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Research article

A high order approach for nonlinear Volterra-Hammerstein integral equations

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Abstract: Here a scheme for solving the nonlinear integral equation of Volterra-Hammerstein type is given. We combine the related theories of homotopy perturbation method (HPM) with the simplified reproducing kernel method (SRKM). The nonlinear system can be transformed into linear equations by utilizing HPM. Based on the SRKM, we can solve these linear equations. Furthermore, we discuss convergence and error analysis of the HPM-SRKM. Finally, the feasibility of this method is verified by numerical examples.

Keywords: reproducing kernel method; homotopy perturbation method; Volterra-Hammerstein integral equations
Mathematics Subject Classification: 45G10, 45J05

1. Introduction

As a classical model of nonlinear integral equation, the nonlinear Volterra-Hammerstein type equations [1–3] can be used in biological models, fluid mechanics, communication theory, etc. In this article, we primarily concentrate on numerical solutions for Volterra-Hammerstein. Generally, these systems can be characterized by:

$$G(u(x)) + \lambda \int_a^x K(x,t)F(u(t))dt = f(x), \quad x, t \in [a,b], \tag{1.1}$$

where nonlinear function F is known. λ is a constant, $G : C^n[a,b] \to C[a,b]$ is a linear operator with boundedness. The choice of constraints H are satisfied that $G(u(x)) = f$ have a unique solution. Several numerical methods of the nonlinear Volterra-Hammerstein type equations have been proffered, for instance, continuous interpolation method [1], iteration method [2], Galerkin method [4], and other methods [5–7]. Mirzaee [8] approximated the nonlinear Hammerstein integral equation by utilizing the least square method based on the Legendre-Bernstein. In [9], Ordokhani proposed a configuration method based on a Walsh function that converts the Hammerstein equations into algebraic equations. Normally, in applying these methods, we have to calculate substantially integrals or employ the iterative method, which is computationally complicated. To address these problems, in [4], Mandal proposed Galerkin methods and acquired the superconvergence results in the uniform norm. Recently, reproducing kernel space theory has commonly applied to solve the nonlinear boundary value problems [10, 11], heat conduction equation [12], interfacial issues [13, 14], the Allen-Cahn equation [15], fractional-order Boussinesq equation [16], and other functional equation models [17–21]. Several methods [22–24] are also proposed for different kinds of integral equations. It is worth mentioning that many scholars improved the RKMs to study various kind of equations which is the case of [25–30]. However, the traditional reproducing kernel method [31] requires orthogonalization in the solution process, and the calculation process is complex and time-consuming. In this work, we apply HPM to do away with the integral term conveniently. We utilize SRKM to effectively avoid the Smith orthogonalization process and economizes the calculation time.

The outline of the work is as follows: We introduce the reproducing kernel theory and the homotopy perturbation theory in section 2. In section 3, we display the HPM-SRKM. Then in section 4, some numerical experiments are presented. Finaly, a conclusion is generalized in the final section.

2. Preliminaries

2.1. Reproducing kernel Hilbert space

The inner product and the norm of two reproducing kernel spaces related to this model are introduced.

Definition 1. *([32]) Let H be the Hilbert space, and the elements in H be complex-valued functions on X. If there is a unique function $K_s(t)$ for $\forall s \in X$ that satisfies*

$$\langle f, K_s \rangle = f(s), \quad f \in H.$$

Then H is defined as a reproducing kernel space, $K(s,t) = K_s(t)$ is defined as a reproducing kernel function.

Definition 2. *([32])*

$$W_2^1[a,b] = \{u(x) \mid u(x) \text{ is an absolutely continuous real value function}, \ \ u'(x) \in L^2[a,b]\}.$$

$$\langle u, v\rangle_{W_2^1} = u(a)v(a) + \int_a^b u'v'dx, \quad u, v \in W_2^1[a,b].$$

$$\|u\|_{W_2^1} = \sqrt{\langle u(x), u(x)\rangle_{W_2^1}}.$$

Definition 3. *([32])*

$$W_2^2[a,b] = \{u(x) \mid u'(x) \text{ is an absolutely continuous real value function}, \ \ u''(x) \in L^2[a,b]\}.$$

$$\langle u, v\rangle_{W_2^2} = u(a)v(a) + u'(a)v'(a) + \int_a^b u''v''dx, \quad u, v \in W_2^2[a,b].$$

$$\|u\|_{W_2^2} = \sqrt{\langle u(x), u(x)\rangle_{W_2^2}}.$$

Theorem 1. *([32]) The reproducing kernel function $r_s(t)$ of $W_2^1[a,b]$ is defined as*

$$r_s(t) = \begin{cases} 1 - a + s, & t \le s, \\ 1 - a + t, & s \le t. \end{cases}$$

Theorem 2. *([32]) The reproducing kernel function $R_s(t)$ of $W_2^2[a,b]$ is defined as*

$$R_s(t) = \begin{cases} st + \frac{st^2}{2} - \frac{t^3}{6}, & t \le s, \\ st + \frac{ts^2}{2} - \frac{s^3}{6}, & s \le t. \end{cases}$$

2.2. Introduction to Homotopy perturbation method (HPM)

The HPM ([33]) is implemented by embedding a small perturbation operator $p(p \in [0,1])$ and a homotopy path is constructed:

$$G(u(x)) + p\lambda \int_a^x K(x,t)F(u(t))dt = f(x), \quad x, t \in [a,b]. \tag{2.1}$$

When the operator $p = 0$, the Eq (2.1) is equivalent to the subsequent initial value problem:

$$G(u(x)) = f(x). \tag{2.2}$$

The Eq (2.1) is the original problem when $p = 1$. The Eqs (2.1) and (2.2) are also subject to condition H. When the operator p changes from 0 to 1, the solution $u(x)$ of Eq (1.1) follows the homotopy path from the initial value Eq (2.2) to the original problem. From the perturbation parameter theory [34], the solution satisfying the homotopy path can be extended into the form of Maclaurin series of p:

$$u(x,p) = \sum_{n=0}^{\infty} p^n u_n(x). \tag{2.3}$$

Therefore, when $p \to 1$, the approximate solution of the homotopy equation is

$$u(x) = \lim_{p \to 1} u(x,p) = \sum_{n=0}^{\infty} u_n(x).$$

Bring Eq (2.3) back Eq (2.1), and taking the k derivatives of function F, the Eq (2.1) is equaled to

$$\sum_{n=0}^{\infty} p^n G(u_n(x)) + \lambda \int_a^x K(x,t) \sum_{k=0}^{\infty} B_k p^{k+1} dt = f(x), \tag{2.4}$$

where

$$\sum_{k=0}^{\infty} B_k p^k = F(\sum_{n=0}^{\infty} p^n u_n(t)) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{dp^k} F(\sum_{n=0}^{k} p^n u_n(t)) \mid_{p=0} \tag{2.5}$$

and B_k is depended on $u_0(t), u_1(t), \cdots, u_k(t)$. By comparing the coefficients of p^k, the solution of Eq (2.1) is equivalent to the following system:

$$\begin{aligned} G(u_0(x)) &= f(x), \\ G(u_{k+1}(x)) &= -\lambda \int_a^x K(x,t) B_k dt. \end{aligned} \tag{2.6}$$

Through the above calculations, we can get the approximate solution of the Eq (1.1) by adding up the solutions of the Eq (2.6).

3. HPM-SRKM for solving equation

3.1. Presenting the HPM-SRKM to solve the Eq (2.6)

Since the system Eq (2.6) are two linear equations, we can equate them to the following problem

$$G(v(x)) = g(x), \tag{3.1}$$

in which $v(x) = u_0(x) + u_{k+1}(x), g(x) = f(x) - \lambda \int_a^x K(x,t)B_k dt$. The constraint H_1 satisfies $v_0(x) = \alpha, v_1(x) = 0, v_2(x) = 0, \cdots, v_n(x) = 0$. The linear operator $G : W_2^2[a,b] \to W_2^1[a,b]$, and G^* is the adjoint operator of G. Let $\phi_i(x) = G^* r_{x_i}(t),\ i = 1, 2, \cdots$, where $G^* r_{x_i}(t) = \langle G^* r_{x_i}, R_t\rangle_{W_2^2} = \langle r_{x_i}, GR_t\rangle_{W_2^1} = GR_{x_i}(t), \{x_i\} \in [a,b]$.

Theorem 3. *If $\{x_i\}_{i=1}^{\infty}$ is dense point set on $[a,b]$, then $\{\phi_i(x)\}_{i=1}^{\infty}$ is linearly independent system completely in $W_2^2[a,b]$.*

Proof. Let

$$\sum_{i=1}^{\infty} c_i \phi_i(x) = 0.$$

For the reversibility of G, we have

$$\sum_{i=1}^{\infty} c_i \phi_i(x) = \sum_{i=1}^{\infty} c_i GR_{x_i}(t) = G(\sum_{i=1}^{\infty} c_i R_{x_i}(t)) = 0.$$

For any $v(x) \in W_2^2[a,b]$, if $\langle v(x), \phi_i\rangle_{W_2^2} = v(x_i) = 0,\ i = 1, 2, \ldots$, then $v(x) \equiv 0$. This proof is completed. □

Let $S_n = \text{Span}\{\phi_1(x), \phi_2(x), \ldots, \phi_n(x)\}$. Define the orthogonal projection operator $P_n : W_2^2[a,b] \to S_n$, $v_n = P_n v$. The approximate solution $v_n \in S_n$ of Eq (3.1) can be expressed as

$$v_n(x) = \sum_{k=1}^{n} q_k \phi_k(x), \tag{3.2}$$

where q_k is the undetermined coefficient.

Theorem 4. *If $v(x)$ is the solution of Eq (3.1), then it is determined by Eq (3.2). The $v_n(x)$ exists and is unique. Meanwhile $v_n(x)$ satisfies*

$$Ab = F, \tag{3.3}$$

where $b = (q_1, q_2, \cdots, q_n)^{\mathrm{T}}, F = (g(x_1, v(x_1)), g(x_2, v(x_2)), \cdots g(x_n, v(x_n)))^{\mathrm{T}}$,

$$A = \begin{pmatrix} \langle\phi_1,\phi_1\rangle & \langle\phi_2,\phi_1\rangle & \ldots & \langle\phi_n,\phi_1\rangle \\ \langle\phi_1,\phi_2\rangle & \langle\phi_2,\phi_2\rangle & \ldots & \langle\phi_n,\phi_2\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle\phi_1,\phi_n\rangle & \langle\phi_2,\phi_n\rangle & \ldots & \langle\phi_n,\phi_n\rangle \end{pmatrix}$$

Proof. (Existence) The matrix A is a Gram matrix, symmetric and positive definite matrix, so that the Eq (3.3) has solutions, then

$$\begin{aligned}\langle P_n v, \phi_k\rangle_{W_2^2} &= \langle v, \phi_k\rangle_{W_2^2} = \langle v, G^* r_{x_k}\rangle_{W_2^2} = \langle Gv, r_{x_k}\rangle_{W_2^1} = Gv(x_k)\\ &= g(x_k, v(x_k)), \quad k = 1, 2, \ldots, n.\end{aligned}$$

(Uniqueness) For each identified $j(j = 1, \cdots, n)$, the solution $v(x)$ of Eq (3.1) satisfies

$$\langle v, \phi_k\rangle_{W_2^2} = Gv(x_j) = g(x_k, v(x_j)).$$

For $v_n(x) = P_n v(x)$, we have

$$\langle P_n v, \phi_k\rangle_{W_2^2} = g(x_k, v(x_j)),$$

then $\langle v_n(x) - v(x), \phi_k(x)\rangle = 0$. Therefore, $v_n(x)$ can be determined by solving the coefficient matrix b in Eq (3.3). □

3.2. Convergence and error estimation

In this subsection, we will argue the convergence of the HPM-SRKM.

Theorem 5. *(Convergence) The approximate solution $v_n(x)$ of Eq (3.2) uniformly convergence to $v(x)$.*

Proof. From $\|v_n - v\| \to 0 (n \to \infty)$ on $W_2^2[a, b]$, we deduce

$$| v_n(x) - v(x) | = | \langle v_n - v, R_x(y)\rangle_{W_2^2} | \le \|v_n - v\|_{W_2^2} \|R_x(y)\|_{W_2^2}.$$

From the continuity of the reproducing kernel function R_x, we obtain

$$| v_n(x) - v(x) | \le M \|v_n - v\|_{W_2^2} \to 0,$$

where M is a nonnegative constant. This proof is completed. □

Theorem 6. *If $K(x, t) \in C^2([a, b] \times [a, b])$ with respect to x, then*

$$\|v_n(x) - v(x)\|_{W_2^2} = O(h^2).$$

Proof. Since $K(x, t) \in C^2([a, b] \times [a, b])$ and $g(x) = f(x) - \lambda \int_a^x K(x, t) B_k dt, \; k = 1, 2, \ldots$, we can obtain $g(x) \in C^2[a, b]$. And $Gv(x) = g(x)$ is continuously. Put

$$r_n(x) = G[v_n(x) - v(x)].$$

On each subinterval $\sigma_i = [x_i, x_{i+1}]$ for $i = 1, 2, \ldots, n$, one gets $r_n(x) \in C^2[x_i, x_{i+1}]$,

$$r_n(x_i) = 0.$$

Let $L_1(x)$ be the linear interpolation polynomial of $r_n(x)$ on $[x_i, x_{i+1}]$. For the error in this interpolation, we can show

$$|r_n(x)| = |r_n(x) - L_1(x)| = O(h^2).$$

The boundness of G^{-1} implies that

$$\|v_n(x) - v(x)\|_{W_2^2} = O(h^2),$$

This proof is completed. □

4. Numerical examples

The theoretical part of the solution is proved in the previous sections. Some numerical examples are given to illustrate its effectiveness. We operate our programs in MATHMATICA 7.0. Meanwhile, the red lines represent the approximate solutions and the blue dots represent the exact solutions in the figure. The absolute errors e_i, the exact and the approximate solutions are listed in the tables. We also use the following formulas to calculate the convergence rate r.

$$r = \log_2 \frac{||e_n||}{||e_{2n}||}.$$

Example 1. *For the following nonlinear HIE:*

$$u(x) = f(x) + \int_0^x [sin(x-t)(1+u^2(t))]dt, \quad x \in [0,1]$$

where $f(x) = \frac{1}{6}(-3 + 8cosx + cos2x)$*, the exact solution is* $u(x) = cosx$*. The numerical results are illustrated in Figure 1. The comparison of the numerical results and the absolute error* e_i *are listed in Table 1. We get an exact solution with higher precision than the method of traditional reproducing kernel method [20] for* $n = 25$*.*

Example 2. *Consider the nonlinear HIE:*

$$u(x) - \int_0^x (x+t)[u(t)]^3 dt = f(x), \quad x \in [0,1]$$

where $f(x) = -\frac{15}{56}x^8 + \frac{13}{14}x^7 - \frac{11}{10}x^6 + \frac{9}{20}x^5 + x^2 - x$*. The exact solution is* $u(x) = x^2 - x$*. The numerical results are illustrated in Figure 2. Table 2 is illustrated the numerical results and the absolute error* e_i*. From the results of Table 2, we can see that our method approximates the exact solution more closely than the Legendre spectral Galerkin and multi-Galerkin methods [4] for* $n = 10$*.*

Table 1. Numerical result and absolute error for Example 1.

x	u(x)	SRKM-HPM	[31]	e_i
0.08	0.996802	0.996801	0.9968	8.52969E-7
0.16	0.987227	0.987224	0.98722	3.40638E-6
0.24	0.971338	0.971330	0.971323	7.64359E-6
0.32	0.949235	0.949222	0.949208	1.35363E-5
0.40	0.921061	0.921040	0.921019	2.10437E-5
0.48	0.886996	0.886965	0.886935	3.01124E-5
0.56	0.847255	0.847214	0.847172	4.06798E-5
0.64	0.802096	0.802043	0.801991	5.2694E-5
0.72	0.751806	0.751740	0.751674	6.61846E-5
0.80	0.696707	0.696625	0.696546	8.14622E-5
0.88	0.637151	0.637052	0.636959	9.95908E-5
0.96	0.547352	0.573397	0.573306	1.23361E-4
r		$r = 2.859$		

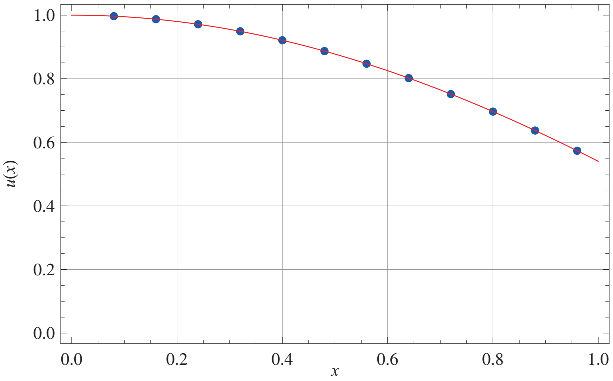

Figure 1. Comparison solutions of Example 1.

Table 2. Numerical result and absolute error for Example 2.

x	u(x)	SRKM-HPM	e_i	e_i in [4]
0.3	-0.21	-0.20998	1.3784E-5	3.2649E-2
0.4	-0.24	-0.23996	3.5960E-5	2.0340E-2
0.5	-0.25	-0.24993	7.0917E-5	1.6981E-2
0.6	-0.24	-0.23988	1.1636E-5	8.2878E-3
0.7	-0.21	-0.20983	1.6496E-5	3.1082E-3
0.8	-0.16	-0.15979	2.0913E-5	1.9572E-3
r		$r = 1.99$		

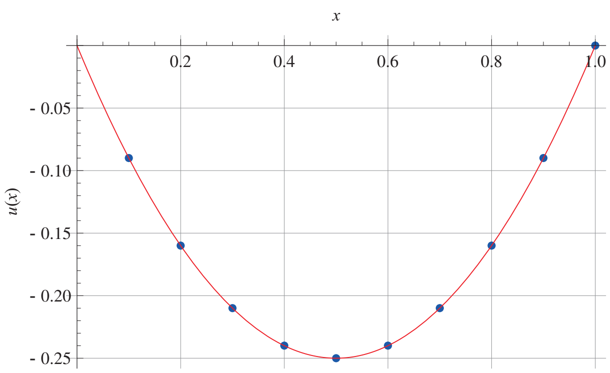

Figure 2. Comparison solutions of Example 2.

5. Conclusions

In this article, the SRKM-HPM was smoothly applied to figure out the nonlinear HIE by getting the approximate uniform solution. Besides, compared with the method of traditional reproducing kernel method [31], Legendre spectral Galerkin, and multi-Galerkin methods [4], the convergence speed and accuracy of solution were better.

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Conflict of interest

The authors declare no conflict of interest.

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