\tag{4.10}$$

By (3.3), (4.2) and (4.5) we have

$$\varepsilon(\eta^t(s),u(t))_{\mu,2}\ge-\frac{\varrho}{4}\|\eta^t\|^2_{\mu,2}-\frac{\varpi\varepsilon^2}{\varrho}\|\Delta u\|^2,\tag{4.11}$$

$$-\int_0^\infty\mu'(s)\|\Delta\eta^t\|^2ds\ge\varrho\|\eta^t\|^2_{\mu,2},\tag{4.12}$$

$$\begin{aligned}&(g(t)+h(t,\cdot,u(t))\zeta_\delta(\theta_t\omega),\varepsilon u+2u_t)\\\le&(\|g(t)\|+\|h(t,\cdot,u(t))\zeta_\delta(\theta_t\omega)\|)(\varepsilon\|u\|+2\|u_t\|)\\\le&\frac{1}{2}\varepsilon\nu\|u\|^2+\alpha\|u_t\|^2+(\alpha^{-1}+\frac{1}{2}\varepsilon\nu^{-1})(\|g(t)\|+\|h(t,\cdot,u(t))\zeta_\delta(\theta_t\omega)\|)^2\\\le&\frac{1}{2}\varepsilon\nu\|u\|^2+\alpha\|u_t\|^2+(2\alpha^{-1}+\varepsilon\nu^{-1})\|g(t)\|^2+(2\alpha^{-1}+\varepsilon\nu^{-1})\|h(t,\cdot,u(t))\zeta_\delta(\theta_t\omega)\|^2\\\le&\frac{1}{2}\varepsilon\nu\|u\|^2+\alpha\|u_t\|^2+(2\alpha^{-1}+\varepsilon\nu^{-1})\|g(t)\|^2+2(2\alpha^{-1}+\varepsilon\nu^{-1})|\zeta_\delta(\theta_t\omega)|^2\|\varphi_6\|^2\\&+2(2\alpha^{-1}+\varepsilon\nu^{-1})|\zeta_\delta(\theta_t\omega)|^2\int_{\mathbb{R}^n}|\varphi_5(x)|^2|u(t,x)|^2dx\\\le&\frac{1}{2}\varepsilon\nu\|u\|^2+\alpha\|u_t\|^2+(2\alpha^{-1}+\varepsilon\nu^{-1})\|g(t)\|^2+c_1|\zeta_\delta(\theta_t\omega)|^2+\frac{1}{2}\varepsilon\gamma\alpha_4\int_{\mathbb{R}^n}|u(t,x)|^{p+2}dx\\&+c_2|\zeta_\delta(\theta_t\omega)|^{2+\frac{4}{p}}\int_{\mathbb{R}^n}|\varphi_5(x)|^{2+\frac{4}{p}}dx\\\le&\frac{1}{2}\varepsilon\nu\|u\|^2+\alpha\|u_t\|^2+(2\alpha^{-1}+\varepsilon\nu^{-1})\|g(t)\|^2+c_1|\zeta_\delta(\theta_t\omega)|^2+\frac{1}{2}\varepsilon\gamma\int_{\mathbb{R}}F(x,u(t,x))dx\\&+\frac{1}{2}\varepsilon\gamma\|\varphi_1\|_{L^1(\mathbb{R}^n)}+c_3|\zeta_\delta(\theta_t\omega)|^{2+\frac{4}{p}}\\\le&\frac{1}{2}\varepsilon\nu\|u\|^2+\alpha\|u_t\|^2+(2\alpha^{-1}+\varepsilon\nu^{-1})\|g(t)\|^2+\frac{1}{2}\varepsilon\gamma\int_{\mathbb{R}}F(x,u(t,x))dx+c_4(1+|\zeta_\delta(\theta_t\omega)|^{2+\frac{4}{p}}),\end{aligned}\tag{4.13}$$

where $c_4>0$ depends on $\alpha,\nu,\gamma,\varepsilon$.

It follows from (4.10)-(4.13) and rewrite the result obtained, we have

$$
\begin{aligned}
&\frac{d}{dt}(\|u_t\|^2+\nu\|u\|^2+\|\Delta u\|^2+\|\eta^t\|^2_{\mu,2}+2\int_{\mathbb{R}^n}F(x,u(t,x))dx+\varepsilon(u,u_t))\\
&+\frac{1}{4}\varepsilon\gamma(\|u_t\|^2+\nu\|u\|^2+\|\Delta u\|^2+\|\eta^t\|^2_{\mu,2}+2\int_{\mathbb{R}^n}F(x,u(t,x))dx+\varepsilon(u,u_t))\\
&+(\alpha-\varepsilon-\frac{1}{4}\varepsilon\gamma)\|u_t\|^2+\varepsilon(1-\frac{1}{4}\gamma-\frac{\varpi\varepsilon}{\varrho})\|\Delta u\|^2+\frac{1}{4}(3\varrho-\varepsilon\gamma)\|\eta^t\|^2_{\mu,2}+\frac{1}{2}\varepsilon\nu(1-\frac{1}{2}\gamma)\|u\|^2\\
\le& c_5(1+\|g(t)\|^2+|\zeta_\delta(\theta_t\omega)|^{2+\frac{4}{p}})-\varepsilon(\alpha-\frac{1}{4}\varepsilon\gamma)(u,u_t)),
\end{aligned}
\tag{4.14}
$$

where $c_5>0$ depends on $\alpha,\nu,\gamma,\varepsilon$.
For the second term on the right-hand side of (4.14) we get

$$
\begin{aligned}
&-\varepsilon(\alpha-\frac{1}{4}\varepsilon\gamma)(u,u_t))\\
\le&\varepsilon(\alpha-\frac{1}{4}\varepsilon\gamma)\|u\|\|u_t\|\\
\le&\frac{1}{2}\varepsilon^2(\alpha-\frac{1}{4}\varepsilon\gamma)\|u\|^2+\frac{1}{2}(\alpha-\frac{1}{4}\varepsilon\gamma)\|u_t\|^2.
\end{aligned}
\tag{4.15}
$$

By (4.14)-(4.15) we get

$$
\begin{aligned}
&\frac{d}{dt}(\|u_t\|^2+\nu\|u\|^2+\|\Delta u\|^2+\|\eta^t\|^2_{\mu,2}+2\int_{\mathbb{R}^n}F(x,u(t,x))dx+\varepsilon(u,u_t))\\
&+\frac{1}{4}\varepsilon\gamma(\|u_t\|^2+\nu\|u\|^2+\|\Delta u\|^2+2\int_{\mathbb{R}^n}F(x,u(t,x))dx+\varepsilon(u,u_t))+(\frac{1}{2}\alpha-\varepsilon-\frac{1}{8}\varepsilon\gamma)\|u_t\|^2\\
&+\varepsilon(1-\frac{1}{4}\gamma-\frac{\varpi\varepsilon}{\varrho})\|\Delta u\|^2+\frac{1}{4}(3\varrho-\varepsilon\gamma)\|\eta^t\|^2_{\mu,2}+\frac{1}{2}\varepsilon(\nu-\frac{1}{2}\nu\gamma-\varepsilon\alpha+\frac{1}{4}\varepsilon^2\gamma)\|u\|^2\\
\le& c_5(1+\|g(t)\|^2+|\zeta_\delta(\theta_t\omega)|^{2+\frac{4}{p}}).
\end{aligned}
\tag{4.16}
$$

Multiplying (4.14) by $e^{\frac{1}{4}\varepsilon\gamma t}$, and then integrating the inequality $[\tau-t,\tau]$, after replacing ω by $\theta_{-\tau}\omega$, we get

$$
\begin{aligned}
&\|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2+\nu\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|\eta^t(\tau,\tau-t,\theta_{-\tau}\omega,\eta^0,s)\|^2_{\mu,2}+2\int_{\mathbb{R}^n}F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0))dx\\
&+\varepsilon(u(\tau,\tau-t,\theta_{-\tau}\omega,u_0),u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0}))\\
&+(\frac{1}{2}\alpha-\varepsilon-\frac{1}{8}\varepsilon\gamma)\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2ds\\
&+\varepsilon(1-\frac{1}{4}\gamma-\frac{\varpi\varepsilon}{\varrho})\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2ds\\
&+\frac{1}{4}(3\varrho-\varepsilon\gamma)\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\|\eta^t(\tau,\tau-t,\theta_{-\tau}\omega,\eta^0,s)\|^2_{\mu,2}ds\\
&+\frac{1}{2}\varepsilon(\nu-\frac{1}{2}\nu\gamma-\varepsilon\alpha+\frac{1}{4}\varepsilon^2\gamma)\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2ds
\end{aligned}
$$

$$\leq e^{-\frac{1}{4}\varepsilon\gamma t}\Big(\|u_{1,0}\|^2+\nu\|u_0\|^2+\|\Delta u_0\|^2+\|\eta^0\|^2_{\mu,2}+2\int_{\mathbb{R}^n}F(x,u_0)dx+\varepsilon(u_0,u_{1,0})\Big)$$
$$+c_5\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\Big(1+\|g(s)\|^2+|\zeta_\delta(\theta_{s-\tau}\omega)|^{2+\frac{4}{p}}\Big)ds. \tag{4.17}$$

For the first term on the right-hand side of (4.17), by (4.4) we get

$$e^{-\frac{1}{4}\varepsilon\gamma t}\Big(\|u_{1,0}\|^2+\nu\|u_0\|^2+\|\Delta u_0\|^2+\|\eta^0\|^2_{\mu,2}+2\int_{\mathbb{R}^n}F(x,u_0)dx+\varepsilon(u_0,u_{1,0})\Big)$$
$$\leq c_6e^{-\frac{1}{4}\varepsilon\gamma t}\Big(1+\|u_{1,0}\|^2+\|u_0\|^2_{H^2(\mathbb{R}^n}+\|u_0\|^{p+2}_{H^2(\mathbb{R}^n)}+\|\eta^0\|^2_{\mu,2}\Big)$$
$$\leq c_7e^{-\frac{1}{4}\varepsilon\gamma t}(1+\|D(\tau-t,\theta_{-t}\omega)\|^{p+2})\to 0,\quad \text{as } t\to\infty. \tag{4.18}$$

By (4.7) we get

$$|\varepsilon(u(\tau,\tau-t,\theta_{-\tau}\omega,u_0),u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0}))|$$
$$\leq\frac{1}{2}\varepsilon\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\frac{1}{2}\varepsilon\|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2$$
$$\leq\frac{1}{2}\nu\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\frac{1}{2}\|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2,$$

which along with (4.2) and (4.18) that for all $t\geq T$,

$$\frac{1}{2}\|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2+\frac{1}{2}\nu\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2$$
$$+\|\eta^t(\tau,\tau-t,\theta_{-\tau}\omega,\eta^0,s)\|^2_{\mu,2}+(\frac{1}{2}\alpha-\varepsilon-\frac{1}{8}\varepsilon\gamma)\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2ds$$
$$+\varepsilon(1-\frac{1}{4}\gamma-\frac{\varpi\varepsilon}{\varrho})\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2ds$$
$$+\frac{1}{4}(3\varrho-\varepsilon\gamma)\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\|\eta^t(s,\tau-t,\theta_{-\tau}\omega,\eta^0,s)\|^2_{\mu,2}ds$$
$$+\frac{1}{2}\varepsilon(\nu-\frac{1}{2}\nu\gamma-\varepsilon\alpha+\frac{1}{4}\varepsilon^2\gamma)\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2ds$$
$$\leq 1+2\|\varphi_1\|_{L^1(\mathbb{R}^n)}+c_5\int_{-\infty}^{0}e^{\frac{1}{4}\varepsilon\gamma s}\Big(1+\|g(s+\tau)\|^2+|\zeta_\delta(\theta_s\omega)|^{2+\frac{4}{p}}\Big)ds.$$

Then the proof is completed. □

Based on Lemma 4.1, we can easily obtain the following Lemma that implies the existence of tempered random absorbing sets of Φ.

Lemma 4.2. *If (3.3)-(3.5), (3.8), (4.1)-(4.2) and (4.5)-(4.9) hold, then the cocycle Φ possesses a closed measurable $\mathcal{D}$-pullback absorbing set $B=\{B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, which is given by*

$$B(\tau,\omega)=\{(u_0,u_{1,0},\eta^0)\in H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times\Re_{\mu,2}:\|u_0\|^2_{H^2(\mathbb{R}^n)}+\|u_{1,0}\|^2+\|\eta^0\|^2_{\mu,2}\leq L(\tau,\omega)\}, \tag{4.19}$$

where

$$L(\tau,\omega)=M_1+M_1\int_{-\infty}^{0}e^{\frac{1}{4}\varepsilon\gamma s}\Big(1+\|g(s+\tau)\|^2+|\zeta_\delta(\theta_s\omega)|^{2+\frac{4}{p}}\Big)ds.$$

In order to derive the uniform tail-estimates of the solutions of (3.2) for large space variables when times is large enough, we need to derive the regularity of the solutions in a space higher than $H^2(\mathbb{R}^n)$.

Lemma 4.3. *Let (3.3)–(3.5), (3.8), (4.1)-(4.2) and (4.5)–(4.8) hold. Then for any $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$, the solution of (3.2) satisfies*

$$
\begin{aligned}
&\|A^{\frac{1}{4}}u_t(\tau, \tau - t, \theta_{-\tau}\omega, u_{1,0})\|^2 + \|A^{\frac{3}{4}}u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)\|^2 + \|A^{\frac{1}{4}}\eta^t(\tau, \tau - t, \theta_{-\tau}\omega, \eta^0, s)\|^2_{\mu,2}\\
&+ \int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}(\|A^{\frac{1}{4}}u_t(s, \tau - t, \theta_{-\tau}\omega, u_{1,0})\|^2 + \|A^{\frac{3}{4}}u(s, \tau - t, \theta_{-\tau}\omega, u_0)\|^2)ds\\
&+ \int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}(\|A^{\frac{1}{4}}\eta^t(s, \tau - t, \theta_{-\tau}\omega, \eta^0, s)\|^2_{\mu,2}\\
&\leq M_2 + M_2 \int_{-\infty}^{0} e^{\frac{1}{4}\varepsilon\gamma s}(1 + \|g(s + \tau)\|_1^2 + |\zeta_\delta(\theta_s\omega)|^2)ds,
\end{aligned}
$$

where $(u_0, u_{1,0}, \eta^0) \in D(\tau - t, \theta_{-\tau}\omega)$ and M_2 is a positive number independent of τ, ω and D.

Proof. Taking the inner product of $(3.2)_1$ with $A^{\frac{1}{2}}u$ in $L^2(\mathbb{R}^n)$, we have

$$
\begin{aligned}
&\frac{d}{dt}(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u) + \alpha(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u) + \|A^{\frac{3}{4}}u\|^2 + (\int_0^\infty \mu(s)\Delta^2\eta(s)ds, A^{\frac{1}{2}}u) + \nu\|A^{\frac{1}{4}}u\|^2 + (f(x, u), A^{\frac{1}{2}}u)\\
&=\|A^{\frac{1}{4}}u_t\|^2 + (g(t) + h(t, \cdot, u)\zeta_\delta(\theta_t\omega), A^{\frac{1}{2}}u) \qquad (4.20)
\end{aligned}
$$

Taking the inner product of $(1.1)_1$ with $A^{\frac{1}{2}}u_t$ in $L^2(\mathbb{R}^n)$, we find that

$$
\begin{aligned}
&\frac{d}{dt}(\|A^{\frac{1}{4}}u_t\|^2 + \nu\|A^{\frac{1}{4}}u\|^2 + \|A^{\frac{3}{4}}u\|^2 + \|A^{\frac{1}{4}}\eta^t\|^2_{\mu,2})\\
&= \int_0^\infty \mu'(s)\|A^{\frac{3}{4}}\eta^t\|^2ds - 2\alpha\|A^{\frac{1}{4}}u_t\|^2 - 2(f(x, u), A^{\frac{1}{2}}u_t) + 2(g(t) + h(t, \cdot, u)\zeta_\delta(\theta_t\omega), A^{\frac{1}{2}}u_t) \qquad (4.21)
\end{aligned}
$$

By (4.20) and (4.21), we get

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|A^{\frac{1}{4}}u_t\|^2 + \nu\|A^{\frac{1}{4}}u\|^2 + \|A^{\frac{3}{4}}u\|^2 + \|A^{\frac{1}{4}}\eta^t\|^2_{\mu,2} + \varepsilon(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u)\Big) + (2\alpha - \varepsilon)\|A^{\frac{1}{4}}u_t\|^2\\
&+ \varepsilon\alpha(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u) + \varepsilon\|A^{\frac{3}{4}}u\|^2 + \varepsilon(\int_0^\infty \mu(s)\Delta^2\eta(s)ds, A^{\frac{1}{2}}u) - \int_0^\infty \mu'(s)\|A^{\frac{3}{4}}\eta^t\|^2ds\\
&+ \varepsilon\nu\|A^{\frac{1}{4}}u\|^2 + \varepsilon(f(x, u), A^{\frac{1}{2}}u) + 2(f(x, u), A^{\frac{1}{2}}u_t)\\
&=(g(t) + h(t, \cdot, u)\zeta_\delta(\theta_t\omega), \varepsilon A^{\frac{1}{2}}u + 2A^{\frac{1}{2}}u_t). \qquad (4.22)
\end{aligned}
$$

By (3.3), (4.5), (4.6) and Lemma 4.1, we have

$$
\varepsilon(\int_0^\infty \mu(s)\Delta^2\eta(s)ds, A^{\frac{1}{2}}u) \geq -\frac{\varrho}{4}\|A^{\frac{1}{4}}\eta^t\|^2_{\mu,2} - \frac{\varpi\varepsilon^2}{\varrho}\|A^{\frac{3}{4}}u\|^2, \qquad (4.23)
$$

$$
- \int_0^\infty \mu'(s)\|A^{\frac{3}{4}}\eta^t\|^2ds \geq \varrho\|A^{\frac{1}{4}}\eta^t\|^2_{\mu,2}, \qquad (4.24)
$$

$$
(g(t) + h(t, \cdot, u(t))\zeta_\delta(\theta_t\omega), \varepsilon A^{\frac{1}{2}}u + 2A^{\frac{1}{2}}u_t)
$$

$$
\begin{aligned}
&\le (\|g(t)\|_1 + \|h(t,\cdot,u(t))\zeta_\delta(\theta_t\omega)\|_1)(\varepsilon\|A^{\frac{1}{4}}u\| + 2\|A^{\frac{1}{2}}u_t\|)\\
&\le \frac{1}{2}\varepsilon\nu\|A^{\frac{1}{4}}u\|^2 + \alpha\|A^{\frac{1}{2}}u_t\|^2 + (\alpha^{-1} + \frac{1}{2}\varepsilon\nu^{-1})(\|g(t)\|_1 + \|h(t,\cdot,u(t))\zeta_\delta(\theta_t\omega)\|_1)^2\\
&\le \frac{1}{2}\varepsilon\nu\|A^{\frac{1}{4}}u\|^2 + \alpha\|A^{\frac{1}{4}}u_t\|^2 + (2\alpha^{-1} + \varepsilon\nu^{-1})\|g(t)\|_1^2 + (2\alpha^{-1} + \varepsilon\nu^{-1})\|h(t,\cdot,u(t))\zeta_\delta(\theta_t\omega)\|_1^2\\
&\le \frac{1}{2}\varepsilon\nu\|A^{\frac{1}{4}}u\|^2 + \alpha\|A^{\frac{1}{4}}u_t\|^2 + (2\alpha^{-1} + \varepsilon\nu^{-1})\|g(t)\|_1^2 + c_8|\zeta_\delta(\theta_t\omega)|^2. \qquad (4.25)
\end{aligned}
$$

From (4.3) and Lemma 4.1 yields

$$
\begin{aligned}
&|\varepsilon(f(x,u),A^{\frac{1}{2}}u) + 2(f(x,u),A^{\frac{1}{2}}u_t)|\\
&\le 2\int_{\mathbb{R}^n}|\frac{\partial f}{\partial u}(x,u)\cdot A^{\frac{1}{4}}u\cdot A^{\frac{1}{4}}u_t + \frac{\partial f}{\partial x}(x,u)\cdot A^{\frac{1}{4}}u_t|dx\\
&\quad + \varepsilon\int_{\mathbb{R}^n}|\frac{\partial f}{\partial u}(x,u)\cdot A^{\frac{1}{4}}u\cdot A^{\frac{1}{4}}u + \frac{\partial f}{\partial x}(x,u)\cdot A^{\frac{1}{4}}u|dx\\
&\le 2\iota\int_{\mathbb{R}^n}|u|^p\cdot|A^{\frac{1}{4}}u|\cdot|A^{\frac{1}{4}}u_t|dx + 2\varsigma\int_{\mathbb{R}^n}|A^{\frac{1}{4}}u|\cdot|A^{\frac{1}{4}}u_t|dx + 2\int_{\mathbb{R}^n}|\varphi_4|\cdot|A^{\frac{1}{4}}u_t|dx\\
&\quad + \varepsilon\iota\int_{\mathbb{R}^n}|u|^p\cdot|A^{\frac{1}{4}}u|\cdot|A^{\frac{1}{4}}u|dx + \varepsilon\varsigma\int_{\mathbb{R}^n}|A^{\frac{1}{4}}u|\cdot|A^{\frac{1}{4}}u|dx + \varepsilon\int_{\mathbb{R}^n}|\varphi_4|\cdot|A^{\frac{1}{4}}u|dx\\
&\le 2\iota\|u\|^p_{L^{\frac{10p}{4}}}\cdot\|A^{\frac{1}{4}}u\|_{L^{10}}\cdot\|A^{\frac{1}{4}}u_t\| + 2\varsigma\|A^{\frac{1}{4}}u\|\cdot\|A^{\frac{1}{4}}u_t\| + \frac{\varepsilon}{4}\|A^{\frac{1}{4}}u_t\|^2 + \frac{4}{\varepsilon}\|\varphi_4\|^2\\
&\quad + \varepsilon\iota\|u\|^p\cdot\|A^{\frac{1}{4}}u\|^2 + \varepsilon\varsigma\|A^{\frac{1}{4}}u\|^2 + \frac{\varepsilon}{2}\|A^{\frac{1}{4}}u\|^2 + \frac{\varepsilon}{2}\|\varphi_4\|^2\\
&\le \varepsilon\|A^{\frac{1}{4}}u_t\|^2 + \frac{2C^{p+1}\iota^2}{\varepsilon}L^p\|A^{\frac{3}{4}}u\|^2 + c_9,
\end{aligned}
$$

where the definition of L see Lemma 4.2, and C is the positive constant satisfying

$$
C\|\Delta u\|^2 \ge \Big(\int_{\mathbb{R}^n}|u|^{10}dx\Big)^{\frac{1}{5}}, \quad C\|u\|_2^2 \ge \Big(\int_{\mathbb{R}^n}|u|^{\frac{10p}{4}}dx\Big)^{\frac{2}{10p}}.
$$

Choosing

$$
0 < \iota^2 \le \frac{\varepsilon^2}{4L^pC^{p+1}},
$$

thus, we get

$$
|\varepsilon(f(x,u),A^{\frac{1}{2}}u) + 2(f(x,u),A^{\frac{1}{2}}u_t)| \le \varepsilon\|A^{\frac{1}{4}}u_t\|^2 + \frac{\varepsilon}{2}\|A^{\frac{3}{4}}u\|^2 + c_9. \qquad (4.26)
$$

By (4.22)–(4.26), we get

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|A^{\frac{1}{4}}u_t\|^2 + \nu\|A^{\frac{1}{4}}u\|^2 + \|A^{\frac{3}{4}}u\|^2 + \|A^{\frac{1}{4}}\eta^t\|^2_{\mu,2} + \varepsilon(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u)\Big) + (\alpha - 2\varepsilon)\|A^{\frac{1}{4}}u_t\|^2\\
&\quad + \varepsilon\alpha(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u) + \varepsilon(\frac{1}{2} - \frac{\varpi\varepsilon}{\varrho})\|A^{\frac{3}{4}}u\|^2 + \frac{3}{4}\varrho\|A^{\frac{1}{4}}\eta^t\|^2_{\mu,2} + \frac{\varepsilon}{2}\nu\|A^{\frac{1}{4}}u\|^2\\
&\le c_{10}(1 + \|g(t)\|_1^2 + |\zeta_\delta(\theta_t\omega)|^2),
\end{aligned}
$$

which can be rewritten as

$$
\frac{d}{dt}\Big(\|A^{\frac{1}{4}}u_t\|^2 + \nu\|A^{\frac{1}{4}}u\|^2 + \|A^{\frac{3}{4}}u\|^2 + \|A^{\frac{1}{4}}\eta^t\|^2_{\mu,2} + \varepsilon(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u)\Big)
$$

$$
\begin{aligned}
&+\frac{1}{4}\varepsilon\gamma\Big(\|A^{\frac{1}{4}}u_t\|^2+\nu\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2+\|A^{\frac{1}{4}}\eta^t\|_{\mu,2}^2+\varepsilon(A^{\frac{1}{4}}u_t,A^{\frac{1}{4}}u)\Big)\\
&+(\alpha-2\varepsilon-\frac{1}{4}\varepsilon\gamma)\|A^{\frac{1}{4}}u_t\|^2+\frac{\varepsilon}{2}(1-\frac{2\varpi\varepsilon}{\varrho}-\frac{\gamma}{2})\|A^{\frac{3}{4}}u\|^2+\frac{3}{4}(\varrho-\frac{1}{3}\varepsilon\gamma)\|A^{\frac{1}{4}}\eta^t\|_{\mu,2}^2+\frac{\varepsilon}{2}\nu(1-\frac{\gamma}{2})\|A^{\frac{1}{4}}u\|^2\\
\leq& c_{10}(1+\|g(t)\|_1^2+|\zeta_\delta(\theta_t\omega)|^2)-\varepsilon(\alpha-\frac{1}{4}\varepsilon\gamma)(A^{\frac{1}{4}}u_t,A^{\frac{1}{4}}u).
\end{aligned}\tag{4.27}
$$

For the last term on the right-hand side of (4.27) we have

$$
\begin{aligned}
&-\varepsilon(\alpha-\frac{1}{4}\varepsilon\gamma)(A^{\frac{1}{4}}u_t,A^{\frac{1}{4}}u)\\
\leq&\varepsilon(\alpha-\frac{1}{4}\varepsilon\gamma)\|A^{\frac{1}{4}}u\|\|A^{\frac{1}{4}}u_t\|\\
\leq&\frac{1}{2}\varepsilon^2(\alpha-\frac{1}{4}\varepsilon\gamma)\|A^{\frac{1}{4}}u\|^2+\frac{1}{2}(\alpha-\frac{1}{4}\varepsilon\gamma)\|A^{\frac{1}{4}}u_t\|^2,
\end{aligned}
$$

which together with (4.27), we get

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|A^{\frac{1}{4}}u_t\|^2+\nu\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2+\|A^{\frac{1}{4}}\eta^t\|_{\mu,2}^2+\varepsilon(A^{\frac{1}{4}}u_t,A^{\frac{1}{4}}u)\Big)\\
&+\frac{1}{4}\varepsilon\gamma\Big(\|A^{\frac{1}{4}}u_t\|^2+\nu\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2+\|A^{\frac{1}{4}}\eta^t\|_{\mu,2}^2+\varepsilon(A^{\frac{1}{4}}u_t,A^{\frac{1}{4}}u)\Big)\\
&+(\frac{\alpha}{2}-2\varepsilon-\frac{1}{8}\varepsilon\gamma)\|A^{\frac{1}{4}}u_t\|^2+\frac{\varepsilon}{2}(1-\frac{2\varpi\varepsilon}{\varrho}-\frac{\gamma}{2})\|A^{\frac{3}{4}}u\|^2+\frac{3}{4}(\varrho-\frac{1}{3}\varepsilon\gamma)\|A^{\frac{1}{4}}\eta^t\|_{\mu,2}^2\\
&+\frac{\varepsilon}{2}(\nu-\frac{\nu}{2}\gamma-\frac{\varepsilon}{2}\alpha+\frac{1}{8}\varepsilon^2\gamma)\|A^{\frac{1}{4}}u\|^2\\
\leq& c_{10}(1+\|g(t)\|_1^2+|\zeta_\delta(\theta_t\omega)|^2).
\end{aligned}
$$

Similar to the remainder of Lemma 4.1, we can obtain the desired result. □

Lemma 4.4. *Let (3.3)–(3.5), (3.8), (4.1)-(4.2) and (4.5)–(4.8) hold. Then for every $\eta>0,\tau\in\mathbb{R},\omega\in\Omega$ and $D\in\mathcal{D}$, there exists $T_0=T_0(\eta,\tau,\omega,D)>0$ and $m_0=m_0(\eta,\tau,\omega)\geq1$ such that for all $t\geq T_0$, $m\geq m_0$ and $(u_0,u_{1,0},\eta^0)\in D(\tau-t,\theta_{-\tau}\omega)$, the solution of (3.2) satisfies*

$$
\begin{aligned}
&\int_{|x|\geq m}(|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})|^2+|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2\\
&+|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+|\eta^t(\tau,\tau-t,\theta_{-\tau}\omega,\eta^0,s)|_{\mu,2}^2)dx<\eta.
\end{aligned}
$$

Proof. Let $\rho:\mathbb{R}^n\to\mathbb{R}$ be a smooth function such that $0\leq\rho(x)\leq1$ for all $x\in\mathbb{R}^n$, and

$$
\rho(x)=0\ \text{ for }\ |x|\leq\frac{1}{2};\quad\text{and}\quad\rho(x)=1\ \text{ for }\ |x|\geq1.
$$

For every $m\in\mathbb{N}$, let

$$
\rho_m(x)=\rho(x/m),\quad x\in\mathbb{R}^n.
$$

Then there exist positive constants c_{11} and c_{12} independent of m such that $|\nabla\rho_m(x)|\leq\frac{1}{m}c_{11}$, $|\Delta\rho_m(x)|\leq\frac{1}{m}c_{12}$ for all $x\in\mathbb{R}^n$ and $m\in\mathbb{N}$.

Similar to the energy equation (3.11), we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\Delta u(t,x)|^2+|\eta^t(s)|^2_{\mu,2}+2F(x,u(t,x)))dx\\
&+2\alpha\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2dx-\int_{\mathbb{R}^n}\rho_m(x)\int_0^\infty\mu'(s)|\Delta\eta^t(s)|^2dsdx\\
=&-4\int_{\mathbb{R}^n}\nabla\rho_m(x)\cdot\Delta u(t,x)\cdot\nabla u_t(t,x)dx-2\int_{\mathbb{R}^n}\Delta\rho_m(x)\cdot\Delta u(t,x)\cdot u_t(t,x)dx\\
&-4\int_{\mathbb{R}^n}\nabla\rho_m(x)\int_0^\infty\mu(s)\Delta\eta^t(s)\nabla u_t(t,x)dsdx-2\int_{\mathbb{R}^n}\Delta\rho_m(x)\int_0^\infty\mu(s)\Delta\eta^t(s)u_t(t,x)dsdx\\
&+2\int_{\mathbb{R}^n}\rho_m(x)g(t,x)u_t(t,x)dx+2\zeta_\delta(\theta_t\omega)\int_{\mathbb{R}^n}\rho_m(x)h(t,x,u(t,x))u_t(t,x)dx. \qquad (4.28)
\end{aligned}$$

Taking the inner product of $(3.2)_1$ with $\rho_m(x)u$ in $L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)u(t,x)u_t(t,x)dx+\alpha\int_{\mathbb{R}^n}\rho_m(x)u(t,x)u_t(t,x)dx+\int_{\mathbb{R}^n}\rho_m(x)|\Delta u(t,x)|^2dx\\
&+\int_{\mathbb{R}^n}\rho_m(x)\int_0^\infty\mu(s)\Delta\eta^t(s)\Delta u(t,x)dsdx+\nu\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2dx+\int_{\mathbb{R}^n}\rho_m(x)f(x,u(t,x))u(t,x)dx\\
=&\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2dx-2\int_{\mathbb{R}^n}\nabla\rho_m(x)\cdot\Delta u(t,x)\cdot\nabla u(t,x)dx-\int_{\mathbb{R}^n}\Delta\rho_m(x)\cdot\Delta u(t,x)\cdot u(t,x)dx\\
&-2\int_{\mathbb{R}^n}\nabla\rho_m(x)\int_0^\infty\mu(s)\Delta\eta^t(s)\nabla u(t,x)dsdx-\int_{\mathbb{R}^n}\Delta\rho_m(x)\int_0^\infty\mu(s)\Delta\eta^t(s)u(t,x)dsdx\\
&+\int_{\mathbb{R}^n}\rho_m(x)(g(t,x)+h(t,x,u(t,x))\zeta_\delta(\theta_t\omega))u(t,x)dx. \qquad (4.29)
\end{aligned}$$

By (4.28)-(4.29) and (4.1), we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)\Big(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\Delta u(t,x)|^2+|\eta^t(s)|^2_{\mu,2}+2F(x,u(t,x))+\varepsilon u(t,x)u_t(t,x)\Big)dx\\
&+(2\alpha-\varepsilon)\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2dx+\varepsilon\alpha\int_{\mathbb{R}^n}\rho_m(x)u(t,x)u_t(t,x)dx+\varepsilon\int_{\mathbb{R}^n}\rho_m(x)|\Delta u(t,x)|^2dx\\
&+\varepsilon\int_{\mathbb{R}^n}\rho_m(x)\int_0^\infty\mu(s)\Delta\eta^t(s)\Delta u(t,x)dsdx-\int_{\mathbb{R}^n}\rho_m(x)\int_0^\infty\mu'(s)|\Delta\eta^t(s)|^2dsdx\\
&+\varepsilon\nu\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2dx+\varepsilon\gamma\int_{\mathbb{R}^n}\rho_m(x)F(x,u(t,x))dx\\
\leq&\int_{\mathbb{R}^n}\rho_m(x)(g(t,x)+h(t,x,u(t,x))\zeta_\delta(\theta_t\omega))(\varepsilon u(t,x)+2u_t(t,x))dx-\varepsilon\int_{\mathbb{R}^n}\rho_m(x)\varphi_3(x)dx\\
&-2\varepsilon\int_{\mathbb{R}^n}\nabla\rho_m(x)\cdot\Delta u(t,x)\cdot\nabla u(t,x)dx-\varepsilon\int_{\mathbb{R}^n}\Delta\rho_m(x)\cdot\Delta u(t,x)\cdot u(t,x)dx\\
&-2\varepsilon\int_{\mathbb{R}^n}\nabla\rho_m(x)\int_0^\infty\mu(s)\Delta\eta^t(s)\nabla u(t,x)dsdx-\varepsilon\int_{\mathbb{R}^n}\Delta\rho_m(x)\int_0^\infty\mu(s)\Delta\eta^t(s)u(t,x)dsdx\\
&-4\int_{\mathbb{R}^n}\nabla\rho_m(x)\cdot\Delta u(t,x)\cdot\nabla u_t(t,x)dx-2\int_{\mathbb{R}^n}\Delta\rho_m(x)\cdot\Delta u(t,x)\cdot u_t(t,x)dx
\end{aligned}$$

$$- 4 \int_{\mathbb{R}^n} \nabla\rho_m(x) \int_0^\infty \mu(s)\Delta\eta^t(s)\nabla u_t(t,x)dsdx - 2\int_{\mathbb{R}^n} \Delta\rho_m(x)\int_0^\infty \mu(s)\Delta\eta^t(s)u_t(t,x)dsdx. \quad (4.30)$$

Similar to the arguments of (4.11)-(4.13), we have the following estimates:

$$\varepsilon \int_{\mathbb{R}^n} \rho_m(x)\int_0^\infty \mu(s)\Delta\eta^t(s)\Delta u(t,x)dsdx \geq -\frac{\varrho}{4}\int_{\mathbb{R}^n}\rho_m(x)|\eta^t|^2_{\mu,2}dx - \frac{\varpi\varepsilon^2}{\varrho}\int_{\mathbb{R}^n}\rho_m(x)|\Delta u|^2dx, \quad (4.31)$$

$$-\int_{\mathbb{R}^n}\rho_m(x)\int_0^\infty \mu'(s)|\Delta\eta^t(s)|^2dsdx \geq \varrho\int_{\mathbb{R}^n}\rho_m(x)|\eta^t|^2_{\mu,2}dx, \quad (4.32)$$

$$\begin{aligned}
&|\int_{\mathbb{R}^n}\rho_m(x)(g(t,x)+h(t,x,u(t,x))\zeta_\delta(\theta_t\omega))(\varepsilon u(t,x)+2u_t(t,x))dx| \\
\leq&\frac{1}{2}\varepsilon\nu\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2dx + \alpha\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2dx + \frac{1}{2}\varepsilon\gamma\int_{\mathbb{R}^n}\rho_m(x)F(x,u(t,x))dx \\
&+ c_{13}\int_{\mathbb{R}^n}\rho_m(x)\Big(|g(t,x)|^2 + |\varphi_1(x)| + |\zeta_\delta(\theta_t\omega)\varphi_6(x)|^2 + |\zeta_\delta(\theta_t\omega)\varphi_5(x)|^{2+\frac{4}{p}}\Big)dx, \quad (4.33)
\end{aligned}$$

where c_{13} depends only on α, ν, γ and ε.

By (4.30)–(4.33) we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)\Big(|u_t(t,x)|^2 + \nu|u(t,x)|^2 + |\Delta u(t,x)|^2 + |\eta^t(s)|^2_{\mu,2} + 2F(x,u(t,x)) + \varepsilon u(t,x)u_t(t,x)\Big)dx \\
&+ (\alpha-\varepsilon)\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2dx + \varepsilon\alpha\int_{\mathbb{R}^n}\rho_m(x)u(t,x)u_t(t,x)dx + \varepsilon(1-\frac{\varpi\varepsilon}{\varrho})\int_{\mathbb{R}^n}\rho_m(x)|\Delta u(t,x)|^2dx \\
&+ \frac{3\varrho}{4}\int_{\mathbb{R}^n}\rho_m(x)|\eta^t|^2_{\mu,2}dx + \frac{1}{2}\varepsilon\nu\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2dx + \frac{1}{2}\varepsilon\gamma\int_{\mathbb{R}^n}\rho_m(x)F(x,u(t,x))dx \\
\leq& c_{14}\int_{\mathbb{R}^n}\rho_m(x)\Big(|g(t,x)|^2 + |\varphi_1(x)| + |\varphi_3(x)| + |\zeta_\delta(\theta_t\omega)\varphi_6(x)|^2 + |\zeta_\delta(\theta_t\omega)\varphi_5(x)|^{2+\frac{4}{p}}\Big)dx \\
&+ \frac{c_{14}}{m}(\|u\| + \|\nabla u\| + \|u_t\| + \|\nabla u_t\| + \|\eta^t(s)\|^2_{\mu,2})\|\Delta u\|, \quad (4.34)
\end{aligned}$$

where $c_{14} > 0$ depends only on α, ν, γ and ε, but not on m.

By (4.34) we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)\Big(|u_t(t,x)|^2 + \nu|u(t,x)|^2 + |\Delta u(t,x)|^2 + |\eta^t(s)|^2_{\mu,2} + 2F(x,u(t,x)) + \varepsilon u(t,x)u_t(t,x)\Big)dx \\
&+ \frac{1}{4}\varepsilon\gamma\int_{\mathbb{R}^n}\rho_m(x)\Big(|u_t(t,x)|^2 + \nu|u(t,x)|^2 + |\Delta u(t,x)|^2 + |\eta^t(s)|^2_{\mu,2} + 2F(x,u(t,x)) + \varepsilon u(t,x)u_t(t,x)\Big)dx \\
&+ (\alpha-\varepsilon-\frac{1}{4}\varepsilon\gamma)\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2dx + \varepsilon(\alpha-\frac{1}{4}\gamma)\int_{\mathbb{R}^n}\rho_m(x)u(t,x)u_t(t,x)dx \\
&+ \varepsilon(1-\frac{1}{4}\gamma-\frac{\varpi\varepsilon}{\varrho})\int_{\mathbb{R}^n}\rho_m(x)|\Delta u(t,x)|^2dx + \frac{1}{4}(3\varrho-\varepsilon\gamma)\int_{\mathbb{R}^n}\rho_m(x)|\eta^t|^2_{\mu,2}dx \\
&+ \frac{1}{2}\varepsilon\nu(1-\frac{1}{2}\gamma)\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2dx + \frac{1}{2}\varepsilon\gamma\int_{\mathbb{R}^n}\rho_m(x)F(x,u(t,x))dx \\
\leq& c_{14}\int_{\mathbb{R}^n}\rho_m(x)\Big(|g(t,x)|^2 + |\varphi_1(x)| + |\varphi_3(x)| + |\zeta_\delta(\theta_t\omega)\varphi_6(x)|^2 + |\zeta_\delta(\theta_t\omega)\varphi_5(x)|^{2+\frac{4}{p}}\Big)dx
\end{aligned}$$

$$+\frac{c_{14}}{m}(\|u\|+\|\nabla u\|+\|u_t\|+\|\nabla u_t\|+\|\eta^t(s)\|_{\mu,2}^2)\|\Delta u\|, \tag{4.35}$$

By Young's inequality we get

$$\begin{aligned}&\left|\varepsilon(\alpha-\frac{1}{4}\gamma)\int_{\mathbb{R}^n}\rho_m(x)u(t,x)u_t(t,x)dx\right|\\ \leq&\frac{1}{2}\varepsilon^2(\alpha-\frac{1}{4}\varepsilon\gamma)\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2dx+\frac{1}{2}(\alpha-\frac{1}{4}\varepsilon\gamma)\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2dx.\end{aligned} \tag{4.36}$$

By (4.35)-(4.36) we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)\Big(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\Delta u(t,x)|^2+|\eta^t(s)|_{\mu,2}^2+2F(x,u(t,x))+\varepsilon u(t,x)u_t(t,x)\Big)dx\\ &+\frac{1}{4}\varepsilon\gamma\int_{\mathbb{R}^n}\rho_m(x)\Big(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\Delta u(t,x)|^2+|\eta^t(s)|_{\mu,2}^2+2F(x,u(t,x))+\varepsilon u(t,x)u_t(t,x)\Big)dx\\ &+(\frac{1}{2}\alpha-\varepsilon-\frac{1}{8}\varepsilon\gamma)\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2dx+\varepsilon(1-\frac{1}{4}\gamma-\frac{\varpi\varepsilon}{\varrho})\int_{\mathbb{R}^n}\rho_m(x)|\Delta u(t,x)|^2dx\\ &+\frac{1}{4}(3\varrho-\varepsilon\gamma)\int_{\mathbb{R}^n}\rho_m(x)|\eta^t|_{\mu,2}^2dx+\frac{1}{2}\varepsilon(\nu-\frac{1}{2}\nu\gamma-\varepsilon\alpha+\frac{1}{4}\varepsilon^2\gamma)\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2dx\\ &+\frac{1}{2}\varepsilon\gamma\int_{\mathbb{R}^n}\rho_m(x)F(x,u(t,x))dx\\ \leq&c_{14}\int_{\mathbb{R}^n}\rho_m(x)\Big(|g(t,x)|^2+|\varphi_1(x)|+|\varphi_3(x)|+|\zeta_\delta(\theta_t\omega)\varphi_6(x)|^2+|\zeta_\delta(\theta_t\omega)\varphi_5(x)|^{2+\frac{4}{p}}\Big)dx\\ &+\frac{c_{14}}{m}(\|u\|+\|\nabla u\|+\|u_t\|+\|\nabla u_t\|+\|\eta^t(s)\|_{\mu,2}^2)\|\Delta u\|,\end{aligned} \tag{4.37}$$

By (4.7) and (4.37) we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)\Big(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\Delta u(t,x)|^2+|\eta^t(s)|_{\mu,2}^2+2F(x,u(t,x))+\varepsilon u(t,x)u_t(t,x)\Big)dx\\ &+\frac{1}{4}\varepsilon\gamma\int_{\mathbb{R}^n}\rho_m(x)\Big(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\Delta u(t,x)|^2+|\eta^t(s)|_{\mu,2}^2+2F(x,u(t,x))+\varepsilon u(t,x)u_t(t,x)\Big)dx\\ \leq&c_{14}\int_{\mathbb{R}^n}\rho_m(x)\Big(|g(t,x)|^2+|\varphi_1(x)|+|\varphi_3(x)|+|\zeta_\delta(\theta_t\omega)\varphi_6(x)|^2+|\zeta_\delta(\theta_t\omega)\varphi_5(x)|^{2+\frac{4}{p}}\Big)dx\\ &+\frac{c_{14}}{m}(\|u\|+\|\nabla u\|+\|u_t\|+\|\nabla u_t\|+\|\eta^t(s)\|_{\mu,2}^2)\|\Delta u\|,\end{aligned} \tag{4.38}$$

Multiplying (4.38) by $e^{\frac{1}{4}\varepsilon\gamma t}$, and then integrating the inequality $[\tau-t,\tau]$, after replacing ω by $\theta_{-\tau}\omega$, we get

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho_m(x)\Big(|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})|^2+\nu|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2\\ &+|\eta^t(\tau,\tau-t,\theta_{-\tau}\omega,\eta^0,s)|_{\mu,2}^2+2F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0))+\varepsilon u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\Big)dx\\ \leq&e^{-\frac{1}{4}\varepsilon\gamma t}\int_{\mathbb{R}^n}\rho_m(x)(|u_{1,0}|^2+\nu|u_0|^2+|\Delta u_0|^2+|\eta^0|_{\mu,2}^2+2F(x,u_0(x))+\varepsilon u_0(x)u_{1,0}(x))dx\end{aligned}$$

$$
\begin{aligned}
&+c_{14}\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{\mathbb{R}^n}\rho_m(x)(|g(s,x)|^2+|\varphi_1(x)|+|\varphi_3(x)|)dxds\\
&+c_{14}\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{\mathbb{R}^n}\rho_m(x)(|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_6(x)|^2+|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_5(x)|^{2+\frac{4}{p}})dxds\\
&+\frac{2c_{14}}{m}\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}(\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2_{H^2(\mathbb{R}^n)}+\|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2_{H^1(\mathbb{R}^n)}\\
&+\|\eta^t(\tau,\tau-t,\theta_{-\tau}\omega,\eta^0,s)\|^2_{\mu,2})ds. \qquad (4.39)
\end{aligned}
$$

Next, we estimate the right-hand side of (4.39). By (4.18), we know that there exists $T_1(\eta,\tau,\omega,D)>0$ such that for all $t\geq T_1$,

$$
e^{-\frac{1}{4}\varepsilon\gamma t}\int_{\mathbb{R}^n}\rho_m(x)(|u_{1,0}|^2+\nu|u_0|^2+|\Delta u_0|^2+|\eta^0|^2_{\mu,2}+2F(x,u_0(x))+\varepsilon u_0(x)u_{1,0}(x))dx<\eta. \qquad (4.40)
$$

For the second and the third terms on the right-hand side of (4.39) we get

$$
\begin{aligned}
&c_{14}\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{\mathbb{R}^n}\rho_m(x)(|g(s,x)|^2+|\varphi_1(x)|+|\varphi_3(x)|)dxds\\
&\quad+c_{14}\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{\mathbb{R}^n}(\rho_m(x)|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_6(x)|^2+|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_5(x)|^{2+\frac{4}{p}})dxds\\
&\leq c_{14}\int_{-\infty}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{|x|\geq\frac{1}{2}m}(|g(s,x)|^2+|\varphi_1(x)|+|\varphi_3(x)|)dxds\\
&\quad+c_{14}\int_{-\infty}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{|x|\geq\frac{1}{2}m}(|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_6(x)|^2+|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_5(x)|^{2+\frac{4}{p}})dxds\\
&\leq c_{14}\int_{-\infty}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{|x|\geq\frac{1}{2}m}(|g(s,x)|^2+|\varphi_1(x)|+|\varphi_3(x)|)dxds\\
&\quad+c_{14}\int_{-\infty}^{0} e^{\frac{1}{4}\varepsilon\gamma s}|\zeta_\delta(\theta_s\omega)|^2ds\int_{|x|\geq\frac{1}{2}m}|\varphi_6(x)|^2dx\\
&\quad+c_{14}\int_{-\infty}^{0} e^{\frac{1}{4}\varepsilon\gamma s}|\zeta_\delta(\theta_s\omega)|^{2+\frac{4}{p}}ds\int_{|x|\geq\frac{1}{2}m}|\varphi_5(x)|^{2+\frac{4}{p}}dx. \qquad (4.41)
\end{aligned}
$$

By (4.8) and the conditions of $\varphi_i(x)(i=1,3,5,6)$ satisfy, we know that there exists $m_1=m_1(\eta,\tau,\omega)\geq 1$ such that for all $m\geq m_1$, the right-hand of side of (4.39) is bounded by η, i.e.,

$$
\begin{aligned}
&c_{14}\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{\mathbb{R}^n}\rho_m(x)(|g(s,x)|^2+|\varphi_1(x)|+|\varphi_3(x)|)dxds\\
&\quad+c_{14}\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{\mathbb{R}^n}(\rho_m(x)|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_6(x)|^2+|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_5(x)|^{2+\frac{4}{p}})dxds\\
&<\eta. \qquad (4.42)
\end{aligned}
$$

For the last term in (4.39), by Lemma 4.1 and Lemma 4.3, we know that there exists $T_2(\eta,\tau,\omega,D)\geq T_1$ such that for all $t\geq T_2$,

$$
\frac{2c_{14}}{m}\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}(\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2_{H^2(\mathbb{R}^n)}+\|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2_{H^1(\mathbb{R}^n)}
$$

$$
\begin{aligned}
&+\|\eta^t(\tau,\tau-t,\theta_{-\tau}\omega,\eta^0,s)\|^2_{\mu,2})ds\\
\leq&\frac{c_{15}}{m},
\end{aligned}
$$

where $c_{15}>0$ depends only on $\alpha,\nu,\gamma,\varepsilon,\tau$ and ω, but not on m. Thus, there exists $m_2=m_2(\eta,\tau,\omega)\geq m_1$ such that for all $m\geq m_2$ and $t\geq T_2$,

$$
\begin{aligned}
&\frac{2c_{14}}{m}\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}(\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2_{H^2(\mathbb{R}^n)}+\|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2_{H^1(\mathbb{R}^n)}\\
&+\|\eta^t(\tau,\tau-t,\theta_{-\tau}\omega,\eta^0,s)\|^2_{\mu,2})ds\\
\leq&\eta,
\end{aligned}
\tag{4.43}
$$

By (4.39), (4.40), (4.42) and (4.43) we see that for all $m\geq m_2$ and $t\geq T_2$,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\rho_m(x)\Big(|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})|^2+\nu|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2\\
&+|\eta^t(\tau,\tau-t,\theta_{-\tau}\omega,\eta^0,s)|^2_{\mu,2}+2F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0))+\varepsilon u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\Big)dx\\
<&3\eta.
\end{aligned}
\tag{4.44}
$$

By (4.7) we have

$$
\begin{aligned}
&\varepsilon\int_{\mathbb{R}^n}\rho_m(x)u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})dx\\
\leq&\frac{1}{2}\nu\int_{\mathbb{R}^n}\rho_m(x)|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2dx+\frac{1}{2}\int_{\mathbb{R}^n}\rho_m(x)|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})|^2dx,
\end{aligned}
$$

which together with (4.2) and (4.44) yields that for all $m\geq m_2$ and $t\geq T_2$,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\rho_m(x)\Big(\frac{1}{2}|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})|^2+\frac{1}{2}\nu|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2\\
&+|\eta^t(\tau,\tau-t,\theta_{-\tau}\omega,\eta^0,s)|^2_{\mu,2})dx\\
\leq&3\eta+2\int_{\mathbb{R}^n}\rho_m(x)\varphi_1(x)dx.
\end{aligned}
\tag{4.45}
$$

Since $\varphi_1\in L^1(\mathbb{R}^n)$, there exists $m_3=m_3(\eta,\tau,\omega)\geq m_2$ such that for all $m\geq m_3$,

$$
2\int_{\mathbb{R}^n}\rho_m(x)\varphi_1(x)dx=2\int_{|x|\geq\frac{1}{2}m}\rho_m(x)\varphi_1(x)dx\leq2\int_{|x|\geq\frac{1}{2}m}|\varphi_1(x)|dx<\eta.
\tag{4.46}
$$

From (4.45)-(4.46) we obtain, for all $m\geq m_3$ and $t\geq T_2$,

$$
\begin{aligned}
&\int_{|x|\geq m}\rho_m(x)\Big(\frac{1}{2}|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})|^2+\frac{1}{2}\nu|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2\\
&+|\eta^t(\tau,\tau-t,\theta_{-\tau}\omega,\eta^0,s)\|^2_{\mu,2}|)dx\\
\leq&\int_{\mathbb{R}^n}\rho_m(x)\Big(\frac{1}{2}|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})|^2+\frac{1}{2}\nu|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2\\
&+|\eta^t(\tau,\tau-t,\theta_{-\tau}\omega,\eta^0,s)\|^2_{\mu,2}|)dx\\
<&4\eta.
\end{aligned}
$$

□

5. Existence of random attractors

In this section, we present the existence and uniqueness of $\mathcal{D}$-pullback random attractors of (3.2).

Let $z = (u, u_t, \eta^t)$ be the solution of (3.2). Denote $u = \tilde{v} + v, \eta^t = \tilde{\eta}^t + \eta$ where $(\tilde{v}, \tilde{\eta}^t)$ and (v, η^t) are the solutions of the following equations, respectively,

$$\begin{cases} \tilde{v}_{tt} + \alpha\tilde{v}_t + \Delta^2\tilde{v} + \int_0^\infty \mu(s)\Delta^2\tilde{\eta}^t(s)ds + \nu\tilde{v} = g(t),\ t > \tau, \\ \tilde{v}(\tau) = u_0,\ \ \tilde{v}_t(\tau) = u_{1,0},\ \ \tilde{\eta}^t(\tau) = \eta^0 \end{cases} \tag{5.1}$$

and

$$\begin{cases} v_{tt} + \alpha v_t + \Delta^2 v + \int_0^\infty \mu(s)\Delta^2\eta^t(s)ds + \nu v = -f(x, u) + h(t, x, u)\zeta_\delta(\theta_t\omega),\ t > \tau, \\ v(\tau) = 0,\ \ v_t(\tau) = 0,\ \ \eta^t(\tau) = 0. \end{cases} \tag{5.2}$$

Lemma 5.1. *Suppose (3.3), (4.7)-(4.8) hold. Then for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$ and $r \in [-t, 0]$, the solution $\tilde{v}$ of (5.1) satisfies*

$$\begin{aligned} &\|\tilde{v}(\tau + r, \tau - t, \theta_{-\tau}\omega, u_0)\|^2_{H^2(\mathbb{R}^n)} + \|\tilde{v}_r(\tau + r, \tau - t, \theta_{-\tau}\omega, u_{1,0})\|^2 + \|\tilde{\eta}^t(\tau + r, \tau - t, \theta_{-\tau}\omega, \eta^0, s)\|^2_{\mu,2} \\ \leq& e^{-\frac{1}{2}\varepsilon r} M_2\Big(1 + \int_\infty^0 e^{\frac{1}{2}\varepsilon s}\|g(s + \tau)\|^2 ds\Big), \end{aligned}$$

where $(u_0, u_{1,0}) \in D(\tau - t, \theta_{-t}\omega)$ and M_2 is a positive number independent of τ, ω and D.

Proof. From (3.10)-(3.11) and (5.1) we see that

$$\begin{aligned} &\frac{d}{dt}(\|\tilde{v}_t\|^2 + \|\Delta\tilde{v}\|^2 + \|\tilde{\eta}^t\|^2_{\mu,2} + \nu\|\tilde{v}\|^2 + \varepsilon(\tilde{v}(t), \tilde{v}_t(t))) + (2\alpha - \varepsilon)\|\tilde{v}_t\|^2 \\ &+ \varepsilon\|\Delta\tilde{v}\|^2 + \varepsilon\nu\|\tilde{v}\|^2 + \varepsilon\alpha(\tilde{v}(t), \tilde{v}_t(t)) + \varepsilon(\tilde{\eta}^t(s), \tilde{v}(t))_{\mu,2} - \int_0^\infty \mu'(s)\|\Delta\tilde{\eta}^t\|^2 ds \\ =&(g(t), \varepsilon\tilde{v}(t) + 2\tilde{v}_t(t)) \\ \leq&\varepsilon\|g(t)\|\|\tilde{v}(t)\| + 2\|g(t)\|\|\tilde{v}_t(t)\| \\ \leq&\frac{1}{2}\varepsilon^2\|\tilde{v}(t)\|^2 + \alpha\|\tilde{v}_t(t)\|^2 + (\frac{1}{2} + \alpha^{-1})\|g(t)\|^2. \end{aligned} \tag{5.3}$$

In addition, we get

$$|(\alpha - \frac{1}{2}\varepsilon)\varepsilon(\tilde{v}(t), \tilde{v}_t(t))| \leq \frac{1}{2}(\alpha - \frac{1}{2}\varepsilon)(\varepsilon^2\|\tilde{v}(t)\|^2 + \|\tilde{v}_t(t)\|^2). \tag{5.4}$$

By (4.11)-(4.12) and (5.3)-(5.4) we have

$$\begin{aligned} &\frac{d}{dt}(\|\tilde{v}_t\|^2 + \|\Delta\tilde{v}\|^2 + \|\tilde{\eta}^t\|^2_{\mu,2} + \nu\|\tilde{v}\|^2 + \varepsilon(\tilde{v}(t), \tilde{v}_t(t))) + (\frac{1}{2}\alpha - \frac{3}{4}\varepsilon)\|\tilde{v}_t\|^2 \\ &+ \varepsilon(1 - \frac{\varpi\varepsilon}{\varrho})\|\Delta\tilde{v}\|^2 + \frac{3\varrho}{4}\|\tilde{\eta}^t\|^2_{\mu,2} + \varepsilon(\nu - \frac{1}{2}\varepsilon - \frac{1}{2}\varepsilon\alpha + \frac{1}{4}\varepsilon^2)\|\tilde{v}\|^2 + \frac{1}{2}\varepsilon^2(\tilde{v}(t), \tilde{v}_t(t)) \\ \leq&(\frac{1}{2} + \alpha^{-1})\|g(t)\|^2, \end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\frac{d}{dt}(\|\tilde{v}_t\|^2+\|\Delta\tilde{v}\|^2+\|\tilde{\eta}^t\|^2_{\mu,2}+\nu\|\tilde{v}\|^2+\varepsilon(\tilde{v}(t),\tilde{v}_t(t)))\\
&+\frac{1}{2}\varepsilon(\|\tilde{v}_t\|^2+\|\Delta\tilde{v}\|^2+\|\tilde{\eta}^t\|^2_{\mu,2}+\nu\|\tilde{v}\|^2+\varepsilon(\tilde{v}(t),\tilde{v}_t(t)))\\
&+(\frac{1}{2}\alpha-\frac{5}{4}\varepsilon)\|\tilde{v}_t\|^2+\frac{1}{2}\varepsilon(1-\frac{2\varpi\varepsilon}{\varrho})\|\Delta\tilde{v}\|^2+\frac{3}{4}(\varrho-\frac{2}{3}\varepsilon)\|\tilde{\eta}^t\|^2_{\mu,2}+\frac{1}{2}\varepsilon(\nu-\varepsilon-\varepsilon\alpha+\frac{1}{2}\varepsilon^2)\|\tilde{v}\|^2\\
\leq&(\frac{1}{2}+\alpha^{-1})\|g(t)\|^2. \qquad (5.5)
\end{aligned}$$

It follows from (4.7) and (5.5) that

$$\begin{aligned}
&\frac{d}{dt}(\|\tilde{v}_t\|^2+\|\Delta\tilde{v}\|^2+\|\tilde{\eta}^t\|^2_{\mu,2}+\nu\|\tilde{v}\|^2+\varepsilon(\tilde{v}(t),\tilde{v}_t(t)))\\
&+\frac{1}{2}\varepsilon(\|\tilde{v}_t\|^2+\|\Delta\tilde{v}\|^2+\|\tilde{\eta}^t\|^2_{\mu,2}+\nu\|\tilde{v}\|^2+\varepsilon(\tilde{v}(t),\tilde{v}_t(t)))\\
\leq&(\frac{1}{2}+\alpha^{-1})\|g(t)\|^2. \qquad (5.6)
\end{aligned}$$

Applying Gronwall's lemma to (5.6), we get for all $\tau\in\mathbb{R}, t\geq 0, r\in[-t,0]$ and $\omega\in\Omega$,

$$\begin{aligned}
&\|\tilde{v}_r(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2+\|\Delta\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\tilde{\eta}^t(\tau+r,\tau-t,\theta_{-\tau}\omega,\eta^0,s)\|^2_{\mu,2}\\
&+\nu\|\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\varepsilon(\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0),\tilde{v}_r(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0}))\\
\leq&e^{-\frac{1}{2}\varepsilon r}e^{-\frac{1}{2}\varepsilon t}(\|u_{1,0}\|^2+\nu\|u_0\|^2+\|\Delta u_0\|^2+\varepsilon(u_0,u_{1,0}))\\
&+(\frac{1}{2}+\alpha^{-1})e^{-\frac{1}{2}\varepsilon r}\int_{\tau-t}^{\tau+r}e^{\frac{1}{2}\varepsilon(s-\tau)}\|g(s)\|^2ds. \qquad (5.7)
\end{aligned}$$

By (4.7) we have

$$\begin{aligned}
&\varepsilon(\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0),\tilde{v}_r(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0}))\\
\leq&\frac{1}{2}\varepsilon\|\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\frac{1}{2}\varepsilon\|\tilde{v}_r(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2\\
\leq&\frac{1}{2}\nu\|\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\frac{1}{2}\|\tilde{v}_r(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2. \qquad (5.8)
\end{aligned}$$

By (5.7)-(5.8) we see that for all $\tau\in\mathbb{R}, t\geq 0, r\in[-t,0]$ and $\omega\in\Omega$,

$$\begin{aligned}
&\frac{1}{2}\|\tilde{v}_r(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2+\|\Delta\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\tilde{\eta}^t(\tau+r,\tau-t,\theta_{-\tau}\omega,\eta^0,s)\|^2_{\mu,2}\\
&+\frac{1}{2}\nu\|\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
\leq&e^{-\frac{1}{2}\varepsilon r}e^{-\frac{1}{2}\varepsilon t}(\|u_{1,0}\|^2+\nu\|u_0\|^2+\|\Delta u_0\|^2+\|\eta^0\|^2_{\mu,2}+\varepsilon(u_0,u_{1,0}))\\
&+(\frac{1}{2}+\alpha^{-1})e^{-\frac{1}{2}\varepsilon r}\int_{\tau-t}^{\tau+r}e^{\frac{1}{2}\varepsilon(s-\tau)}\|g(s)\|^2ds. \qquad (5.9)
\end{aligned}$$

Similar to (4.16), one can verify that

$$e^{-\frac{1}{2}\varepsilon t}(\|u_{1,0}\|^2+\nu\|u_0\|^2+\|\Delta u_0\|^2+\|\eta^0\|^2_{\mu,2}+\varepsilon(u_0,u_{1,0}))\to 0,\ \text{ as }\ t\to\infty,$$

which along with (5.9) yields the desire result. □

Based on Lemma 5.1, we infer that system (5.1) has a tempered pullback random absorbing set.

Lemma 5.2. *Suppose (3.3), (4.8)-(4.9) hold, then (5.1) possesses a closed measurable $\mathcal{D}$-pullback absorbing set $B_1 = \{B_1(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, which is given by*

$$B_1(\tau,\omega) = \{(u_0, u_{1,0}, \eta^0) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \Re_{\mu,2} : \|u_0\|^2_{H^2(\mathbb{R}^n)} + \|u_{1,0}\|^2 + \|\eta^0\|^2_{\mu,2} \leq L_1(\tau,\omega)\}, \quad (5.10)$$

where

$$L_1(\tau,\omega) = M_2 + M_2 \int_{-\infty}^{0} e^{\frac{1}{2}\varepsilon s}\|g(s+\tau)\|^2 ds.$$

Lemma 5.3. *Suppose (4.8)-(4.9) hold, then the sequence of the solutions to (5.1)*

$$\{\tilde{v}(\tau, \tau - t_n, \theta_{-\tau}\omega, u_0^{(n)}), \tilde{v}_t(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{1,0}^{(n)}), \tilde{\eta}^t(\tau, \tau - t_n, \theta_{-\tau}\omega, \eta^{(0n)})\}_{n=1}^{\infty}$$

converges in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \Re_{\mu,2}$ for any $\tau \in \mathbb{R}, \omega \in \Omega, D \in \mathcal{D}, t_n \to \infty$ monotonically, and $(u_0^{(n)}, u_{1,0}^{(n)}, \eta^{(0n)}) \in D(\tau - t_n, \theta_{-t_n}\omega)$.

Proof. Let $m > n$ and

$$\begin{aligned}
&v_{n,m}(t, \tau - t_n, \theta_{-\tau}\omega)\\
=&\tilde{v}(t, \tau - t_n, \theta_{-\tau}\omega, u_0^{(n)}) - \tilde{v}(t, \tau - t_m, \theta_{-\tau}\omega, u_0^{(m)})\\
=&\tilde{v}(t, \tau - t_n, \theta_{-\tau}\omega, u_0^{(n)}) - \tilde{v}(t, \tau - t_n, \theta_{-\tau}\omega, \tilde{v}(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, u_0^{(m)})\\
&\eta^t_{n,m}(t, \tau - t_n, \theta_{-\tau}\omega, s)\\
=&\tilde{\eta}^t(t, \tau - t_n, \theta_{-\tau}\omega, \eta^{(0n)}, s) - \tilde{\eta}^t(t, \tau - t_m, \theta_{-\tau}\omega, \eta^{(0m)}, s)\\
=&\tilde{\eta}^t(t, \tau - t_n, \theta_{-\tau}\omega, \eta^{(0n)}, s) - \tilde{\eta}^t(t, \tau - t_n, \theta_{-\tau}\omega, s, \tilde{\eta}^t(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, \eta^{(0m)}, s).
\end{aligned} \quad (5.11)$$

for $t \geq \tau - t_n$.

by (5.1) we get

$$\begin{cases}
\partial^2_{tt}v_{n,m}(t) + \alpha\partial_t v_{n,m}(t) + \Delta^2 v_{n,m}(t) + \int_0^{\infty}\mu(s)\Delta^2\eta^t_{n,m}ds + \nu v_{n,m}(t) = 0, \ t > \tau - t_n,\\
v_{n,m}(\tau - t_n) = u_0^{(n)} - \tilde{v}(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, u_0^{(m)}), \ \ \partial_t v_{n,m}(\tau - t_n) = u_{1,0}^{(n)} - \tilde{v}_t,\\
\eta^{\tau}_{n,m}(\tau - t_n, s) = \eta^{(0n)} - \tilde{\eta}^t(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, \eta^{(0m)}, s).
\end{cases} \quad (5.12)$$

Similar to (5.9) with $r = 0, t = t_n$ and $g = 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\|\partial_t v_{n,m}(\tau, \tau - t_n, \theta_{-\tau}\omega)\|^2 + \|\Delta v_{n,m}(\tau, \tau - t_n, \theta_{-\tau}\omega)\|^2 + \|\eta^t_{n,m}(\tau, \tau - t_n, \theta_{-\tau}\omega, s)\|^2_{\mu,2}\\
&+ \frac{1}{2}\nu v_{n,m}(\tau, \tau - t_n, \theta_{-\tau}\omega)\|^2\\
\leq& e^{-\frac{1}{2}\varepsilon t_n}(\|\partial_t v_{n,m}(\tau - t_n)\|^2 + \|v_{n,m}(\tau - t_n)\|^2 + \|\Delta v_{n,m}(\tau - t_n)\|^2 + \|\eta^{\tau}_{n,m}(\tau - t_n, s)\|^2_{\mu,2}),
\end{aligned} \quad (5.13)$$

which together with $(5.12)_2$, we get

$$\|\partial_t v_{n,m}(\tau, \tau - t_n, \theta_{-\tau}\omega)\|^2 + 2\|\Delta v_{n,m}(\tau, \tau - t_n, \theta_{-\tau}\omega)\|^2 + \|\eta^t_{n,m}(\tau, \tau - t_n, \theta_{-\tau}\omega, s)\|^2_{\mu,2}$$

$$
\begin{aligned}
&+ \nu v_{n,m}(\tau, \tau - t_n, \theta_{-\tau}\omega)\|^2 \\
\leq & 2e^{-\frac{1}{2}\varepsilon t_n}(\|\tilde{v}_t(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, u_{1,0}^{(m)}\|^2 + \|\tilde{v}(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, u_0^{(m)}\|_{H^2}^2) \\
&+ \|\tilde{\eta}^t(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, \eta^{(0m)}, s)\|_{\mu,2}^2) \\
&+ 2e^{-\frac{1}{2}\varepsilon t_n}(\|u_{1,0}^{(n)}\|^2 + \|u_0^{(n)}\|^2 + \|\Delta u_0^{(n)}\|^2 + \|\eta^{(0n)}\|_{\mu,2}^2). \qquad (5.14)
\end{aligned}
$$

By (5.9) with $r = -t_n$, and $t = t_m$, we obtain

$$
\begin{aligned}
&\|\tilde{v}_t(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, u_{1,0}^{(m)})\|^2 + 2\|\Delta\tilde{v}(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, u_0^{(m)})\|^2 \\
&+ \|\tilde{\eta}^t(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, \eta^{(0m)})\|_{\mu,2}^2 + \nu\|\tilde{v}(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, u_0^{(m)})\|^2 \\
\leq & 2e^{\frac{1}{2}\varepsilon t_n}e^{-\frac{1}{2}\varepsilon t_m}(\|u_{1,0}^{(n)}\|^2 + \nu\|u_0^{(n)}\|^2 + \|\Delta u_0^{(n)}\|^2 + \|\eta^{(0n)}\|_{\mu,2}^2 + \varepsilon(u_0^{(n)}, u_{1,0}^{(n)})) \\
&+ (1 + 2\alpha^{-1})e^{\frac{1}{2}\varepsilon t_n}\int_{\tau - t_m}^{\tau - t_n} e^{\frac{1}{2}\varepsilon(s-\tau)}\|g(s)\|^2 ds. \qquad (5.15)
\end{aligned}
$$

It follows from (5.14)-(5.15) that for $m > n \to \infty$,

$$
\|\partial_t v_{n,m}(\tau, \tau - t_n, \theta_{-\tau}\omega)\|^2 + \|v_{n,m}(\tau, \tau - t_n, \theta_{-\tau}\omega)\|_{H^2(\mathbb{R}^n)}^2 + \|\eta_{n,m}^t(\tau, \tau - t_n, \theta_{-\tau}\omega, s)\|_{\mu,2}^2 \to 0,
$$

together with (5.11) implies $\{\tilde{v}(\tau, \tau - t_n, \theta_{-\tau}\omega, u_0^{(n)}), \tilde{v}_t(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{1,0}^{(n)}), \tilde{\eta}^t(\tau, \tau - t_n, \theta_{-\tau}\omega, \eta^{(0n)})\}_{n=1}^\infty$ is a Cauchy sequence in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_{\mu,2}$. This complete the proof. □

Lemma 5.4. *Suppose (3.3), (4.8)-(4.9) hold, then (5.1) has a unique $\mathcal{D}$-pullback random attractor $\mathcal{A}_1 = \{\mathcal{A}_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_{\mu,2}$, which is actually a singleton; that is, $\mathcal{A}_1(\tau, \omega)$ consisting of a single point for all $\tau \in \mathbb{R}, \omega \in \Omega$.*

Proof. From Lemmas 5.2 and 5.3 by applying the abstract results in [29], we can get the existence and uniqueness of the $\mathcal{D}$-pullback random attractor $\mathcal{A}_1 \in \mathcal{D}$ of (5.1) in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_{\mu,2}$ immediately.

Next, we prove $\mathcal{A}_1$ is a singleton. Suppose $\{t_n\}_{n=1}^\infty$ 1 be a sequence of numbers such that $t_n \to \infty$ as $n \to \infty$. Given $\tau \in \mathbb{R}, \omega \in \Omega$, let $(z_0^{(n)}, z_{1,0}^{(n)}, \eta^{(0n)}), (y_0^{(n)}, y_{1,0}^{(n)}, y^{(0n)}) \in \mathcal{A}_1(\tau - t_n, \theta_{-t_n}\omega)$.

Similar to (5.13) we have

$$
\begin{aligned}
&\|\tilde{v}_t(\tau, \tau - t_n, \theta_{-\tau}\omega, z_{1,0}^{(n)}) - \tilde{v}_t(\tau, \tau - t_n, \theta_{-\tau}\omega, y_{1,0}^{(n)})\|^2 \\
&+ 2\|\Delta\tilde{v}(\tau, \tau - t_n, \theta_{-\tau}\omega, z_0^{(n)}) - \Delta\tilde{v}(\tau, \tau - t_n, \theta_{-\tau}\omega, y_0^{(n)})\|^2 \\
&+ \|\tilde{\eta}^t(\tau, \tau - t_n, \theta_{-\tau}\omega, \eta^{(0n)}) - \tilde{\eta}^t(\tau, \tau - t_n, \theta_{-\tau}\omega, y^{(0n)})\|_{\mu,2}^2 \\
&+ \nu\|\tilde{v}(\tau, \tau - t_n, \theta_{-\tau}\omega, z_0^{(n)}) - \tilde{v}(\tau, \tau - t_n, \theta_{-\tau}\omega, y_0^{(n)})\|^2 \\
\leq & e^{-\frac{1}{2}\varepsilon t_n}(\|z_{1,0}^{(n)} - y_{1,0}^{(n)}\|^2 + \|z_0^{(n)} - y_0^{(n)}\|^2 + \|\Delta z_0^{(n)} - \Delta y_0^{(n)}\|^2 + \|\eta^{(0n)} - y^{(0n)}\|_{\mu,2}^2) \\
\leq & 2e^{-\frac{1}{2}\varepsilon t_n}(\|z_{1,0}^{(n)}\|^2 + \|z_0^{(n)}\|_{H^2(\mathbb{R}^n)}^2 + \|y_{1,0}^{(n)}\|^2 + \|y_{1,0}^{(n)}\|_{H^2(\mathbb{R}^n)}^2 + \|\eta^{(0n)}\|_{\mu,2}^2 + \|y^{(0n)}\|_{\mu,2}^2) \\
\leq & 4e^{-\frac{1}{2}\varepsilon t_n}\|\mathcal{A}_1(\tau - t_n, \theta_{-t_n}\omega)\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times\mathfrak{R}_{\mu,2}}^2. \qquad (5.16)
\end{aligned}
$$

Due to $\mathcal{A}_1 \in \mathcal{D}$, we see that the right-hand side of (5.16) tends to zero as $n \to \infty$, and thus we get

$$
\begin{aligned}
&\lim_{n\to\infty}(\tilde{v}_t(\tau, \tau - t_n, \theta_{-\tau}\omega, z_{1,0}^{(n)}) - \tilde{v}_t(\tau, \tau - t_n, \theta_{-\tau}\omega, y_{1,0}^{(n)})) = 0 \quad \text{in } L^2(\mathbb{R}^n), \\
&\lim_{n\to\infty}(\tilde{v}(\tau, \tau - t_n, \theta_{-\tau}\omega, z_0^{(n)}) - \tilde{v}(\tau, \tau - t_n, \theta_{-\tau}\omega, y_0^{(n)})) = 0 \quad \text{in } H^2(\mathbb{R}^n),
\end{aligned}
$$

$$\lim_{n\to\infty}(\tilde{\eta}^t(\tau,\tau-t_n,\theta_{-\tau}\omega,\eta^{(0n)})-\tilde{\eta}^t(\tau,\tau-t_n,\theta_{-\tau}\omega,y^{(0n)}))=0 \quad \text{in } \Re_{\mu,2}.$$

which together with the invariance of $\mathcal{A}_1$, we know that the $\mathcal{D}$-pullback random attractor $\mathcal{A}_1$ is a singleton. This complete the proof. □

To obtain the asymptotic compactness of the solutions of (5.2), we need the following Lemma.

Lemma 5.5. *Let $u_0 \in H^2(\mathbb{R}^n)$, $u_{1,0} \in L^2(\mathbb{R}^n)$, $\eta^0 \in \Re_{\mu,2}, \tau \in \mathbb{R}, \omega \in \Omega$ and $T > 0$. If (3.3)-(3.5), (3.8), (4.1)-(4.2) and (4.5)-(4.8) hold, then the solution of (5.2) satisfies, for all $t \in [\tau, \tau+T]$,*

$$\|A^{\frac{3}{4}}v(t,\tau,\omega)\|+\|A^{\frac{1}{4}}v_t(t,\tau,\omega)\|+\|A^{\frac{1}{4}}\eta^t(t,\tau,\omega,s)\|_{\mu,2}\le C,$$

where C is a positive number depending on τ, ω, T and R when $\|(u_0,u_{1,0},\eta^0)\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times\Re_{\mu,2}} \le R$.

Proof. This is an immediate consequence of Lemma 4.3. □

Lemma 5.6. *Let (3.3)–(3.5), (3.6), (4.1)–(4.3) and (4.5)–(4.9) hold. Then the cocycle Φ is $\mathcal{D}$-pullback asymptotically compact in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times\Re_{\mu,2}$; that is, the sequence $\{\Phi(t_n,\tau-t_n,\theta_{-t_n}\omega,(u_0^{(n)},u_{1,0}^{(n)}),\eta^{(0n)}\}_{n=1}^{\infty}$ has a convergent subsequence in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times\Re_{\mu,2}$ for any $\tau\in\mathbb{R},\omega\in\Omega, D\in\mathcal{D}, t_n\to\infty$ and $(u_0^{(n)},u_{1,0}^{(n)},\eta^{(0n)})\in D(\tau-t_n,\theta_{-t_n}\omega)$.*

Proof. Given $t\in\mathbb{R}^+,\tau\in\mathbb{R},\omega\in\Omega$ and $(u_0,u_{1,0},\eta^0)\in H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times\Re_{\mu,2}$, define

$$\Phi_1(t,\tau,\omega,(u_0,u_{1,0},\eta^0))=(\tilde{v}(t+\tau,\tau,\theta_{-\tau}\omega,u_0),\tilde{v}_t(t+\tau,\tau,\theta_{-\tau}\omega,u_{1,0}),\tilde{\eta}^t(t+\tau,\tau,\theta_{-\tau}\omega,\eta^0,s)),$$
$$\Phi_2(t,\tau,\omega,(u_0,u_{1,0},\eta^0))=(v(t+\tau,\tau,\theta_{-\tau}\omega,u_0),v_t(t+\tau,\tau,\theta_{-\tau}\omega,u_{1,0}),\eta^t(t+\tau,\tau,\theta_{-\tau}\omega,\eta^0,s)),$$

where $(\tilde{v},\tilde{\eta}^t)$ and (v,η^t) are the solutions of (5.1) and (5.2), respectively.

By(3.78) we have

$$\Phi(t,\tau,\omega,(u_0,u_{1,0},\eta^0))=\Phi_1(t,\tau,\omega,(u_0,u_{1,0},\eta^0))+\Phi_2(t,\tau,\omega,(u_0,u_{1,0},\eta^0)). \tag{5.17}$$

Let $B\in\mathcal{D}$ be the $\in\mathcal{D}$-pullback absorbing set of Φ given by (4.19). From Lemmas 4.2, 4.4 and 5.4 we see that for every $\delta>0$ there exists $t_0=t_0(\delta,\tau,\omega,B)>0$ and $k_0=k_0(\delta,\tau,\omega)\ge1$ such that for all $(u_0,u_{1,0},\eta^0)\in B(\tau-t_0,\theta_{-t_0}\omega)$,

$$\|\Phi(t_0,\tau-t_0,\theta_{-t_0}\omega,(u_0,u_{1,0},\eta^0))|_{\tilde{O}_{k_0}}\|_{H^2(\tilde{O}_{k_0})\times L^2(\tilde{O}_{k_0})\times\Re_{\mu,2}}<\delta, \tag{5.18}$$

with $\tilde{O}_{k_0}=\{x\in\mathbb{R}^n:|x|>k_0\}$, and

$$\Phi_1(t_0,\tau-t_0,\theta_{-t_0}\omega,B(\tau-t_0,\theta_{-t_0}\omega)) \quad \text{is covered by a ball of radius} \quad \delta \tag{5.19}$$

in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times\Re_{\mu,2}$.

In addition, by Lemma 5.5 we know that for every $t\in\mathbb{R}^+,\tau\in\mathbb{R},\omega\in\Omega$ and $k\in\mathbb{N}$,

$$\Phi_2(t,\tau-t,\theta_{-t}\omega,B(\tau-t,\theta_{-t}\omega)) \quad \text{is bounded in} \quad H^3(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times\Re_{\mu,3},$$

and thus for each $k\in\mathbb{N}$,

$$\Phi_2(t,\tau-t,\theta_{-t}\omega,B(\tau-t,\theta_{-t}\omega))|_{O_k} \quad \text{is precompact} \quad H^2(O_k)\times L^2(O_k)\times\Re_{\mu,2}, \tag{5.20}$$

with $O_k=\{x\in\mathbb{R}^n:|x|<k\}$.

It follows from (5.17)–(5.20) we get that all conditions of Theorem 2.1 are satisfied, so Φ is $\mathcal{D}$-pullback asymptotically compact in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_{\mu,2}$. □

Since Lemma 4.2 implies a closed measurable $\mathcal{D}$-pullback absorbing set for Φ, and Φ is $\mathcal{D}$-pullback asymptotically compact in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_{\mu,2}$ from Lemma 5.6, we immediately get the following existence theorem by Theorem 2.2.

Theorem 5.1. *Let (3.3)–(3.5), (3.6), (4.1)–(4.3) and (4.5)–(4.9) hold. Then the cocycle Φ has a unique $\mathcal{D}$-pullback random attractor in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_{\mu,2}$.*

6. The discussion of the proposed method's theoretical analysis

In this paper, we use the uniform estimates on the tails of solutions and the splitting technique to obtained the existence and uniqueness of $\mathcal{D}$-pullback attractor for the problem (1.1). The method used in this paper is proposed by P. W. Bates et al [3], they applied the method to deal with the asymptotic behavior of the non-automatous random system on unbounded domains. More precisely, one first need to show that the tails of the solutions of (1.1) are uniformly small outside a bounded domain for large time, and then derive the asymptotic compactness of solutions in bounded domains by splitting the solutions as two parts: one part has trivial dynamics in the sense that it possesses a unique tempered attracting random solution; and the other part has regularity higher than $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_{\mu,2}$ based on the estimates of solutions (see Lemma 4.3).

7. Conclusions

Using the uniform estimates on the tails of solutions and the splitting technique, we obtained the existence and uniqueness of $\mathcal{D}$-pullback attractor for the problem (1.1). It is well-known that the pullback random attractors are employed to describe long-time behavior for an non-autonomous dynamical system with random term, while the $\mathcal{D}$-pullback attractor that we obtained can characterize the asymptotic behavior of the equation like (1.1), which is featured with both stochastic term and non-autonomous term.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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