Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(1): 1399–1415.
DOI:10.3934/math.2022082
Received: 26 September 2021
Accepted: 18 October 2021
Published: 25 October 2021

Research article

A robust family of exponential attractors for a time semi-discretization of the Ginzburg-Landau equation

Narcisse Batangouna*

Faculté des Sciences et Techniques, Université Marien Ngouabi, BP: 69, Brazzaville, Congo

* **Correspondence:** Email: banarcissess@yahoo.fr; Tel: +242066552703.

Abstract: We consider a time semidiscretization of the Ginzburg-Landau equation by the backward Euler scheme. For each time step τ, we build an exponential attractor of the dynamical system associated to the scheme. We prove that, as τ tends to 0, this attractor converges for the symmetric Hausdorff distance to an exponential attractor of the dynamical system associated to the Allen-Cahn equation. We also prove that the fractal dimension of the exponential attractor and of the global attractor is bounded by a constant independent of τ.

Keywords: Allen-Cahn equation; Ginzburg-Landau equation; exponential attractor; global attractor; backward Euler scheme
Mathematics Subject Classification: 35Gxx, 65Kxx

1. Introduction

We consider the following system: find $u : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}^m \quad (m \geqslant 1)$ such that

$$\frac{\partial u}{\partial t} - \Delta u = (1 - |u|^2)u, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

where Ω is a bounded subset of $\mathbb{R}^d$ $(d \geq 1)$ with smooth boundary $\partial\Omega$. This system is endowed with homogeneous Dirichlet boundary conditions and an initial condition.

This problem arises in the study of superconductivity of liquids. The unknown u is an order parameter and when $m = 2$ or 3, it can be interpreted as the preferential orientation vector of molecules (see, e.g., [4, 11] and references therein). The set Ω is the region occupied by the liquid. We note that (1.1) is a system of reaction-diffusion equations. Indeed, by noting $u = (u_1, \ldots, u_m)$, it can be written as

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \Delta u_1 = (1 - \sum_{i=1}^m u_i^2)u_1, \quad x \in \Omega, t > 0 \\ \vdots \\ \dfrac{\partial u_m}{\partial t} - \Delta u_m = (1 - \sum_{i=1}^m u_i^2)u_m, \quad x \in \Omega, t > 0. \end{cases} \tag{1.2}$$

The boundary condition reads

$$u_i = 0 \quad \text{on} \quad x \in \partial\Omega,\ t > 0, \quad \forall i \in \{1, \dots, m\}.$$

When $m = 2$, the system (1.1) is known as the Ginzburg-Landau equation. When $m = 1$, the system reduces to a single equation called the Allen-Cahn equation [1].

Problem (1.1) has been extensively studied. In particular, starting with an initial value in $L^\infty(\Omega)^m$, it is easy to derive an L^∞ bound on the solution and to obtain global existence. This problem illustrates the case of reaction-diffusion systems with an invariant region. In [18], Temam proved the existence of a global attractor associated to this problem. He also gave an upper bound for its Hausdorff dimension and for its fractal dimension, using the method of Lyapunov exponents. In this paper, we consider a time semidiscretization of problem (1.1) and we prove the convergence, as the time step goes to 0, of a family of exponential attractors associated to the discrete problem.

For a dissipative dynamical system, an exponential attractor is a compact positively invariant set which contains the global attractor, has finite fractal dimension and attracts exponentially the trajectories. In comparison with the global attractor, an exponential attractor is expected to be more robust to perturbations: global attractors are generally upper semicontinuous with respect to perturbations, but the lower semicontinuity can be proved only in some particular cases (see e.g. [2, 13, 18]). This includes perturbations which are obtained by time and/or space discretizations of the governing equations [17, 20]. We note that, contrary to the global attractor, an exponential attractor is generally not unique.

The notion of inertial manifold, an exponential attractor which is also a manifold, was introduced in [9]. In relation with (1.1), we note that families of inertial manifolds which are robust wich respect to time and space discretization were built in [12] for the complex Ginzburg-Landau equation in one space dimension. However, all known constructions of inertial manifolds are based on a restrictive condition, the so-called spectral gap condition. As a consequence, the existence of inertial manifolds is not known for many physically important equations, such as the two-dimensional Navier-Stokes equations.

Eden *et al.* gave in [6] a construction of exponential attractions based on a "squeezing property". They proved the continuity of exponential attractors for classical Galerkin approximations, but only up to a time shift (see also [8, 10]). In [7], Efendiev, Miranville and Zelik proposed a construction of exponential attractors based on a "smoothing property" and on an appropriate error estimate, where the continuity holds without time shift. Their construction, which is valid in Banach spaces, has been adapted to many situations, including singular perturbations. We refer the reader to the review [13] and the references therein for more details.

In [14], Pierre used the result of Efendiev, Miranville and Zelig to analyze the case where the perturbation is a time semidiscretization of the underlying equation, and when the time step goes to 0. He first proposed an abstract construction of a robust family of exponential attractors, and then he applied it to the backward Euler time semidiscretization of the Allen-Cahn equation with a polynomial nonlinearity. The abstract result in [14] was also applied in [3] to the case of a time-splitting discretization of the Caginalp phase-field system. The construction was adapted in [15] to a finite element space semidiscretization of the Allen-Cahn equation.

Our purpose in this note is to show that the abstract result in [14] can also be applied to the model problem (1.1), for every space dimension d and for every m. We use a backward Euler scheme for the time discretization. The analysis is comparable to the case of the Allen-Cahn equation ($m = 1$) performed in [14], but the estimates are somewhat simpler here thanks to the L^∞ estimates. Our paper is outlined as follows. We first derive in Section 2 the estimates for the continuous-in-time problem. Then, in Section 3, we derive their discrete-in-time counterparts and we give the error estimate between the discrete and countinuous problems. In the last section, we are in position to apply the abstract result. For every time step τ, we build an exponential attractor $\mathcal{M}_\tau$ for the discrete-in-time problem and we show that $\mathcal{M}_\tau$ converges for the symmetric Hausdorff distance to an exponential attractor $\mathcal{M}_0$ of the continuous problem, as τ tends to 0. We also show that the fractal dimension of $\mathcal{M}_\tau$ is bounded by a constant independent of τ. As a corollary of our analysis, we obtain the upper semicontinuity of the global attractor $\mathcal{A}_\tau$ as τ tends to 0. Since $\mathcal{A}_\tau \subset \mathcal{M}_\tau$, the fractal dimension of the global attractors is also bounded by a constant independent of τ.

2. The continuous problem

2.1. The continuous semigroup

Let $H = L^2(\Omega)^m$ be equipped with the usual norm

$$|v|_0^2 = \sum_{i=1}^m \int_\Omega v_i^2 dx$$

for $v = (v_1, \ldots, v_m)$ and the associated inner product $(\cdot,\cdot)_0$. We set $V = H_0^1(\Omega)^m$ endowed with the norm

$$\|v\|^2 = \sum_{i=1}^m \int_\Omega |\nabla v_i|^2 dx.$$

We recall the Poincaré inequality: there exists a constant $c_0 = c_0(\Omega)$ such that

$$|v|_0 \leq c_0\|v\|, \quad \forall v \in V. \tag{2.1}$$

We define the function $g : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ by

$$g(w) = (|w|^2 - 1)w, \quad \forall w \in \mathbb{R}^m.$$

Using the coordinates $g(w) = (g_1(w), \ldots, g_m(w))$, we have

$$g_i(w) = g_i(w_1, \ldots, w_m) = (\sum_{j=1}^m w_j^2 - 1)w_i.$$

The problem (1.1) with homogeneous Dirichlet condition and initial condition can be written as:

$$\frac{\partial u}{\partial t} - \Delta u + g(u) = 0, \quad x \in \Omega, \quad t > 0 \tag{2.2}$$

$$u = 0, \quad x \in \partial\Omega, \quad t > 0 \tag{2.3}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{2.4}$$

Throughout this paper, we shall assume that $\delta \geq 1$ and we denote

$$\mathcal{B}_\delta = \{w \in \mathbb{R}^m, |w| \leq \delta\}$$

the ball of $\mathbb{R}^m$ centered at 0 and with radius δ. In order to derive L^∞ estimates, the basic idea is that for $w \in \partial\mathcal{B}_\delta$, we have $|w| = \delta$ and so the vector $-g(w) = (1-\delta^2)w$ points inside $\mathcal{B}_\delta$. The set $\mathcal{B}_\delta$ is an invariant region. Let

$$L^2(\Omega;\mathcal{B}_\delta) = \left\{v \in L^2(\Omega)^m, \quad v(x) \in \mathcal{B}_\delta \quad \text{for a.e.} \quad x \in \Omega\right\}.$$

Note that $L^2(\Omega;\mathcal{B}_\delta) \subset L^\infty(\Omega)^m$ and that $L^2(\Omega;\mathcal{B}_\delta)$ is a closed convex subset of $L^2(\Omega)^m$. More precisely, we have the following well-posedness result [18]:

Theorem 2.1. *For all $u_0 \in L^2(\Omega;\mathcal{B}_\delta)$, the problem (2.2)-(2.4) has a unique solution for all time, $u(t) \in L^2(\Omega;\mathcal{B}_\delta), \forall t$, $u(t) \in L^2(0,T;H_0^1), \forall T > 0$, and the mapping $S_0(t) : u_0 \longmapsto u(t)$ is continuous in $L^2(\Omega)^m$, $\forall t \geq 0$.*

Furthermore, if $u_0 \in H_0^1(\Omega)^m \cap L^2(\Omega;\mathcal{B}_\delta)$, then

$$u \in C([0,T];H_0^1(\Omega)^m) \cap L^2(0,T;H^2(\Omega)), \forall T > 0.$$

This result defines the semigroup $S_0(t) : u_0 \longmapsto u(t)$ on $L^2(\Omega;\mathcal{B}_\delta)$.

2.2. Dissipative a priori estimates

We first derive the dissipative estimates. Multiplying (1.1) by u, integrating over Ω, and using the following result

$$g(u)u = (|u|^2-1)|u|^2 = (|u|^2-\frac{1}{2})^2-\frac{1}{4}, \tag{2.5}$$

we obtain

$$\frac{1}{2}\frac{d}{dt}|u|_0^2 + \|u\|^2 + \int_\Omega (|u|^2-\frac{1}{2})^2 dx \leq \frac{1}{4}|\Omega|, \tag{2.6}$$

where $|\Omega| = \int_\Omega 1dx$. Using the Poincaré inequality (2.1), we obtain

$$\frac{d}{dt}|u|_0^2 + \frac{2}{c_0^2}|u|_0^2 \leq \frac{1}{2}|\Omega|$$

that is

$$\frac{d}{dt}|u|_0^2 + c_1|u|_0^2 \leq c_1^{'},$$

with $c_1 = 2/c_0^2$ and $c_1^{'} = |\Omega|/2$. Using Gronwall's lemma, we obtain

$$|u(t)|_0^2 \leq |u_0|_0^2 e^{-c_1 t} + \frac{c_1^{'}}{c_1}(1-e^{-c_1 t}).$$

This yields:

Proposition 2.2 (Absorbing set in H)**.** *If* $|u_0|_0 \le R$*, then*

$$|u(t)|_0 \le \rho_0, \quad \forall t \ge t_0(R),$$

where $\rho_0^2 = 1 + \frac{c_1'}{c_1}$ *and* $t_0(R) = \dfrac{1}{c_1}\log(R^2)$.

In the following, we set $r > 0$. Integrating (2.6) from t to $t + r$ yields

$$2\int_t^{t+r} \|u\|^2 ds \le |u(t)|_0^2 + rc_1', \forall t \ge 0$$

If $|u_0|_0 \le R$ and $t \ge t_0(R)$, then we have

$$2\int_t^{t+r} \|u\|^2 ds \le \rho_0^2 + rc_1'. \tag{2.7}$$

Using the fact that $\forall w \in \mathcal{B}_\delta$,

$$|g(w)| = |(|w|^2 - 1)||w| \le (\delta^2 + 1)\delta = C_\delta, \tag{2.8}$$

and multiplying (2.2) by $-\Delta u$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + |\Delta u|_0^2 = \int_\Omega g(u)\Delta u dx \le \frac{|\Omega|}{2}C_\delta^2 + \frac{1}{2}|\Delta u|_0^2.$$

Thus, we have

$$\frac{d}{dt}\|u\|^2 \le |\Omega|C_\delta^2, \quad \forall t \ge 0. \tag{2.9}$$

Using (2.7), (2.9) and the uniform Gronwall lemma [18], we find:

Proposition 2.3 (Absorbing set in V)**.** *If* $|u_0|_0 \le R$*, then*

$$\|u(t)\| \le \rho_1, \quad \forall t \ge t_0(R) + r,$$

where

$$\rho_1^2 = \frac{\rho_0^2 + rc_1'}{2r} + r|\Omega|C_\delta^2$$

is independent of R.

Next, we show that u is Hölder continuous in time. Multiplying (2.2) by $\frac{\partial u}{\partial t}$ and integrating over Ω, we obtain

$$\left|\frac{du}{dt}\right|_0^2 + \frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{d}{dt}\int_\Omega \frac{1}{4}(|u|^2 - 1)^2 dx = 0.$$

Integrating over $[0, t]$, we have

$$\int_0^t \left|\frac{du}{dt}\right|_0^2 + \frac{1}{2}\|u(t)\|^2 + \int_\Omega G(u(t))dx \le \frac{1}{2}\|u_0\|^2 + \int_\Omega G(u_0)dx, \quad \forall t \ge 0,$$

where

$$G(w) = \frac{1}{4}(|w|^2 - 1)^2 \quad \text{for} \quad w \in \mathbb{R}^m.$$

In particular if $u_0 \in L^2(\Omega; \mathcal{B}_\delta) \cap V$, we have for all $t_1, t_2 \ge 0$,

$$|u(t_1) - u(t_2)|_0^2 = \left|\int_{t_1}^{t_2} \frac{du}{ds}(s)ds\right|_0^2 \le |t_2 - t_1|C(\|u_0\|, \delta). \tag{2.10}$$

2.3. Estimate for the difference of solutions

We consider two solutions u and $\bar{u}$ of (2.2)-(2.4) with values in $\mathcal{B}_\delta$. Let $w = u - \bar{u}$, which satisfies

$$\frac{\partial w}{\partial t} - \Delta w + g(u) - g(\bar{u}) = 0. \tag{2.11}$$

We multiply (2.11) by w and we integrate over Ω. We find

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|w|_0^2 + \|\nabla w\|^2 &= -(g(u) - g(\bar{u}), u - \bar{u})_0 \\ &\leq c_\delta |w|_0^2. \end{aligned} \tag{2.12}$$

In the last line, we have used the Cauchy-Schwarz inequality and the mean inequality, which reads

$$\forall\ v, \bar{v} \in \mathcal{B}_\delta, \quad |g(v) - g(\bar{v})|_0 \leq c_\delta |v - \bar{v}|_0, \tag{2.13}$$

with

$$c_\delta = \sup_{w \in \mathcal{B}_\delta} \|D_g(w)\| < +\infty.$$

Using Gronwall's lemma, we deduce from (2.12) that

$$|w(t)|_0^2 + 2\int_0^t \|w(s)\|^2 ds \leq |w(0)|_0^2 \exp(2c_\delta t), \quad t \geq 0. \tag{2.14}$$

Next, we multiply (2.11) by $\frac{\partial w}{\partial t}$ and we integrate over Ω. We obtain

$$\left|\frac{\partial w}{\partial t}\right|_0^2 + \frac{1}{2}\frac{d}{dt}\|w\|^2 = -\left(g(u) - g(\bar{u}), \frac{\partial w}{\partial t}\right)_0,$$

which implies

$$\left|\frac{\partial w}{\partial t}\right|_0^2 + \frac{1}{2}\frac{d}{dt}\|w\|^2 \leq c_\delta |w|_0 \left|\frac{\partial w}{\partial t}\right|_0 \leq \left|\frac{\partial w}{\partial t}\right|_0^2 + \frac{c_\delta^2}{4}|w|_0^2.$$

Thus, we have

$$\frac{d}{dt}\|w\|^2 \leq \frac{c_\delta^2}{2}|w|_0^2, \quad \forall t \geq 0. \tag{2.15}$$

Multiplying (2.15) by t and adding $\|w\|^2$, we find

$$\frac{d}{dt}(t\|w\|^2) \leq \frac{c_\delta^2}{2} t|w|_0^2 + \|w\|^2.$$

Thus

$$t\|w(t)\|^2 \leq \int_0^t \frac{c_\delta^2}{2} t|w(s)|_0^2 ds + \int_0^t \|w(s)\|^2 ds, \quad \forall t \geq 0.$$

Using (2.14), we obtain that $\forall t \in [0, T]$

$$t\|w(t)\|^2 \leq c_1(\delta, T)|w(0)|_0^2. \tag{2.16}$$

This is a H-V smoothing property.

3. The discrete problem

3.1. The discrete semigroup

We use the backward Euler scheme. Let $\tau > 0$ be the time step. The scheme can be written as: let $u_0 \in L^2(\Omega; \mathcal{B}_\delta)$, and for $n = 0, 1, \ldots$, find $u^{n+1} \in L^2(\Omega; \mathcal{B}_\delta) \cap V$ which solves

$$\frac{u^{n+1} - u^n}{\tau} - \Delta u^{n+1} + g(u^{n+1}) = 0 \quad in \quad V'. \tag{3.1}$$

We have the following well-posedness result.

Theorem 3.1. *Assume that $\delta t \leq 1/c_\delta$. Then for all $u \in L^2(\Omega; \mathcal{B}_\delta)$, there exists a unique $v = v_{\tau,u} \in L^2(\Omega; \mathcal{B}_\delta) \cap V$ such that*

$$\frac{v - u}{\tau} - \Delta v + g(v) = 0 \quad in \quad V' \tag{3.2}$$

In addition, the mapping $S_\tau : u \longmapsto v_{\tau,u}$ is Lipschitz continuous from $L^2(\Omega; \mathcal{B}_\delta)$ into $L^2(\Omega; \mathcal{B}_\delta) \cap V$, with

$$\|S_\tau u - S_\tau \bar{u}\| \leq \frac{c_0}{\tau} |u - \bar{u}|_0, \quad \forall u, \bar{u} \in L^2(\Omega; \mathcal{B}_\delta). \tag{3.3}$$

This result defines for each $\tau \in (0, 1/c_\delta]$ a discrete semigroup $\{S_\tau^n, n \in \mathbb{N}\}$ on $L^2(\Omega; \mathcal{B}_\delta)$. In the remainder of the manuscript, we will assume that $\tau \in (0, 1/c_\delta]$.

Proof. We first prove the existence of a solution. We assume that $d = 1, 2$ or 3 so that $H_0^1(\Omega) \subset L^6(\Omega)$. The general case can be obtained on replacing g with g_δ, where for $w \in \mathbb{R}^m$,

$$\begin{cases} g_\delta(w) = g(w) & \text{if} \quad |w| \leq \delta, \\ g_\delta(w) = (\delta^2 - 1)w & \text{if} \quad |w| \geq \delta. \end{cases} \tag{3.4}$$

Let $u \in L^2(\Omega; \mathcal{B}_\delta)$. We minimize the functional

$$F(w) = \frac{1}{2\tau} |w - u|_0^2 + \frac{1}{2} \|w\|^2 + \int_\Omega G(w(x)) dx$$

in V, and we get a solution $v \in V$ of the Euler-Lagrange equation (3.2). Since $H_0^1(\Omega) \subset L^6(\Omega)$, we have $g(v) \in L^2(\Omega)^m$ and by elliptic regularity,

$$v \in W^{2,2}(\Omega)^m \subset C(\bar{\Omega})^m.$$

It remains to show that $v(x) \in \mathcal{B}_\delta$ for all $x \in \Omega$. Since $\mathcal{B}_\delta$ is a closed convex subset of $\mathbb{R}^m$, it is sufficient to show that for any hyperplane $\mathcal{H}$ containing $\mathcal{B}_\delta$, we have $v(x) \in \mathcal{H}$, $\forall x \in \Omega$. Let $\mathcal{H} = \{w \in \mathbb{R}^m, \langle n, w \rangle \leq \beta\}$ be such a hyperplane, where $n = (n_1, \ldots, n_m)$ is a vector of $\mathbb{R}^m$ of norm 1 and $\beta \geq \delta$. Here, $\langle n, w \rangle = \sum_{i=1}^m n_i w_i$ is the usual inner product of $\mathbb{R}^m$. The partial differential equation (3.2) can be written componentwise

$$v_i - \tau \Delta v_i = \tau(1 - |v|^2) v_i + u_i$$

in $L^2(\Omega)$, for $i = 1, \ldots, m$. Multiplying the above equality by n_i and summing over i, we have

$$\bar{v} - \tau \Delta \bar{v} = \tau(1 - |v|^2) \bar{v} + \bar{u}$$

in $L^2(\Omega)$, where $\bar{v} = \sum_{i=1}^{m} n_i v_i$ and $\bar{u} = \sum_{i=1}^{m} n_i u_i$. Since $u \in L^2(\Omega; \mathcal{B}_\delta)$, we have $\bar{u}(x) \leq \beta$ a.e. $x \in \Omega$ and so

$$\bar{v} - \beta - \tau\Delta\bar{v} \leq \tau(1 - |v|^2)\bar{v} \text{ a.e. in } \Omega.$$

Multiplying by $(\bar{v} - \beta)_+$, which belongs to $H_0^1(\Omega)$, we have

$$\int_\Omega (\bar{v} - \beta)_+^2 + \tau \int_\Omega |\nabla(\bar{v} - \beta)_+|^2 \leq \int_\Omega \tau(1 - |v|^2)\bar{v}(\bar{v} - \beta)_+ \leq 0$$

since $\beta \geq \delta \geq 1$. Thus, $\bar{v}(x) \leq \beta$, $\forall x \in \Omega$ as claimed.

Next, we prove the uniqueness of the solution. Let $u, \bar{u} \in L^2(\Omega; \mathcal{B}_\delta)$ and let $v, \bar{v}$ be the corresponding solutions of (3.2) in $L^2(\Omega; \mathcal{B}_\delta) \cap V$. The difference $w = v - \bar{v}$ satisfies

$$\frac{w}{\tau} - \Delta w + g(v) - g(\bar{v}) = \frac{u - \bar{u}}{\tau}.$$

Multiplying the above equality by w and using (2.13), we get

$$\frac{1}{\tau}|w|_0^2 + ||w||^2 - c_\delta |w|_0^2 \leq \frac{1}{\tau}|u - \bar{u}|_0 |w|_0.$$

Using (2.1) and dividing by $|w|_0$, we find

$$(\frac{1}{\tau} - c_\delta)|w|_0 + \frac{1}{c_0}||w|| \leq \frac{1}{\tau}|u - \bar{u}|_0.$$

If $\tau \leq 1/c_\delta$, we find (3.3), which implies the uniqueness of the solution. □

3.2. Uniform dissipative estimates

Proposition 3.2 (Absorbing set in H)**.** *Assume that $\tau \leq c_0^2/2$. If $|u^0|_0 \leq R$, then*

$$|u^0|_0 \leq \rho_0, \quad \forall n\tau \geq 2t_0(R),$$

where ρ_0 and $t_0(R)$ are as in Proposition 2.2.

Proof. We multiply (3.1) by u^{n+1} in H, we use (2.5) and the identity

$$(a - b, a)_0 = \frac{1}{2}\left(|a|_0^2 - |b|_0^2 + |a - b|_0^2\right). \tag{3.5}$$

This yields

$$\frac{1}{2\tau}\left(|u^{n+1}|_0^2 - |u^n|_0^2 + |u^{n+1} - u^n|_0^2\right) + ||u^{n+1}||^2 \leq \frac{|\Omega|}{4}. \tag{3.6}$$

From the Poincaré inequality (2.1), we deduce that

$$\frac{1}{2\tau}\left(|u^{n+1}|_0^2 - |u^n|_0^2\right) + \frac{1}{c_0^2}|u^{n+1}|_0^2 \leq \frac{|\Omega|}{4},$$

that is

$$(1 + \frac{2\tau}{c_0^2})|u^{n+1}|_0^2 \leq |u^n|_0^2 + c_1^{'}\tau, \tag{3.7}$$

where $c_1^{'} = |\Omega|/2$. We set $\alpha = 1 + \dfrac{2\tau}{c_0^2}$. By induction, we deduce from (3.7) that

$$|u^n|_0^2 \leq \alpha^{-n}|u_0|_0^2 + c_1^{'}\tau\frac{1-\alpha^{-n}}{\alpha-1}, \quad \forall n \geq 0.$$

Using the inequality $1 + x \geq \exp(x/2)$, valid for all $x \in [0,1]$, we see that if $2\tau \leq c_0^2$, then we have $\alpha^{-1} \leq \exp(-\tau/c_0^2)$. Thus, we get

$$|u^n|^2 \leq \exp(-n\tau/c_0^2)|u^0|^2 + \frac{c_1^{'}c_0^2}{2}, \quad \forall n \geq 0.$$

The claim follows, by setting $\rho_0^2 = 1 + \frac{c_1^{'}c_0^2}{2}$, as above. □

We recall the following discrete uniform Gronwall lemma from [16].

Lemma 3.3. *Let $n_0, n \in \mathbb{N}$ and let a_1, a_2, a_3, τ be positive constants. Assume that (d^n), (g^n) and (h^n) are three sequences of nonnegative real numbers which satisfy*

$$\frac{d^{n+1}-d^n}{\tau} \leq g^n d^n + h^n, \quad \forall n \geq n_0,$$

and

$$\tau\sum_{n=k_0}^{k_0+N} g^n \leq a_1, \quad \tau\sum_{n=k_0}^{k_0+N} h^n \leq a_2, \quad \tau\sum_{n=k_0}^{k_0+N} d^n \leq a_3,$$

for all $k_0 \geq n_0$. Then

$$d^n \leq \left(a_2 + \frac{a_3}{r'}\right)\exp(a_1), \quad \forall n \geq n_0 + N,$$

where $r' = \tau N$.

We have:

Proposition 3.4 (Absorbing set in V)**.** *We assume that $\tau > 0$ satisfies $\tau \leq c_0^2/2$ and $\tau \leq r/2$. If $|u^0|_0 \leq R$, then for all $n \in \mathbb{N}$ such that $n\tau \geq 2t_0(R) + 2r$, we have*

$$\|u^n\|_0 \leq 2\rho_1. \tag{3.8}$$

Proof. First, we multiply (3.1) by $-\Delta u^{n+1}$ and we integrate over Ω. This yields

$$\frac{1}{2\tau}(\|u^{n+1}\|^2 - \|u^n\|^2 + \|u^{n+1}-u^n\|^2) + |\Delta u^{n+1}|_0^2 = \int_\Omega g(u)\Delta u,$$

where $\int_\Omega g(u)\Delta u = \int_\Omega \sum_{i=1}^m g_i(u)\Delta u_i$. By the Cauchy-Schwarz inequality, we have

$$\frac{1}{2\tau}(\|u^{n+1}\|^2 - \|u^n\|^2 + \|u^{n+1}-u^n\|^2) + |\Delta u^{n+1}|_0^2 \leq \int_\Omega |g(u^{n+1})||\Delta u^{n+1}|.$$

Since $|u^{n+1}| \leq \delta$ for a.e. $x \in \Omega$, we have $|g(u^{n+1})| \leq C_\delta$ (cf. (2.8)). Young's inequality yields

$$\frac{1}{2\tau}(\|u^{n+1}\|^2 - \|u^n\|^2 + \|u^{n+1}-u^n\|^2) + |\Delta u^{n+1}|_0^2 \leq \frac{1}{2}|\Omega|C_\delta^2 + \frac{1}{2}|\Delta u^{n+1}|_0^2.$$

Thus, we have

$$\frac{1}{\tau}(\|u^{n+1}\|^2 - \|u^n\|^2) + \frac{1}{\tau}\|u^{n+1} - u^n\|^2 \leq |\Omega| C_\delta^2, \quad \forall n > 0. \tag{3.9}$$

Next, let $k_0, n \in \mathbb{N} \setminus \{0\}$. By summing inequality (3.6) for n varying from $k_0 - 1$ to $k_0 + N - 1$, we find

$$|u^{k_0+N}|_0^2 - |u^{k_0-1}|_0^2 + 2\tau \sum_{n=k_0-1}^{k_0+N-1} \|u^{n+1}\|^2 \leq \frac{|\Omega|}{2}\tau(N+1).$$

If $k_0\tau \geq 2t_0(R) + \tau$, we deduce from Proposition 3.2 that

$$2\tau \sum_{n=k_0}^{k_0+N} \|u^n\|^2 \leq \frac{|\Omega|}{2}\tau(N+1) + \rho_0^2. \tag{3.10}$$

Let $n_0 \in \mathbb{N}$ such that $n_0\tau \geq 2t_0(R) + \tau$ and let $N = [r/\tau]$. We assume that $\tau \leq r/2$ so that $N \geq 2$. We set $r' = N\tau \in [r - \tau, r]$ and

$$k_1 = |\Omega|(r' + r)/4 + \rho_0^2/2.$$

We apply Lemma 3.3 with $d^n = \|u^n\|^2$, $g^n = 0$ and $h^n = |\Omega| C_\delta^2$. Using the estimates (3.9) and (3.10), we obtain

$$\|u^n\|^2 \leq \frac{k_1}{r'} + |\Omega| C_\delta^2 (r' + r), \quad \forall n \geq n_0 + N.$$

Since $r' \in [r/2, r]$, this implies the inequality (3.8). □

3.3. Estimates for the difference of solutions

In this section, (u^n) and $(\bar{u}^n)$ are two sequences generated by the scheme (3.1). We denote by $v^n = u^n - \bar{u}^n$ their difference, which satisfies

$$\frac{1}{\tau}(v^{n+1} - v^n) - \Delta v^{n+1} + g(u^{n+1}) - g(\bar{u}^{n+1}) = 0 \quad \text{in} \quad V'. \tag{3.11}$$

We first derive a discrete version of estimate 2.14.

Lemma 3.5. *We assume that $\tau \leq 1/4c_\delta$. Then for all $n \geq 0$, we have*

$$|v^n|_0^2 + 2\tau \sum_{k=0}^{n-1} \|v^{k+1}\|^2 \leq \exp(4c_\delta n\tau)|v^0|_0^2.$$

Proof. We multiply Eq (3.11) by v^{n+1}, we integrate over Ω, we use the Cauchy-Scwharz inequality and estimate (2.13). This yields

$$\frac{1}{2\tau}(|v^{n+1}|_0^2 - |v^n|_0^2) + \|v^{n+1}\|^2 \leq c_\delta |v^{n+1}|_0^2,$$

that is

$$(1 - 2\tau c_\delta)|v^{n+1}|_0^2 + 2\tau\|v^{n+1}\|^2 \leq |v^n|_0^2,$$

for all $n \geq 0$. Since

$$1 \leq \frac{1}{1-s} \leq 1 + 2s, \quad \forall s \in [0, \frac{1}{2}],$$

it follows that

$$|v^{n+1}|_0^2 + 2\tau\|v^{n+1}\|^2 \le (1+4\tau c_\delta)|v^n|_0^2, \quad \forall n \ge 0.$$

By induction, we obtain

$$|v^n|_0^2 + 2\tau\sum_{k=0}^{n-1}\|v^{k+1}\|^2 \le (1+4\tau c_\delta)^n|v^0|_0^2, \quad \forall n \ge 0.$$

Finally, we use that $1+s \le \exp(s)$ with $s = 4\tau c_\delta$ and the claim is proved. □

Next, we derive an H-V smoothing property, a discrete analog of (2.16).

Lemma 3.6. *Let $T > 0$. For all $0 < n\tau \le T$, we have*

$$n\tau\|v^n\|^2 \le C(T,\delta)|v^0|_0^2.$$

Proof. Let $n \ge 1$. We know by Theorem (3.1) that v^n and v^{n+1} both belong to V. We multiply (3.11) by $v^{n+1} - v^n$ and integrate over Ω. Using the identity (3.5) (with the norm $|\cdot|_0$ replaced by $\|\cdot\|$), we find

$$\begin{aligned}
\frac{1}{\tau}|v^{n+1} - v^n|_0^2 &+ \frac{1}{2}\|v^{n+1}\|^2 - \frac{1}{2}\|v^n\|^2 + \frac{1}{2}\|v^{n+1} - v^n\|^2 \\
&= -(g(u^{n+1}) - g(\bar{u}^{n+1}), v^{n+1} - v^n)_0 \\
&\le c_\delta|v^{n+1}|_0|v^{n+1} - v^n|_0 \\
&\le \frac{1}{\tau}|v^{n+1} - v^n|_0^2 + \frac{c_\delta^2\tau}{4}|v^n|_0^2.
\end{aligned}$$

In the second line, we used the Cauchy-Schwarz inequality and the estimate (2.13). Thus,

$$\|v^{n+1}\|^2 - \|v^n\|^2 \le \frac{c_\delta^2\tau}{2}|v^n|_0^2, \quad \forall n \ge 1.$$

We multiply this by n and we add $\|v^{n+1}\|^2$. This yields

$$(n+1)\|v^{n+1}\|^2 - n\|v^n\|^2 \le \frac{c_\delta^2}{2}n\tau|v^n|_0^2 + \|v^{n+1}\|^2, \quad \forall n \ge 1.$$

We set $\alpha_n = n\|v^n\|^2$ ($\alpha_n = 0$ if $n = 0$) and $\beta_n = \frac{c_\delta^2}{2}n\tau|v^n|_0^2 + \|v^{n+1}\|^2$. From what precedes, we have (note that the case $n = 0$ is obvious)

$$\alpha_{n+1} \le \alpha_n + \beta_n, \quad \forall n \ge 0.$$

By induction, $\alpha_n \le \alpha_0 + \sum_{k=0}^{n-1}\beta_k$, for all $n \ge 1$. Thus,

$$n\tau\|v^n\|^2 \le \frac{c_\delta^2}{2}\tau\sum_{k=0}^{n-1}k\tau|v^k|_0^2 + \tau\sum_{k=0}^{n-1}\|v^{k+1}\|^2.$$

Applying Lemma 3.5, we find that for all $n \ge 1$,

$$n\tau\|v^n\|^2 \le \frac{c_\delta^2}{2}(n\tau)^2\exp(4c_\delta n\tau)|v^0|_0^2 + \frac{1}{2}\exp(4c_\delta n\tau)|v^0|_0^2.$$

Thus, if $0 < n\tau \leq T$, we may choose

$$C(T,\delta) = \frac{c_\delta^2}{2} T^2 \exp(4c_\delta T) + \frac{1}{2}\exp(4c_\delta T),$$

and this concludes the proof. □

Proposition 3.7. *Assume that $\tau \leq 1/4c_\delta$. For each $T > 0$ and each $R > 0$, there is a constant $C(T,R)$ independent of τ such that $|u^0|_0 \leq R$ and $0 \leq n\tau \leq T$ imply*

$$|u^n|_0 \leq C(T,R).$$

Proof. We set $u^n = (u^n - \bar{u}^n) + (\bar{u}^n - \bar{u}^0)$, where $(\bar{u}^n)$ is the solution of the scheme (3.1) with initial value $\bar{u}^0 = 0$. We write $\bar{u}^n - \bar{u}^0 = \sum_{k=0}^{n-1}(\bar{u}^{k+1} - \bar{u}^k)$ and by the triangle inequality, we have

$$|u^n|_0 \leq |u^n - \bar{u}^n|_0 + \sum_{k=0}^{n-1} \left|\bar{u}^{k+1} - \bar{u}^k\right|_0 .$$

For the first term in the right-hand side, we use Lemma 3.5 with $|v^0|_0 \leq R$, and for the second term, we apply the Cauchy-Schwarz inequality. This yields

$$|u^n|_0 \leq \exp(2c_\delta n\tau)R + (n\tau)^{1/2}\left(\frac{1}{\tau}\sum_{k=0}^{n-1} |\bar{u}^{k+1} - \bar{u}^k|_0\right)^{1/2}, \tag{3.12}$$

for all $n \geq 0$. Now, we estimate the last term above. We write (3.1) for the sequence $(\bar{u}^n)$ and we multiply this by $\bar{u}^{n+1} - \bar{u}^n$ for the $L^2(\Omega)$ scalar product. This yields

$$\begin{aligned} \frac{|\bar{u}^{n+1} - \bar{u}^n|_0^2}{\tau} + \frac{1}{2}\|\bar{u}^{n+1}\|^2 - \frac{1}{2}\|\bar{u}^n\|^2 &\leq |(g(\bar{u}^{n+1}), \bar{u}^{n+1} - \bar{u}^n)_0| \\ &\leq C_\delta |\Omega|^{1/2} |\bar{u}^{n+1} - \bar{u}^n|_0 \\ &\leq \frac{1}{2\tau}|\bar{u}^{n+1} - \bar{u}^n|_0^2 + \frac{C_\delta^2 \tau}{2}|\Omega|, \end{aligned}$$

where C_δ is the constant from (2.8). By summing these estimates, we obtain

$$\frac{1}{2\tau}\sum_{k=0}^{n-1} |\bar{u}^{k+1} - \bar{u}^k|_0^2 + \frac{1}{2}\|\bar{u}^n\|^2 \leq \frac{1}{2}\|\bar{u}^0\|^2 + \frac{C_\delta^2}{2} n\tau|\Omega|,$$

for all $n \geq 0$. Since $\bar{u}^0 = 0$, we find that

$$\frac{1}{\tau}\sum_{k=0}^{n-1} |\bar{u}^{k+1} - \bar{u}^k|_0^2 \leq C_\delta^2 n\tau|\Omega|.$$

Using (3.12), we see that if $0 \leq n\tau \leq T$, we have $|u^n|_0 \leq C(R,T)$ with

$$C(R,T) = \exp(2c_\delta T)R + C_\delta T|\Omega|^{1/2}.$$

This concludes the proof. □

3.4. Uniform error estimate on a finite time interval

To estimate the error over a finite time interval, we follow the methodology described in [19]. We consider a sequence (u^n) generated by (3.1). For each $\tau > 0$, we associate to this sequence the functions $u_\tau, \bar{u}_\tau : \mathbb{R}_+ \longrightarrow L^2(\Omega)$ defined by

$$\begin{aligned} u_\tau(t) &= u^n + \frac{t - n\tau}{\tau}(u^{n+1} - u^n), \quad t \in [n\tau, (n+1)\tau), \\ \bar{u}_\tau(t) &= u^{n+1}, \quad t \in [n\tau, (n+1)\tau). \end{aligned}$$

We assume that $u^0 \in L^2(\Omega; \mathcal{B}_\delta) \cap V$. By Theorem 3.1, for each n, u^n belongs to $L^2(\Omega; \mathcal{B}_\delta) \cap V$. Thus, $u_\tau \in C^0(\mathbb{R}_+; L^2(\Omega; \mathcal{B}_\delta) \cap V)$, $\partial_t u_\tau \in L^\infty_{loc}(\mathbb{R}_+; V)$ and $\bar{u}_\tau \in L^\infty_{loc}(\mathbb{R}_+; L^2(\Omega; \mathcal{B}_\delta) \cap V)$. The scheme (3.1) can be rewritten

$$\partial_t u_\tau - \Delta \bar{u}_\tau + g(\bar{u}_\tau) = 0 \quad \text{in } V', \quad \text{for a.e. } t \geq 0,$$

or in the same way,

$$\partial_t u_\tau - \Delta u_\tau + g(u_\tau) = -\Delta(u_\tau - \bar{u}_\tau) + [g(u_\tau) - g(\bar{u}_\tau)], \text{ for a.e. } t \geq 0. \tag{3.13}$$

Let u be solution of (2.2)-(2.4) with $u_0 \in L^2(\Omega; \mathcal{B}_\delta) \cap V$. We define $e_\tau = u_\tau - u$. The following error estimate holds:

Theorem 3.8. *Let $T > 0$ and $R_1 > 0$. There exists a constant $C(T, R_1)$ independent of τ such that $u^0 = u_0$ and $\|u^0\| \leq R_1$ imply*

$$\sup_{t \in [0, N\tau]} |e_\tau(t)|_0 \leq C(T, R_1)\tau^{1/2},$$

where $N = \lfloor T/\tau \rfloor$ (here, $\lfloor \cdot \rfloor$ denotes the floor function).

Proof. By subtracting (2.2) from (3.13), we find

$$\partial_t e_\tau - \Delta e_\tau + g(u_\tau) - g(u) = -\Delta(u_\tau - \bar{u}_\tau) + [g(u_\tau) - g(\bar{u}_\tau)], \text{ for a.e. } t \geq 0.$$

Multiplying by e_τ,and integrating over Ω, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|e_\tau|_0^2 + \|e_\tau\|^2 &\leq |(g(u_\tau) - g(u), e_\tau)_0| + \|u_\tau - \bar{u}_\tau\|\|e_\tau\| + |(g(u_\tau) - g(\bar{u}_\tau), e_\tau)_0| \\ &\leq c_\delta |e_\tau|_0^2 + \|u_\tau - \bar{u}_\tau\|\|e_\tau\| + c_\delta |u_\tau - \bar{u}_\tau|_0 |e_\tau|_0, \end{aligned} \tag{3.14}$$

where c_δ is the constant from (2.13). From the Poincaré inequality (2.1) and Young's inequality, we derive that

$$\frac{d}{dt}|e_\tau|_0^2 \leq (2c_\delta + c_\delta^2 c_0^2)|e_\tau|_0^2 + \|u_\tau - \bar{u}_\tau\|^2, \quad \text{for a.e. } t \geq 0.$$

Let $T > 0$ and define $N = \lfloor T/\tau \rfloor$. Using $e_\tau(0) = 0$, the classic Gronwall lemma yields

$$|e_\tau(t)|_0^2 \leq \exp((2c_\delta + c_0^2 c_\delta^2)T) \int_0^{N\tau} \|u_\tau - \bar{u}_\tau\|^2 ds, \quad \forall t \in [0, N\tau].$$

On $[n\tau, (n+1)\tau]$, we have $\|u_\tau - \bar{u}_\tau\| \leq \|u^{n+1} - u^n\|$, so that

$$\int_0^{N\tau} \|u_\tau - \bar{u}_\tau\|_V^2 ds \leq \tau \sum_{n=0}^{N-1} \|u^{n+1} - u^n\|_0^2.$$

By summing estimate (3.9) from $n = 0$ to $n = N - 1$ (we note that (3.9) is also valid for $n = 0$ since $u^0 \in V$), we find that

$$\sum_{n=0}^{N-1} \|u^{n+1} - u^n\|^2 \leq \|u^0\|^2 + N\tau|\Omega|C_\delta^2.$$

Hence,

$$|e_\tau(t)|_0^2 \leq \exp((2c_\delta + c_0^2 c_\delta^2)T)(R_1^2 + T|\Omega|C_\delta^2)\tau, \quad \forall t \in [0, N\tau].$$

This proves the claim. □

4. Convergence of exponential attractors

4.1. Definitions

We recall some standard definitions (see e.g. [13, 18]). Throughout Section 4.1, $\mathcal{K}$ denotes a closed bounded subset of the Hilbert space H. A continuous-in-time semigroup $\{S(t),\ t \in \mathbb{R}_+\}$ on $\mathcal{K}$ is a family of (nonlinear) operators such that $S(t)$ is a continuous operator from $\mathcal{K}$ into itself, for all $t \geq 0$, with $S(0) = Id$ (identity in $\mathcal{K}$) and

$$S(t+s) = S(t) \circ S(s), \quad \forall s, t \geq 0.$$

A discrete-in-time semigroup $\{S(t),\ t \in \mathbb{N}\}$ on $\mathcal{K}$ is a family of (nonlinear) operators which satisfy these properties with $\mathbb{R}_+$ replaced by $\mathbb{N}$. A discrete-in-time semigroup is usually denoted $\{S^n,\ n \in \mathbb{N}\}$, where $S (= S(1))$ is a continuous (nonlinear) operator from $\mathcal{K}$ into itself. The term "dynamical system" will sometimes be used instead of "semigroup".

Definition 4.1 (Global attractor)**.** Let $\{S(t),\ t \geq 0\}$ be a continuous or discrete semigroup on $\mathcal{K}$. A set $\mathcal{A} \subset \mathcal{K}$ is called the global attractor of the dynamical system if the following three conditions are satisfied:

1. $\mathcal{A}$ is compact in H;
2. $\mathcal{A}$ is invariant, i.e. $S(t)\mathcal{A} = \mathcal{A}$, for all $t \geq 0$;
3. $\mathcal{A}$ attracts $\mathcal{K}$, i.e.

$$\lim_{t \to +\infty} dist_H(S(t)\mathcal{K}, \mathcal{A}) = 0.$$

Here, $dist_H$ denotes the non-symmetric Hausdorff semidistance in H between two subsets, which is defined as

$$dist_H(A, B) = \sup_{a \in A} \inf_{b \in B} |a - b|_H.$$

It is easy to see, thanks to the invariance and the attracting property, that the global attractor, when it exists, is unique [18].

Let $A \subset H$ be a (relatively compact) subset of H. For $\epsilon > 0$, we denote $N_\epsilon(A, H)$ the minimum number of balls of H of radius $\epsilon > 0$ which are necessary to cover A. The *fractal dimension of* A (see e.g. [6, 18]) is the number

$$dim_F(A, H) = \limsup_{\epsilon \to 0} \frac{\log_2(N_\epsilon(A, H))}{\log_2(1/\epsilon)} \in [0, +\infty].$$

Definition 4.2 (Exponential attractor)**.** Let $\{S(t),\ t \geq 0\}$ be a continuous or discrete semigroup on $\mathcal{K}$. A set $\mathcal{M} \subset \mathcal{K}$ is an exponential attractor of the dynamical system if the following three conditions are satisfied:

1. $\mathcal{M}$ is compact in H and has finite fractal dimension;
2. $\mathcal{M}$ is positively invariant, i.e. $S(t)\mathcal{M} \subset \mathcal{M}$, for all $t \geq 0$;
3. $\mathcal{M}$ attracts $\mathcal{K}$ exponentially, i.e.

$$dist_H(S(t)\mathcal{K}, \mathcal{M}) \leq Ce^{-\alpha t}, \quad t \geq 0,$$

for some positive constants C and α.

The exponential attractor, if it exists, contains the global attractor (actually, the existence of an exponential attractor yields the existence of the global attractor, see [2, 5]).

4.2. Convergence of attractors

We may now state our main result. We recall that $\mathcal{K}_\delta = L^2(\Omega; \mathcal{B}_\delta)$ is a closed convex subset of H and that $V = H_0^1(\Omega)^m$ is compactly imbedded into H. We also note that $\mathcal{K}_\delta$ is a bounded subset of H since for each $v \in \mathcal{K}_\delta$, we have

$$|v|_0 \leq \delta |\Omega|^{1/2}.$$

Theorem 2.1 shows that $\{S_0(t)\ ;\ t \in \mathbb{R}_+\}$ is a continuous-in-time semigroup on $\mathcal{K}_\delta$ and Theorem 3.1 shows that $\{S_\tau^n\ ;\ n \in \mathbb{N}\}$ is a discrete-in-time semigroup on $\mathcal{K}_\delta$. We have:

Theorem 4.3. *Let $\tau_0 > 0$ be small enough. For every $\tau \in (0, \tau_0]$, $\{S_\tau^n\ ;\ n \in \mathbb{N}\}$ possesses an exponential attractor $\mathcal{M}_\tau$ on $\mathcal{K}_\delta$ and $\{S_0(t)\ ;\ t \in \mathbb{N}_+\}$ possesses an exponential attractor $\mathcal{M}_0$ on $\mathcal{K}_\delta$ such that:*

1. *The fractal dimension of $\mathcal{M}_\tau$ is bounded, uniformly with respect to $\tau \in [0, \tau_0]$,*

$$\dim_F \mathcal{M}_\tau \leq C_1,$$

where C_1 is independent of τ;

2. *$\mathcal{M}_\tau$ attracts $\mathcal{K}_\delta$ uniformly with respect to $\tau \in (0, \tau]$, i.e.*

$$\forall \tau \in (0, \tau_0],\ dist_H(S_\tau^n \mathcal{K}_\delta, \mathcal{M}_\tau) \leq C_2 e^{-C_3 n\tau},\ n \in \mathbb{N},$$

where the positive constants C_2 and C_3 are independent of τ;

3. *the family $\{\mathcal{M}_\tau\ ; \tau \in [0, \tau_0]\}$ is continuous at 0,*

$$dist_{sym}(\mathcal{M}_\tau, \mathcal{M}_0) \leq C_4 \tau^{C_5},$$

where $C_4 > 0$ and $C_5 \in (0, 1)$ are independent of τ and $dist_{sym}$ is the symmetric Hausdorff distance between subsets of H, defined by

$$dist_{sym}(A, B) = \max(dist_H(A, B), dist_H(B, A)).$$

Proof. We apply Theorem 2 in [14] with the spaces H and V and the set

$$\mathcal{B} = \{v \in \mathcal{K}_\delta \ : \ \|v\| \leq 2\rho_1\},$$

and we choose $\tau_0 > 0$ small enough so that all the estimates from Section 3 are valid. By Proposition 2.3 and Proposition 3.4, $\mathcal{B}$ is an absorbing set in $\mathcal{K}_\alpha$, uniformly with respect to $\tau \in [0, \tau_0]$. The estimates derived for the continuous problem in Section 2 show that assumptions (H1)-(H4) from [14, Theorem 2] are satisfied. The estimates from Section 3 show that assumptions (H5)-(H9) are also satisfied. The conclusion follows (we note that Theorem 2 in [14] is stated for a family of semigroups which act on the whole space H, but with a minor modification of the proof, it can be applied to our situation where the semigroups act on $\mathcal{K}_\delta$). □

By arguing as in the proof of Corollary 1 in [14], we have:

Proposition 4.4. *For each $\tau \in [0, \tau_0]$, the semigroup $\{S_\tau(t), t \geq 0\}$ possesses a global attractor $\mathcal{A}_\tau$ in $\mathcal{K}_\delta$ which is bounded in V, compact and connected in H. In addition, $dist_H(\mathcal{A}_\tau, \mathcal{A}_0) \to 0$ when $\tau \to 0^+$ and the fractal dimension of $\mathcal{A}_\tau$ is bounded by a constant independent of τ.*

Conflict of interest

The author declares no conflict of interest.

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