Mathematics

Research article

Generalized inequalities involving fractional operators of the Riemann-Liouville type

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Abstract: In this paper, we present a general formulation of the well-known fractional drifts of Riemann-Liouville type. We state the main properties of these integral operators. Besides, we study Ostrowski, Székely-Clark-Entringer and Hermite-Hadamard-Fejér inequalities involving these general fractional operators.

1. Introduction

The idea of fractional calculus is as old as traditional calculus (see [1]). Until recently, research on fractional calculus was confined to the field of pure mathematics but, in the last two decades, many applications of fractional calculus appeared in several fields of engineering, applied sciences, physics, economy, etc.

For a complementary study on the recent developments in the field of fractional calculus as well as its applications see [2–7].

It is important to note that the global fractional derivatives (e.g., Caputo and Riemann-Liouville) are not collecting mere local information. By contrast, fractional operators keep track of the history

of the process being studied; this feature allows modeling the non-local and distributed responses that commonly appear in natural and physical phenomena. On the other side, one has to recognize that these fractional derivatives D^α show some drawbacks.

This paper relies on the introduction and use of new differential operators, depending on a general kernel function, which include at once several fractional derivatives earlier introduced and studied in many different sources.

As we know, by manipulating simple algebraic identities, we can follow the idea of fractional differential operators of Riemann-Liouville or Caputo type [8–10]. In this paper we will use a general kernel T in order to define general integral and differential operators of Riemann-Liouville type:

$$^{RL}D^\alpha f(t) = \frac{d}{dt}\left\{ J_{T,a}^{1-\alpha}(f)(t) \right\}, \tag{1.1}$$

We state the main properties of these integral operators. Furthermore, we study Ostrowski, Székely-Clark-Entringer and Hermite-Hadamard-Fejér inequalities involving these general fractional operators.

2. Preliminaries

One of the first operators that can be called fractional is the Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{C}$, with $Re(\alpha) > 0$, defined as follows (see [11]).

Definition 1. *Let $a < b$ and $f \in L^1((a,b); \mathbb{R})$. The right and left side Riemann-Liouville fractional integrals of order α, with $Re(\alpha) > 0$, are defined, respectively, by*

$$^{RL}J_{a^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \tag{2.1}$$

and

$$^{RL}J_{b^-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\, ds, \tag{2.2}$$

with $t \in (a,b)$.

When $\alpha \in (0,1)$, their corresponding Riemann-Liouville fractional derivatives are given by

$$\left(^{RL}D_{a^+}^\alpha f\right)(t) = \frac{d}{dt}\left(^{RL}J_{a^+}^{1-\alpha} f(t)\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^\alpha}\, ds,$$

$$\left(^{RL}D_{b^-}^\alpha f\right)(t) = -\frac{d}{dt}\left(^{RL}J_{b^-}^{1-\alpha} f(t)\right) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(s)}{(s-t)^\alpha}\, ds.$$

Other definitions of fractional operators are the following ones.

Definition 2. *Let $a < b$ and $f \in L^1((a,b); \mathbb{R})$. The right and left side Hadamard fractional integrals of order α, with $Re(\alpha) > 0$, are defined, respectively, by*

$$H_{a^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s}\, ds, \tag{2.3}$$

and

$$H_{b^-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log \frac{s}{t}\right)^{\alpha-1} \frac{f(s)}{s}\, ds, \tag{2.4}$$

with $t \in (a,b)$.

When $\alpha \in (0, 1)$, Hadamard fractional derivatives are given by the following expressions:

$$(^H D_{a^+}^\alpha f)(t) = t \frac{d}{dt} (H_{a^+}^{1-\alpha} f(t)) = \frac{1}{\Gamma(1-\alpha)} t \frac{d}{dt} \int_a^t \left(\log \frac{t}{s} \right)^{-\alpha} \frac{f(s)}{s} \, ds,$$

$$(^H D_{b^-}^\alpha f)(t) = -t \frac{d}{dt} (H_{b^-}^{1-\alpha} f(t)) = \frac{-1}{\Gamma(1-\alpha)} t \frac{d}{dt} \int_t^b \left(\log \frac{s}{t} \right)^{-\alpha} \frac{f(s)}{s} \, ds,$$

with $t \in (a, b)$.

In [12], the author introduced new fractional integral operators, called the Katugampola fractional integrals, in the following way.

Definition 3. *Let $0 < a < b$, $f : [a, b] \to \mathbb{R}$ an integrable function, and $\alpha \in (0, 1)$, $\rho > 0$ two fixed real numbers. The right and left side Katugampola fractional integrals of order α are defined, respectively, by*

$$K_{a^+}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(s) \, ds, \tag{2.5}$$

and

$$K_{b^-}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{t^{\rho-1}}{(s^\rho - t^\rho)^{1-\alpha}} f(s) \, ds, \tag{2.6}$$

with $t \in (a, b)$.

Some generalizations of the Riemann-Liouville and Hadamard fractional derivatives appeared in [13]. These generalizations, called Katugampola fractional derivatives, are defined as

$$(^K D_{a^+}^{\alpha,\rho} f)(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} t^{1-\rho} \frac{d}{dt} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^\alpha} f(s) \, ds,$$

$$(^K D_{b^-}^{\alpha,\rho} f)(t) = \frac{-\rho^\alpha}{\Gamma(1-\alpha)} t^{1-\rho} \frac{d}{dt} \int_t^b \frac{s^{\rho-1}}{(s^\rho - t^\rho)^\alpha} f(s) \, ds,$$

with $t \in (a, b)$.

The relations between these two fractional operators are the following:

$$(^K D_{a^+}^{\alpha,\rho} f)(t) = t^{1-\rho} \frac{d}{dt} K_{a^+}^{1-\alpha,\rho} f(t),$$

$$(^K D_{b^-}^{\alpha,\rho} f)(t) = -t^{1-\rho} \frac{d}{dt} K_{b^-}^{1-\alpha,\rho} f(t).$$

Definition 4. *Let $0 < a < b$, $g : [a, b] \to \mathbb{R}$ an increasing positive function on (a, b) with continuous derivative on (a, b), $f : [a, b] \to \mathbb{R}$ an integrable function, and $\alpha \in (0, 1)$ a fixed real number. The right and left side Kilbas-Marichev-Samko fractional integrals of order α of f with respect to g are defined, respectively (see [14]), by*

$$I_{g,a^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g'(s)f(s)}{(g(t) - g(s))^{1-\alpha}} \, ds, \tag{2.7}$$

and

$$I_{g,b^-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{g'(s)f(s)}{(g(s) - g(t))^{1-\alpha}} \, ds, \tag{2.8}$$

with $t \in (a, b)$.

There are other definitions of integral operators in the global case, but they are slight modifications of the previous ones, some include non-singular kernel and others incorporate different terms.

3. General fractional integral of Riemann-Liouville type

Now, we give the definition of a general fractional integral.

Definition 5. *Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g : [a, b] \to \mathbb{R}$ be a positive function on (a, b) with continuous positive derivative on (a, b), and $G : [0, g(b) - g(a)] \times (0, \infty) \to \mathbb{R}$ a continuous function which is positive on $(0, g(b) - g(a)] \times (0, \infty)$. Let us define the function $T : [a, b] \times [a, b] \times (0, \infty) \to \mathbb{R}$ by*

$$T(t, s, \alpha) = \frac{G(|g(t) - g(s)|, \alpha)}{g'(s)}.$$

The right and left integral operators, denoted respectively by J^{α}_{T,a^+} and J^{α}_{T,b^-}, are defined for each measurable function f on $[a, b]$ as

$$J^{\alpha}_{T,a^+} f(t) = \int_a^t \frac{f(s)}{T(t, s, \alpha)} \, ds, \tag{3.1}$$

$$J^{\alpha}_{T,b^-} f(t) = \int_t^b \frac{f(s)}{T(t, s, \alpha)} \, ds, \tag{3.2}$$

with $t \in [a, b]$.

We say that $f \in L^1_T[a, b]$ if $J^{\alpha}_{T,a^+} |f|(t), J^{\alpha}_{T,b^-} |f|(t) < \infty$ for every $t \in [a, b]$.

Note that these operators generalize the integral operators in Definitions 1–4:

(*A*) If we choose

$$g(t) = t,$$
$$G(x, \alpha) = \Gamma(\alpha) \, x^{1-\alpha},$$
$$T(t, s, \alpha) = \Gamma(\alpha) \, |t - s|^{1-\alpha},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the right and left Riemann-Liouville fractional integrals ${}^{RL}J^{\alpha}_{a^+}$ and ${}^{RL}J^{\alpha}_{b^-}$ in (2.1) and (2.2), respectively. Its corresponding right and left Riemann-Liouville fractional derivatives are

$$({}^{RL}D^{\alpha}_{a^+} f)(t) = \frac{d}{dt} \left({}^{RL}J^{1-\alpha}_{a^+} f(t) \right),$$
$$({}^{RL}D^{\alpha}_{b^-} f)(t) = -\frac{d}{dt} \left({}^{RL}J^{1-\alpha}_{b^-} f(t) \right).$$

(*B*) If we choose

$$g(t) = \log t,$$
$$G(x, \alpha) = \Gamma(\alpha) \, x^{1-\alpha},$$
$$T(t, s, \alpha) = \Gamma(\alpha) \, t \left| \log \frac{t}{s} \right|^{1-\alpha},$$

then J^α_{T,a^+} and J^α_{T,b^-} are the right and left Hadamard fractional integrals $H^\alpha_{a^+}$ and $H^\alpha_{b^-}$ in (2.3) and (2.4), respectively. Its corresponding right and left Hadamard fractional derivatives are

$$({}^H D^\alpha_{a^+} f)(t) = t \frac{d}{dt}(H^{1-\alpha}_{a^+} f(t)),$$

$$({}^H D^\alpha_{b^-} f)(t) = -t \frac{d}{dt}(H^{1-\alpha}_{b^-} f(t)).$$

(*C*) If we choose

$$g(t) = t^\rho,$$
$$G(x, \alpha) = \Gamma(\alpha)\rho^\alpha x^{1-\alpha},$$
$$T(t, s, \alpha) = \frac{\Gamma(\alpha)}{\rho^{1-\alpha}} \frac{|t^\rho - s^\rho|^{1-\alpha}}{s^{\rho-1}},$$

then J^α_{T,a^+} and J^α_{T,b^-} are the right and left Katugampola fractional integrals $K^{\alpha,\rho}_{a^+}$ and $K^{\alpha,\rho}_{b^-}$ in (2.5) and (2.6), respectively. Its corresponding right and left Katugampola fractional derivatives are

$$({}^K D^{\alpha,\rho}_{a^+} f)(t) = t^{1-\rho} \frac{d}{dt}(K^{1-\alpha,\rho}_{a^+} f(t)),$$

$$({}^K D^{\alpha,\rho}_{b^-} f)(t) = -t^{1-\rho} \frac{d}{dt}(K^{1-\alpha,\rho}_{b^-} f(t)).$$

(*D*) If we choose a function g with the properties in Definition 5 and

$$G(x, \alpha) = \Gamma(\alpha) x^{1-\alpha},$$
$$T(t, s, \alpha) = \Gamma(\alpha) \frac{|g(t) - g(s)|^{1-\alpha}}{g'(s)},$$

then J^α_{T,a^+} and J^α_{T,b^-} are the right and left Kilbas-Marichev-Samko fractional integrals I^α_{g,a^+} and I^α_{g,b^-} in (2.7) and (2.8), respectively.

Definition 6. *Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g : [a, b] \to \mathbb{R}$ be a positive function on (a, b) with continuous positive derivative on (a, b), and $G : [0, g(b) - g(a)] \times (0, \infty) \to \mathbb{R}$ a continuous function which is positive on $(0, g(b) - g(a)] \times (0, \infty)$. For each function $f \in L^1_T[a, b]$, its right and left generalized derivative of order α are defined, respectively, by*

$$D^\alpha_{T,a^+} f(t) = \frac{1}{g'(t)} \frac{d}{dt}\left(J^{1-\alpha}_{T,a^+} f(t)\right),$$

$$D^\alpha_{T,b^-} f(t) = \frac{-1}{g'(t)} \frac{d}{dt}\left(J^{1-\alpha}_{T,b^-} f(t)\right).$$

(3.3)

for each $t \in (a, b)$.

Note that if we choose

$$g(t) = t,$$

$$G(x, \alpha) = \Gamma(\alpha) \, x^{1-\alpha},$$

$$T(t, s, \alpha) = \Gamma(\alpha) \, |t - s|^{1-\alpha},$$

then

$$D_{T,a^+}^\alpha f(t) = {}^{RL}D_{a^+}^\alpha f(t)$$

and

$$D_{T,b^-}^\alpha f(t) = {}^{RL}D_{b^-}^\alpha f(t).$$

Also, we can obtain Hadamard and Katugampola fractional derivatives as particular cases of this generalized derivative.

3.1. Properties of the integral operators

The following result collects some elementary properties of J_{T,a^+}^α and J_{T,b^-}^α.

Proposition 7. *Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g : [a, b] \to \mathbb{R}$ be a positive function on $(a, b]$ with continuous positive derivative on (a, b), and $G : [0, g(b) - g(a)] \times (0, \infty) \to \mathbb{R}$ a continuous function which is positive on $(0, g(b) - g(a)] \times (0, \infty)$. Then the right and left integral operators J_{T,a^+}^α and J_{T,b^-}^α have the following properties:*

(1) For every functions $f_1, f_2 \in L_T^1[a, b]$, $c_1, c_2 \in \mathbb{R}$ and $t \in [a, b]$, we have

$$J_{T,a^+}^\alpha (c_1 f_1 + c_2 f_2)(t) = c_1 J_{T,a^+}^\alpha f_1(t) + c_2 J_{T,a^+}^\alpha f_2(t),$$

$$J_{T,b^-}^\alpha (c_1 f_1 + c_2 f_2)(t) = c_1 J_{T,b^-}^\alpha f_1(t) + c_2 J_{T,b^-}^\alpha f_2(t).$$

(2) For every functions $f_1, f_2 \in L_T^1[a, b]$ with $f_1 \leq f_2$ and $t \in [a, b]$, we have

$$J_{T,a^+}^\alpha f_1(t) \leq J_{T,a^+}^\alpha f_2(t),$$

$$J_{T,b^-}^\alpha f_1(t) \leq J_{T,b^-}^\alpha f_2(t).$$

(3) For every function $f \in L_T^1[a, b]$ and $t \in [a, b]$, we have

$$\left| J_{T,a^+}^\alpha f(t) \right| \leq J_{T,a^+}^\alpha |f|(t),$$

$$\left| J_{T,b^-}^\alpha f(t) \right| \leq J_{T,b^-}^\alpha |f|(t).$$

(4) For every function $f \in L_T^1[a, b]$ and $t \in [a, b]$, we have

$$J_{T,a^+}^\alpha f(t) + J_{T,b^-}^\alpha f(t) = \int_a^b \frac{f(s)}{T(t, s, \alpha)} \, ds.$$

Theorem 8. *Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g : [a, b] \to \mathbb{R}$ be a positive function on $(a, b]$ with continuous positive derivative on (a, b), and $G : [0, g(b) - g(a)] \times (0, \infty) \to \mathbb{R}$ a continuous function which is positive on $(0, g(b) - g(a)] \times (0, \infty)$.*

(1) If for some $1 < p \leq \infty$ and $M > 0$, we have

$$\int_a^b \frac{1}{T(t, s, \alpha)^{p/(p-1)}} \, ds \leq M$$

for every $t \in [a, b]$, *then* $L^p[a, b] \subseteq L^1_T[a, b]$ *(we take* $p/(p-1) = 1$ *if* $p = \infty$). *Furthermore,* J^α_{T,a^+} *and* J^α_{T,b^-} *are linear bounded operators from* $L^p[a, b]$ *to* $L^\infty[a, b]$ *with norm at most* $M^{(p-1)/p}$.

(2) *If* $T(t, s, \alpha) \geq c > 0$ *for every* $t, s \in [a, b]$, *then* $L^1[a, b] \subseteq L^1_T[a, b]$. *Furthermore,* J^α_{T,a^+} *and* J^α_{T,b^-} *are linear bounded operators from* $L^1[a, b]$ *to* $L^\infty[a, b]$ *with norm at most* $1/c$.

Proof. By using Hölder inequality, since $T(t, s, \alpha) \geq 0$, we have for each $t \in [a, b]$

$$\left| J^\alpha_{T,a^+} f(t) \right| \leq J^\alpha_{T,a^+} |f|(t) = \int_a^t \frac{|f(s)|}{T(t, s, \alpha)} \, ds \leq \int_a^b \frac{|f(s)|}{T(t, s, \alpha)} \, ds$$

$$\leq \Big(\int_a^b |f(s)|^p ds \Big)^{1/p} \Big(\int_a^b \frac{1}{T(t, s, \alpha)^{p/(p-1)}} \, ds \Big)^{(p-1)/p},$$

$$\left\| J^\alpha_{T,a^+} f \right\|_\infty \leq M^{(p-1)/p} \|f\|_p,$$

$$\left| J^\alpha_{T,b^-} f(t) \right| \leq J^\alpha_{T,b^-} |f|(t) = \int_t^b \frac{|f(s)|}{T(t, s, \alpha)} \, ds \leq \int_a^b \frac{|f(s)|}{T(t, s, \alpha)} \, ds$$

$$\leq \Big(\int_a^b |f(s)|^p ds \Big)^{1/p} \Big(\int_a^b \frac{1}{T(t, s, \alpha)^{p/(p-1)}} \, ds \Big)^{(p-1)/p}$$

$$\left\| J^\alpha_{T,b^-} f \right\|_\infty \leq M^{(p-1)/p} \|f\|_p.$$

Hence, $J^\alpha_{T,a^+} |f|(t), J^\alpha_{T,b^-} |f|(t) < \infty$ for every $t \in [a, b]$, and so, $L^p[a, b] \subseteq L^1_T[a, b]$. Furthermore, J^α_{T,a^+} and J^α_{T,b^-} are linear bounded operators from $L^p[a, b]$ to $L^\infty[a, b]$ with norm at most $M^{(p-1)/p}$.

Assume now that $p = 1$ and $T(t, s, \alpha) \geq c > 0$ for every $t, s \in [a, b]$. We have for each $t \in [a, b]$

$$\left| J^\alpha_{T,a^+} f(t) \right| \leq J^\alpha_{T,a^+} |f|(t) = \int_a^t \frac{|f(s)|}{T(t, s, \alpha)} \, ds \leq \frac{1}{c} \int_a^b |f(s)| ds,$$

$$\left\| J^\alpha_{T,a^+} f \right\|_\infty \leq \frac{1}{c} \|f\|_1,$$

$$\left| J^\alpha_{T,b^-} f(t) \right| \leq J^\alpha_{T,b^-} |f|(t) = \int_t^b \frac{|f(s)|}{T(t, s, \alpha)} \, ds \leq \frac{1}{c} \int_a^b |f(s)| ds,$$

$$\left\| J^\alpha_{T,b^-} f \right\|_\infty \leq \frac{1}{c} \|f\|_1.$$

Hence, $J^\alpha_{T,a^+} |f|(t), J^\alpha_{T,b^-} |f|(t) < \infty$ for every $t \in [a, b]$, and so, $L^1[a, b] \subseteq L^1_T[a, b]$. Furthermore, J^α_{T,a^+} and J^α_{T,b^-} are linear bounded operators from $L^1[a, b]$ to $L^\infty[a, b]$ with norm at most $1/c$. $\square$

Theorem 9. *Let* $a < b$, $\alpha \in \mathbb{R}^+$. *Let* $g : [a, b] \to \mathbb{R}$ *be a positive function on* (a, b) *with continuous positive derivative on* (a, b), *and* $G : [0, g(b) - g(a)] \times (0, \infty) \to \mathbb{R}$ *a continuous function which is positive on* $(0, g(b) - g(a)] \times (0, \infty)$.

(1) *If there exist constants* $1 \leq p < \infty$ *and* M_1 *such that*

$$\int_s^b \frac{1}{T(t, s, \alpha)^p} \, dt \leq M_1$$

for every $s \in [a, b]$, *then* J^α_{T,a^+} *is a linear bounded operator from* $L^1[a, b]$ *to* $L^p[a, b]$ *with norm at most* $M_1^{1/p}$.

(2) *If there exist constants $1 \le p < \infty$ and M_2 such that*

$$\int_a^s \frac{1}{T(t,s,\alpha)^p}\, dt \le M_2$$

for every $s \in [a,b]$, then J^α_{T,b^-} is a linear bounded operator from $L^1[a,b]$ to $L^p[a,b]$ with norm at most $M_2^{1/p}$.

Proof. By using Minkowski's integral inequality, since $T(t,s,\alpha) \ge 0$ and $I_{[a,t]}(s) = I_{[s,b]}(t)$ for every $t,s \in [a,b]$

$$\begin{aligned}
\left\| J^\alpha_{T,a^+} f \right\|_p &= \left(\int_a^b \left| \int_a^b \frac{f(s)I_{[a,t]}(s)}{T(t,s,\alpha)}\, ds \right|^p dt \right)^{1/p} \\
&\le \int_a^b \left(\int_a^b \frac{|f(s)|^p I_{[s,b]}(t)}{T(t,s,\alpha)^p}\, dt \right)^{1/p} ds \\
&\le \int_a^b \left(\int_s^b \frac{1}{T(t,s,\alpha)^p}\, dt \right)^{1/p} |f(s)|\, ds \\
&\le M_1^{1/p}\|f\|_1.
\end{aligned}$$

In a similar way, since $I_{[t,b]}(s) = I_{[a,s]}(t)$ for every $t,s \in [a,b]$,

$$\begin{aligned}
\left\| J^\alpha_{T,b^-} f \right\|_p &= \left(\int_a^b \left| \int_a^b \frac{f(s)I_{[t,b]}(s)}{T(t,s,\alpha)}\, ds \right|^p dt \right)^{1/p} \\
&\le \int_a^b \left(\int_a^b \frac{|f(s)|^p I_{[a,s]}(t)}{T(t,s,\alpha)^p}\, dt \right)^{1/p} ds \\
&\le \int_a^b \left(\int_a^s \frac{1}{T(t,s,\alpha)^p}\, dt \right)^{1/p} |f(s)|\, ds \\
&\le M_2^{1/p}\|f\|_1.
\end{aligned}$$

$\square$

Proposition 10. *Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g : [a,b] \to \mathbb{R}$ be a positive function on (a,b) with continuous positive derivative on (a,b), and $G : [0, g(b) - g(a)] \times (0, \infty) \to \mathbb{R}$ a continuous function which is positive on $(0, g(b) - g(a)] \times (0, \infty)$.*
   (1) *If*

$$\int_a^b \int_a^t \frac{1}{T(t,s,\alpha)^2}\, ds\, dt < \infty,$$

then J^α_{T,a^+} is a Hilbert-Schmidt integral operator on $L^2[a,b]$, and so, a continuous and compact operator.

   (2) *If*

$$\int_a^b \int_t^b \frac{1}{T(t,s,\alpha)^2}\, ds\, dt < \infty,$$

then J^α_{T,b^-} is a Hilbert-Schmidt integral operator on $L^2[a,b]$, and so, a continuous and compact operator.

Proof. Denote by I_B the characteristic function of the set B (i.e., the function such that $I_B(t) = 1$ if $t \in B$ and $I_B(t) = 0$ if $t \notin B$). Then

$$J^\alpha_{T,a^+} f(t) = \int_a^t \frac{f(s)}{T(t, s, \alpha)} \, ds = \int_a^b A(t, s) \, f(s) \, ds,$$

where

$$A(t, s) = \frac{I_{[a,t]}(s)}{T(t, s, \alpha)} \, .$$

We have

$$\int_a^b \int_a^b A(t, s)^2 \, ds \, dt = \int_a^b \int_a^t \frac{1}{T(t, s, \alpha)^2} \, ds \, dt < \infty.$$

Therefore, J^α_{T,a^+} is a Hilbert-Schmidt integral operator on $L^2[a, b]$, thus, it is a linear compact operator. This finishes the proof of the first item. The proof of the second one is similar. $\qquad\square$

3.2. On the Ostrowski inequality in the generalized framework

The utility of inequalities, particularly integral inequalities involving convex functions, is widely recognized as one of the main elements supporting the development of several modern branches of mathematics, and so, it has received considerable attention in recent years.

Ostrowski proved in [15] the following interesting inequality:

Theorem 11. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. If $f' \in L^\infty[a, b]$, then*

$$\left| f(x) - \frac{1}{b - a} \int_a^b f(s) \, ds \right| \leq \frac{1}{b - a} \left[\left(\frac{b - a}{2} \right)^2 + \left(x - \frac{a + b}{2} \right)^2 \right] \|f'\|_\infty.$$

Since then, there are a lot of generalizations and applications of this inequality (see, e.g., [16]). In particular, Dragomir and Wang generalized this inequality to $L^p[a, b]$ $(p > 1)$ in [17] as follows:

Theorem 12. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. If $p > 1$, $1/p + 1/q = 1$ and $f' \in L^p[a, b]$, then*

$$\left| f(x) - \frac{1}{b - a} \int_a^b f(s) \, ds \right| \leq \left[\frac{(x - a)^{q+1} + (b - x)^{q+1}}{(q + 1)(b - a)^q} \right]^{1/q} \|f'\|_p \, .$$

In this paper we prove a version of this inequality involving our kernel $1/T$. The main improvement is to consider this general weight, but also, we prove the inequality for a larger class of functions, and we include the case $p = 1$.

Theorem 13. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function, and $u : [a, b] \to [0, \infty)$ given by*

$$u_{t_0,\alpha}(s) = \frac{1}{T(t_0, s, \alpha)}$$

for each fixed $t_0 \in [a, b]$ and $\alpha > 0$. Assume that

$$\int_0^{g(b)-g(a)} \frac{dx}{G(x, \alpha)} < \infty$$

for some $\alpha > 0$. Then

$$\int_a^b u_{t_0,\alpha}(s)\,ds < \infty$$

for every $t_0 \in [a, b]$. Also, we have:

(1) *If $1 < p \leq \infty$ and $1/p + 1/q = 1$, then*

$$\left| f(x) - \frac{1}{\int_a^b u_{t_0,\alpha}(s)\,ds} \int_a^b f(s)\,u_{t_0,\alpha}(s)\,ds \right| \leq \left(\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right)^{1/q} \|f'\|_p$$

for every $x, t_0 \in [a, b]$.

(2) *If $p = 1$, then*

$$\left| f(x) - \frac{1}{\int_a^b u_{t_0,\alpha}(s)\,ds} \int_a^b f(s)\,u_{t_0,\alpha}(s)\,ds \right| \leq \|f'\|_1$$

for every $x, t_0 \in [a, b]$.

Proof. First of all, let us check that $u_{t_0,\alpha} \in L^1[a, b]$:

$$\begin{aligned}
\int_a^b u_{t_0,\alpha}(s)\,ds &= \int_a^b \frac{ds}{T(t_0, s, \alpha)} = \int_a^b \frac{g'(s)\,ds}{G(|g(t_0) - g(s)|, \alpha)} \\
&= \int_a^{t_0} \frac{g'(s)\,ds}{G(g(t_0) - g(s), \alpha)} + \int_{t_0}^b \frac{g'(s)\,ds}{G(g(s) - g(t_0), \alpha)} \\
&= \int_0^{g(t_0)-g(a)} \frac{dx}{G(x, \alpha)} + \int_0^{g(b)-g(t_0)} \frac{dx}{G(x, \alpha)} \\
&\leq 2 \int_0^{g(b)-g(a)} \frac{dx}{G(x, \alpha)} < \infty.
\end{aligned}$$

Note that $f u_{t_0,\alpha} \in L^1[a, b]$, since $f \in L^\infty[a, b]$ and $u_{t_0,\alpha} \in L^1[a, b]$. We can assume that $f' \in L^p[a, b]$, since otherwise the inequality trivially holds.

Let us define m and M as the minimum and maximum values of f on $[a, b]$, respectively. Thus, we have

$$m \leq \frac{1}{\int_a^b u_{t_0,\alpha}(s)\,ds} \int_a^b f(s)\,u_{t_0,\alpha}(s)\,ds \leq M.$$

The intermediate values theorem gives that there exists $x_0 \in [a, b]$ with

$$f(x_0) = \frac{1}{\int_a^b u_{t_0,\alpha}(s)\,ds} \int_a^b f(s)\,u_{t_0,\alpha}(s)\,ds.$$

Assume first $1 < p < \infty$. Hölder inequality gives

$$\left| f(x) - \frac{1}{\int_a^b u_{t_0,\alpha}(s)\,ds} \int_a^b f(s)\,u_{t_0,\alpha}(s)\,ds \right|$$

$$= |f(x) - f(x_0)| = \left| \int_{x_0}^x f'(s)\,ds \right|$$

$$\leq \left| \int_{x_0}^x |f'(s)|^p ds \right|^{1/p} \left| \int_{x_0}^x 1^q\,ds \right|^{1/q}$$

$$= |x - x_0|^{1/q} \left| \int_{x_0}^x |f'(s)|^p ds \right|^{1/p}$$

$$\leq \max\{x - a,\, b - x\}^{1/q} \|f'\|_p.$$

The desired inequality holds since

$$\max\{x - a,\, b - x\} = \frac{b - a}{2} + \left| x - \frac{a + b}{2} \right|.$$

If $p = 1$ or $p = \infty$, then a similar and simpler argument gives the inequalities. □

3.3. On the Székely-Clark-Entringer inequality

The following Székely-Clark-Entringer inequality appears in [18].

Proposition 14. *If $p \geq 1$ is an integer and $0 \leq x_1, \ldots, x_n \leq n - 1$, then*

$$\left(\sum_{j=1}^n x_j^p \right)^{1/p} \leq (n - 1)^{1 - 1/p} \sum_{j=1}^n x_j^{1/p}.$$

In this section we are going to prove a Székely-Clark-Entringer-type inequality for generalized integrals.

Theorem 15. *Consider real numbers $a < b$, $\alpha > 0$, $0 < r \leq p$ and $f : [a, b] \to \mathbb{R}$ a measurable function. Then the following inequality for fractional integrals holds:*

$$\left(\int_a^b \frac{|f(s)|^p}{T(b, s, \alpha)}\,ds \right)^{1/p} \leq \|f\|_\infty^{1 - r/p} \left(\int_a^b \frac{|f(s)|^r}{T(b, s, \alpha)}\,ds \right)^{1/p}. \tag{3.4}$$

Proof. We have

$$\int_a^b \frac{|f(s)|^p}{T(b, s, \alpha)}\,ds = \int_a^b \frac{|f(s)|^{p-r}|f(s)|^r}{T(b, s, \alpha)}\,ds \leq \|f\|_\infty^{p-r} \int_a^b \frac{|f(s)|^r}{T(b, s, \alpha)}\,ds,$$

$$\left(\int_a^b \frac{|f(s)|^p}{T(b, s, \alpha)}\,ds \right)^{1/p} \leq \|f\|_\infty^{1 - r/p} \left(\int_a^b \frac{|f(s)|^r}{T(b, s, \alpha)}\,ds \right)^{1/p}.$$

□

The argument in the proof of Theorem 15 allows to obtain a strongly improvement of Proposition 14 (it appears in [19], with a more complicated proof).

Corollary 16. *Consider real numbers* $0 < r \le 1 \le p$, $\Delta > 0$ *and* $0 \le x_1, \ldots, x_n \le \Delta$. *Then*

$$\Big(\sum_{j=1}^{n} x_j^p\Big)^{1/p} \le \Delta^{1-r} \sum_{j=1}^{n} x_j^r.$$

Proof. The argument in the proof of Theorem 15 gives

$$\|f\|_{L^p(\mu)} \le \|f\|_{L^\infty(\mu)}^{1-r/p} \|f\|_{L^r(\mu)}^{r/p}$$

for any measure μ and $0 < r \le p$. In particular, since $\|(x_1, \ldots, x_n)\|_\infty \le \Delta$, we have

$$\sum_{j=1}^{n} x_j \le \Delta^{1-r} \sum_{j=1}^{n} x_j^r.$$

Thus, the following known inequality

$$\Big(\sum_{j=1}^{n} x_j^p\Big)^{1/p} \le \sum_{j=1}^{n} x_j$$

finishes the proof. $\qquad\square$

3.4. On the Hermite-Hadamard inequality

The following double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{3.5}$$

holds for any convex function f on $[a, b]$.

This inequality was published by Hermite in 1883 and, independently, by Hadamard in 1893. It gives an estimation of the mean value of a convex function and note that it also provides a refinement of Jensen inequality. Probably the most important extension of this inequality is the so called Hermite-Hadamard-Fejér inequality

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le \frac{f(a)+f(b)}{2} \int_a^b g(x)\,dx$$

for any convex function f on $[a, b]$ and any non-negative integrable function g which is symmetric with respect to $(a + b)/2$.

The motivated reader is referred to [20] and references therein for more information and other extensions of Hermite-Hadamard inequality.

In [21] the authors proved the following variant of Hermite-Hadamard inequality for Riemann-Liouville fractional integrals:

Theorem 17. *Let $0 \le a < b$, $\alpha > 0$ and $f : [a, b] \to \mathbb{R}$ a convex positive function. Then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\alpha}{2(b-a)^\alpha}\left(\int_a^b (b-t)^{\alpha-1} f(t)\, dt + \int_a^b (t-a)^{\alpha-1} f(t)\, dt\right) \le \frac{f(a)+f(b)}{2}.$$

Also, in [22] appear general inequalities which are a version of Hermite-Hadamard-Fejér inequality in the context of fractional calculus.

We now present a slight improvement of Hermite-Hadamard-Fejér inequality, that we will apply in the context of fractional calculus.

Theorem 18. *Let $a < b$, $f : [a, b] \to \mathbb{R}$ a convex function and μ a finite measure on $[a, b]$ which is symmetric with respect to $(a + b)/2$. Then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\mu([a,b]) \le \int_a^b f(s)\, d\mu(s) \le \frac{f(a)+f(b)}{2}\mu([a,b]).$$

Proof. Since f is a convex function, we have for $s \in [a, b]$

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{a+b-s+s}{2}\right) \le \frac{f(a+b-s)+f(s)}{2}.$$

Hence, we obtain

$$f\left(\frac{a+b}{2}\right)\mu([a,b]) = \int_a^b f\left(\frac{a+b}{2}\right) d\mu(s) \le \frac{1}{2}\int_a^b f(a+b-s)\, d\mu(s) + \frac{1}{2}\int_a^b f(s)\, d\mu(s).$$

Since μ is a symmetric measure with respect to $(a + b)/2$, we have

$$\int_a^b f(a+b-s)\, d\mu(s) = \int_a^b f(s)\, d\mu(s),$$

which gives the first inequality.

In order to prove the second one, the convexity of f gives for $s \in [a, b]$

$$f(s) = f\left(\frac{b-s}{b-a}a + \frac{s-a}{b-a}b\right) \le \frac{b-s}{b-a}f(a) + \frac{s-a}{b-a}f(b),$$

$$f(a+b-s) = f\left(\frac{s-a}{b-a}a + \frac{b-s}{b-a}b\right) \le \frac{s-a}{b-a}f(a) + \frac{b-s}{b-a}f(b),$$

$$f(s) + f(a+b-s) \le f(a) + f(b).$$

Thus,

$$\frac{1}{2}\int_a^b f(s)\, d\mu(s) + \frac{1}{2}\int_a^b f(a+b-s)\, d\mu(s)$$
$$\le \int_a^b \frac{f(a)+f(b)}{2}\, d\mu(s)$$
$$= \frac{f(a)+f(b)}{2}\mu([a,b]).$$

Since μ is a symmetric measure with respect to $(a + b)/2$, we have

$$\int_a^b f(a + b - s)\,d\mu(s) = \int_a^b f(s)\,d\mu(s),$$

which gives the second inequality. $\qquad\square$

The argument in the proof of Theorem 18 gives the following result.

Proposition 19. *Let $a < b$, $f : [a, b] \to \mathbb{R}$ a convex function and μ a finite measure on $[a, b]$. Then the following inequalities hold:*

$$f\left(\frac{a + b}{2}\right)\mu([a, b]) \le \frac{1}{2}\left(\int_a^b f(a + b - s)\,d\mu(s) + \int_a^b f(s)\,d\mu(s)\right) \le \frac{f(a) + f(b)}{2}\mu([a, b]).$$

Proposition 19 has the following consequence in the context of fractional calculus.

Proposition 20. *Let $a < b$, $\alpha > 0$ and $f : [a, b] \to \mathbb{R}$ a convex function. Then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a + b}{2}\right) \le \frac{1}{2\mathbb{T}(\alpha)}\left(\int_a^b \frac{f(a + b - s)}{T(b, s, \alpha)}\,ds + \int_a^b \frac{f(s)}{T(b, s, \alpha)}\,ds\right) \le \frac{f(a) + f(b)}{2}, \qquad (3.6)$$

with

$$\mathbb{T}(\alpha) = \int_a^b \frac{1}{T(b, s, \alpha)}\,ds = \int_0^{g(b) - g(a)} \frac{dx}{G(x, \alpha)}.$$

4. Conclusions

In this paper, we present a general formulation of the well-known fractional drifts of Riemann-Liouville type. Our approach includes the Riemann-Liouville, Hadamard, Katugampola, and Kilbas-Marichev-Samko fractional derivatives. We state the main properties of these integral operators in Subsection 3.1. In particular, we provide sufficient conditions to ensure that these operators are bounded and are Hilbert-Schmidt operators. Also, we study Ostrowski inequality involving these general fractional operators. Finally, we obtain a kind of Székely-Clark-Entringer and Hermite-Hadamard-Fejér inequalities for these operators.

Acknowledgments

Conflict of interest

The authors declare no conflict of interest.

References

1. K. Oldham, J. Spanier, *The fractional calculus, theory and applications of differentiation and integration of arbitrary order*, 1 Ed., New York: Academic Press, 1974.

2. A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel, theory and application to heat transfer model, *arXiv*. Available from: `https://arxiv.org/abs/1602.03408`.

3. D. Baleanu, A. Fernandez, On fractional operators and their classifications, *Mathematics*, **7** (2019), 830. doi: 10.3390/math7090830.

4. A. Fernandez, M. A. Özarslan, D. Baleanu, On fractional calculus with general analytic kernels, *Appl. Math. Comput.*, **354** (2019), 248–265. doi: 10.1016/j.amc.2019.02.045.

5. L. Huang, D. Baleanu, G. Wu, S. Zeng, A new application of the fractional logistic map, *Rom. J. Phys.*, **61** (2016), 1172–1179.

6. D. Kumar, J. Singh, M. Al Qurashi, D. Baleanu, Analysis of logistic equation pertaining to a new fractional derivative with non-singular kernel, *Adv. Mechan Eng.*, **9** (2017), 1–8. doi: 10.1177/1687814017690069.

7. B. Shiri, D. Baleanu, System of fractional differential algebraic equations with applications, *Chaos Solitons Fractals,* **120** (2019), 203–212. doi: 10.1016/j.chaos.2019.01.028.

8. M. Caputo, A linear model of dissipation whose Q is almost frequency independent II, *Geophys. J. Int.*, **13** (1967), 529–539. doi: 10.1111/j.1365-246X.1967.tb02303.x.

9. M. Caputo, *Elasticitá e dissipazione*, 1 Ed., Bologna: Zanichelli, 1969.

10. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 73–85. doi: 10.12785/pdfa/010201.

11. A. Carpinteri, F. Mainardi, *Fractals and fractional calculus in continuum mechanics*, Vienna: Springer, 1997.

12. U. N. Katugampola, New approach to a generalized fractional integral, *Appl. Math. Comput.*, **218** (2011), 860–865. doi: 10.1016/j.amc.2011.03.062.

13. U. N. Katugampola, A new approach to generalized fractional derivatives, *arXiv*. Available from: `https://arxiv.org/abs/1106.0965`.

14. A. A. Kilbas, O. I. Marichev, S. G. Samko, *Fractional integrals and derivatives, theory and applications*, Pennsylvania: Gordon & Breach, 1993.

15. A. Ostrowski, Über die Absolutabweichung einer di erentienbaren funktionen von ihren integralimittelwert, *Comment. Math. Hel.*, **10** (1938), 226–227.

16. S. S. Dragomir, T. M. Rassias, *Ostrowski type inequalities and applications in numerical integration*, Dordrecht: Kluwer Academic, 2002. doi: 10.1007/978-94-017-2519-4.

17. S. S. Dragomir, S. Wang, A new inequality of Ostrowski's type in L_p norm, *Indian J. Math.*, **40** (1998), 299–304.

18. L. A. Székely, L. H. Clark, R. C. Entringer, An inequality for degree sequences, *Discrete Math.*, **103** (1992), 293–300. doi: 10.1016/0012-365X(92)90321-6.

19. J. M. Rodríguez, J. L. Sánchez, J. M. Sigarreta, CMMSE-on the first general Zagreb index, *J. Math. Chem.*, **56** (2018), 1849–1864. doi: 10.1007/s10910-017-0816-y.

20. *Selected Topics on Hermite-Hadamard Inequalities*, RGMIA Monographs, Victoria University, 2000. Available from: `https://rgmia.org/papers/monographs/Master.pdf`.

21. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.*, **57** (2013), 2403–2407. doi: 10.1016/j.mcm.2011.12.048.

22. M. Vivas-Cortez, P. Kórus, J. E. Nápoles Valdés, Some generalized Hermite-Hadamard-Fejér inequality for convex functions, *Adv. Differ. Equations*, **199** (2021), 199. doi: 10.1186/s13662-021-03351-7.