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Research article

Transitivity and sensitivity for the p-periodic discrete system via Furstenberg families

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Abstract: The consistency and implication relation of chaotic properties of p-periodic discrete system and its induced autonomous discrete system are obtained. The chaotic properties discussed involve several types of transitivity and some stronger forms of sensitivity in the sense of Furstenberg families.

Keywords: p-periodic discrete system; Furstenberg families; transitivity; sensitivity; mixing
Mathematics Subject Classification: 37B45, 37B55, 54H20

1. Introduction

Since Kolyada and Snoha [1] proposed non-autonomous discrete systems, many practical problems can be described by non-autonomous discrete systems, such as biology, computer science and technology, control science and engineering and so on [2–5]. The non-autonomous discrete system is formed by iterating several mappings in a certain order. This is more complicated and difficult compared to the composition of an autonomous system involving only one mapping. Therefore, most scholars study non-autonomous discrete systems by approximating them to autonomous systems. In recent years, there have been relatively few studies on the chaos of non-autonomous systems. For example, Sanchez, Sanchi, and Villanueva [6] studied the interaction of some dynamical properties of a nonautonomous discrete dynamical system and its induced hyperspace nonautonomous discrete dynamical system. Abu-Saris [7] investigated the asymptotic behavior of nonautonomous discrete dynamical systems governed by the system of difference equations (recursive equations). Lan and Peris [8] characterized the set of weak stable points and showed that the set of weak stable points is residual, and investigated the relation between weak stability and shadowing property. Canovas [9] presented some recent results concerning non-autonomous discrete systems and explained some problems in this field. Shi [10] showed some close relationships between chaotic dynamical behaviors of the original system and its induced system. Lan [11] studied transitivity, periodic density and sensitivity in an original nonautonomous system and its connections with the same ones in its fuzzified system. Vasisht and Das [12] introduced various properties of non-autonomous dynamical systems having periodic shadowing property and local weak specification property. Salman, Wu, and Das [13] expressed some sufficient conditions under which topological transitivity and dense periodic points imply sensitivity for nonautonomous systems on Hausdorff uniform spaces.

Denote that $\mathbb{N}$ be the set of natural number and H be a complete metric space with the metric d. $h_n : H \to H(n \in \mathbb{N})$ be a continuous mapping sequence. Denote $h_{1,\infty} = (h_1, h_2, \cdots)$. $(H, h_{1,\infty})$ is called a non-autonomous discrete system. For any $x \in H$, the orbit of point x under $h_{1,\infty}$ is $Orb(x, h_{1,\infty}) = \{x, h_1(x), h_2 \circ h_1(x), \cdots, h_1^n(x), \cdots\}$. Where, $h_1^n = h_n \circ \cdots \circ h_1$, h_1^0 is the identity mapping.

If a non-autonomous discrete system satisfies :

$$h_{n+p}(x) = h_n(x), \quad \forall x \in H, \quad n \geq 0, \quad p \in \mathbb{N}$$

then, $(H, h_{1,\infty})$ is called a p-periodic discrete system. Particularly, if $p = 1$, $(H, h_{1,\infty})$ is an autonomous discrete system. Let $\widehat{h} = h_p \circ \ldots \circ h_1$, then $(H, \widehat{h})$ is said to be an autonomous discrete system induced by the p-periodic discrete system $(H, h_{1,\infty})$.

In 1989 Devaney [14] introduced the concept of Devaneys chaos. A continuous mapping $h : H \to H$ is Devaney chaotic if:

(1) h has sensitive dependence on initial condition;

(2) h is topologically transitive;

(3) h is periodically dense.

Sensitively dependent on initial conditions means that in a system that changes deterministically or with weak randomness over time, two almost identical states become inconsistent after a sufficient period of time. In 1971, the first precise definition for sensitivity was proposed by Ruelle and Takens [15]. Afterwards, some definitions related to sensitivity were successfully introduced. For example, Li-Yorke sensitive, syndetically sensitive, infinite sensitive, collectively sensitive and $\mathcal{F}$-sensitive see [16–20].

The transitivity is the key factor of dynamic system. Its intuitive explanation is that the orbit of a certain point runs through every field of every point in the metric space. In recent years, the topological properties of the dynamic system, such as syndetically transitive, totally transitive, transitive, mixing and weakly mixing (see [14,21–24]), has attracted the attention of many scholars.

In 2012, Huang [25] studied the sensitivity of a special non-autonomous discrete system-periodical discrete system. This paper generalizes the results in [25], and studies the stronger form of sensitivity relationship between the p-periodic discrete system and its induced autonomous discrete system in the sense of Furstenberg families. At the same time, the transitivity and mixing of p-periodic discrete system are discussed.

The structure of this paper is as follows: In Section 2, some concepts related to sensitivity, transitivity and mixing are introduced. In Section 3, the transitivity and mixing of the p-periodic discrete system will be discussed. In Section 4, some necessary and sufficient conditions or sufficient conditions for the chaotic properties of p-periodic discrete system will be given.

2. Preliminaries

A subset $K \subset \mathbb{N}$ is syndetic if there exists a $T \in \mathbb{N}$ such that $[t, t + T] \cap K \neq \emptyset$ for any $t \in \mathbb{N}$. Let H is a complete metric space with the metric d. For a p-periodic discrete system $(H, h_{1,\infty})$, $h_{1,\infty}$ is called transitive if there is an $x \in H$ such that $\overline{Orb(x, h_{1,\infty})} = H$. For any $\lambda > 0$ and any nonempty open subsets $A, B \subset H$, denote

$$N_{h_{1,\infty}}(A, \lambda) = \{n \in \mathbb{N} : \quad there \quad exist \quad x, y \in A \quad with \quad d(h_1^n(x), h_1^n(y)) > \lambda\};$$

$$N_{h_{1,\infty}}(A, B) = \{n \in \mathbb{N} : h_1^n(A) \cap B \neq \emptyset\}.$$

Through the above definition, we can get another equivalent definition of transitivity, i.e., for any nonempty open subsets $A, B \subset H$, one has $N_{h_{1,\infty}}(A, B) \neq \emptyset$; $h_{1,\infty}$ is called totally transitive if for any integer $n \geq 1$ such that h_1^n is transitive; $h_{1,\infty}$ is called weakly mixing if $h_{1,\infty} \times h_{1,\infty}$ is transitive on $H \times H$, i.e., for any nonempty open sets $A_i, B_i \subset H(i = 1, 2)$ there exists an integer $n \geq 1$ such that $h_1^n(A_1) \cap B_1 \neq \emptyset$ and $h_1^n(A_2) \cap B_2 \neq \emptyset$; $h_{1,\infty}$ is called mixing if for any nonempty open subsets $A, B \subset H$, there exists a $N > 0$ such that $h_1^n(A) \cap B \neq \emptyset$ for any $n \geq N$; $h_{1,\infty}$ is called syndetically transitive if for any nonempty open sets $A, B \subset H$, $N_{h_{1,\infty}}(A, B)$ is syndetic.

Let $\mathbb{Z}^+$ be the set of positive integers and $\mathcal{P}$ be the collection of all subsets of $\mathbb{Z}^+$. A collection $\mathcal{R} \subset \mathcal{P}$ is a Furstenberg family if and only if it is hereditary upwards, i.e., $R_1 \subset R_2$ and $R_1 \in \mathcal{R}$ imply $R_2 \in \mathcal{R}$; A Furstenberg family $\mathcal{R}$ is p-invariant if $pR \in \mathcal{R}$ for any $R \in \mathcal{R}$; A Furstenberg family $\mathcal{R}$ is called positive shift-invariant if for any $R \in \mathcal{R}$ and any integer $r \geq 0$ such that $R + r \in \mathcal{R}$; A Furstenberg family $\mathcal{R}$ is called negative shift-invariant if for any $R \in \mathcal{R}$ and any integer $r \geq 0$ such that $R - r \in \mathcal{R}$; Where, $R + r = \{b + r : b \in R\}$ and $R - r = \{b - r \geq 1 : b \in R\}$. A Furstenberg family $\mathcal{R}$ is called shift-invariant if it is both negative shift-invariant and positive shift-invariant.

Definition 2.1. [26] Let $\mathcal{F}$ be a Furstenberg family. The system $(H, h_{1,\infty})$ (or the mapping sequence $h_{1,\infty}$) is $\mathcal{F}$-sensitive with the sensitivity constant $\lambda > 0$ if for any nonempty open set $V \in H$ with $N_{h_{1,\infty}}(V, \lambda) \in \mathcal{F}$.

Definition 2.2. [26] Let $\mathcal{F}_1, \mathcal{F}_2$ be two Furstenberg family. The system $(H, h_{1,\infty})$ (or the mappings sequence $h_{1,\infty}$) is $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive with the sensitivity constant $\lambda > 0$ if for any $x \in H$ and any $\varepsilon > 0$, there is $y \in H$ with $d(x, y) < \varepsilon$ such that

$$\{n \in \mathbb{N} : d(h_1^n(x), h_1^n(y)) > \delta\} \in \mathcal{F}_2 \text{ and } \{n \in \mathbb{N} : d(h_1^n(x), h_1^n(y)) < \delta\} \in \mathcal{F}_1$$

for some $\delta > 0$.

Definition 2.3. [26] Let $\mathcal{F}$ be a Furstenberg family. The system $(H, h_{1,\infty})$ (or the mapping sequence $h_{1,\infty}$) is $\mathcal{F}$-collectively sensitivity with the collective sensitivity constant $\lambda > 0$. If for any $\varepsilon > 0$ and any finitely many distinct points $x_1, x_2, \cdots, x_s \in H$ and there exist s distinct points $y_1, y_2, \cdots, y_s$ in H such that the following two conditions are satisfied.

(1) $d(x_i, y_j) < \varepsilon$ for all $1 \leq i, j \leq s$;

(2) there exist i_0 and j_0 satisfying $1 \leq i_0, j_0 \leq s$ such that

$$\{n \in \mathbb{N} : d(h_1^n(x_i), h_1^n(y_{j_0})) > \lambda (1 \leq i \leq s) \quad or \quad d(h_1^n(x_{i_0}), h_1^n(y_j)) > \lambda (1 \leq j \leq s)\} \in \mathcal{F}.$$

Definition 2.4. [27] The system $(H, h_{1,\infty})$ (or the mapping sequence $h_{1,\infty}$) is said to be collectively sensitivity with the collective sensitivity constant $\lambda > 0$. If for any $\varepsilon > 0$ and any finitely many distinct points $x_1, x_2, \cdots, x_s \in H$ and there exist s distinct points $y_1, y_2, \cdots, y_s$ in H such that the following two conditions are satisfied.

(1) $d(x_i, y_j) < \varepsilon$ for all $1 \leq i, j \leq s$;

(2) there exist $1 \leq i_0, j_0 \leq s$ and $n \in \mathbb{N}$ such that

$$d(h_1^n(x_i), h_1^n(y_{j_0})) > \lambda(1 \leq i \leq s) \quad or \quad d(h_1^n(x_{i_0}), h_1^n(y_j)) > \lambda(1 \leq j \leq s).$$

Definition 2.5. The mapping sequence $h_n(n \geq 1)$ is said to be uniformly continuous if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for any $x, y \in H$ with $d(x, y) < \delta(\varepsilon)$, $d(h_1^n(x), h_1^n(y)) < \varepsilon$ for any $n \geq 1$.

Definition 2.6. [27] The mapping sequence $h_n(n \geq 1)$ is said to be collectively uniformly continuous, if for any $\varepsilon > 0$, there exist $\delta > 0$ such that for any finitely distinct points $x_1, x_2, \cdots, x_s \in H$ and $y_1, y_2, \cdots, y_s \in H$ with $d(x_i, y_j) < \delta(\forall 1 \leq i, j \leq s)$, there are $1 \leq i_0, j_0 \leq s$ such that $d(h_n(x_{i_0}), h_n(y_j)) < \varepsilon(1 \leq j \leq s)$ or $d(h_n(x_i), h_n(y_{j_0})) < \varepsilon(1 \leq i \leq s)$ for any $n \geq 1$.

Lemma 2.7. [27] If $h_n(n \geq 1)$ is uniform continuous, then it is collectively uniform continuous.

3. Transitivity and mixing of the p-periodic discrete system $(H, h_{1,\infty})$

In this section, the syndetically transitivity, totally transitivity, transitivity, mixing and weakly mixing of the p-period discrete system will be discussed.

Theorem 3.1. Let mappings $h_n(n \geq 1)$ be continuous bijections on H. The autonomous discrete system $(H, \widehat{h})$ induced by the p-periodic discrete system $(H, h_{1,\infty})$ is topologically transitive (resp. syndetically transitive) if and only if the p-periodic discrete system $(H, h_{1,\infty})$ is topologically transitive (resp. syndetically transitive).

Proof. (i) Since the autonomous discrete system $(H, \widehat{h})$ is topologically transitive, then for any nonempty open subsets $A, B \subset H$, $N_{\widehat{h}}(A, B) \neq \emptyset$. That is

$$N_{\widehat{h}}(A, B) = \{n \in \mathbb{N} : \widehat{h}^n(A) \cap B \neq \emptyset\} = \{n \in \mathbb{N} : h_1^{pn}(A) \cap B \neq \emptyset\} \neq \emptyset.$$

So, $pN_{\widehat{h}}(A, B) \subset N_{h_{1,\infty}}(A, B)$, which implies that $N_{h_{1,\infty}}(A, B) \neq \emptyset$. Thus, the p-periodic discrete system $(H, h_{1,\infty})$ is topologically transitive.

Suppose that the p-periodic discrete system $(H, h_{1,\infty})$ is topologically transitive, then for any nonempty open subsets $A, B \subset H$ there exists a $n > 0$ such that $h_1^n(A) \cap B \neq \emptyset$.

Take $m \geq 0$ such that $n = pm + k, 0 \leq k \leq p - 1$. Then, one has that $\emptyset \neq h_1^n(A) \cap B = h_1^{pm+k}(A) \cap B$. Obviously we can get $\emptyset \neq (h_{pm+1}^{pm+k})^{-1}(h_{pm+1}^{pm+k} \circ h_1^{pm}(A) \cap B)$. And since $h_n(n \geq 1)$ are continuous bijections, then equation

$$(h_{pm+1}^{pm+k})^{-1}(h_{pm+1}^{pm+k} \circ h_1^{pm}(A) \cap B) = (h_{pm+1}^{pm+k})^{-1}(h_{pm+1}^{pm+k} \circ h_1^{pm}(A)) \cap (h_{pm+1}^{pm+k})^{-1}(B) \neq \emptyset$$

holds. Set $(h_{pm+1}^{pm+k})^{-1}(B) = B'$ which implies that $\widehat{h}^m(A) \cap B' \neq \emptyset$. Thus, the autonomous discrete system $(H, \widehat{h})$ is topologically transitive.

(ii) Since the autonomous discrete system $(H, \widehat{h})$ is syndetically transitive, then for any nonempty subsets $A, B \subset H$, $N_{\widehat{h}}(A, B)$ is syndetic. And since

$$N_{\widehat{h}}(A, B) = \{n \in \mathbb{N} : \widehat{h}^n(A) \cap B \neq \emptyset\} = \{n \in \mathbb{N} : h_1^{pn}(A) \cap B \neq \emptyset\}.$$

So, $pN_{\widehat{h}}(A,B) \subset N_{h_{1,\infty}}(A,B)$. This implies the p-periodic discrete system $(H, h_{1,\infty})$ is syndetically transitive.

Suppose that the p-periodic discrete system $(H, h_{1,\infty})$ is syndetically transitive, then for any nonempty subsets A, B of H, $N_{h_{1,\infty}}(A,B) = \{n \in \mathbb{N} : h_1^n(A) \cap B \neq \varnothing\}$ is syndetic. For any $n \in N_{h_{1,\infty}}(A,B)$, we pick $m \geq 0$ such that $n = pm + k, 1 \leq k \leq p-1$. Similar to the proof of sufficiency in Theorem 3.1 (i), we can get $\widehat{h}^m(A) \cap B' \neq \varnothing$, where $B' = (h_{pm+1}^{pm+k})^{-1}(B)$. And hence by the Theorem 2.4.4 (3) in [25], it follows that $\{m_i\}_{i=1}^{\infty}$ is an infinite set. And according to the previous derivation, we can get $\{m_i\}_{i=1}^{\infty} \subset N_{\widehat{h}}(A, B')$. So, $N_{\widehat{h}}(A, B')$ is syndetic. This implies the autonomous discrete system $(H, \widehat{h})$ is syndetically transitive.

This complete the proof.

Theorem 3.2. Let mappings $h_n (n \geq 1)$ be the continuous self-mapping on H.

(i) The autonomous discrete system $(H, \widehat{h})$ induced by the p-periodic discrete system $(H, h_{1,\infty})$ is topologically weakly mixing, then the p-periodic discrete system $(H, h_{1,\infty})$ is topologically weakly mixing.

(ii) The autonomous discrete system $(H, \widehat{h})$ induced by the p-periodic discrete system $(H, h_{1,\infty})$ is topologically mixing, then the p-periodic discrete system $(H, h_{1,\infty})$ is topologically weakly mixing.

(iii) The autonomous discrete system $(H, \widehat{h})$ induced by the p-periodic discrete system $(H, h_{1,\infty})$ is totally transitive, then the p-periodic discrete system $(H, h_{1,\infty})$ is syndetically transitive.

Proof. (i) Suppose that the autonomous discrete system $(H, \widehat{h})$ is topologically weakly mixing, then for any nonempty subsets $A_i, B_i \subset H (i = 1, 2)$, there exists an integer $n \geq 1$ such that $\widehat{h}^n(A_1) \cap B_1 \neq \varnothing$ and $\widehat{h}^n(A_2) \cap B_2 \neq \varnothing$. As the autonomous discrete system $(H, \widehat{h})$ is induced by the p-periodic discrete system $(H, h_{1,\infty})$, then

$$\widehat{h}^n(A_1) \cap B_1 = h_1^{pn}(A_1) \cap B_1 \neq \varnothing \text{ and } \widehat{h}^n(A_2) \cap B_2 = h_1^{pn}(A_2) \cap B_2 \neq \varnothing.$$

Let $m = pn$, then one has

$$h_1^m(A_1) \cap B_1 \neq \varnothing \quad and \quad h_1^m(A_2) \cap B_2 \neq \varnothing.$$

So, the p-periodic discrete system $(H, h_{1,\infty})$ is topologically weakly mixing.

(ii) Suppose that the autonomous discrete system $(H, \widehat{h})$ is topologically mixing, then for any nonempty subsets $U_i, V_i \subset H (i = 1, 2)$, there exists a $N_1 > 0$ and $N_2 > 0$ such that $\widehat{h}^{n_1}(U_1) \cap V_1 \neq \varnothing$ and $\widehat{h}^{n_2}(U_2) \cap V_2 \neq \varnothing$ for any $n_1 \geq N_1$ and $n_2 \geq N_2$. Choose $N = \max N_1, N_2$, then for any $n \geq N$, one has that $\widehat{h}^n(U_1) \cap V_1 \neq \varnothing$ and $\widehat{h}^n(U_2) \cap V_2 \neq \varnothing$. As the autonomous discrete system $(H, \widehat{h})$ is induced by the p-periodic discrete system $(H, h_{1,\infty})$, then $h_1^{np}(U_1) \cap V_1 \neq \varnothing$ and $h_1^{np}(U_2) \cap V_2 \neq \varnothing$. Let $m = np$. It is obviously the p-periodic discrete system $(H, h_{1,\infty})$ is topologically weakly mixing.

(iii) Since the autonomous discrete system $(H, \widehat{h})$ is totally transitive, then for any nonempty open subsets $A, B \subset H$ and for every integer $n \geq 1$ there exists a $s \in \mathbb{N}$ such that $(\widehat{h}^n)^s(A) \cap B \neq \varnothing$. And since $(\widehat{h}^n)^s(A) \cap B = h_1^{pns}(A) \cap B \neq \varnothing$. Let $m = pns$, then $h_1^m(A) \cap B \neq \varnothing$. It is easy to find an integer $M > 0$ with $m \in [n, n+M] \cap N_{h_1^m}(A, B)$. Thus, the p-periodic discrete system $(H, h_{1,\infty})$ is syndetically transitive.

This complete the proof.

Theorem 3.3. Let mappings $h_n (n \geq 1)$ be the continuous self-mapping on H.

(i) If the autonomous discrete system $(H, \widehat{h})$ induced by the p-periodic discrete system $(H, h_{1,\infty})$ is topologically mixing, then it is topologically weakly mixing.

(ii) If the p-periodic discrete system $(H, h_{1,\infty})$ is topologically mixing, then it is topologically weakly mixing.

Proof. (i) Let the autonomous discrete system $(H, \widehat{h})$ be topologically mixing. Given any nonempty subsets A_1, A_2, B_1, B_2 of H. Then exist $N_1, N_2 > 0$ such that for any $n_1 > N_1$, $\widehat{h}^{n_1}(A_1) \cap B_1 \neq \emptyset$ and for any $n_2 > N_2$, $\widehat{h}^{n_2}(A_2) \cap B_2 \neq \emptyset$. Set $N = max\{N_1, N_2\}$, then for any $n > N$, it follows that $\widehat{h}^n(A_1) \cap B_1 \neq \emptyset$ and $\widehat{h}^n(A_2) \cap B_2 \neq \emptyset$. Thus, the autonomous discrete system $(H, \widehat{h})$ is topologically weakly mixing.

(ii) This proof is similar to (i), so omitted here.

This complete the proof.

Obviously, through the above theorem, we can get the following Corollary 3.4.

Corollary 3.4. Let mappings $h_n (n \geq 1)$ be the continuous bijections on H.

(i) If the autonomous discrete system $(H, \widehat{h})$ is mixing, then the p-periodic discrete system $(H, h_{1,\infty})$ is transitive.

(ii) If the autonomous discrete system $(H, \widehat{h})$ is totally transitive, then it is syndetically transitive.

(iii) If the autonomous discrete system $(H, \widehat{h})$ is weakly mixing, then the p-periodic discrete system $(H, h_{1,\infty})$ is weakly mixing.

(iv) If the autonomous discrete system $(H, \widehat{h})$ is mixing, then the p-periodic discrete system $(H, h_{1,\infty})$ is weakly mixing.

From the above discussion and the relationship between concepts, one can get the relationship of syndetically transitive, totally transitive, transitive, mixing and weakly mixing (see Figure 1).

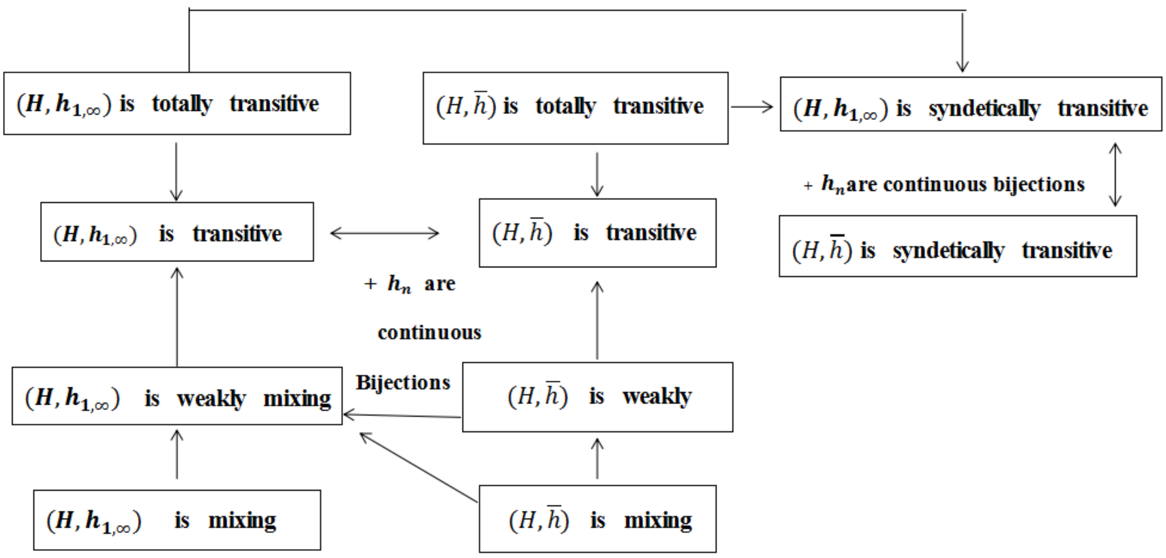

Figure 1. Relationship of conceptions.

Remark 3.5. According to the above figure, we can see that there are still some relationships that cannot be deduced back. For example,

(i) If the p-periodic discrete system $(H, h_{1,\infty})$ is topologically weakly mixing, then is the autonomous discrete system $(H, \widehat{h})$ topologically weakly mixing?

(ii) If the p-periodic discrete system $(H, h_{1,\infty})$ is topologically mixing, then is the autonomous discrete system $(H, \widehat{h})$ topologically mixing?

(iii) Does the syndetically transitivity of the p-periodic discrete system $(H, h_{1,\infty})$ imply totally transitivity?

(iv) If the p-periodic discrete system $(H, h_{1,\infty})$ is syndetically transitive, then is the autonomous discrete system $(H, \widehat{h})$ totally transitive?

These questions can continue to be studied in depth.

4. Sensitivity of the p-periodic discrete system $(H, h_{1,\infty})$

In this section, some necessary and sufficient conditions, or sufficient conditions for p-periodic discrete system to be $\mathcal{F}$-sensitive, $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive, $\mathcal{F}$-collectively sensitive, $\mathcal{F}$-collectively infinity sensitive, sensitive or collectively sensitive will be given.

Theorem 4.1. Let mappings $f_n (n \geq 1)$ be continuous mapping sequence on H and the Furstenberg family $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ be p-invariant. The autonomous discrete system $(H, \widehat{h})$ induced by the p-periodic discrete system $(H, h_{1,\infty})$ is $\mathcal{F}$-sensitive (resp., $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive, $\mathcal{F}$-collectively sensitive), then the p-periodic discrete system $(H, h_{1,\infty})$ is $\mathcal{F}$-sensitive (resp., $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive, $\mathcal{F}$-collectively sensitive, $\mathcal{F}$-collectively infinity sensitive).

Proof. (i) **$\mathcal{F}$-sensitive.** Let the autonomous discrete system $(H, \widehat{h})$ be $\mathcal{F}$-sensitive with the sensitive constant $\lambda > 0$. Then for any open set $A \subset H$, $N_{\widehat{h}}(A, \lambda) \in \mathcal{F}$. By the p-invariantity of $\mathcal{F}$, $pN_{\widehat{h}}(A, \lambda) \in \mathcal{F}$ is established for any positive integer $p \in \mathbb{N}$. And since

$$\begin{aligned} N_{\widehat{h}}(A, \lambda) &= \{n \in \mathbb{N} : \textit{there are } x, y \in A \textit{ such that } d(\widehat{h}^n(x), \widehat{h}^n(y)) > \lambda\} \\ &= \{n \in \mathbb{N} : \textit{there are } x, y \in A \textit{ such that } d(h_1^{pn}(x), h_1^{pn}(y)) > \lambda\}, \end{aligned}$$

then $pN_{\widehat{h}}(A, \lambda) \subset N_{h_{1,\infty}}(A, \lambda)$. And hence according to the upward heredity of $\mathcal{F}$ and the p-periodic discrete system $(H, h_{1,\infty})$ is $\mathcal{F}$-sensitive.

(ii) $(\mathcal{F}_1, \mathcal{F}_2)$**-sensitive.** Assume that the autonomous discrete system $(H, \widehat{h})$ is $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive with the sensitive constant $\lambda > 0$. For any $x \in H$ and $\varepsilon > 0$, there is a $y \in H$ with $d(x, y) < \varepsilon$ such that

$$\{n \in \mathbb{N} : d(\widehat{h}^n(x), \widehat{h}^n(y)) > \delta\} \in \mathcal{F}_2 \quad \textit{and} \quad \{n \in \mathbb{N} : d(\widehat{h}^n(x), \widehat{h}^n(y)) < \delta\} \in \mathcal{F}_1.$$

By the p-invariantity of $(\mathcal{F}_1, \mathcal{F}_2)$, $p\{n \in \mathbb{N} : d(\widehat{h}^n(x), \widehat{h}^n(y)) > \delta\} \in \mathcal{F}_2$ and $p\{n \in \mathbb{N} : d(\widehat{h}^n(x), \widehat{h}^n(y)) < \delta\} \in \mathcal{F}_1$ are established for any positive integer $p \in \mathbb{N}$. And since

$$\{n \in \mathbb{N} : d(\widehat{h}^n(x), \widehat{h}^n(y)) > \delta\} = \{n \in \mathbb{N} : d(h_1^{pn}(x), h_1^{pn}(y)) > \delta\}$$

and

$$\{n \in \mathbb{N} : d(\widehat{h}^n(x), \widehat{h}^n(y)) < \delta\} = \{n \in \mathbb{N} : d(h_1^{pn}(x), h_1^{pn}(y)) < \delta\},$$

then

$$p\{n \in \mathbb{N} : d(\widehat{h}^n(x), \widehat{h}^n(y)) > \delta\} \subset \{m \in \mathbb{N} : d(h_1^m(x), h_1^m(y)) > \delta\}$$

and

$$p\{n \in \mathbb{N} : d(\widehat{h}^n(x), \widehat{h}^n(y)) < \delta\} \subset \{m \in \mathbb{N} : d(h_1^m(x), h_1^m(y)) < \delta\}.$$

And hence according to the upward heredity of Furstenberg families $\mathcal{F}_1, \mathcal{F}_2$ and the p-periodic discrete system $(H, h_{1,\infty})$ is $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive.

(iii) $\mathcal{F}$**-collectively sensitive**. Let the autonomous discrete system $(H, \widehat{h})$ be $\mathcal{F}$-collectively sensitive with the sensitive constant $\lambda > 0$. For any $\varepsilon > 0$ and any finitely many distinct points $x_1, x_2, \cdots, x_s \in H$, there exist s distinct points $y_1, y_2, \cdots, y_s$ in H such that the following two conditions are satisfied.

(1) $d(x_i, y_j) < \varepsilon$ for all $1 \leq i, j \leq s$;

(2) there exist i_0 and j_0 satisfying $1 \leq i_0, j_0 \leq s$ such that

$$\{n \in \mathbb{N} : d(\widehat{h}^n(x_i), \widehat{h}^n(y_{j_0})) > \lambda(1 \leq i \leq s) \quad or \quad d(\widehat{h}^n(x_{i_0}), \widehat{h}^n(y_j)) > \lambda(1 \leq j \leq s)\} \in \mathcal{F}.$$

Then,

$$p\{n \in \mathbb{N} : d(\widehat{h}^n(x_i), \widehat{h}^n(y_{j_0})) > \lambda(1 \leq i \leq s) \quad or \quad d(\widehat{h}^n(x_{i_0}), \widehat{h}^n(y_j)) > \lambda(1 \leq j \leq s)\} \in \mathcal{F}$$

is established for any positive integer $p \in \mathbb{N}$. And since

$$\{n \in \mathbb{N} : d(\widehat{h}^n(x_i), \widehat{h}^n(y_{j_0})) > \lambda(1 \leq i \leq s)$$

or

$$d(\widehat{h}^n(x_{i_0}), \widehat{h}^n(y_j)) > \lambda(1 \leq j \leq s)\} = \{n \in \mathbb{N} : d(h_1^{pn}(x_i), h_1^{pn}(y_{j_0})) > \lambda(1 \leq i \leq s)$$

or

$$d(h_1^{pn}(x_{i_0}), h_1^{pn}(y_j)) > \lambda(1 \leq j \leq s)\},$$

then

$$p\{n \in \mathbb{N} : d(\widehat{h}^n(x_i), \widehat{h}^n(y_{j_0})) > \lambda(1 \leq i \leq s)$$

or

$$d(\widehat{h}^n(x_{i_0}), \widehat{h}^n(y_j)) > \lambda(1 \leq j \leq s)\} \subset \{m \in \mathbb{N} : d(h_1^m(x_i), h_1^m(y_{j_0})) > \lambda(1 \leq i \leq s)$$

or

$$d(h_1^m(x_{i_0}), h_1^m(y_j)) > \lambda(1 \leq j \leq s)\}.$$

According to the upward heredity of Furstenberg family $\mathcal{F}$ and the p-periodic discrete system $(H, h_{1,\infty})$ is $\mathcal{F}$-collectively sensitive.

This complete the proof.

Theorem 4.2. Let mappings $h_n (n \geq 1)$ be uniformly continuous on H. The autonomous discrete system $(H, \widehat{h})$ induced by the p-periodic discrete system $(H, h_{1,\infty})$ is sensitive (resp., collectively sensitive) if and only if the p-periodic discrete system $(H, h_{1,\infty})$ is sensitive (resp., collectively sensitive).

Proof. (i) **sensitive.** Assume that autonomous discrete system $(H, \widehat{h})$ be sensitive with the sensitive constant $\lambda > 0$. Then for any open set $A \subset H$, $N_{\widehat{h}}(A, \lambda) \neq \emptyset$. And since

$$\begin{aligned} N_{\widehat{h}}(A, \lambda) &= \{n \in \mathbb{N} : there \quad are \quad x, y \in A \quad such \quad that \quad d(\widehat{h}^n(x), \widehat{h}^n(y)) > \lambda\} \\ &= \{n \in \mathbb{N} : there \quad are \quad x, y \in A \quad such \quad that \quad d(h_1^{pn}(x), h_1^{pn}(y)) > \lambda\}, \end{aligned}$$

then $pN_{\widehat{h}}(A, \lambda) \subset N_{h_{1,\infty}}(A, \lambda)$. This implies $N_{h_{1,\infty}}(A, \lambda) \neq \emptyset$. Thus, the p-period discrete system $(H, h_{1,\infty})$ is sensitive.

Let the p-period discrete system $(H, h_{1,\infty})$ be sensitive with the sensitive constant $\lambda > 0$ and let B be any nonempty open set of H. Then $N_{h_{1,\infty}}(B, \lambda) \neq \emptyset$. Since h_n is uniformly continuous on H. Then there exists $\lambda_1 > 0$ such that for any $x, y \in B$ with $d(x, y) < \lambda_1$,

$$d(h_1^n(x), h_1^n(y)) < \lambda_1 \qquad (4-1)$$

for any $n \geq 1$. Pick $n_0 \in N_{h_{1,\infty}}(B, \lambda)$, one has that $d(h_1^{n_0}(x), h_1^{n_0}(y)) > \lambda$. Then there must be $m \geq 0$ such that $n_0 = mp + k, 0 \leq k \leq p-1$. So,

$$d(h_1^{n_0}(x), h_1^{n_0}(y)) = d(h_1^k(\widehat{h}^m(x)), h_1^k(\widehat{h}^m(y))) > \lambda.$$

Thus, $d(\widehat{h}^m(x), \widehat{h}^m(y)) > \lambda_1$. Otherwise, the result will conflict with (4-1). So, $n_0 - k \in N_{\widehat{h}}(B, \lambda_1) \neq \emptyset$. This implies that the autonomous discrete system $(H, \widehat{h})$ is $\mathcal{F}$-sensitive.

(ii) **collectively sensitive**. Assume that the autonomous discrete system $(H, \widehat{h})$ is collectively sensitive with the sensitive constant $\lambda > 0$. For any $\varepsilon > 0$ and any finitely many distinct points $x_1, x_2, \cdots, x_s \in H$,there exist s distinct points $y_1, y_2, \cdots, y_s$ in H such that the following two conditions are satisfied.

(1) $d(x_i, y_j) < \varepsilon$ for all $1 \leq i, j \leq s$;

(2) there exist $1 \leq i_0, j_0 \leq s$ and $n \in \mathbb{N}$ such that

$$d(\widehat{h}^n(x_i), \widehat{h}^n(y_{j_0})) > \lambda (1 \leq i \leq s) \quad or \quad d(\widehat{h}^n(x_{i_0}), \widehat{h}^n(y_j)) > \lambda (1 \leq j \leq s).$$

And since

$$d(\widehat{h}^n(x_i), \widehat{h}^n(y_{j_0})) = d(h_1^{pn}(x_i), h_1^{pn}(y_{j_0})) > \lambda (1 \leq i \leq s)$$

and

$$d(\widehat{h}^n(x_{i_0}), \widehat{h}^n(y_j)) = d(h_1^{pn}(x_{i_0}), h_1^{pn}(y_j)) > \lambda (1 \leq j \leq s).$$

Set $m = pn$, then

$$d(h_1^m(x_i), h_1^m(y_{j_0})) > \lambda (1 \leq i \leq s) \quad or \quad d(h_1^m(x_{i_0}), h_1^m(y_j)) > \lambda (1 \leq j \leq s).$$

Thus, the p-periodic discrete system $(H, h_{1,\infty})$ is collectively sensitive.

Let the p-periodic discrete system $(H, h_{1,\infty})$ be collectively sensitive with the sensitive constant $\lambda > 0$. By Lemma 2.8, the uniformly continuity of h_n can imply collectively uniformly continuity of h_n. That is there exists a $\lambda_1 > 0$ and for any finitely distinct points $x_1, x_2, \cdots, x_s \in H, y_1, y_2, \cdots, y_s \in H$ with $d(x_i, y_j) < \lambda_1 (\forall 1 \leq i, j \leq s)$ such that there are $1 \leq i_0, j_0 \leq s$ with

$$d(h_1^n(x_{i_0}), h_1^n(y_j)) < \lambda \quad or \quad d(h_1^n(x_i), h_1^n(y_{j_0})) < \lambda \qquad (4-2)$$

for any $n \geq 1$. According to the collectively sensitive of $(H, h_{1,\infty})$, there exists a $n_0 \in \mathbb{N}$ such that

$$d(h_1^{n_0}(x_i), h_1^{n_0}(y_{j_0})) > \lambda (1 \leq i \leq s) \quad or \quad d(h_1^{n_0}(x_{i_0}), h_1^{n_0}(y_j)) > \lambda (1 \leq j \leq s).$$

Without loss of generality, we only consider the following case, the other case is similar.

There exists an $1 \leq i_0 \leq s$ such that

$$d(h_1^{n_0}(x_{i_0}), h_1^{n_0}(y_j)) > \lambda (\forall 1 \leq j \leq s).$$

Then there must be $m \geq 0$ such that $n_0 = mp + k, 0 \leq k \leq p-1$. So, one has that

$$d(h_1^{n_0}(x_{i_0}), h_1^{n_0}(y_j)) = d(h_1^k(\widehat{h}^m(x_{i_0})), h_1^k(\widehat{h}^m(y_j))) > \lambda(\forall 1 \leq j \leq s).$$

Thus, $d(\widehat{h}^m(x_{i_0}), \widehat{h}^m(y_j)) > \lambda_1(\forall 1 \leq j \leq s)$. Otherwise, the result will conflict with (4-2). This implies that

$$d(\widehat{h}^m(x_i), \widehat{h}^m(y_{j_0})) > \lambda_1(1 \leq i \leq s) \quad or \quad d(\widehat{h}^m(x_{i_0}), \widehat{h}^m(y_j)) > \lambda_1(1 \leq j \leq s)\}.$$

Thus, the autonomous discrete system $(H, \widehat{h})$ is $\mathcal{F}$-collectively sensitive.

In fact, by the proof of Theorem 4.2, we can see that in general, the sensitivity of an autonomous system $(H, \widehat{h})$ implies the sensitivity of a periodic discrete system $(H, h_{1,\infty})$. However, the converse is only valid under the conditions that "$f_n(n \geq 1)$ be uniformly continuous". A counterexample is given below for illustration.

Example 4.3. Let $I = [0, 1]$ be the closed unit interval and f, g be defined by

$$f(x) = \begin{cases} 1 - 2x & for \quad x \in [0, \frac{1}{2}) \\ x & for \quad x \in [\frac{1}{2}, 1] \end{cases}$$

$$g(x) = \begin{cases} 0 & when \quad x \quad is \quad an \quad irrational \quad number \quad in \quad I \\ 1 & when \quad x \quad is \quad a \quad rational \quad number \quad in \quad I \end{cases}$$

Where, g is the Dirichlet function. In the following, the relationship of the sensitivity between the three-period system $f, g, f, f, g, f, f, g, f, \cdots$ and the autonomous system induced by it will be considered.

Let $\lambda = \frac{1}{4}$, for any $\varepsilon > 0$ and $x \in [0, 1]$ there must be $y \in B(x, \varepsilon)$ (When x is a rational number, take y as an irrational number; when x is an irrational number, take y as a rational number) such that

$$d(g \circ f(x), g \circ f(y)) = 1 > \lambda.$$

Thus, the three-period system $f, g, f, f, g, f, f, g, f, \cdots$ is sensitive. Next, we consider the sensitivity of the autonomous system induced by the three-periodic system. Let $h = f \circ g \circ f$, then for any $x \in [0, 1]$, $h(x) = f \circ g \circ f(x) \equiv 1$. So, the autonomous system is not sensitive.

5. Conclusions

The induced autonomous discrete system $(H, \widehat{h})$ is $\mathcal{F}$-sensitive (resp., $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive, $\mathcal{F}$-collectively sensitive) implies that the p-periodic discrete system $(H, h_{1,\infty})$ is $\mathcal{F}$-sensitive (resp., $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive, $\mathcal{F}$-collectively sensitive). Under the condition that the mappings $f_n(n \geq 1)$ are uniformly continuous, the p-periodic discrete system $(H, h_{1,\infty})$ is sensitive (resp., collectively sensitive) if and only if the induced autonomous discrete system $(H, \widehat{h})$ is sensitive (resp., collectively sensitive). What's more, some sufficient and necessary conditions or sufficient conditions for the p-period discrete system $(H, h_{1,\infty})$ to be syndetically transitive, totally transitive, transitive, mixing and weakly mixing are obtained.

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Conflict of interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

1. S. Kolyada, L. Snoha, Topological entropy of nonautonomous dynamical systems, *Random Comput. Dyn.,* **4** (1996), 205–233.
2. W. Qian, F. Meng, Periodic solutions of a class of nonautonomous discrete time semi-ratio-dependent predator-prey systems, *Discrete Cont. Dyn. Syst.,* **4** (2004), 563–574. doi: 10.3934/dcdsb.2004.4.563.
3. C. Guo, Y. Chen, J. Shu, Dynamical behaviors of non-autonomous fractional FitzHugh-Nagumo system driven by additive noise in unbounded domains, *Front. Math. China,* **16** (2021), 59–93. doi: 10.1007/s11464-021-0896-7.
4. Z. Wang, J. Zhang, M. Chen, A unified approach to periodic solutions for a class of non-autonomous second order Hamiltonian systems, *Nonlinear Anal-Real.,* **58** (2021), 103218. doi: 10.1016/j.nonrwa.2020.103218.
5. J. M. Cushing, S. M. Henson, The effcct of periodic habit fluctuations on a nonlincar insect population model, *J. Math. Biol.,* **36** (1997), 201–226. doi: 10.1007/s002850050098 .
6. I. Sanchez, M. Sanchi, H. Villanueva, Chaos in hyperspaces of nonautonomous discrete systems, *Chaos Soliton. Fract.,* **94** (2017), 68–74. doi: 10.1016/j.chaos.2016.11.009.
7. R. M. Abu-Saris, On nonautonomous discrete dynamical systems driven by means, *Adv. Differ. Equ-Ny.,* **13** (2006), 1–7. doi: 10.1155/ADE/2006/43470.
8. Y. Lan, A. Peris, Weak stability of non-autonomous discrete dynamical systems, *Topol. Appl.,* **250** (2018), 53–60. doi: 10.1016/j.topol.2018.10.006.
9. J. S. Canovas, Recent results on non-autonomous discrete systems, *SeMA J.,* **51** (2010), 33–40. doi: 10.1007/BF03322551.
10. Y. Shi, Chaos in nonautonomous discrete dynamical systems approached by their induced systems, *Int. J. Bifurcat. Chaos,* **22** (2012), 1250284. doi: 10.1142/S0218127412502847.
11. Y. Lan, Chaos in nonautonomous discrete fuzzy dynamical systems, *J. Nonlinear Sci. Appl.,* **9** (2016), 404–412. doi: 10.22436/jnsa.009.02.06.
12. R. Vasisht, R. Das, Specification and shadowing properties for non-autonomous systems, J. Dyn. Control Syst., **2021** (2021), 1–12. doi: 10.1007/s10883-021-09535-4.

13. M. Salman, X. Wu, R. Das, Sensitivity of nonautonomous dynamical systems on uniform spaces, *Int. J. Bifurcat. Chaos,* **31** (2021), 2150017. doi: 10.1142/S0218127421500176.

14. R. Devaney, L. Robert, An introduction to chaotic dynamical systems, *Acta Appl. Math.,* **19** (1990), 204–205.

15. D. Ruelle, F. Takens, On the nature of turbulence, *Commun. Math. Phys.,* **20** (1971), 178–188. doi: 10.1007/BF01646553.

16. H. Liu, L. Liao, L. Wang, Thickly syndetical sensitivity of topological dynamical system, *Discrete Dyn. Nat. Soc.,* **2014** (2014), 583431. doi: 10.1155/2014/583431.

17. R. Li, The large deviations theorem and ergodic sensitivity, *Commun. Nonlinear Sci.,* **18** (2013), 819–825. doi: 10.1016/j.cnsns.2012.09.008.

18. T. Moothathu, Stronger forms of sensitivity for dynamical systems, *Nonlinearity,* **20** (2007), 2115–2126. doi: 10.1088/0951-7715/20/9/006.

19. R. Li, T. Lu, G. Chen, X. Yang, Further discussion on Kato's chaos in set-valued discrete systems, *J. Appl. Anal. Comput.,* **10** (2020), 2491–2505. doi: 10.11948/20190388.

20. H. Wang, J. Xiong, F. Tan, Furstenberg families and sensitivity, *Discrete Dyn. Nature Soc.,* **12** (2010), 649348. doi: 10.1155/2010/649348.

21. R. Li, T. Lu, G. Chen, G. Liu, Some stronger forms of topological transitivity and sensitivity for a sequence of uniformly convergent continuous maps, *J. Math. Anal. Appl.,* **494** (2020), 124443. doi: 10.1016/j.jmaa.2020.124443.

22. L. Alseda, M. A. D. Rio, J. A. Rodriguez, A note on the totally transitive graph maps stability of pwl cellular, *Int. J. Bifurcat. Chaos,* **11** (2001), 841–843. doi: 10.1142/S0218127401002365.

23. M. Murillo-Arcila, A. Peris, Mixing properties for nonautonomous linear dynamics and invariant sets, *Appl. Math. Lett.,* **26** (2013), 215–218. doi: 10.1016/j.aml.2012.08.014.

24. L. Wang, J. Liang, Z. Chu, Weakly mixing property and chaos, *Arch. Math.,* **109** (2017), 83–89. doi: 10.1007/s00013-017-1044-1.

25. Q. Huang, Chaos theory and application of discrete dynamic system, *Diss, Shandong Univ.,* 2012 (In Chinese).

26. R. Li, Z. Yu, H. Wang, Stronger forms of transitivity and sensitivity for nonautonomous discrete dynamical systems and furstenberg families, *J. Dyn. Control Syst.,* **26** (2020), 109–126. doi: 10.1007/s10883-019-09437-6.

27. X. Yang, X. Tang, T. Lu, The collectively sensitivity and accessible in non-autonomous composite systems, *Acta Math. Sci.,* (In Chinese, in press).