Research article

Elastic transformation method for solving ordinary differential equations with variable coefficients

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Abstract: Aiming at the problem of solving nonlinear ordinary differential equations with variable coefficients, this paper introduces the elastic transformation method into the process of solving ordinary differential equations for the first time. A class of first-order and a class of third-order ordinary differential equations with variable coefficients can be transformed into the Laguerre equation through elastic transformation. With the help of the general solution of the Laguerre equation, the general solution of these two classes of ordinary differential equations can be obtained, and then the curves of the general solution can be drawn. This method not only expands the solvable classes of ordinary differential equations, but also provides a new idea for solving ordinary differential equations with variable coefficients.

1. Introduction

As we all know, elasticity is a very important concept in many fields such as economy, physics, biology, etc. In the economic field, Marshall [1] proposed the elasticity for the first time. Zhang [19] further introduced the concept of elastic coefficient comprehensively. Woods et al. [10] gave an expression of the elastic relationship between the independent variable and the dependent variable. Li et al. [5, 6] applied elasticity to the boundary value problem of oil and gas models, so that the result can more accurately reflect the actual situation.

Ordinary differential equation (ODE) is a very important part of mathematics, and it is widely used in technical engineering, natural sciences and other fields [8, 14] but it is very difficult to seek the general solution of high-order ODE with variable coefficients. Only a small part of ODEs in special forms can find general solution in the knowledge of mathematical theory [17] and many

scholars studied its properties of solutions [2, 15]. Alfaqih et al. [13] studied the existence and uniqueness of solution for first-order ordinary differential equations by applying relation-theoretic fixed point theorems. Schmitt et al. [16] studied the existence for almost periodic solutions of nonlinear second order ODEs to extend existence results for periodic solutions of periodic equations. Li et al. [11] explored a class of integrable n-th order linear ODEs with variable coefficients, and the general solution of this class of equations can be directly obtained by elementary integral method. Fang [3] utilized the theory of linear differential operator decomposition to research the operator solution of third-order linear ODE with variable coefficients and deduced some integrable types. Guo [7] applied Maple software to convert the third-order linear differential equations with variable coefficients to a third-order linear differential equations with constant coefficients, and found their numerical solution. Li et al. [9] utilized method of reduction of order to convert three special third-order nonlinear differential equations with variable coefficients to solvable second-order nonlinear differential equations and find general solution. Wu [18] transformed a class of first-order linear differential equations into solvable second-order homogeneous linear differential equations with constant coefficients to find their general solution by using the method of upgrading order.

Based on the above research, this paper proposes for the first time an elastic transformation method for solving general solution of nonlinear ODEs with variable coefficients. A class of first-order and a class of third-order ODEs with variable coefficients can finally be converted to second-order Laguerre equation through elastic upgrading transformation method or elastic reduced transformation method. And then with the help of the general solution of the Laguerre equation, the general solution of these two classes ODEs can be obtained.

The organization of this paper is as follows: In Section 2, the basic knowledge of elasticity and the general solution of the Laguerre equation are shown. In Section 3, we give two main theorems and a flow chart for solving two classes equations. In Section 4, we apply the two theorems to give concrete examples and draw curves of the general solution to discuss the influence of the coefficients. A brief discussion is given in the last section.

2. Preliminary knowledge

Definition 2.1. When the increment of the independent variable x approaches 0, the limit of the ratio of the change range $\frac{\Delta y(x)}{y(x)}$ of the differentiable non-zero function $y(x)$ to the change range $\frac{\Delta x}{x}$ of the independent variable x is called elasticity, which is denoted as $\eta(x)$:

$$\eta(x) = \lim_{\Delta x \to 0} \frac{\frac{\Delta y(x)}{y(x)}}{\frac{\Delta x}{x}} = \frac{x}{y(x)} \frac{dy(x)}{dx} = \frac{d \ln y(x)}{d \ln x} \triangleq \frac{Ey(x)}{Ex}. \tag{2.1}$$

2.1. Mixed operation of elasticity

For any two differentiable functions $y_1(x)$, $y_2(x)$, constant k_1, k_2, when $y_1^2(x) + y_2^2(x) \neq 0$, $k_1^2 + k_2^2 \neq 0$, the following formula holds:

$$\frac{E\left[k_1 y_1(x) \pm k_2 y_2(x)\right]}{Ex} = \frac{k_1 y_1(x) \frac{Ey_1(x)}{Ex} \pm k_2 y_2(x) \frac{Ey_2(x)}{Ex}}{k_1 y_1(x) \pm k_2 y_2(x)}.$$

Proof. According to the definition of elasticity Eq (2.1):

$$\frac{E\left[k_1 y_1(x) \pm k_2 y_2(x)\right]}{Ex} = \frac{x}{k_1 y_1(x) \pm k_2 y_2(x)} \frac{d\left[k_1 y_1(x) \pm k_2 y_2(x)\right]}{dx}$$

$$= \frac{1}{k_1 y_1(x) \pm k_2 y_2(x)} \left[k_1 y_1(x) \frac{x}{y_1(x)} \frac{dy_1(x)}{dx} \pm k_2 y_2(x) \frac{x}{y_2(x)} \frac{dy_2(x)}{dx}\right]$$

$$= \frac{k_1 y_1(x) \frac{Ey_1(x)}{Ex} \pm k_2 y_2(x) \frac{Ey_2(x)}{Ex}}{k_1 y_1(x) \pm k_2 y_2(x)}.$$

□

Lemma 2.2. *The original function of the elastic solution function of the ODE is:*

$$y(x) = e^{\int \frac{\eta(x)}{x} dx},$$

where $\eta(x)$ is the general solution function of the ODE and also the elasticity of $y(x)$.

Proof. For Eq (2.1), use the method of separation of variables [17, 20] to find $y(x)$:

$$y(x) = e^{\int \frac{\eta(x)}{x} dx}.$$

□

2.2. General solution of Laguerre equation

The equation of the form

$$xy + (1 - x)y' + ny = 0 \quad (n=0,\ 1,\ 2\ldots,\ x > 0) \tag{2.2}$$

is called Laguerre equation, and its general solution is [12]:

$$y(x) = AL_n(x) + BG(-n, 1, x), \tag{2.3}$$

where A, B are arbitrary constants, $L(x)$ and $G(x)$ are given below.

2.2.1. Laguerre polynomial [12]

In Eq (2.3), $L_n(x)$ is a Laguerre polynomial of degree n:

$$L_n(x)=\Gamma(n + 1)F(-n, 1, x)$$
$$= \sum_{k=0}^{n} (-1)^k \frac{(n!)^2}{(k!)^2 (n - k)!} x^k, \tag{2.4}$$

where $\Gamma(x)$ is the gamma function, when x is a positive integer, the expression is:

$$\Gamma(n + 1) = n! \quad (n = 0, 1, 2, \cdots),$$

$F(-n, 1, x)$ is the first type of confluent hypergeometric function and has the property [4]:

$$\frac{d}{dx}F(\alpha, \gamma, x) = \frac{\alpha}{\gamma}F(\alpha + 1, \gamma + 1, x),$$

so

$$\frac{d}{dx}L_n(x) = \frac{d}{dx}\left[n!F(-n, 1, x)\right]$$
$$= n!(-n)F(1 - n, 2, x). \tag{2.5}$$

2.2.2. The second type of confluent hypergeometric function [4, 12]

In Eq (2.3), $G(-n, 1, x)$ is the second type of confluent hypergeometric function:

$$G(-n, 1, x) = F(-n, 1, x) \ln x - 1$$
$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{(k!)^2 (n-k)!} x^k \ln x - 1, \tag{2.6}$$

and has the property:

$$\frac{d}{dx} G(-n, 1, x) = \frac{d}{dx} [F(-n, 1, x) \ln x - 1]$$
$$= \ln x(-n) F(1-n, 2, x) + \frac{1}{x} F(-n, 1, x). \tag{2.7}$$

3. Main theorems and proofs

Theorem 3.1. *The general solution theorem for a class of first-order nonlinear ODEs with variable coefficients.*

For a class of first-order nonlinear ODEs with variable coefficients:

$$xy' + y^2 - xy + nx = 0 \quad (n=0, 1, 2\ldots, x > 0), \tag{3.1}$$

the general solution is:

$$y(x) = \frac{x \left[n!(-n) F(1-n, 2, x) \right] + Cx \left[\ln x(-n) F(1-n, 2, x) + \frac{1}{x} F(-n, 1, x) \right]}{L_n(x) + CG(-n, 1, x)}$$

or

$$y(x) = \frac{Dx \left[n!(-n) F(1-n, 2, x) \right] + x \left[\ln x(-n) F(1-n, 2, x) + \frac{1}{x} F(-n, 1, x) \right]}{DL_n(x) + G(-n, 1, x)},$$

where C, D are arbitrary constants.

Proof. Obviously, the Eq (3.1) only has non-zero solutions: $y(x) \neq 0$.

Firstly, make an elastic upgrading transformation on the function $y(x)$:

$$y(x) = \frac{x}{\eta(x)} \frac{d\eta(x)}{dx} = \frac{E\eta(x)}{Ex}. \tag{3.2}$$

Where $\eta(x)$ is a function related to x, $y(x)$ is the elasticity of $\eta(x)$ and $\eta(x) \neq 0$.

Secondly, find the first derivative of elasticity $y(x)$:

$$y'(x) = \frac{x}{\eta(x)} \eta''(x) - \frac{x}{\eta^2(x)} [\eta'(x)]^2 + \frac{1}{\eta(x)} \eta'(x).$$

Thirdly, according to the Eq (3.2) and $y'(x)$, the Eq (3.1) is simplified to:

$$x\eta'' + (1-x)\eta' + n\eta = 0. \tag{3.3}$$

This is the Laguerre equation with x as the independent variable, η as the dependent variable and $n=0, 1, 2 \ldots$. Eqs (3.1) and (3.3) have the same domain and $\eta(x) \neq 0$ when $y(x) \neq 0$, so all the solutions of Eq (3.1) can be obtained from the solution of Eq (3.3). From Eq (2.3), the general solution of the above equation is:

$$\eta(x) = AL_n(x) + BG(-n, 1, x), \tag{3.4}$$

where A, B are arbitrary constants, $A^2 + B^2 \neq 0$.

Finally, according to (3.2)–(3.4) and the mixed operation of elasticity, the general solution of the original first-order equation Eq (3.1) is:

$$
\begin{aligned}
y(x) &= \frac{E\eta(x)}{Ex} \\
&= \frac{E\left[AL_n(x) + BG(-n, 1, x)\right]}{Ex} \\
&= \frac{Ax\frac{d}{dx}\left[L_n(x)\right] + Bx\frac{d}{dx}\left[G(-n, 1, x)\right]}{AL_n(x) + BG(-n, 1, x)},
\end{aligned}
$$

where A, B are arbitrary constants, $A^2 + B^2 \neq 0$. To simplify further, the general solution of Eq (3.1) is:

$$y(x) = \frac{Ax\left[n!(-n)F(1 - n, 2, x)\right] + Bx\left[\ln x(-n)F(1 - n, 2, x) + \frac{1}{x}F(-n, 1, x)\right]}{AL_n(x) + BG(-n, 1, x)}$$

$$
= \begin{cases}
\dfrac{x[n!(-n)F(1-n,2,x)]+Cx\left[\ln x(-n)F(1-n,2,x)+\frac{1}{x}F(-n,1,x)\right]}{L_n(x)+CG(-n,1,x)} \\
(A \neq 0, C = \frac{B}{A}) \\[1em]
\dfrac{Dx[n!(-n)F(1-n,2,x)]+x\left[\ln x(-n)F(1-n,2,x)+\frac{1}{x}F(-n,1,x)\right]}{DL_n(x)+G(-n,1,x)} \\
(B \neq 0, D = \frac{A}{B})
\end{cases}, \tag{3.5}
$$

where C, D are arbitrary constants. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

According to (3.1)–(3.5), the steps of solving a class of first-order nonlinear ODEs with variable coefficients by elastic upgrading transformation method are summarized as follows:

Step 1: Perform an elastic upgrading transformation on the function $y(x)$:

$$y(x) = \frac{x}{\eta(x)} \frac{d\eta(x)}{dx} \quad (\eta(x) \neq 0).$$

Step 2: Find the first derivative of the elasticity $y(x)$.

Step 3: Upgrade the order of the first-order equation to Laguerre equation and find the general solution of Laguerre equation.

Step 4: Obtain the general solution of these first-order ODEs according to the general solution of Laguerre equation.

Theorem 3.2. *The general solution theorem for a class of third-order nonlinear ODEs with variable coefficients.*

For a class of third-order nonlinear ODEs with variable coefficients:

$$(xy)^2 y''' + \left[2xy^2 - 3x^2yy' + xy^2 - (xy)^2\right]y'' + 2x^2(y')^3$$
$$- (3x - x^2)y(y')^2 + (1 - x + nx)y^2 y' = 0 \quad (n=0, 1, 2\ldots, x > 0), \tag{3.6}$$

the general solution is:

$$y = Ce^{A \int \frac{L_n(x)}{x} dx + B \int \frac{G(-n,1,x)}{x} dx}, \tag{3.7}$$

where A, B, C are arbitrary constants.

Proof. $y(x) = 0$ is a zero solution of Eq (3.6) and satisfies Eq (3.7). When $y(x) \neq 0$, the three following steps are done to find the non-zero solution of Eq (3.6):

Firstly, perform an elastic reduced transformation on the function $y(x)$:

$$\eta(x) = \frac{x}{y(x)} \frac{dy(x)}{dx}, \tag{3.8}$$

where $\eta(x)$ is a function related to x and the elasticity of $y(x)$, hence $\eta(x) \neq 0$ clearly. According to Eq (3.8):

$$y'(x) = \frac{y(x)}{x} \eta(x). \tag{3.9}$$

Secondly, find the second and third derivatives of $y(x)$:

$$y''(x) = \frac{y(x)}{x} \left\{ \eta'(x) + \frac{1}{x} \eta(x) \left[\eta(x) - 1\right] \right\},$$
$$y'''(x) = \frac{y(x)}{x} \left[\eta''(x) + \frac{1}{x} \left[3\eta(x) - 2\right] \eta'(x) + \frac{1}{x^2} \eta(x) \left[\eta(x) - 1\right] \left[\eta(x) - 2\right] \right].$$

Thirdly, put $y'(x)$, $y''(x)$, $y'''(x)$ into Eq (3.6) and simplify the equation to:

$$x\eta'' + (1 - x)\eta' + n\eta = 0.$$

This is the Laguerre equation with x as the independent variable, η as the dependent variable and $n=0, 1, 2\ldots$. From Eq (2.3), the general solution of the above equation is:

$$\eta(x) = AL_n(x) + BG(-n, 1, x), \tag{3.10}$$

where A, B are arbitrary constants, $A^2 + B^2 \neq 0$.

Finally, substituting Eq (3.10) into Lemma 2.2, the non-zero solution of Eq (3.6) is obtained as:

$$y(x) = e^{\int \frac{\eta(x)}{x} dx}$$
$$= Ce^{A \int \frac{L_n(x)}{x} dx + B \int \frac{G(-n,1,x)}{x} dx}, \tag{3.11}$$

where A, B, C are arbitrary constants, $A^2 + B^2 \neq 0$.

In summary, the general solution of Eq (3.6) is:

$$y(x) = Ce^{A \int \frac{L_n(x)}{x} dx + B \int \frac{G(-n,1,x)}{x} dx}, \tag{3.12}$$

in this case, A, B, C are arbitrary constants. $\qquad\qquad\Box$

According to (3.6)–(3.12), the steps for solving a class of third-order nonlinear ODEs with variable coefficients by elastic reduced transformation method are summarized as follows:

Step 1: Verify whether the equation has a zero solution.

Step 2: When $y(x) \neq 0$. Perform an elastic reduced transformation on the function $y(x)$ and find $y'(x)$.

Step 3: Obtain $y''(x)$, $y'''(x)$ expressed by elasticity $\eta(x)$.

Step 4: Reduce the order of the third-order ODE to Laguerre equation and find the general solution of Laguerre equation.

Step 5: Obtain the non-zero general solution of these third-order ODEs according to the general solution of Laguerre equation. Further obtain all general solutions of these third-order ODEs.

Flow chart

The flow chart of the steps for solving these two classes ODEs using elastic transformation is shown in Figure 1:
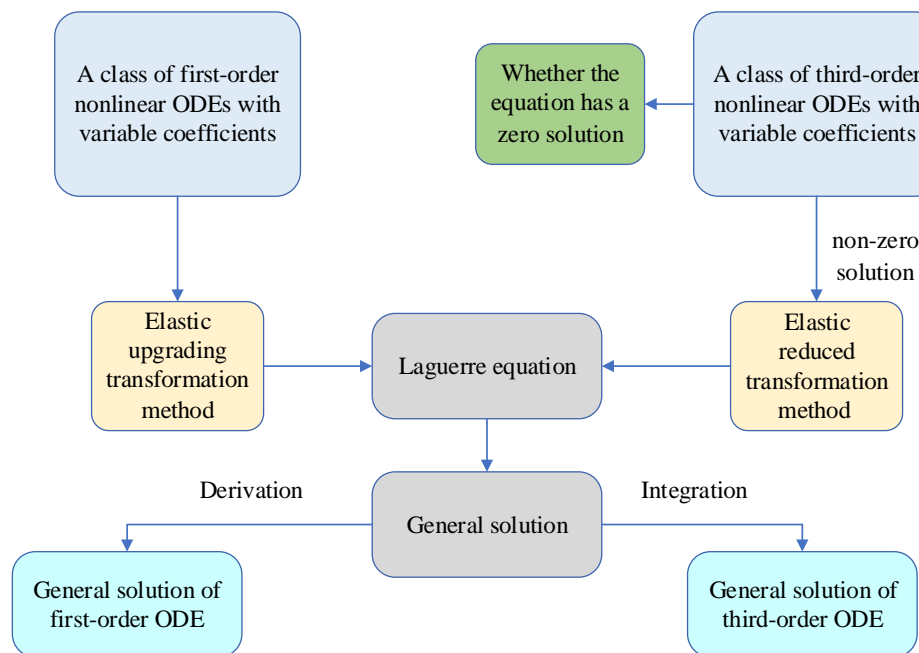

Figure 1. The flow chart of solving two classes of variable coefficients equations.

4. Application

4.1. The application of elastic upgrading transformation method to solving a class of first-order nonlinear ODEs

4.1.1. Solving process

Example 4.1. Find the general solution of the following first-order nonlinear ODE with variable coefficients.

$$xy' + y^2 - xy + 2x = 0 \quad (x > 0). \tag{4.1}$$

Solve: Step 1: Obviously, Eq (4.1) only has non-zero solutions. Make an elastic upgrading transformation on the function $y(x)$:

$$y(x) = \frac{x}{\eta(x)} \frac{d\eta(x)}{dx} \quad (\eta(x) \neq 0).$$

Step 2: Find the first derivative of $y(x)$:

$$y'(x) = \frac{x}{\eta(x)} \eta''(x) - \frac{x}{\eta^2(x)} [\eta'(x)]^2 + \frac{1}{\eta(x)} \eta'(x).$$

Step 3: Substitute $y(x)$, $y'(x)$ into Eq (4.1), and then simplify the equation to get the Laguerre equation $x\eta'' + (1 - x)\eta' + 2\eta = 0$. According to (2.4)–(2.7):

$$L_2(x) = x^2 - 4x + 2,$$

$$G(-1, 2, x) = (\frac{1}{2}x^2 - 2x + 1) \ln x - 1,$$

$$\frac{d}{dx} L_2(x) = 2x - 4,$$

$$\frac{d}{dx} G(-1, 2, x) = \frac{\frac{1}{2}x^2 - 2x + 1}{x} + (x - 2) \ln x.$$

Then the general solution of Laguerre equation is:

$$\eta(x) = AL_2(x) + BG(-1, 2, x)$$

$$= A(x^2 - 4x + 2) + B\left[(\frac{1}{2}x^2 - 2x + 1) \ln x - 1\right],$$

where A, B are arbitrary constants, $A^2 + B^2 \neq 0$.

Step 4: According to the general solution of the Laguerre equation and Theorem 1, the general solution of the Eq (4.1) is:

$$y(x) = \frac{Ax(2x - 4) + Bx\left[\frac{\frac{1}{2}x^2 - 2x + 1}{x} + (x - 2) \ln x\right]}{A(x^2 - 4x + 2) + B\left[(\frac{1}{2}x^2 - 2x + 1) \ln x - 1\right]}$$

$$= \begin{cases} \frac{x(2x-4)+Cx\left[\frac{\frac{1}{2}x^2-2x+1}{x}+(x-2)\ln x\right]}{(x^2-4x+2)+C\left[(\frac{1}{2}x^2-2x+1)\ln x-1\right]} & (A \neq 0, C = \frac{B}{A}) \\ \frac{Dx(2x-4)+x\left[\frac{\frac{1}{2}x^2-2x+1}{x}+(x-2)\ln x\right]}{D(x^2-4x+2)+\left[(\frac{1}{2}x^2-2x+1)\ln x-1\right]} & (B \neq 0, D = \frac{A}{B}), \end{cases} \tag{4.2}$$

where $C, \ D$ are arbitrary constants.

It has been verified that the Eq (4.2) is the general solution of the Eq (4.1).

4.1.2. Images analysis

Figures 2 and 3 show the curves of general solution $y(x)$ at $C = -1, \ 0, \ 1$ and $D = -1, \ 0, \ 1$ from left to right, respectively. The basic characteristics of the curves are consistent, and the fluctuations of $C, \ D = -1$ are relatively large. Clearly, the coefficients $C, \ D$ have a greater influence on the general solution.

1. When $A \neq 0$, the curves of general solution $y(x)$ with different values of C is shown in Figure 2:
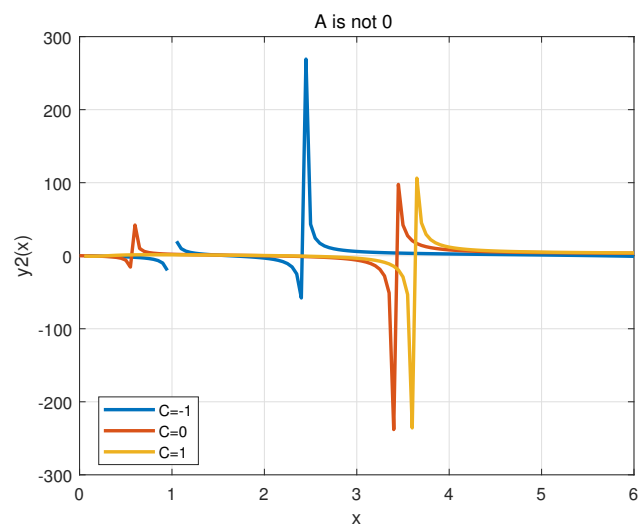

Figure 2. The changes of $y(x)$ when C takes different values.

2. When $B \neq 0$, the curves of general solution $y(x)$ with different values of D as shown in Figure 3:
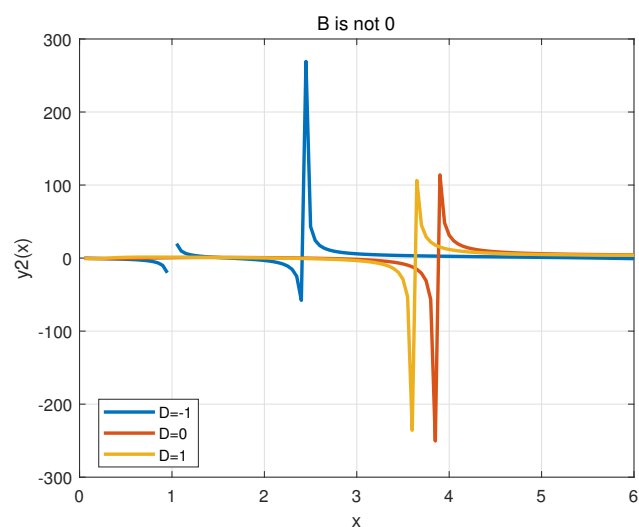

Figure 3. The changes of $y(x)$ when D takes different values.

4.2. The application of elastic reduced transformation method to solving a class of third-order nonlinear ODEs

4.2.1. Solving process

Example 4.2. Find the general solution of the following third-order nonlinear ODE with variable coefficients.

$$(xy)^2 y''' + \left[2xy^2 - 3x^2yy' + xy^2 - (xy)^2\right]y'' + 2x^2(y')^3$$
$$- (3x - x^2)y(y')^2 + (1 + x)y^2 y' = 0 \quad (n = 0, 1, 2, ..., x > 0). \tag{4.3}$$

Solve: Step 1: Obviously, Eq (4.3) has a zero solution.

Step 2: When $y(x) \neq 0$. Perform an elastic reduced transformation on the function $y(x)$: $\eta(x) = \frac{x}{y(x)}\frac{dy(x)}{dx}$, find $y'(x) = \frac{y(x)}{x}\eta(x)$.

Step 3: Obtain $y''(x)$, $y'''(x)$:

$$y''(x) = \frac{y(x)}{x}\left\{\eta'(x) + \frac{1}{x}\eta(x)\left[\eta(x) - 1\right]\right\},$$

$$y'''(x) = \frac{y(x)}{x}\left\{\eta''(x) + \frac{1}{x}\left[3\eta(x) - 2\right]\eta'(x) + \frac{1}{x^2}\eta(x)\left[\eta(x) - 1\right]\left[\eta(x) - 2\right]\right\}.$$

Step 4: On the basis of $y'(x)$, $y''(x)$, $y'''(x)$, the Eq (4.3) is converted to the Laguerre equation $x\eta'' + (1 - x)\eta' + 2\eta = 0$, its general solution is:

$$\eta(x) = AL_2(x) + BG(-1, 2, x),$$

where A, B are arbitrary constants, $A^2 + B^2 \neq 0$, and

$$L_2(x) = x^2 - 4x + 2,$$
$$G(-1, 2, x) = (\frac{1}{2}x^2 - 2x + 1)\ln x - 1.$$

Step 5: According to the general solution of the Laguerre equation and Theorem 2, the non-zero solution of the Eq (4.3) is:

$$y(x) = Ce^{A\left(\frac{x^2}{2} - 4x + 2\ln x\right) + B\left[-\frac{x^2}{8} + \left(\frac{x^2}{4} - 2x + \frac{1}{2}\ln x - 1\right)\ln x + 2x\right]}, \tag{4.4}$$

where A, B, C are arbitrary constants, $A^2 + B^2 \neq 0$.

Obviously, $y(x) = 0$ also satisfies the Eq (4.4), so the general solution of the Eq (4.3) is:

$$y(x) = Ce^{A\left(\frac{x^2}{2} - 4x + 2\ln x\right) + B\left[-\frac{x^2}{8} + \left(\frac{x^2}{4} - 2x + \frac{1}{2}\ln x - 1\right)\ln x + 2x\right]}, \tag{4.5}$$

in this case, A, B, C are all arbitrary constants.

It has been verified Eq (4.5) is the general solution of Eq (4.3).

4.2.2. Images analysis

Figure 4 shows the curves of $y(x)$ when C = 1.4, 1.2, 1, 0.8, −0.8, −1, −1.2, −1.4 from top to bottom, the basic characteristics of this eight curves in each coordinate axis are consistent. Apparently, after fixing the values of A, B the change of the coefficient C has less influence on the general solution $y(x)$. Figure 5 shows the curves of $y(x)$ when B = −0.8, −1, −1.2, −1.4 from top to bottom. The general trend of the curves is the same, and the maximum values are almost overlapped. It is easy to see that after fixing the values of A, C the change of the coefficient B has less influence on the general solution $y(x)$. Figure 6, the solid lines represent the curves of $y(x)$ at A = 0.8, 1, 1.2, 1.4, and the dashed lines represent the curves of $y(x)$ at A = −0.8, −1, −1.2, −1.4 from top to bottom. Obviously, after fixing the values of B, C when A is a positive number, the effect on the general solution $y(x)$ is small, when A is a negative number, the effect on the general solution $y(x)$ is large.

1. After A, B are fixed, the curves of $y(x)$ with different values of C is shown in Figure 4:
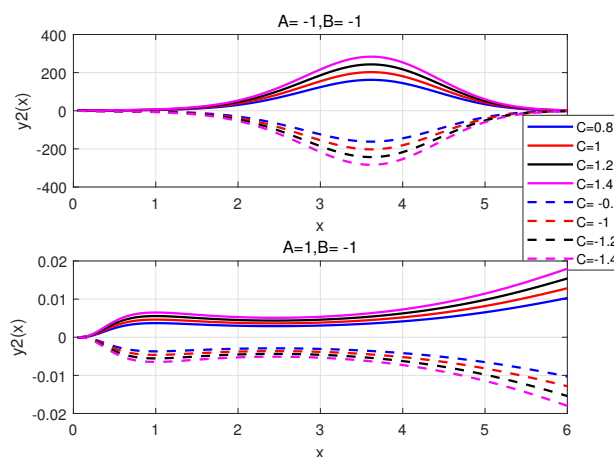

Figure 4. The changes of $y(x)$ with C after A, B are fixed.

2. After A, C are fixed, the curves of $y(x)$ with different values of B as shown in Figure 5:
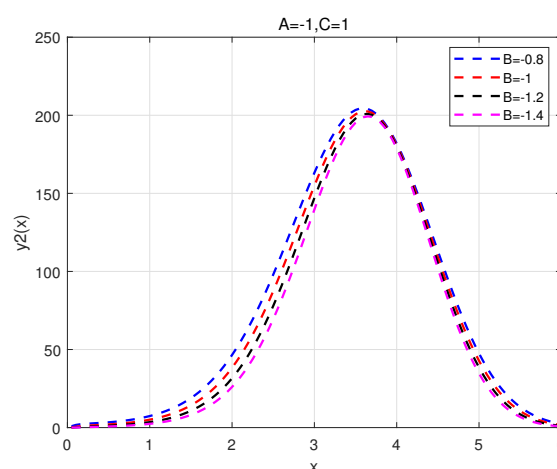

Figure 5. The changes of $y(x)$ with B after A, C are fixed.

3. After B, C are fixed, the curves of $y(x)$ with different values of A is shown in Figure 6:
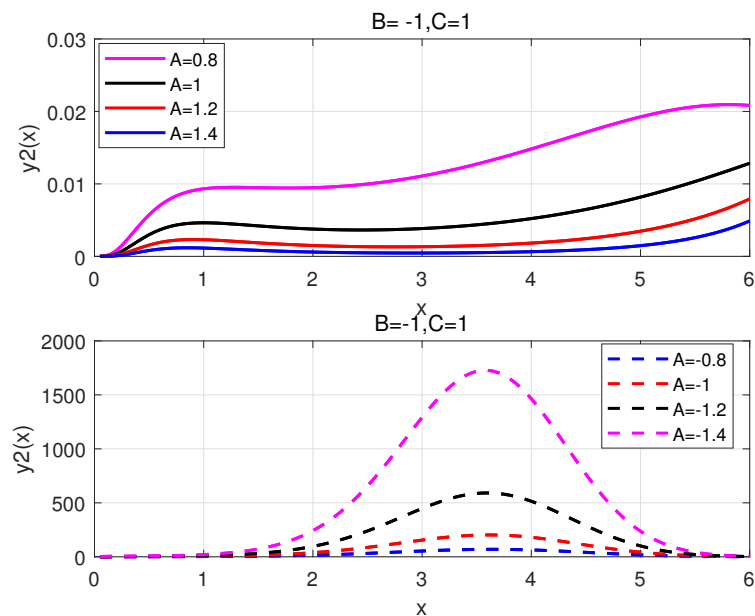

Figure 6. The changes of $y(x)$ with A after B, C are fixed.

5. Conclusions

1) A class of first-order ODEs with variable coefficients are simplified to the Laguerre equation by using the elastic upgrading transformation method, and then the general solution of the first-order ODEs is obtained through the general solution of the Laguerre equation. So the order of the low-order ODE is appropriately increased to make the ODE solvable by using the elastic upgrading transformation.

2) The elastic reduced transformation method is used to convert a class of third-order ODEs with variable coefficients into the Laguerre equation, and obtain the general solution of the third-order differential equation with the help of the general solution of the Laguerre equation. The elastic reduced transformation is also suitable for appropriately decreasing the order of high-order ODEs to make them solvable.

3) The elastic transformation method not only provides a new idea for solving the general solution of ODEs with variable coefficients, but also expands the solvable classes of ODEs.

4) The solution process of the application examples in the article further promotes the application of the general solution of the Laguerre equation.

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Conflict of interest

The authors declare that there are no conflict of interest in this paper.

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