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Research article

Toeplitz operators between large Fock spaces in several complex variables

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Abstract: Let ω belong to the weight class $\mathcal{W}$, the large Fock space $\mathcal{F}^p_\omega$ consists of all holomorphic functions f on $\mathbb{C}^n$ such that the function $f(\cdot)\omega(\cdot)^{1/2}$ is in $L^p(\mathbb{C}^n, dv)$. In this paper, given a positive Borel measure μ on $\mathbb{C}^n$, we characterize the boundedness and compactness of Toeplitz operator T_μ between two large Fock spaces F^p_ω and F^q_ω for all possible $0 < p, q < \infty$.

Keywords: large Fock space; boundedness; compactness; Toeplitz operator
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1. Introduction

Let $\mathbb{C}^n$ be the n-dimensional Euclidean space. For any two points $z = (z_1, ..., z_n)$ and $\xi = (\xi_1, ..., \xi_n)$ in $\mathbb{C}^n$, we write $\langle z, \xi\rangle = z_1\overline{\xi_1} + ... + z_n\overline{\xi_n}$ and $|z| = \sqrt{\langle z, z\rangle}$. Given $z \in \mathbb{C}^n$ and $r > 0$, set the Euclidean ball $B(z, r) = \{\xi \in \mathbb{C}^n : |\xi - z| < r\}$. Let dV to be the ordinary Lebesgue volume measure on $\mathbb{C}^n$. Suppose $\varphi : \mathbb{C}^n \to \mathbb{R}$ is a C^2-plurisubharmonic function and set

$$\omega(z) = \exp(-2\varphi(z)), \qquad z \in \mathbb{C}^n.$$

We say that ω belongs to the weight class $\mathcal{W}$ if φ satisfies the following conditions:
(I) There exists $c > 0$ such that

$$\inf_{z\in\mathbb{C}^n} \sup_{\xi\in B(z,c)} \Delta\varphi(\xi) > 0; \tag{1.1}$$

(II) $\Delta\varphi$ satisfies the reverse-Hölder inequality

$$\|\Delta\varphi\|_{L^\infty(B(z,r))} \le Cr^{-2n} \int_{B(z,r)} \Delta\varphi(\xi)dV(\xi), \quad \forall z \in \mathbb{C}^n,\ r > 0$$

for some $0 < C < +\infty$;

(III) The eigenvalues of H_φ are comparable, that is, there exists $\delta_0 > 0$ such that

$$(H_\varphi(z)u, u) \geq \delta_0 \Delta\varphi(z)|u|^2, \quad \forall z, u \in \mathbb{C}^n,$$

where

$$H_\varphi = \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z_k}} \right)_{j,k}.$$

Suppose $0 < p < \infty, \omega \in \mathcal{W}$. The space L^p_ω consists of all Lebesgue measurable functions f on $\mathbb{C}^n$ for which

$$\|f\|_{p,\omega} = \left(\int_{\mathbb{C}^n} |f(z)|^p \omega(z)^{p/2} dV(z) \right)^{1/p} < \infty.$$

It is obvious that $L^p_\omega = L^p(\mathbb{C}^n, \omega^{p/2} dV)$. We use L^p to stand the usual p-th Lebesgue space with the norm $\|\cdot\|_{L^p} = \left(\int_{\mathbb{C}^n} |\cdot|^p dV(z) \right)^{1/p}$.

Let $H(\mathbb{C}^n)$ be the family of all entire functions on $\mathbb{C}^n$. The weighted Fock space F^p_ω is defined as

$$F^p_\omega = L^p_\omega \cap H(\mathbb{C}^n).$$

It is clear that F^p_ω is a Banach space under $\|\cdot\|_{p,\omega}$ if $1 \leq p \leq \infty$, and F^p_ω is an F-space under the metric $d(f,g) = \|f-g\|^p_{p,\omega}$ if $0 < p < 1$.

Taking $\varphi(z) = \frac{1}{2}|z|^2$, F^p_ω is the classical Fock space, which has been studied by many authors, see [5, 7, 10, 11, 18] and the references therein. The weight function ω with the restriction that $dd^c\varphi \simeq dd^c|z|^2$ in [8] and [16] belongs to $\mathcal{W}$ as well. Notice that the class $\mathcal{W}$ also contains nonradial weights, an example is given by

$$\omega_{p,f}(z) = |f(z)|^p e^{-|z|^2/2},$$

where $p > 0$ and f is a nonvanishing analytic function in F^p_ω. This class of weights have attracted many attention in the setting of Fock spaces, see [1, 6, 14] for instance.

For $\omega \in \mathcal{W}$, the mapping $f \mapsto f(z)$ is a bounded linear functional on F^2_ω for each $z \in \mathbb{C}^n$. By the Riesz representation theorem in functional analysis, there exists a unique function $K_z \in F^2_\omega$ such that $f(z) = \langle f, K_z \rangle_\omega$ for all $f \in F^2_\omega$, where

$$\langle f, g \rangle_\omega = \int_{\mathbb{C}^n} f(z)\overline{g(z)}\omega(z) dV(z), \quad f, g \in F^2_\omega.$$

The function $K(\cdot, z) = K_z(\cdot)$ is called the reproducing kernel of F^2_ω. For $0 < p < \infty$ and $z \in \mathbb{C}^n$, we let

$$k_{p,z}(\cdot) = K(\cdot, z)/\|K(\cdot, z)\|_{p,\omega}$$

denote the normalized reproducing kernel for F^p_ω. Notice that the set $\{k_{p,z} : z \in \mathbb{C}^n\}$ is bounded in F^p_ω and $k_{p,z} \to 0$ uniformly on every compact subset of $\mathbb{C}^n$ when $|z| \to \infty$.

Suppose μ is a Borel measure on $\mathbb{C}^n$, the Toeplitz operator T_μ induced by μ is defined as

$$T_\mu f(\cdot) = \int_{\mathbb{C}^n} f(\xi) K(\cdot, \xi) \omega(\xi) d\mu(\xi)$$

if it is well (densely) defined.

During the past few decades much effort has been devoted to the study of Toeplitz operators on Fock spaces. In the case $n = 1$, when $d\mu = gdA$ for some restricted function g, for example g is bounded or $g \in BMO$, the induced Toeplitz operator T_μ has been studied in [2–5, 8, 17]. For $\mu \geq 0$, Isralowitz and Zhu in [11] characterized the mapping properties of T_μ on $F^2_{|z|^2/2}$. In [7], Hu an Lv obtained sufficient and necessary conditions on μ for which T_μ is bounded (or compact) from $F^p_{|z|^2/2}$ to $F^q_{|z|^2/2}$ for $1 < p, q < \infty$. Denote $d = \partial + \overline{\partial}$ and $d^c = \frac{\sqrt{-1}}{4}\left(\overline{\partial} - \partial\right)$. With the restriction that $dd^c\varphi \simeq dd^c|z|^2$ on the weight φ in $\mathbb{C}^n$, Schuster and Varolin in [16] studied the boundedness and compactness of Toeplitz operators in terms of averaging functions and Berezin transforms. Later on, the corresponding problems were discussed from F^p_φ to F^q_φ for $0 < p, q < \infty$ in [8], between F^p_φ and F^∞_φ for $0 < p \leq \infty$ in [12]. In 2015, Oliver and Pascuas in [15] characterized the boundedness and compactness of positive Toeplitz operators on doubling Fock space F^p_ϕ for $1 \leq p < \infty$. In [9], the authors discussed the corresponding problems from F^p_ϕ to F^q_ϕ for $0 < p, q < \infty$.

The purpose of this work is to extend some results of [7–9, 11, 12, 15] concerning Toeplitz operators to large Fock spaces. In Section 2, we will give some lemmas would be used in the following sections. Section 3 is devoted to characterize those $\mu \geq 0$ for which the induced operators T_μ are bounded (or compact) from F^p_ω to F^q_ω for $0 < p, q < \infty$. Our approach depends on whether $0 < p \leq q < \infty$ or $0 < q < p < \infty$. We summarize the main results of the paper as below:

Theorem 1.1. *Suppose* $0 < p \leq q < \infty, \mu \geq 0$. *Let* α *be as defined in* (2.6) *below. Suppose also* $\omega \in \mathcal{W}$. *Then the following statements are equivalent:*

(A) $T_\mu : F^p_\omega \to F^q_\omega$ *is bounded;*

(B) $\widetilde{\mu}_t(\cdot)\tau^{\frac{2n(p-q)}{pq}}(\cdot) \in L^\infty$ *for some (equivalent: any)* $t > 0$*;*

(C) $\widehat{\mu}_\delta(\cdot)\tau^{\frac{2n(p-q)}{pq}}(\cdot) \in L^\infty$ *for some (equivalent: any)* $0 < \delta \leq \alpha$*;*

(D) The sequence $\left\{\widehat{\mu}_r(a_k)\tau(a_k)^{\frac{2n(p-q)}{pq}}\right\}_k$ *is bounded for some (equivalent: any)* (τ, r)*-lattice* $\{a_k\}_k$ *with* $0 < r \leq \alpha$.

Furthermore,

$$\|T_\mu\|_{F^p_\omega \to F^q_\omega} \simeq \left\|\widetilde{\mu}_t\tau^{\frac{2n(p-q)}{pq}}\right\|_{L^\infty} \simeq \left\|\widehat{\mu}_\delta\tau^{\frac{2n(p-q)}{pq}}\right\|_{L^\infty} \simeq \left\|\left\{\widehat{\mu}_r(a_k)\tau(a_k)^{\frac{2n(p-q)}{pq}}\right\}_k\right\|_{l^\infty}. \tag{1.2}$$

Theorem 1.2. *Suppose* $0 < p \leq q < \infty, \mu \geq 0$. *Let* α *be as defined in* (2.6) *below. Suppose also* $\omega \in \mathcal{W}$. *Then the following statements are equivalent:*

(A) $T_\mu : F^p_\omega \to F^q_\omega$ *is compact;*

(B) $\widetilde{\mu}_t(z)\tau^{\frac{2n(p-q)}{pq}}(z) \to 0$ *as* $z \to \infty$ *for some (equivalent: any)* $t > 0$*;*

(C) $\widehat{\mu}_\delta(z)\tau^{\frac{2n(p-q)}{pq}}(z) \to 0$ *as* $z \to \infty$ *for some (equivalent: any)* $0 < \delta \leq \alpha$*;*

(D) $\widehat{\mu}_r(a_k)\tau(a_k)^{\frac{2n(p-q)}{pq}} \to 0$ *as* $k \to \infty$ *for some (equivalent: any)* (τ, r)*-lattice* $\{a_k\}_k$ *with* $0 < r \leq \alpha$.

Theorem 1.3. *Suppose* $0 < q < p < \infty, \mu \geq 0$. *Let* α *be as defined in* (2.6) *below. Suppose also* $\omega \in \mathcal{W}$. *Then the following statements are equivalent:*

(A) $T_\mu : F^p_\omega \to F^q_\omega$ *is bounded;*

(B) $T_\mu : F^p_\omega \to F^q_\omega$ *is compact;*

(C) $\widetilde{\mu}_t \in L^{\frac{pq}{p-q}}$ *for some (equivalent: any)* $t > 0$*;*

(D) $\widehat{\mu}_s \in L^{\frac{pq}{p-q}}$ *for some (equivalent: any)* $0 < s \leq \alpha$*;*

(E) $\left\{\widehat{\mu}_\delta(a_k)\tau(a_k)^{\frac{2n(p-q)}{pq}}\right\}_k \in l^{\frac{pq}{p-q}}$ *for some (equivalent: any)* (τ, δ)*-lattice* $\{a_k\}_k$ *with* $0 < \delta \leq \alpha$*.*

Furthermore,

$$\|T_\mu\|_{F^p_\omega \to F^q_\omega} \simeq \|\widetilde{\mu}_t\|_{L^{\frac{pq}{p-q}}} \simeq \left\|\widehat{\mu}_s\right\|_{L^{\frac{pq}{p-q}}} \simeq \left\|\left\{\widehat{\mu}_\delta(a_k)\tau(a_k)^{\frac{2n(p-q)}{pq}}\right\}_k\right\|_{l^{\frac{pq}{p-q}}}. \tag{1.3}$$

In what follows, we use the notation $A \lesssim B$ to indicate that there is a constant $C > 0$ with $A \leq CB$. A and B are called equivalent, denoted by "$A \simeq B$", if there exists some C such that $A \lesssim B \lesssim A$.

2. Preliminaries

In this section, we will give some basic estimates which would be used in the following sections. For $z \in \mathbb{C}^n$, set

$$\tau_\varphi(z) = \sup\{r > 0 : \sup_{\xi \in B(z,r)} \Delta\varphi(\xi) \leq r^{-2}\}.$$

Throughout this paper, we simply write $\tau(z)$ instead of $\tau_\varphi(z)$. Let φ be as in (1.1), then there exist $A, B > 0$ such that

$$|z|^{-A} \lesssim \tau(z) \lesssim |z|^B, \quad \text{for} \quad |z| > 1. \tag{2.1}$$

See [1], given $\delta > 0$, write $B^\delta(z) = B(z, \delta\tau(z))$, and $B(z) = B^1(z)$ for short. By [14], there exists some $C > 0$ such that for $z \in \mathbb{C}$,

$$C^{-1}\tau(\xi) \leq \tau(z) \leq C\tau(\xi) \tag{2.2}$$

for $\xi \in B^\delta(z)$.

From (2.2) and the triangle inequality, for $\delta > 0$ we have $m_1 = m_1(\delta)$, $m_2 = m_2(\delta)$ that

$$B^\delta(z) \subseteq B^{m_1\delta}(\xi) \text{ and } B^\delta(\xi) \subseteq B^{m_2\delta}(z) \text{ whenever } \xi \in B^\delta(z). \tag{2.3}$$

Clearly, $m_j > 1$ for $j = 1, 2$. Furthermore,

$$\rho = \sup_{0<\delta\leq 1} [m_1(\delta) + m_2(\delta)] < \infty. \tag{2.4}$$

Given $\delta > 0$, we call a sequence $\{a_k\}_{k=1}^\infty$ in $\mathbb{C}^n$ is a (τ, δ)-lattice if $\left\{B^\delta(a_k)\right\}_k$ covers $\mathbb{C}^n$ and the balls $\left\{B^{\delta/5}(a_k)\right\}_k$ are pairwise disjoint. For $\delta > 0$, the existence of some δ-lattice comes from a standard covering lemma, see Proposition 7 in [6] for details. Given a (τ, δ)-lattice $\{a_k\}_k$ and $m > 0$, there exists some integer N such that each $z \in \mathbb{C}^n$ can be in at most N disks of $\left\{B^{m\delta}(a_k)\right\}_k$. Equivalently,

$$\sum_{k=1}^\infty \chi_{B^{m\delta}(a_k)}(z) \leq N \tag{2.5}$$

for $z \in \mathbb{C}^n$, see [6].

Arroussi and Tong in [1] obtained the pointwise and the L^p_ω-norm estimates of the reproducing kernel $K(\cdot, \cdot)$ as follows:

Lemma 2.1. *Let K_z be the reproducing kernel of F^2_ω. Then*
(a) For $\omega \in \mathcal{W}$, there exists $\alpha \in (0,1]$ such that

$$|K_z(\zeta)| \simeq \|K_z\|_{2,\omega} \cdot \|K_\zeta\|_{2,\omega}, \quad \zeta \in B^\alpha(z). \tag{2.6}$$

(b) For $\omega \in \mathcal{W}$ and $0 < p < \infty$, one has

$$\|K_z\|_{p,\omega} \simeq \omega(z)^{-1/2}\tau(z)^{2n(1-p)/p}, \quad z \in \mathbb{C}^n. \tag{2.7}$$

The following result gives the boundedness of the point evaluation functional on F^p_ω, which can be seen in [1].

Lemma 2.2. *Let $\omega \in \mathcal{W}, \mu \geq 0$ and $0 < p < \infty$. Then for any $f \in H(\mathbb{C}^n)$:*
(a) For any $\delta \in (0,1]$, there exists $C > 0$ such that

$$|f(z)|^p \omega(z)^{p/2} \leq \frac{C}{\delta^{2n}\tau(z)^{2n}} \int_{B^\delta(z)} |f(\zeta)|^p \omega(\zeta)^{p/2} dV(\zeta), \ \ z \in \mathbb{C}^n.$$

(b) For any $\delta > 0$, there exists C depending only on n, p and δ such that

$$\int_{\mathbb{C}^n} |f(z)|^p \omega(z)^{p/2} d\mu(z) \leq C \int_{\mathbb{C}^n} |f(z)|^p \omega(z)^{p/2} \widehat{\mu}_\delta(z) dV(z).$$

For our later use, we need the concepts of averaging functions and Berezin transforms. The average of μ is defined as

$$\widehat{\mu}_\delta(z) = \mu(B^\delta(z)) \cdot \tau(z)^{-2n}, \qquad z \in \mathbb{C}^n.$$

Given $t > 0$, we set the general Berezin transform of μ to be

$$\widetilde{\mu}_t(z) = \int_{\mathbb{C}^n} \left|k_{t,z}(\zeta)\right|^t \omega(\zeta)^{t/2} d\mu(\zeta), \ \ z \in \mathbb{C}^n.$$

Lemma 2.3. *Let α be as defined in* (2.6)*. Suppose $0 < p < \infty, \mu \geq 0$. Then the following statements are equivalent:*
(A) $\widetilde{\mu}_t(\cdot) \in L^p$ for any $t > 0$;
(B) $\widehat{\mu}_\delta(\cdot) \in L^p$ for any $0 < \delta \leq \alpha$;
(C) The sequence $\left\{\tau(a_k)^{2n/p}\widehat{\mu}_r(a_k)\right\}_k \in l^p$ for any (τ, r)-lattice $\{a_k\}_k$ with $0 < r \leq \alpha$.
Furthermore,

$$\|\widetilde{\mu}_t\|_{L^p} \simeq \left\|\widehat{\mu}_\delta\right\|_{L^p} \simeq \left\|\left\{\tau(a_k)^{2n/p}\widehat{\mu}_r(a_k)\right\}_k\right\|_{l^p}. \tag{2.8}$$

Proof. The equivalence between (A) and (B) follows from Lemma 6.1 in [1]. The proof of the equivalence between (B) and (C) is similar to that of Lemma 2.5 in [9] and we omit the details.

3. Results

In this section, we are going to characterize those $\mu \geq 0$ for which the induced Toeplitz operator T_μ is bounded (or compact) from one large Fock space F^p_ω to another F^q_ω. To this purpose, we need the relatively compact subsets in F^p_ω. With the same proof as that of Lemma 3.2 in [8], we know a bounded subset $E \subset F^p_\omega$ is relatively compact if and only if for each $\varepsilon > 0$ there is some $S > 0$ such that

$$\sup_{f\in E} \int_{|z|\geq S} |f(z)|^p \omega(z)^{p/2} dV(z) < \varepsilon. \tag{3.1}$$

This observation on the compact subsets in Fock spaces is crucial to our study on the compactness of T_μ from F^p_ω to F^q_ω. Because the inclusion between any two spaces F^p_ω and F^q_ω is no longer valid while $p \neq q$, and also F^p_ω is not a Banach space with $0 < p < 1$, the approach in [7, 8, 11, 12, 15, 16] does not work here.

Proof of Theorem 1.1. We show $(C) \Rightarrow (D)$ first. Similar to the proof of (2.13) in [9], for given $0 < p < \infty$, $s \in \mathbb{R}$ and (τ, r)-lattice $\{a_j\}_j$, (τ, δ)-lattice $\{b_j\}_j$, we get

$$\left\| \left\{ \widehat{\mu}_r(a_j)\tau(a_j)^{s+2n/p} \right\}_j \right\|_{l^p} \simeq \left\| \left\{ \widehat{\mu}_\delta(b_j)\tau(b_j)^{s+2n/p} \right\}_j \right\|_{l^p}.$$

Then (D) follows from (C) immediately, moreover

$$\left\| \left\{ \widehat{\mu}_r(a_k)\tau(a_k)^{\frac{2n(p-q)}{pq}} \right\}_k \right\|_{l^\infty} \leq \left\| \widehat{\mu}_\delta \tau^{\frac{2n(p-q)}{pq}} \right\|_{L^\infty}. \tag{3.2}$$

Next we prove $(B) \Rightarrow (C)$. Taking $0 < r_0 \leq \alpha$ as α in (2.6), then

$$\widehat{\mu}_{r_0}(z) \lesssim \widetilde{\mu}_2(z).$$

This tells us (B) implies (C) for r_0. By Lemma 2.3, for fixed $\delta, r > 0$ we obtain

$$\|\widehat{\mu}_\delta\|_{L^p} \simeq \|\widehat{\mu}_r\|_{L^p}.$$

Notice that this formula is still true for $p = \infty$. These imply

$$\left\| \widehat{\mu}_\delta \tau^{\frac{2n(p-q)}{pq}} \right\|_{L^\infty} \simeq \left\| \widehat{\mu}_{r_0} \tau^{\frac{2n(p-q)}{pq}} \right\|_{L^\infty} \lesssim \left\| \widetilde{\mu}_t \tau^{\frac{2n(p-q)}{pq}} \right\|_{L^\infty} \tag{3.3}$$

for all $\delta > 0$.

Now we prove that (D) implies (B). By (2.3), we have some $m > 0$ such that $B^r(z) \subset B^{mr}(a)$ for $z \in B^r(a)$ and $a \in \mathbb{C}^n$. For any $t > 0$, set $s = \frac{tpq}{pq-p+q}$. Lemma 2.2 tells us, for $f \in F^s_\omega$,

$$\sup_{z\in B^r(a)} |f(z)|^s \omega(z)^{s/2} \leq \frac{C}{\tau(a)^{2n}} \int_{B^{mr}(a)} |f(\zeta)|^s \omega(\zeta)^{s/2} dV(\zeta). \tag{3.4}$$

By Lemma 2.1, we know

$$|k_{t,z}(\zeta)|^t \tau(z)^{\frac{2n(p-q)}{pq}} \simeq |k_{s,z}(\zeta)|^t.$$

Then from (3.4) and (2.5) we obtain

$$\begin{aligned}
&\widetilde{\mu}_t(z)\tau(z)^{\frac{2n(p-q)}{pq}}\\
&\simeq \int_{\mathbb{C}^n} \left|k_{s,z}(\zeta)\right|^t \omega(\zeta)^{t/2} d\mu(\zeta)\\
&\leq \sum_{k=1}^{\infty} \int_{B^r(a_k)} \left|k_{s,z}(\zeta)\right|^t \omega(\zeta)^{t/2} d\mu(\zeta)\\
&\leq \sum_{k=1}^{\infty} \mu(B^r(a_k)) \left(\sup_{\zeta\in B^r(a_k)} \left|k_{s,z}(\zeta)\right|^s \omega(\zeta)^{s/2} \right)^{t/s}\\
&\lesssim \sum_{k=1}^{\infty} \widehat{\mu}_r(a_k)\tau(a_k)^{\frac{2n(p-q)}{pq}} \left(\int_{B^{mr}(a_k)} \left|k_{s,z}(\zeta)\right|^s \omega(\zeta)^{s/2} dV(\zeta) \right)^{t/s}\\
&\lesssim \sup_k \widehat{\mu}_r(a_k)\tau(a_k)^{\frac{2n(p-q)}{pq}} \left(\sum_{k=1}^{\infty} \int_{B^{mr}(a_k))} \left|k_{s,z}(\zeta)\right|^s \omega(\zeta)^{s/2} dV(\zeta) \right)^{t/s}\\
&\lesssim N^{t/s} \sup_k \widehat{\mu}_r(a_k)\tau(a_k)^{\frac{2n(p-q)}{pq}} \left\|k_{s,z}\right\|_{s,\omega}^t .
\end{aligned}$$

This gives

$$\left\| \widetilde{\mu}_t \tau^{\frac{2n(p-q)}{pq}} \right\|_{L^\infty} \lesssim \left\| \left\{ \widehat{\mu}_r(a_k)\tau(a_k)^{\frac{2n(p-q)}{pq}} \right\}_k \right\|_{l^\infty} . \tag{3.5}$$

That is, (D) indicates (B).

Now we prove that $(A) \Rightarrow (B)$. We suppose the statement (A) is valid. Since $\|k_{p,z}\|_{p,\omega} = 1$, we have

$$\begin{aligned}
\|T_\mu\|_{F^p_\omega \to F^q_\omega} \geq \|T_\mu k_{p,z}\|_{q,\omega} &= \left(\int_{\mathbb{C}^n} |T_\mu k_{p,z}(\zeta)|^q \omega(\zeta)^{q/2} dV(\zeta) \right)^{1/q}\\
&\gtrsim \left(\int_{B(z)} |T_\mu k_{p,z}(\zeta)|^q \omega(\zeta)^{q/2} dV(\zeta) \right)^{1/q}\\
&\gtrsim \tau(z)^{2n/q} |T_\mu k_{p,z}(z)| \omega(z)^{1/2}.
\end{aligned}$$

The last inequality above follows from Lemma 2.2(a). Meanwhile, by Lemma 2.1 we obtain

$$\begin{aligned}
|T_\mu k_{p,z}(z)| &\geq \int_{\mathbb{C}^n} k_{p,z}(\zeta) K(z,\zeta) \omega(\zeta) d\mu(\zeta)\\
&= \frac{1}{\|K(\cdot,z)\|_{p,\omega}} \int_{\mathbb{C}^n} |K(z,\zeta)|^2 \omega(\zeta) d\mu(\zeta)\\
&= \frac{\|K(\cdot,z)\|^2_{2,\omega}}{\|K(\cdot,z)\|_{p,\omega}} \int_{\mathbb{C}^n} |k_{2,z}(\zeta)|^2 \omega(\zeta) d\mu(\zeta)\\
&\simeq \tau(z)^{-2n/p} \omega(z)^{-1/2} \widetilde{\mu}_2(z).
\end{aligned}$$

Therefore,

$$\widetilde{\mu}_2(z)\tau(z)^{\frac{2n(p-q)}{pq}} \lesssim \|T_\mu\|_{F^p_\omega \to F^q_\omega}. \tag{3.6}$$

This and the equivalence between (B) and (C) shows the estimate (3.6) remains true when $\widetilde{\mu}_2$ is replaced by $\widetilde{\mu}_t$ for any $t > 0$. That is, (A) implies (B).

Now we are going to prove the implication $(C) \Rightarrow (A)$. Given $\delta > 0$, we claim there is some positive constant C such that

$$\|T_\mu f\|_{q,\omega}^q \le C \int_{\mathbb{C}^n} |f(\zeta)|^q \omega(\zeta)^{q/2} \widehat{\mu}_\delta(\zeta)^q dV(\zeta) \tag{3.7}$$

for $f \in F_\omega^p$. In fact, when $q > 1$, by applying Lemma 2.2(a) with $\delta = 1$ to the weight ω^2 and the holomorphic function $K(\cdot, z) f(\cdot)$ to get

$$|T_\mu f(z)| \lesssim \int_{\mathbb{C}^n} |K(\zeta, z)||f(\zeta)|\omega(\zeta)\widehat{\mu}_\delta(\zeta) dV(\zeta).$$

This and Hölder's inequality tell us

$$\begin{aligned}
&\left|T_\mu f(z)\right|^q \omega(z)^{q/2} \\
\lesssim &\left(\int_{\mathbb{C}^n} \widehat{\mu}_\delta(\zeta)|f(\zeta)||K(\zeta, z)|\omega(\zeta)\omega(z)^{1/2} dV(\zeta)\right)^q \\
\lesssim &\int_{\mathbb{C}^n} |f(\zeta)|^q \omega(\zeta)^{q/2} \widehat{\mu}_\delta(\zeta)^q \left|K(\zeta, z)\omega(\zeta)^{1/2}\omega(z)^{1/2}\right| dV(\zeta) \\
&\quad \times \left(\int_{\mathbb{C}^n} \left|K(\zeta, z)\omega(\zeta)^{1/2}\omega(z)^{1/2}\right| dV(\zeta)\right)^{\frac{q}{q'}} \\
\lesssim &\int_{\mathbb{C}^n} |f(\zeta)|^q \omega(\zeta)^{q/2} \widehat{\mu}_\delta(\zeta)^q \left|K(\zeta, z)\omega(\zeta)^{1/2}\omega(z)^{1/2}\right| dV(\zeta).
\end{aligned}$$

Integrating both sides above, applying Fubini's theorem and (2.7) to get (3.7). When $q \le 1$, for given $\delta > 0$ we pick some $r > 0$ so that $\rho^2 r \le \min\{\delta, 1\}$ with ρ as in (2.4), and let $\{a_k\}_k$ be some (τ, r)-lattice. Then for $f \in F_\omega^p$,

$$\begin{aligned}
\left|T_\mu f(z)\right|^q &\le \left(\sum_{k=1}^\infty \int_{B^r(a_k)} |f(\zeta)K(\zeta, z)|\omega(\zeta) d\mu(\zeta)\right)^q \\
&\le \sum_{k=1}^\infty \left(\int_{B^r(a_k)} |f(\zeta)K(\zeta, z)|\omega(\zeta) d\mu(\zeta)\right)^q \\
&\le \sum_{k=1}^\infty \widehat{\mu}_r(a_k)^q \tau(a_k)^{2nq} \left(\sup_{\zeta \in B^r(a_k)} |f(\zeta)K(\zeta, z)|\omega(\zeta)\right)^q.
\end{aligned}$$

Apply Lemma 2.2(a), there is some constant $C > 0$ such that $\left|T_\mu f(z)\right|^q$ is not more than C times

$$\sum_{k=1}^\infty \widehat{\mu}_r(a_k)^q \tau(a_k)^{2nq-2n} \int_{B^{\rho r}(a_k)} |f(\zeta)K(\zeta, z)|^q \omega(\zeta)^q dV(\zeta).$$

From (2.3) and (2.4), we have $B^r(a_k) \subseteq B^{\rho^2 r}(\zeta)$ if $\zeta \in B^{\rho r}(a_k)$. This, together with (2.2) and (2.5), implies

$$\begin{aligned}\left|T_\mu f(z)\right|^q &\leq C\sum_{k=1}^{\infty}\int_{B^{\rho r}(a_k)}\widehat{\mu}_{\rho^2 r}(\zeta)^q\tau(\zeta)^{2nq-2n}|f(\zeta)|^q|K(\zeta,z)|^q\omega(\zeta)^q dV(\zeta)\\ &\leq CN\int_{\mathbb{C}^n}\widehat{\mu}_{\rho^2 r}(\zeta)^q\tau(\zeta)^{2nq-2n}|f(\zeta)|^q|K(\zeta,z)|^q\omega(\zeta)^q dV(\zeta)\\ &\leq C\int_{\mathbb{C}^n}\widehat{\mu}_{\delta}(\zeta)^q\tau(\zeta)^{2nq-2n}|f(\zeta)|^q|K(\zeta,z)|^q\omega(\zeta)^q dV(\zeta).\end{aligned}$$

Similarly, integrating both sides of the above with respect to $\omega(z)^{q/2}dV(z)$ and applying Fubini's theorem to get (3.7).

Now we suppose (C) is true, by $p \leq q$, (3.7) and the fact that

$$|f(z)|\omega(z)^{1/2} \lesssim \tau(z)^{-2n/p}\|f\|_{p,\omega} \ \text{ for } \ f \in F^p_\omega,$$

we obtain

$$\begin{aligned}\|T_\mu f\|^q_{q,\omega} &\lesssim \int_{\mathbb{C}^n}|f(\zeta)|^p\omega(\zeta)^{p/2}\widehat{\mu}_\delta(\zeta)^q\left(\tau(\zeta)^{-2n/p}\|f\|_{p,\omega}\right)^{q-p}dV(\zeta)\\ &\lesssim \left\|\widehat{\mu}_\delta\tau^{\frac{2n(p-q)}{pq}}\right\|^q_{L^\infty}\|f\|^q_{p,\omega}\end{aligned}$$

for $f \in F^p_\omega$. Therefore, T_μ is bounded from F^p_ω to F^q_ω and

$$\|T_\mu\|_{F^p_\omega\to F^q_\omega} \lesssim \left\|\widehat{\mu}_\delta\tau^{\frac{2n(p-q)}{pq}}\right\|_{L^\infty}. \tag{3.8}$$

The estimates of (1.2) come from (3.2), (3.3), (3.5), (3.6) and (3.8). The proof is finished.

Proof of Theorem 1.2. The proof of the implications $(B) \Rightarrow (C)$ and $(C) \Rightarrow (D)$ can be carried out as the same part of Theorem 1.1.

Now we assume μ satisfies condition (D) for some (τ, r)-lattice $\{a_k\}_k$. Then, for $\varepsilon > 0$ there exists some integer $K > 0$ such that $\widehat{\mu}_r(a_k)\tau(a_k)^{\frac{2n(p-q)}{pq}} < \varepsilon$ whenever $k > K$. Notice that, $\bigcup_{k=1}^{K}\overline{B^{mr}(a_k)}$ is a compact subset of $\mathbb{C}^n$, and $\{k_{s,z} : z \in \mathbb{C}^n\} \subseteq F^s_\omega$ uniformly converges to 0 on $\bigcup_{k=1}^{K}\overline{B^{mr}(a_k)}$ as $z \to \infty$, where $s = \frac{tpq}{pq-p+q}$. From Lemma 2.1, (2.5) and (3.4), when $|z|$ is sufficiently large, we have

$$\begin{aligned}&\widetilde{\mu}_t(z)\tau(z)^{\frac{2n(p-q)}{pq}}\\ \simeq &\int_{\mathbb{C}^n}\left|k_{s,z}(\zeta)\right|^t\omega(\zeta)^{t/2}d\mu(\zeta)\\ \leq &\int_{\bigcup_{k=1}^{K}\overline{B^{mr}(a_k)}}\left|k_{s,z}(\zeta)\right|^t\omega(\zeta)^{t/2}d\mu(\zeta)\\ &+\sum_{k=K+1}^{\infty}\mu(B^r(a_k))\left(\sup_{\zeta\in B^r(a_k)}\left|k_{s,z}(\zeta)\right|^s\omega(\zeta)^{s/2}d\mu(\zeta)\right)^{t/s}\end{aligned}$$

$$
\begin{aligned}
&< \varepsilon + C \sum_{k=K+1}^{\infty} \widehat{\mu}_r(a_k)\tau(a_k)^{\frac{2n(p-q)}{pq}} \left(\int_{B^{mr}(a_k)} \left|k_{s,z}(\zeta)\right|^s \omega(\zeta)^{s/2} dV(\zeta) \right)^{t/s} \\
&< \varepsilon + C \sup_{k \geq K+1} \widehat{\mu}_r(a_k)\tau(a_k)^{\frac{2n(p-q)}{pq}} \left(\sum_{k=K+1}^{\infty} \int_{B^{mr}(a_k))} \left|k_{s,z}(\zeta)\right|^s \omega(\zeta)^{s/2} dV(\zeta) \right)^{t/s} \\
&< \varepsilon + CN^{\frac{pq-p+q}{pq}} \left\|k_{s,z}\right\|_{s,\omega}^t \varepsilon = C\varepsilon,
\end{aligned}
$$

where C is independent of ε. This yields that $\widetilde{\mu}_t(z)\tau(z)^{\frac{2n(p-q)}{pq}} \to 0$ as $z \to \infty$. So, μ satisfies (B) for any $t > 0$.

To prove $(A) \Rightarrow (B)$, we suppose T_μ is compact from F^p_ω to F^q_ω. Since $\{k_{p,z} : z \in \mathbb{C}^n\}$ is bounded in F^p_ω, $\left\{T_\mu k_{p,z} : z \in \mathbb{C}^n\right\}$ is relatively compact in F^q_ω. By (3.1), for any $\varepsilon > 0$ there exists some $S > 0$ such that

$$
\sup_{z \in \mathbb{C}^n} \int_{|\zeta|>S} \left|T_\mu k_{p,z}(\zeta)\right|^q \omega(\zeta)^{q/2} dV(\zeta) < \varepsilon^q.
$$

When $|z|$ is sufficiently large and $\zeta \in B(z)$,

$$
|\zeta| \geq |z| - |\zeta - z| \geq |z| - \tau(z) \gtrsim |z| - |z|^B \geq |z|^B > S,
$$

where $B \in (0, 1)$ as in (2.1). Hence, $B(z) \subseteq \{\zeta : |\zeta| > S\}$. By the proof of $(A) \Rightarrow (B)$ in Theorem 1.1, we obtain

$$
\widetilde{\mu}_2(z)\tau(z)^{\frac{2n(p-q)}{pq}} \lesssim \left(\int_{B(z)} \left|T_\mu k_{p,z}(\zeta)\right|^q \omega(\zeta)^{q/2} dV(\zeta) \right)^{1/q} < \varepsilon
$$

when $|z|$ is sufficiently large. Hence,

$$
\lim_{z \to \infty} \widetilde{\mu}_2(z)\tau(z)^{\frac{2n(p-q)}{pq}} = 0.
$$

The equivalence between (B) and (C) shows the above limit is still valid if μ_2 is replaced by μ_t for any $t > 0$.

Finally, we suppose the statement (C) is true. For $R > 0$, set μ_R to be $\mu_R(V) = \mu\left(V \cap \overline{B(0, R)}\right)$ for $V \subseteq \mathbb{C}^n$ measurable. Then a similar way to that of Lemma 3.1 in [9] shows T_{μ_R} is compact from F^p_ω to F^q_ω. And also, $\mu - \mu_R \geq 0$. By (C) and (1.2), for $\delta > 0$ fixed, we have

$$
\|T_\mu - T_{\mu_R}\|_{F^p_\omega \to F^q_\omega} \simeq \left\| (\widehat{\mu - \mu_R})_\delta \tau^{\frac{2n(p-q)}{pq}} \right\|_{L^\infty} \to 0
$$

as $R \to \infty$. Therefore, T_μ is compact from F^p_ω to F^q_ω. The proof is finished.

Now we are in the position to prove Theorem 1.3. For our purpose, we recall Khinchine's inequality. Let r_s be the Rademacher function defined by

$$
r_0(t) = \begin{cases} 1, & if\ 0 \leq t - [t] < \frac{1}{2} \\ -1, & if\ \frac{1}{2} \leq t - [t] < 1 \end{cases}
$$

and $r_s(t) = r_0(2^s t)$ for $s = 1, 2, \ldots$, where $[t]$ denotes the largest integer less than or equal to t. For $0 < l < \infty$, there exists some positive constants C_1 and C_2 depending only on l such that

$$
C_1 \left(\sum_{s=1}^m |b_s|^2 \right)^{\frac{l}{2}} \leq \int_0^1 \left| \sum_{s=1}^m b_s r_s(t) \right|^l dt \leq C_2 \left(\sum_{s=1}^m |b_s|^2 \right)^{\frac{l}{2}}
$$

for all $m \geq 1$ and complex numbers $b_1, b_2, \ldots, b_m$. More details can be found in [13].

Proof of Theorem 1.3. The equivalence among the statements $(C), (D)$ and (E) follows from Lemma 2.4. It is trivial that $(B) \Rightarrow (A)$. To finish our proof, we are going to prove the implications $(A) \Rightarrow (E)$, $(D) \Rightarrow (A)$ and $(D) \Rightarrow (B)$.

To get $(A) \Rightarrow (E)$, fix $\delta = \delta_0$ with δ_0 in for any (τ, δ_0)-lattice $\{a_s\}_s$ and sequence $\{\lambda_s\}_s \in l^p$, we consider

$$f(z) = \sum_{n=0}^{\infty} \lambda_n k_{p,a_s}.$$

By Proposition 2.3 in [1] we know $f \in F^p_\omega$ with $\|f\|_{p,\omega} \lesssim \|\{\lambda_s\}_s\|_{l^p}$. Since $T_\mu : F^p_\omega \to F^q_\omega$ is bounded, we obtain

$$T_\mu(f) = \sum_{s=0}^{\infty} \lambda_s T_\mu k_{p,a_s} \in F^q_\omega.$$

By Khinchine's inequality we have

$$\left(\sum_{s=1}^{\infty} \left| \lambda_s T_\mu k_{p,a_s}(z) \right|^2 \right)^{q/2} \lesssim \int_0^1 \left| \sum_{s=1}^{\infty} \lambda_s r_s(t) T_\mu k_{p,a_s}(z) \right|^q dt.$$

This and Fubini's theorem give

$$\begin{aligned}
&\int_{\mathbb{C}^n} \left(\sum_{s=1}^{\infty} \left| \lambda_s T_\mu k_{p,a_s}(z) \right|^2 \right)^{q/2} \omega(z)^{q/2} dV(z) \\
&\lesssim \int_0^1 dt \int_{\mathbb{C}^n} \left| \sum_{s=1}^{\infty} \lambda_s r_s(t) T_\mu k_{p,a_s}(z) \right|^q \omega(z)^{q/2} dV(z) \\
&= \int_0^1 \left\| T_\mu \left(\sum_{s=1}^{\infty} \lambda_s r_s(t) k_{p,a_s} \right) \right\|_{q,\omega}^q dt \\
&\lesssim \|T_\mu\|^q_{F^p_\omega \to F^q_\omega} \|\{\lambda_s\}_s\|^q_{l^p}.
\end{aligned}$$

Meanwhile, there is

$$\begin{aligned}
&\int_{\mathbb{C}^n} \left(\sum_{s=1}^{\infty} \left| \lambda_s T_\mu k_{p,a_s}(z) \right|^2 \right)^{q/2} \omega(z)^{q/2} dV(z) \\
&\gtrsim \sum_{j=1}^{\infty} \int_{B^{\delta_0}(a_j)} \left(\sum_{s=1}^{\infty} \left| \lambda_s T_\mu k_{p,a_s}(z) \right|^2 \right)^{q/2} \omega(z)^{q/2} dV(z) \\
&\gtrsim \sum_{j=1}^{\infty} |\lambda_j|^q \int_{B^{\delta_0}(a_j)} |T_\mu k_{p,a_j}(z)|^q \omega(z)^{q/2} dV(z) \\
&\gtrsim \sum_{j=1}^{\infty} |\lambda_j|^q \tau(a_j)^{2n} |T_\mu k_{p,a_j}(a_j)|^q \omega(a_j)^{q/2}
\end{aligned}$$

$$\gtrsim \sum_{j=1}^{\infty} |\lambda_j|^q \tau(a_j)^{2n+2nq-2nq/p} \left| \int_{B^{\delta_0}(a_j)} |K(a_j,\zeta)|^2 \omega(\zeta) d\mu(\zeta) \right|^q \omega(a_j)^q$$
$$\gtrsim \sum_{j=1}^{\infty} |\lambda_j|^q \tau(a_j)^{2n-2nq/p} \widehat{\mu}_{\delta_0}(a_j)^q,$$

therefore,

$$\sum_{j=1}^{\infty} |\lambda_j|^q \tau(a_j)^{2n-2nq/p} \widehat{\mu}_{\delta_0}(a_j)^q \lesssim \|T_\mu\|^q_{F^p_\omega \to F^q_\omega} \|\{\lambda_j\}_j\|^q_{l^p}$$
$$= \|T_\mu\|^q_{F^p_\omega \to F^q_\omega} \|\{|\lambda_j|^q\}_j\|_{l^{p/q}}.$$

Since $p > q$, the conjugate exponent of $\frac{p}{q}$ is $\frac{p}{p-q}$, the duality argument shows

$$\left\{\tau(a_j)^{2n-2nq/p} \widehat{\mu}_{\delta_0}(a_j)^q\right\}_{j=1}^{\infty} \in l^{\frac{p}{p-q}},$$

and

$$\left\|\left\{\tau(a_j)^{2n-2nq/p} \widehat{\mu}_{\delta_0}(a_j)^q\right\}_j\right\|_{l^{\frac{p}{p-q}}} \lesssim \|T_\mu\|^q_{F^p_\omega \to F^q_\omega}.$$

This and Lemma 2.4 imply

$$\left\|\left\{\tau(a_j)^{\frac{2n(p-q)}{pq}} \widehat{\mu}_\delta(a_j)\right\}_j\right\|_{l^{\frac{pq}{p-q}}} \lesssim \|T_\mu\|_{F^p_\omega \to F^q_\omega} \tag{3.9}$$

for any (τ, δ)-lattice $\{a_j\}$. From this, the conclusion (E) follows.

Now we prove $(D) \Rightarrow (A)$. Suppose $\widehat{\mu}_s \in L^{\frac{pq}{p-q}}$ for some $s > 0$. Similar to that in Theorem 4.4 of [11], we know $\left\{\widehat{\mu}_s(a_k)\tau(a_k)^{\frac{2n(p-q)}{pq}}\right\}_k \in l^\infty$ for some (τ, s)-lattice $\{a_k\}_k$. Theorem 1.1 gives $\widehat{\mu}_s \tau^{\frac{2n(p-q)}{pq}} \in L^\infty$, which shows that T_μ is well-defined on F^p_ω. Notice that $p/q > 1$. By (3.7), Hölder's inequality and (2.7), we obtain

$$\|T_\mu f\|^q_{q,\omega} \lesssim \left\{\int_{\mathbb{C}^n} \left(|f(\zeta)|^q \omega(\zeta)^{q/2}\right)^{p/q} dV(\zeta)\right\}^{q/p} \left\{\int_{\mathbb{C}^n} \widehat{\mu}_s(\zeta)^{\frac{pq}{p-q}} dV(\zeta)\right\}^{\frac{p-q}{p}}$$
$$\lesssim \|\widehat{\mu}_s\|^q_{L^{\frac{pq}{p-q}}} \|f\|^q_{p,\omega}$$

for $f \in F^p_\omega$. Hence, T_μ is bounded from F^p_ω to F^q_ω with

$$\|T_\mu\|_{F^p_\omega \to F^q_\omega} \lesssim \|\widehat{\mu}_s\|_{L^{\frac{pq}{p-q}}}. \tag{3.10}$$

To prove $(D) \Rightarrow (B)$, we take μ_R as $\mu_R(V) = \mu\left(V \cap \overline{B(0,R)}\right)$ for $V \subseteq \mathbb{C}^n$ measurable. Then $\mu - \mu_R \geq 0$, and for $s > 0$ we have $\left\|(\widehat{\mu - \mu_R})_s\right\|_{L^{\frac{pq}{p-q}}} \to 0$ as $R \to \infty$. By (3.10),

$$\|T_\mu - T_{\mu_R}\|_{F^p_\omega \to F^q_\omega} = \|T_{(\mu-\mu_R)}\|_{F^p_\omega \to F^q_\omega} \lesssim \left\|(\widehat{\mu - \mu_R})_s\right\|_{L^{\frac{pq}{p-q}}} \to 0$$

whenever $R \to \infty$. Since T_{μ_R} is compact from F^p_ω to F^q_ω, the operator $T_\mu : F^p_\omega \to F^q_\omega$ is compact as well.

The norm equivalence (1.3) comes from Lemma 2.3, (3.9) and (3.10). The proof is finished.

4. Conclusions

In this paper, we study those $\mu \geq 0$ for which the induced Toeplitz operators T_μ are bounded (or compact) between two large Fock spaces F^p_ω and F^q_ω for all possible $0 < p, q < \infty$. Our approach depends on whether $0 < p \leq q < \infty$ or $0 < q < p < \infty$. The boundedness (or compactness) of $T_\mu : F^p_\omega \to F^q_\omega$ is characterized in terms of the average or the general Berezin transforms of μ.

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Conflict of interest

The authors declare no conflict of interest in this paper.

References

1. H. Arroussi, C. Z. Tong, Weighted composition operators between large Fock spaces in several complex variables, *J. Funct. Anal.*, **277** (2019), 3436–3466. doi: 10.1016/j.jfa.2019.04.008.
2. C. A. Berger, L. A. Coburn, Toeplitz operators on the Segal-Bargmann space, *Trans. Amer. Math. Soc.*, **301** (1987), 813–829. doi: 10.2307/2000671.
3. W. Bauer, L. A. Coburn, J. Isralowitz, Heat flow, BMO, and the compactness of Toeplitz operators, *J. Funct. Anal.*, **259** (2010), 57–78. doi: 10.1016/j.jfa.2010.03.016.
4. W. Bauer, J. Isralowitz, Compactness characterization of operators in the Toeplitz algebra of the Fock space F^p_α, *J. Funct. Anal.*, **263** (2012), 1323–1355. doi: 10.1016/j.jfa.2012.04.020.
5. L. A. Coburn, J. Isralowitz, B. Li, Toeplitz operators with BMO symbols on the Segal-Bargmann space, *Trans. Amer. Math. Soc.*, **363** (2011), 3015–3030. doi: 10.1090/S0002-9947-2011-05278-5.
6. G. M. Dall'Ara, Pointwise eatimates of weighted Bergman kernels in several complex variables, *Adv. Math.*, **285** (2015), 1706–1740. doi: 10.1016/j.aim.2015.06.024.
7. Z. J. Hu, X. F. Lv, Toeplitz operators from one Fock space to another, *Integr. Equat. Oper. Th.*, **70** (2011), 541–559. doi: 10.1007/s00020-011-1887-y.
8. Z. J. Hu, X. F. Lv, Toeplitz operators on Fock spaces $F^p(\varphi)$, *Integr. Equat. Oper. Th.*, **80** (2014), 33–59. doi: 10.1007/s00020-014-2168-3.
9. Z. J. Hu, X. F. Lv, Positive Toeplitz operators between different doubling Fock spaces, *Taiwan. J. Math.*, **21** (2017), 467–487. doi: 10.11650/tjm/7031.
10. Z. J. Hu, E. M. Wang, Hankel operators between Fock spaces, *Integr. Equat. Oper. Th.*, **90** (2018), 1–20. doi: 10.1007/s00020-018-2459-1.

11. J. Isralowitz, K. H. Zhu, Toeplitz operators on the Fock space, *Integr. Equat. Oper. Th.*, **66** (2010), 593–611. doi: 10.1007/s00020-010-1768-9.

12. J. Lu, X. F. Lv, Toeplitz operators between Fock spaces, *B. Aust. Math. Soc.*, **92** (2015), 316–324. doi: 10.1017/S0004972715000477.

13. D. H. Luecking, Embedding theorems for spaces of analytic functions via Khinchine's inequality, *Mich. Math. J.*, **40** (1993), 333–358. doi: 10.1307/mmj/1029004756.

14. X. F. Lv, Bergman projections on weighted Fock spaces in several complex variables, *J. Inequal. Appl.*, **2017** (2017), 1–10. doi: 10.1186/s13660-017-1560-3.

15. R. Oliver, D. Pascuas, Toeplitz operators on doubling Fock spaces, *J. Math. Anal. Appl.*, **435** (2016), 1426–1457. doi: 10.1016/j.jmaa.2015.11.023.

16. A. P. Schuster, D. Varolin, Toeplitz operators and Carleson measures on generalized Bargmann-Fock spaces, *Integr. Equat. Oper. Th.*, **72** (2012), 363–392. doi: 10.1007/s00020-011-1939-3.

17. E. M. Wang, Toeplitz operators with BMO and IMO symbols between Fock spaces, *Arch. Math.*, **114** (2020), 541–551. doi: 10.1007/s00013-020-01445-4.

18. K. H. Zhu, *Analysis on Fock spaces*, New York: Springer, 2012. doi: 10.1007/978-1-4419-8801-0.