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Research article

On a Langevin equation involving Caputo fractional proportional derivatives with respect to another function

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Abstract: In this work, we introduce and study a class of Langevin equation with nonlocal boundary conditions governed by a Caputo fractional order proportional derivatives of an unknown function with respect to another function. The qualitative results concerning the given problem are obtained with the aid of the lower regularized incomplete Gamma function and applying the standard fixed point theorems. In order to homologate the theoretical results we obtained, we present two examples.

Keywords: Caputo fractional proportional order derivative with respect to another function; Langevin equation; incomplete Gamma function; fixed point theorems
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1. Introduction

The classical calculus connected to the traditional integrals and derivatives is considered to be the core of modern mathematics. The fractional calculus is the generalization of this calculus as it deals with the integrals and derivatives of any order. There has been a great deal of interest in such type of generalizing calculus because of the findings obtained by some researchers who utilized the fractional integrals and derivatives being at the receiving end of modeling some real world problems that arise in variety of disciplines [1–15]. What makes the fractional calculus distinctive is the fact there are variety of fractional integrals and derivatives and thus a researcher can choose the best fractional operator which suited to the problem under investigation. Moreover, there are two kinds of fractional operators. The first type consist of non-local fractional operators. The second type contains local ones. The local fractional derivatives were initiated first by Khalil et al. [16, 17]. The derivatives proposed in these two works were modified by [18, 19]. The modified derivative was used by Jarad et al. [20] to generate a new class of fractional operators called fractional proportional operators which contain two parameters and give rise to known fractional operators when one of these parameters tend to certain values. And even more, these operators were generalized in [21, 22] and fractional proportional operators with respect to an increasing function were proposed.

The Langevin equation embodying integer order derivative was proposed by Langevin in 1908 [23]. This well known equation delineates the evolution of certain physical phenomena in fluctuating environments [24] and describes anomalous transport [25]. It was extended to the fractional order by Lim et al. [26] who proposed a version of Langevin equations involving two fractional order for the sake of depicturing the viscoelastic anomalous diffusion in the complex liquids. In [27], the authors considered a generalized Langevin equation that lims mechanical random forces. Lozinski et al. [28] considered applications of the mentioned equation in polymer rheology and stochastic simulation. In [29], Laadjal et al. discussed some qualitative properties of solutions to multi-term fractional Langevin equation with boundary conditions.

Recently, Laadjal et al. [30] have studied the existence and uniqueness of solutions to fractional proportional differential equation with the help of incomplete Gamma function.

Motivated and inspired by the aforementioned works, in this article, we deliberate the existence and uniqueness of solutions to the following class of Langevin differential equations:

$$ {}_{p}^{C}\mathfrak{D}_{a}^{\alpha,\rho,v}\left({}_{p}^{C}\mathfrak{D}_{a}^{\beta,\rho,v}+\lambda\right)x(t)=f(t,x(t)),\quad t\in[a,b], \tag{1.1}$$

$$ x(a)=0,\quad x(b)=\xi x(\eta), \tag{1.2}$$

where $\rho\in(0,1]$, $0<\alpha,\beta\leq 1, a<\eta<b$, $\lambda,\xi\in\mathbb{R}$, $f:[a,b]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given nonlinear function, $v(t)$ is a strictly increasing continuous function on $[a,b]$ and ${}_{p}^{C}\mathfrak{D}_{a}^{i,\rho,v}$ denotes the Caputo fractional proportional derivative (CFPD) with respect to the function v of order i $(i=\alpha,\beta)$.

Note that from Eq (1.1), we have the following special cases (with the nonlocal boundary conditions (1.2)):

Case 1. *If $v(t)=t$ for all $t\in[a,b]$, Eq (1.1) reduces to a Langevin equation involving two v-CFPDs.*

$$ {}_{p}^{C}\mathfrak{D}_{a}^{\alpha,\rho}\left({}_{p}^{C}\mathfrak{D}_{a}^{\beta,\rho}+\lambda\right)x(t)=f(t,x(t)). \tag{1.3}$$

Case 2. *If $\rho=1$, Eq (1.1) reduces to a Langevin equation involving two v-Caputo fractional derivatives*

$$ {}^{C}D_{a}^{\alpha,v}\left({}^{C}D_{a}^{\beta,v}+\lambda\right)x(t)=f(t,x(t)). \tag{1.4}$$

Case 3. *If $\rho=1$ and $v(t)=t$, Eq (1.1) reduces to a Langevin equation involving the usual Caputo fractional derivatives*

$$ {}^{C}D_{a}^{\alpha}\left({}^{C}D_{a}^{\beta}+\lambda\right)x(t)=f(t,x(t)). \tag{1.5}$$

Case 4. *If $\rho=1$ and $v(t)=\ln t$ for all $t\in[a,b], a>0$, (1.1) reduces to a Langevin equation involving Caputo-Hadamard fractional derivatives*

$$ {}^{CH}D_{a}^{\alpha}\left({}^{CH}D_{a}^{\beta}+\lambda\right)x(t)=f(t,x(t)). \tag{1.6}$$

Case 5. *If $\rho=1$ and $v(t)=\frac{t^{\mu}}{\mu}$, (1.1) reduces to a Langevin equation involving the Katugampola fractional derivatives*

$$ {}^{CK}D_{a}^{\alpha}\left({}^{CK}D_{a}^{\beta}+\lambda\right)x(t)=f(t,x(t)). \tag{1.7}$$

Moreover, other several special cases can be obtained as well.

2. Preliminaries

In this section, we present some definitions, propositions, lemmas and theorems needed through the whole article.

For $\theta > 0$ (with $n-1 < \theta \leq n$, $n \in \mathbb{N}$) and $\psi \in L^1[a,b]$, we have the following definitions [3]:

The fractional integral of Reimann-Liouville type of the function ψ is defined by [3]

$$(I_a^{\theta}\psi)(t) = \frac{1}{\Gamma(\theta)} \int_a^t (t-\tau)^{\theta-1} \psi(\tau)\, d\tau. \tag{2.1}$$

The fractional derivative of Reimann-Liouville type of the function ψ is defined by

$$\begin{aligned}({}^{R}D_a^{\theta}\psi)(t) &= \frac{d^n}{dt^n} I_a^{n-\theta}\psi(t) \\ &= \frac{1}{\Gamma(n-\theta)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\theta-1} \psi(\tau)\, d\tau.\end{aligned} \tag{2.2}$$

The fractional derivative of Caputo type of the function $\psi \in C^{(n)}[a,b]$. is defined by [3]

$$\begin{aligned}({}^{C}D_a^{\theta}\psi)(t) &= (I_a^{n-\theta}\psi^{(n)})(t) \\ &= \frac{1}{\Gamma(n-\theta)} \int_a^t (t-\tau)^{n-\theta-1} \psi^{(n)}(\tau)\, d\tau.\end{aligned} \tag{2.3}$$

The fractional integral of Katugampola type of the function ψ is defined by [31]

$$({}^{K}I_a^{\theta,\mu}\psi)(t) = \frac{1}{\Gamma(\theta)} \int_a^t \left(\frac{t^{\mu}-\tau^{\mu}}{\mu}\right)^{\theta-1} \psi(\tau) \frac{d\tau}{\tau^{1-\mu}}. \tag{2.4}$$

The Caputo-Katugampola fractional derivative of the function $\psi \in C^{(n)}[a,b]$ is defined by [32]

$$\begin{aligned}({}^{CK}D_a^{\theta,\mu}\psi)(t) &= ({}^{K}I_a^{n-\theta}\zeta^n\psi)(t) \\ &= \frac{1}{\Gamma(n-\theta)} \int_a^t \left(\frac{t^{\mu}-\tau^{\mu}}{\mu}\right)^{\theta-1} \zeta^n\psi(\tau) \frac{d\tau}{\tau^{1-\mu}}.\end{aligned} \tag{2.5}$$

where $\zeta = t^{1-\mu}\frac{d}{dt}$.

The fractional integral of Haramard type of the function ψ is defined by [3]

$$({}^{H}I_a^{\theta}\psi)(t) = \frac{1}{\Gamma(\theta)} \int_a^t \left(\ln\frac{t}{\tau}\right)^{\theta-1} \psi(\tau) \frac{d\tau}{\tau}. \tag{2.6}$$

The Caputo-Hadamard fractional derivative of the function $\psi \in C^{(n)}[a,b]$ is defined by [33]

$$\begin{aligned}({}^{CH}D_a^{\theta,\rho}\psi)(t) &= ({}^{H}I_a^{n-\theta}\gamma^n\psi)(t) \\ &= \frac{1}{\Gamma(n-\theta)} \int_a^t \left(\ln\frac{t}{\tau}\right)^{n-\theta-1} \gamma^n\psi(\tau) \frac{d\tau}{\tau}.\end{aligned} \tag{2.7}$$

where $\gamma = t\frac{d}{dt}$.

Let $\rho \in (0,1]$ and v be strictly increasing continuously differentiable function. The Reimann-Liouville fractional proportional integral (RLFPI) of the function $\psi \in L^1[a,b]$ with respect to the function v is defined by [20]

$$(\mathcal{J}_a^{\theta,\rho,v}\psi)(t) = \frac{1}{\rho^\theta \Gamma(\theta)} \int_a^t (v(t) - v(\tau))^{\theta-1} e^{\frac{\rho-1}{\rho}(v(t)-v(\tau))} \psi(\tau) v'(\tau) d\tau. \tag{2.8}$$

Let $\rho \in (0,1]$. The Caputo fractional proportional derivative (CFPD) of the function $\psi \in C^{(n)}[a,b]$ with respect to the function $v \in C^{(n)}[a,b]$ is defined by [20]

$$\begin{aligned}({}_p^C\mathfrak{D}_a^{\theta,\rho,v}\psi)(t) &= \mathcal{J}_a^{n-\theta,\rho,v}(D^{n,\rho,v}\psi)(t) \\ &= \frac{1}{\rho^\theta \Gamma(n-\theta)} \int_a^t (v(t) - v(\tau))^{n-\theta-1} e^{\frac{\rho-1}{\rho}(v(t)-v(\tau))} (D^{n,\rho,v}\psi)(\tau) v'(\tau) d\tau. \end{aligned} \tag{2.9}$$

where

$$(D^{n,\rho,v}\psi)(t) = (\underbrace{D^{\rho,v}D^{\rho,v}\cdots D^{\rho,v}}_{n\text{-times}}\psi)(t), \tag{2.10}$$

with

$$(D^{\rho,v}\psi)(t) = (1-\rho)\psi(t) + \rho\frac{\psi'(t)}{v'(t)}. \tag{2.11}$$

Let $\rho \in (0,1]$. The Reimann-Liouville fractional proportional derivative (RLFPD) of the function ψ with respect to the function v is defined by [20]

$$\begin{aligned}({}_p^{\mathcal{R}}\mathfrak{D}_a^{\theta,\rho,v}\psi)(t) &= D^{n,\rho,v}(\mathcal{J}_a^{n-\theta,\rho,v}\psi)(t) \\ &= \frac{D^{n,\rho,v}}{\rho^{n-\theta}\Gamma(n-\theta)} \int_a^t (v(t) - v(\tau))^{n-\theta-1} e^{\frac{\rho-1}{\rho}(v(t)-v(\tau))} \psi(\tau) v'(\tau) d\tau. \end{aligned} \tag{2.12}$$

Remark 6. *Note that, for $\rho = 1$ and $v(t) = t$, the definitions of the RLFPD and CFPD reduce to the usual definitions of Riemann-Liouville fractional derivative and Caputo fractional derivative, respetively. On other hand note that ${}_p^{\mathcal{R}}\mathfrak{D}_a^{-\theta,\rho,v} = \mathcal{J}_a^{\theta,\rho,v}$.*

Proposition 7 ([20]). *Let $\rho \in (0,1], \beta > 0$ and $\theta > 0$ with $n-1 < \theta \leq n$, and $\psi \in L^1[a,b]$, we have the following properties:*

$$(\mathcal{J}_a^{\theta,\rho,v}(v(\cdot) - v(a))^{\beta-1} e^{\frac{\rho-1}{\rho}v(\cdot)})(t) = \frac{\Gamma(\beta)}{\rho^\theta\Gamma(\theta+\beta)}(v(t) - v(a))^{\theta+\beta-1} e^{\frac{\rho-1}{\rho}v(t)}; \tag{2.13}$$

$$({}_p^{\mathcal{R}}\mathfrak{D}_a^{\theta,\rho,v}(v(\cdot) - v(a))^{\beta-1} e^{\frac{\rho-1}{\rho}v(\cdot)})(t) = \frac{\rho^\theta\Gamma(\beta)}{\Gamma(\beta-\theta)}(v(t) - v(a))^{\beta-\theta-1} e^{\frac{\rho-1}{\rho}v(t)}; \tag{2.14}$$

$$\mathcal{J}_a^{\theta,\rho,v}(\mathcal{J}_a^{\beta,\rho,v}\psi)(t) = \mathcal{J}_a^{\beta,\rho,v}(\mathcal{J}_a^{\theta,\rho,v}\psi)(t) = (\mathcal{J}_a^{\theta+\beta,\rho,v}\psi)(t); \tag{2.15}$$

$${}_p^C\mathfrak{D}_a^{\theta,\rho,v}(\mathcal{J}_a^{\theta,\rho,v}\psi)(t) = \psi(t); \tag{2.16}$$

$${}_p^{\mathcal{R}}\mathfrak{D}_a^{\theta,\rho,v}(\mathcal{J}_a^{\theta,\rho,v}\psi)(t) = \psi(t). \tag{2.17}$$

Proposition 8 ([21]). *We have*

$$\mathcal{J}_{a}^{\theta,\rho,v}({}_{p}^{C}\mathfrak{D}_{a}^{\theta,\rho,v}\psi)\,(t) = \psi\,(t) - \sum_{k=0}^{n-1} c_k\,(v(t)-v(a))^k\, e^{\frac{\rho-1}{\rho}(v(t)-v(\tau))},\ \psi \in C^{(n)}[a,b], \tag{2.18}$$

where $c_k = \frac{(D^{k,\rho,v}\psi)(a)}{\rho^k k!}$;

$$\mathcal{J}_{a}^{\theta,\rho}({}_{p}^{\mathcal{R}}\mathfrak{D}_{a}^{\theta,\rho,v}\psi)\,(t) = \psi\,(t) - \sum_{k=1}^{n} q_k\,(v(t)-v(\tau))^{\theta-k}\, e^{\frac{\rho-1}{\rho}(v(t)-v(\tau))}, \tag{2.19}$$

where $q_k = \frac{(\mathcal{J}_a^{k-\theta,\rho,v}\psi)(a)}{\rho^{\theta-k}\Gamma(\theta-k+1)}$.

Definition 9 ([34,35]). *Let* $\theta \in \mathbb{C}$ $(\Re(\theta) > 0)$, *we have the following definitions:*
The upper incomplete Gamma function is defined by

$$\Gamma(\theta,t) = \int_t^{+\infty} y^{\theta-1}e^{-y}dy, \quad t \geq 0. \tag{2.20}$$

The lower incomplete Gamma function is defined by

$$\gamma(\theta,t) = \int_0^{t} y^{\theta-1}e^{-y}dy, \quad t \geq 0. \tag{2.21}$$

The upper regularized incomplete Gamma function is defined by

$$Q(\theta,t) = \frac{\Gamma(\theta,t)}{\Gamma(\theta)}. \tag{2.22}$$

The lower regularized incomplete Gamma function is defined by

$$\mathcal{P}(\theta,t) = 1 - Q(\theta,t) = \frac{\gamma(\theta,t)}{\Gamma(\theta)}. \tag{2.23}$$

The functions $\mathcal{P}$ *and* Q *are also called "Incomplete Gamma functions ratios".*

Lemma 10 ([34]). *Let* $\theta \geq 0$, *For all* $t \geq 0$ *we have the following properties:*

$$\Gamma(\theta+1,t) = \theta\Gamma(\theta,t) + t^{\theta}e^{-t}; \tag{2.24}$$

$$\gamma(\theta,t) = \Gamma(\theta) - \Gamma(\theta,t); \tag{2.25}$$

$$\gamma(\theta+1,t) = \theta\gamma(\theta,t) - t^{\theta}e^{-t}; \tag{2.26}$$

$$\int_{t_1}^{t_2} y^{\theta-1}e^{-y}dy = \gamma(\theta,t_2) - \gamma(\theta,t_1), t_2 \geq t_1 > 0. \tag{2.27}$$

Lemma 11 ([30]). *Let* $\theta,\mu \in \mathbb{R}^+$. *It is clear that* $\mathcal{P}(\theta,\mu(t-a))$ *is a non-decreasing function with respect to* $t \in [a,b]$. *And moreover*

$$\mathcal{P}(\theta,\mu(t-a)) \in [0,1]\, \textit{for all}\ t \geq a; \tag{2.28}$$

$$\max_{t\in[a,b]}\mathcal{P}(\theta,\mu(t-a)) = \mathcal{P}(\theta,\mu(t-a))|_{t=b} = \mathcal{P}(\theta,\mu(b-a)); \tag{2.29}$$

$$\min_{t\in[a,b]}\mathcal{P}(\theta,\mu(t-a)) = \mathcal{P}(\theta,\mu(t-a))|_{t=a} = 0. \tag{2.30}$$

3. Incomplete Gamma functions vs RLFPIs with respect to another function

In this section, we present new essential lemmas related to the incomplete Gamma functions. These lemmas will be helpful in proving our main results about the existence and uniqueness of solutions for the considered problem.

Remark 12. *In all the following results, we assume that $v : [a, b] \longrightarrow \mathbb{R}$ is a continuous, differentiable and strictly increasing function.*

Lemma 13. *Let $\rho \in (0, 1]$, $\theta > 0$, and $\psi(t) = 1$ for all $t \in [a, b]$. Then*

$$(\mathcal{J}_a^{\theta,\rho,v} 1)(t) = \begin{cases} \frac{\mathcal{P}(\theta, \frac{1-\rho}{\rho}(v(t)-v(a)))}{(1-\rho)^\theta}, & for \quad \rho \in (0, 1), \\ \frac{(v(t)-v(a))^\theta}{\Gamma(\theta+1)}, & for \quad \rho = 1, \end{cases} \tag{3.1}$$

where function $\mathcal{P}$ is defined by (2.23). Moreover,

$$\lim_{\rho \to 1^-} (\mathcal{J}_a^{\theta,\rho,v} 1)(t) = \left(I_a^{\theta,v} 1\right)(t) = \frac{(v(t) - v(a))^\theta}{\Gamma(\theta + 1)}, \tag{3.2}$$

and

$$\max_{t \in [a,b]} [\lim_{\rho \to 1^-} (\mathcal{J}_a^{\theta,\rho,v} 1)(t)] = \frac{(v(b) - v(a))^\theta}{\Gamma(\theta + 1)}. \tag{3.3}$$

Proof. For $\rho \in (0, 1)$, from Definition 2.8, we have

$$(\mathcal{J}_a^{\theta,\rho,v} 1)(t) = \frac{1}{\rho^\theta \Gamma(\theta)} \int_a^t (v(t) - v(\tau))^{\theta-1} e^{\frac{\rho-1}{\rho}(v(t)-v(\tau))} v'(\tau) d\tau. \tag{3.4}$$

Let $y = \frac{1-\rho}{\rho}(v(t) - v(\tau))$, then $dy = -\frac{1-\rho}{\rho} v'(\tau) d\tau$, So $d\tau = -\frac{\rho}{1-\rho} \frac{1}{v'(\tau)} dy$. Hence, we have

$$\begin{aligned} (\mathcal{J}_a^{\theta,\rho,v} 1)(t) &= \frac{1}{\rho^\theta \Gamma(\theta)} \int_a^t \left(\frac{\rho}{1-\rho} y\right)^{\theta-1} e^{\frac{\rho-1}{\rho}\left(\frac{\rho}{1-\rho} y\right)} v'(\tau) \left(-\frac{\rho}{1-\rho} \frac{1}{v'(\tau)} dy\right) \\ &= \frac{-1}{\rho^\theta \Gamma(\theta)} \int_{\frac{1-\rho}{\rho}(v(t)-v(a))}^0 \left(\frac{\rho}{1-\rho} y\right)^{\theta-1} e^{-y} \frac{\rho}{1-\rho} dy \\ &= \frac{1}{(1-\rho)^\theta \Gamma(\theta)} \int_0^{\frac{1-\rho}{\rho}(v(t)-v(a))} y^{\theta-1} e^{-y} dy \\ &= \frac{\gamma(\theta, \frac{1-\rho}{\rho} (v(t) - v(a)))}{(1-\rho)^\theta \Gamma(\theta)} \\ &= \frac{\mathcal{P}(\theta, \frac{1-\rho}{\rho} (v(t) - v(a)))}{(1-\rho)^\theta}. \end{aligned}$$

For $\rho = 1$ we have

$$(\mathcal{J}_a^{\theta,\rho,v} 1)(t) = \frac{1}{\Gamma(\theta)} \int_a^t (v(t) - v(\tau))^{\theta-1} v'(\tau) d\tau.$$

$$= \frac{(v(t)-v(a))^{\theta}}{\Gamma(\theta+1)}.$$

Concerning the limit formula (3.2) , we have

$$\begin{aligned}
\lim_{\rho\to 1^-}\left(\mathcal{J}_a^{\theta,\rho,v}1\right)(t) &= \lim_{\rho\to 1^-}\frac{1}{\rho^{\theta}\Gamma(\theta)}\int_a^t (v(t)-v(\tau))^{\theta-1}\, e^{\frac{\rho-1}{\rho}(t-\tau)}v'(\tau)d\tau \\
&= \frac{1}{\Gamma(\theta)}\int_a^t (v(t)-v(\tau))^{\theta-1}\, v'(\tau)d\tau \\
&= \frac{(v(t)-v(a))^{\theta}}{\Gamma(\theta+1)}.
\end{aligned}$$

Finally, formula (3.3) is immediate and hence the proof is completed. □

Lemma 14. *Let $X = C([a,b],\mathbb{R})$ be the Banach space of all continuous functions from $[a,b]$ to $\mathbb{R}$ endowed with the norm $\|\psi\| = \sup_{t\in[a,b]}|\psi(t)|$, and let $\rho\in(0,1]$, $\theta>0$ and $\psi\in X$. Then*

$$|(\mathcal{J}_a^{\theta,\rho,v}\psi)(t)| \le \begin{cases} \frac{\mathcal{P}(\theta,\frac{1-\rho}{\rho}(v(t)-v(a)))}{(1-\rho)^{\theta}}\,\|\psi\|, & \text{for } \rho\in(0,1), \\ \frac{(v(t)-v(a))^{\theta}}{\Gamma(\theta+1)}\,\|\psi\|, & \text{for } \rho=1, \end{cases} \tag{3.5}$$

for all $t\in[a,b]$. Moreover, for $\eta\in[a,b]$, we have

$$\sup_{t\in[a,\eta]}|(\mathcal{J}_a^{\theta,\rho,v}\psi)(t)| \le \begin{cases} \frac{\mathcal{P}(\theta,\frac{1-\rho}{\rho}(v(\eta)-v(a)))}{(1-\rho)^{\theta}}\,\|\psi\|, & \text{for } \rho\in(0,1), \\ \frac{(v(\eta)-v(a))^{\theta}}{\Gamma(\theta+1)}\,\|\psi\|, & \text{for } \rho=1. \end{cases} \tag{3.6}$$

Proof. The proof can be carried out by following the same steps as in Lemma 13. □

Lemma 15. *Let $\rho\in(0,1]$, $t_1,t_2\in[a,b]$ $(t_1\le t_2)$, and $\delta>0$. Then*

$$\begin{aligned}
&\int_{t_1}^{t_2}(v(b)-v(\tau))^{\delta-1}\, e^{\frac{\rho-1}{\rho}(v(b)-v(\tau))}v'(\tau)d\tau \qquad (3.7)\\
&= \frac{\rho^{\delta}\Gamma(\delta)}{(1-\rho)^{\delta}}\left[\mathcal{P}(\delta,\frac{1-\rho}{\rho}(v(b)-v(t_1))) - \mathcal{P}(\delta,\frac{1-\rho}{\rho}(v(b)-v(t_2)))\right],
\end{aligned}$$

where the function $\mathcal{P}$ is given by (2.23).

Proof. The proof can be accomplished by trailing the same steps as in Lemma 3.3 of [30] and Lemma 13. □

Lemma 16. *Let $\rho\in(0,1]$, $\delta>0$ and $a\le\tau\le t_1<t_2\le b$. Then*

$$\lim_{t_2\to t_1}\int_a^{t_1}|(V_{\delta}(t_2,\tau)-V_{\delta}(t_1,\tau))\,v'(\tau)|\,d\tau = 0, \tag{3.8}$$

where

$$V_{\delta}(t,\tau) = (v(t)-v(\tau))^{\delta-1}\, e^{\frac{\rho-1}{\rho}(v(t)-v(\tau))}. \tag{3.9}$$

Proof. To calculate the above limit, the sign of the term inside the absolute value must be studied.

From Remark 12, $v'(\tau) > 0$ for all $\tau \in [a,b]$, and thus for any $s_1, s_2 \in [a,b]$ such that $s_2 > s_1$, we have $v(s_2) > v(s_1)$.

For $\rho = 1$, we look at the three cases $\delta = 1, \delta < 1$ and $\delta > 1$ as follows

$$\begin{aligned}
&\int_a^{t_1} |(V_\delta(t_2,\tau) - V_\delta(t_1,\tau))\, v'(\tau)|_{\rho=1}\, v'(\tau)d\tau \\
= &\int_a^{t_1} \left|(v(t_2) - v(\tau))^{\delta-1} - (v(t_1) - v(\tau))^{\delta-1}\right| v'(\tau)d\tau \\
= &\begin{cases} 0, \ \text{ for } \delta = 1, \\ \frac{1}{\delta}\left((v(t_2) - v(t_1))^\delta - (v(t_2) - v(a))^\delta + (v(t_1) - v(a))^\delta\right), \ \text{ for } 0 < \delta < 1, \\ -\frac{1}{\delta}\left((v(t_2) - v(t_1))^\delta - (v(t_2) - v(a))^\delta + (v(t_1) - v(a))^\delta\right), \ \text{ for } \delta > 1, \end{cases} .
\end{aligned}$$

hence the integral has the value zero as $t_2 \to t_1$.

Next, for $\rho \in (0,1)$ and $0 < \delta \le 1$: because $\delta - 1 \le 0$, $\frac{\rho-1}{\rho}(v(t_2) - v(\tau)) \le 0$, and $\frac{\rho-1}{\rho}(v(t_1) - v(\tau)) \le 0$, we conclude that

$$\begin{aligned}
V_\delta(t_2,\tau) - V_\delta(t_1,\tau) &= (v(t_2) - v(\tau))^{\delta-1}\, e^{\frac{\rho-1}{\rho}(v(t_2)-v(\tau))} - (v(t_1) - v(\tau))^{\delta-1}\, e^{\frac{\rho-1}{\rho}(v(t_1)-v(\tau))} \\
&\le 0.
\end{aligned}$$

Then, we get

$$\begin{aligned}
\int_a^{t_1} |(V_\delta(t_2,\tau) - V_\delta(t_1,\tau))\, v'(\tau)|\, d\tau &= \int_a^{t_1} -(v(t_2) - v(\tau))^{\delta-1}\, e^{\frac{\rho-1}{\rho}(v(t_2)-v(\tau))} v'(\tau)d\tau \\
&\quad + \int_a^{t_1} (v(t_1) - v(\tau))^{\delta-1}\, e^{\frac{\rho-1}{\rho}(v(t_1)-v(\tau))} v'(\tau)d\tau.
\end{aligned}$$

From Lemma 15, we obtain

$$\begin{aligned}
&\int_a^{t_1} |(V_\delta(t_2,\tau) - V_\delta(t_1,\tau))\, v'(\tau)|\, d\tau \\
= \ &\frac{\rho^\delta \Gamma(\delta)}{(1-\rho)^\delta} \left\{ -\mathcal{P}(\delta, \frac{1-\rho}{\rho}(v(t_2) - v(a))) + \mathcal{P}(\delta, \frac{1-\rho}{\rho}(v(t_2) - v(t_1))) \right. \\
&\left. + \mathcal{P}(\delta, \frac{1-\rho}{\rho}(v(t_1) - v(a))) - 0 \right\} \\
\to \ &0 \text{ as } t_2 \to t_1.
\end{aligned}$$

Now, for $\rho \in (0,1)$, and $\delta > 1$: since $V_\delta(t,\tau)$ is continuous function on $[a,b] \times [a,b]$, it is uniformly continuous and hence for any $\epsilon > 0$ there exists a constant $\omega = \omega(\epsilon) > 0$ such that

$$|V_\delta(t_2,\tau) - V_\delta(t_1,\tau)| < \epsilon,$$

for all $t_1, t_2, \tau_1, \tau_2 \in [a,b]$ and $|t_2 - t_1| < \omega$, $|\tau_2 - \tau_1| < \omega$.

Therefore,

$$\begin{aligned}\int_a^{t_1} |V_\delta(t_2,\tau) - V_\delta(t_1,\tau)|\, v'(\tau)d\tau &\le \epsilon \int_a^{t_1} v'(\tau)d\tau \\ &= (v(t_1) - v(a))\,\epsilon \\ &\le (v(b) - v(a))\epsilon.\end{aligned}$$

Thus, we conclude that

$$\int_a^{t_1} |(V_\delta(t_2,\tau) - V_\delta(t_1,\tau))|\, v'(\tau)d\tau \to 0 \text{ uniformly as } t_2 \to t.$$

The proof is completed. □

4. Equivalence of problem (1.1) and (1.2) to an integral equation

In this section, we prove the equivalence of the considered boundary value problem to an equation involving fractional proportional integral. In all the following results, we assume that:

$$e^{\frac{\rho-1}{\rho}v(b)}(v(b) - v(a))^\beta \neq \xi e^{\frac{\rho-1}{\rho}v(\eta)}(v(\eta) - v(a))^\beta.$$

Lemma 17. *Let $\rho \in (0,1]$, $0 < \alpha,\beta \le 1$. For $\psi \in C([a,b],\mathbb{R})$. The solution of the following linear problem*

$${}_p^C\mathfrak{D}_a^{\alpha,\rho,v}\left({}_p^C\mathfrak{D}_a^{\beta,\rho,v} + \lambda\right)x(t) = \psi(t), \tag{4.1}$$

with the nonlocal boundary conditions (1.2) and the solution of the following integral equation

$$\begin{aligned} x(t) &= -\lambda\left(\mathcal{J}_a^{\beta,\rho,v}x\right)(t) + \left(\mathcal{J}_a^{\alpha+\beta,\rho,v}\psi\right)(t) + Qe^{\frac{\rho-1}{\rho}v(t)}(v(t)-v(a))^\beta \\ &\quad \times\left[\lambda\left(\mathcal{J}_a^{\beta,\rho,v}x\right)(b) - \left(\mathcal{J}_a^{\alpha+\beta,\rho,v}\psi\right)(b) - \lambda\xi\left(\mathcal{J}_a^{\beta,\rho,v}x\right)(\eta) + \xi\left(\mathcal{J}_a^{\alpha+\beta,\rho,v}\psi\right)(\eta)\right], \end{aligned} \tag{4.2}$$

where

$$Q = \left[e^{\frac{\rho-1}{\rho}v(b)}(v(b)-v(a))^\beta - \xi e^{\frac{\rho-1}{\rho}v(\eta)}(v(\eta)-v(a))^\beta\right]^{-1} \tag{4.3}$$

are equivalent.

Proof. Applying the operator $\mathcal{J}_a^{\alpha,\rho,v}$ to both sides of Eq (4.1) and using the first property of Propostion 8, we get

$${}_p^C\mathfrak{D}_a^{\beta,\rho,v}x(t) + \lambda x(t) - c_0 e^{\frac{\rho-1}{\rho}(v(t)-v(a))} = \mathcal{J}_a^{\alpha,\rho,v}\psi(t).$$

Next, applying the operator $J_a^{\beta,\rho,v}$ on both sides of the previous equation yields

$$x(t) = \overline{c}_0 e^{\frac{\rho-1}{\rho}(v(t)-v(a))} + c_0\mathcal{J}_a^{\beta,\rho,v}e^{\frac{\rho-1}{\rho}(v(t)-v(a))} - \lambda\mathcal{J}_a^{\beta,\rho,v}x(t) + \mathcal{J}_a^{\beta,\rho,v}\mathcal{J}_a^{\alpha,\rho,v}\psi(t),$$

so,

$$x(t) = \overline{c}_0 e^{\frac{\rho-1}{\rho}(v(t)-v(a))} + \frac{c_0}{\Gamma(\beta+1)\rho^\beta}e^{\frac{\rho-1}{\rho}(v(t)-v(a))}(v(t)-v(a))^\beta - \lambda\mathcal{J}_a^{\beta,\rho,v}x(t) + \mathcal{J}_a^{\alpha+\beta,\rho,v}\psi(t). \tag{4.4}$$

From the boundary condition $x(a) = 0$, we get $\overline{c}_0 = 0$.

Now, using the boundary condition $x(b) = \xi x(\eta)$, we obtain

$$c_0 = \frac{\Gamma(\beta+1)\rho^\beta \left[\lambda \left(\mathcal{J}_a^{\beta,\rho,\nu} x\right)(b) - \left(\mathcal{J}_a^{\alpha+\beta,\rho,\nu}\psi\right)(b) - \lambda\xi \left(\mathcal{J}_a^{\beta,\rho,\nu} x\right)(\eta) + \xi \left(\mathcal{J}_a^{\alpha+\beta,\rho,\nu}\psi\right)(\eta)\right]}{e^{\frac{\rho-1}{\rho}(\nu(b)-\nu(a))}(\nu(b)-\nu(a))^\beta - \xi e^{\frac{\rho-1}{\rho}(\nu(\eta)-\nu(a))}(\nu(\eta)-\nu(a))^\beta}. \tag{4.5}$$

Substituting the values of c_0 and $\overline{c}_0$ in (4.4) we obtain formula (4.2).

Now, to prove the other way, it is enough to replace t by a and b to get the boundary conditions (1.2) and to obtain (4.1) it is adequate to apply operators ${}^{C}_{p}\mathfrak{D}_a^{\beta,\rho,\nu}$ and ${}^{C}_{p}\mathfrak{D}_a^{\alpha,\rho,\nu}$ consecutively to both sides of (4.2). □

5. Uniqueness result

In this section we hold out the uniqueness of solutions to problem (1.1) and (1.2).

Let $X = C([a,b],\mathbb{R})$ be a Banach space of all continuous functions from $[a,b]$ to $\mathbb{R}$ endowed with the norm $\|x\| = \sup_{t\in[a,b]} |x(t)|$.

Associated with the problem (1.1) and (1.2), we define a fixed point operator $T : X \to X$ by

$$\begin{aligned} Tx(t) &= -\lambda \left(\mathcal{J}_a^{\beta,\rho,\nu} x\right)(t) + \left(\mathcal{J}_a^{\alpha+\beta,\rho,\nu} f(\cdot, x(\cdot))\right)(t) + Qe^{\frac{\rho-1}{\rho}\nu(t)}(\nu(t)-\nu(a))^\beta \\ &\quad \times\Big[\lambda \left(\mathcal{J}_a^{\beta,\rho,\nu} x\right)(b) - \left(\mathcal{J}_a^{\alpha+\beta,\rho,\nu} f(\cdot, x(\cdot))\right)(b) - \lambda\xi \left(\mathcal{J}_a^{\beta,\rho,\nu} x\right)(\eta) \\ &\quad +\xi \left(\mathcal{J}_a^{\alpha+\beta,\rho,\nu} f(\cdot, x(\cdot))\right)(\eta)\Big]. \end{aligned} \tag{5.1}$$

and we define the constants

$$\begin{aligned} S_\delta &= \left(\frac{\mathcal{P}(\delta, \frac{1-\rho}{\rho}(\nu(b)-\nu(a)))}{(1-\rho)^\delta}\right)\left(1 + |Q|\, e^{\frac{\rho-1}{\rho}\nu(a)}(\nu(b)-\nu(a))^\beta\right) \\ &\quad + |Q|\, e^{\frac{\rho-1}{\rho}\nu(a)}(\nu(b)-\nu(a))^\beta\, |\xi| \left(\frac{\mathcal{P}(\delta, \frac{1-\rho}{\rho}(\nu(\eta)-\nu(a)))}{(1-\rho)^\delta}\right), \quad \delta \in \{\beta, \alpha+\beta\}. \end{aligned} \tag{5.2}$$

We should remark that the fixed point of operator T is the solution of the integral Eq (4.4) and consequently the solution of problem (1.1) and (1.2).

Theorem 18. *Let $\rho \in (0,1)$ and assume that $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

(H_1) *There exists $K > 0$ such that $|f(t,z_1) - f(t,z_2)| \le K|z_1 - z_2|$, for all $t \in [a,b]$, $z_1, z_2 \in \mathbb{R}$, and $|f(t,0)| \le \Omega(t)$, with Ω is a continuous and non-negative function where $\sup_{t\in[a,b]} \Omega(t) = \varrho$.*

Then problem (1.1) and (1.2) has a unique solution on $[a,b]$ if

$$KS_{\alpha+\beta} + |\lambda|\, S_\beta < 1, \tag{5.3}$$

where $S_{\alpha+\beta}$ and S_β are given by (5.2).

Proof. Let us choose $r > 0$ satisfying

$$r \geq \frac{\varrho S_{\alpha+\beta}}{1 - \left(K S_{\alpha+\beta} + |\lambda| S_{\beta}\right)}, \tag{5.4}$$

and consider $B_r = \{x \in X : \|x\| \leq r\}$. We first show that $TB_r \subset B_r$.

Let $x \in B_r$, for any $t \in [a, b]$ we have

$$\begin{aligned} |Tx(t)| &= \Big| -\lambda \left(\mathcal{J}_a^{\beta,\rho,v} x\right)(t) + \left(\mathcal{J}_a^{\alpha+\beta,\rho,v} f(\cdot, x(\cdot))\right)(t) + Q e^{\frac{\rho-1}{\rho} v(t)} (v(t) - v(a))^{\beta} \\ &\quad \times \Big[\lambda \left(\mathcal{J}_a^{\beta,\rho,v} x\right)(b) - \left(\mathcal{J}_a^{\alpha+\beta,\rho,v} f(\cdot, x(\cdot))\right)(b) - \lambda \xi \left(\mathcal{J}_a^{\beta,\rho,v} x\right)(\eta) \\ &\quad + \xi \left(\mathcal{J}_a^{\alpha+\beta,\rho,v} f(\cdot, x(\cdot))\right)(\eta) \Big] \Big| \\ &\leq |\lambda| \left| \left(\mathcal{J}_a^{\beta,\rho,v} x\right)(t) \right| + \left| \left(\mathcal{J}_a^{\alpha+\beta,\rho,v} f(\cdot, x(\cdot))\right)(t) \right| + |Q| e^{\frac{\rho-1}{\rho} v(t)} (v(t) - v(a))^{\beta} \\ &\quad \times \Big[|\lambda| \left| \left(\mathcal{J}_a^{\beta,\rho,v} x\right)(b) \right| + \left| \left(\mathcal{J}_a^{\alpha+\beta,\rho,v} f(\cdot, x(\cdot))\right)(b) \right| + |\lambda| |\xi| \left| \left(\mathcal{J}_a^{\beta,\rho,v} x\right)(\eta) \right| \\ &\quad + |\xi| \left| \left(\mathcal{J}_a^{\alpha+\beta,\rho,v} f(\cdot, x(\cdot))\right)(\eta) \right| \Big]. \end{aligned}$$

Using (H_1) and Lemma 14 we get

$$\begin{aligned} |Tx(t)| &\leq \frac{|\lambda| \mathcal{P}(\beta, \frac{1-\rho}{\rho}(v(b) - v(a))) \|x\|}{(1-\rho)^{\beta}} \\ &\quad + \frac{\mathcal{P}(\alpha+\beta, \frac{1-\rho}{\rho}(v(b) - v(a)))}{(1-\rho)^{\alpha+\beta}} (K \|x\| + \varrho) \\ &\quad + |Q| e^{\frac{\rho-1}{\rho} v(a)} (v(b) - v(a))^{\beta} \Big[\frac{|\lambda| \mathcal{P}(\beta, \frac{1-\rho}{\rho}(v(b) - v(a)))}{(1-\rho)^{\beta}} \|x\| \\ &\quad + \frac{\mathcal{P}(\alpha+\beta, \frac{1-\rho}{\rho}(v(b) - v(a)))}{(1-\rho)^{\alpha+\beta}} (K \|x\| + \varrho) + \frac{|\lambda| |\xi| \mathcal{P}(\beta, \frac{1-\rho}{\rho}(v(\eta) - v(a)))}{(1-\rho)^{\beta}} \|x\| \\ &\quad + \frac{|\xi| \mathcal{P}(\alpha+\beta, \frac{1-\rho}{\rho}(v(\eta) - v(a)))}{(1-\rho)^{\alpha+\beta}} (K \|x\| + \varrho) \Big]. \end{aligned}$$

After simplifications, we reach that

$$|Tx(t)| \leq \left(K S_{\alpha+\beta} + |\lambda| S_{\beta}\right) \|x\| + \varrho S_{\alpha+\beta},$$

where $S_{\alpha+\beta}$ and S_{β} are given by (5.2). Thus

$$\|Tx\| \leq \left(K S_{\alpha+\beta} + |\lambda| S_{\beta}\right) r + \varrho S_{\alpha+\beta} \leq r,$$

we obtain $TB_r \subset B_r$.

Next, we prove that the operator T is a contraction mapping. For $x, y \in X$, for all $t \in [a, b]$ we have

$$|Tx(t) - Ty(t)| = \Big| -\lambda \left(\mathcal{J}_a^{\beta,\rho,v} (x - y)\right)(t) + \left(\mathcal{J}_a^{\alpha+\beta,\rho,v} (f(\cdot, x(\cdot)) - f(\cdot, y(\cdot)))\right)(t)$$

$$+Qe^{\frac{\rho-1}{\rho}\nu(t)}(\nu(t)-\nu(a))^{\beta}\Big[\lambda\left(\mathcal{J}_a^{\beta,\rho,\nu}(x-y)\right)(b)$$
$$-\left(\mathcal{J}_a^{\alpha+\beta,\rho,\nu}\left(f\left(\cdot,x\left(\cdot\right)\right)-f\left(\cdot,y\left(\cdot\right)\right)\right)\right)(b)-\lambda\xi\left(\mathcal{J}_a^{\beta,\rho,\nu}(x-y)\right)(\eta)$$
$$+\xi\left(\mathcal{J}_a^{\alpha+\beta,\rho,\nu}\left(f\left(\cdot,x\left(\cdot\right)\right)-f\left(\cdot,y\left(\cdot\right)\right)\right)\right)(\eta)\Big]\Big|.$$

From (H_1) and Lemma 14 we get

$$\begin{aligned}|Tx(t)-Ty(t)| \leq\ & \frac{|\lambda|\mathcal{P}(\beta,\frac{1-\rho}{\rho}(\nu(b)-\nu(a)))\,\|x-y\|}{(1-\rho)^{\beta}}+\frac{\mathcal{P}(\alpha+\beta,\frac{1-\rho}{\rho}(\nu(b)-\nu(a)))}{(1-\rho)^{\alpha+\beta}}K\,\|x-y\| \\ &+Qe^{\frac{\rho-1}{\rho}\nu(t)}(\nu(t)-\nu(a))^{\beta}\Bigg[\frac{|\lambda|\mathcal{P}(\beta,\frac{1-\rho}{\rho}(\nu(b)-\nu(a)))\,\|x-y\|}{(1-\rho)^{\beta}} \\ &+\frac{\mathcal{P}(\alpha+\beta,\frac{1-\rho}{\rho}(\nu(b)-\nu(a)))}{(1-\rho)^{\alpha+\beta}}K\,\|x-y\|+\frac{|\lambda|\,|\xi|\,\mathcal{P}(\beta,\frac{1-\rho}{\rho}(\nu(\eta)-\nu(a)))\,\|x-y\|}{(1-\rho)^{\beta}} \\ &+\frac{|\xi|\,\mathcal{P}(\alpha+\beta,\frac{1-\rho}{\rho}(\nu(\eta)-\nu(a)))}{(1-\rho)^{\alpha+\beta}}K\,\|x-y\|\Bigg].\end{aligned}$$

Then, after simplifications, we conclude that

$$|Tx(t)-Ty(t)|\leq\left(KS_{\alpha+\beta}+|\lambda|\,S_{\beta}\right)\|x-y\|,$$

which on taking the norm for $t\in[a,b]$ produces

$$\|Tx-Ty\|\leq\left(KS_{\alpha+\beta}+|\lambda|\,S_{\beta}\right)\|x-y\|.$$

From the condition (5.3) the operator T is a contraction. Hence, by Banach fixed point theorem the problem (1.1) and (1.2) has a unique solution on $[a,b]$. The proof is completed. □

6. Existence result

In this section, by using Leray-Schauder alternative fixed point theorem [36], we present the following result about the existence of the solutions for the given problem.

Consider the following hypothesis:

(H_2) $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ are continuous functions and there exist a real positive constants ς_0 and ς_1 such that

$$|f(t,z)|\leq\varsigma_0+\varsigma_1|z|,$$

for all $(t,z)\in[a,b]\times\mathbb{R}$.

Theorem 19. *Let $\rho\in(0,1)$ and assume that (H_2) holds. If*

$$\varsigma_1 S_{\alpha+\beta}+|\lambda|\,S_{\beta}<1, \tag{6.1}$$

then the boundary value problem (1.1) and (1.2) has at least one solution on $[a,b]$.

Proof. We first show that the operator T is completely continuous.

It is clear that the continuity of f implies the continuity of the operator T. Now, let Υ be any nonempty bounded subset of X. Then, there exists $N > 0$ such that for any $x \in \Upsilon$, $\|x\| \leq N$. Notice that from condition (H_2) for all $x \in \Upsilon$ we have

$$|f(t, x(t))| \leq \varsigma_0 + \varsigma_1 N. \tag{6.2}$$

Next we prove that $T(\Upsilon)$ is uniformly bounded. Let $x \in \Upsilon$. Then, for any $t \in [a, b]$ we have

$$\begin{aligned} |Tx(t)| &= \Big| -\lambda \left(\mathcal{J}_a^{\beta,\rho,v} x\right)(t) + \left(\mathcal{J}_a^{\alpha+\beta,\rho,v} f(\cdot, x(\cdot))\right)(t) + Q e^{\frac{\rho-1}{\rho} v(t)} (v(t) - v(a))^\beta \\ &\quad \times \Big[\lambda \left(\mathcal{J}_a^{\beta,\rho,v} x\right)(b) - \left(\mathcal{J}_a^{\alpha+\beta,\rho,v} f(\cdot, x(\cdot))\right)(b) - \lambda\xi \left(\mathcal{J}_a^{\beta,\rho,v} x\right)(\eta) \\ &\quad + \xi \left(\mathcal{J}_a^{\alpha+\beta,\rho,v} f(\cdot, x(\cdot))\right)(\eta) \Big] \Big| \\ &\leq |\lambda| \left| \left(\mathcal{J}_a^{\beta,\rho,v} x\right)(t) \right| + \left| \left(\mathcal{J}_a^{\alpha+\beta,\rho,v} f(\cdot, x(\cdot))\right)(t) \right| + |Q| e^{\frac{\rho-1}{\rho} v(t)} (v(t) - v(a))^\beta \\ &\quad \times \Big[|\lambda| \left| \left(\mathcal{J}_a^{\beta,\rho,v} x\right)(b) \right| + \left| \left(\mathcal{J}_a^{\alpha+\beta,\rho,v} f(\cdot, x(\cdot))\right)(b) \right| + |\lambda| |\xi| \left| \left(\mathcal{J}_a^{\beta,\rho,v} x\right)(\eta) \right| \\ &\quad + |\xi| \left| \left(\mathcal{J}_a^{\alpha+\beta,\rho,v} f(\cdot, x(\cdot))\right)(\eta) \right| \Big]. \end{aligned}$$

Benefiting from (H_1) and Lemma 14 we notch up that

$$\begin{aligned} |Tx(t)| &\leq \frac{|\lambda| \mathcal{P}(\beta, \frac{1-\rho}{\rho}(v(b) - v(a)))}{(1-\rho)^\beta} N + \frac{\mathcal{P}(\alpha+\beta, \frac{1-\rho}{\rho}(v(b) - v(a)))}{(1-\rho)^{\alpha+\beta}} (\varsigma_0 + \varsigma_1 N) \\ &\quad + |Q| e^{\frac{\rho-1}{\rho} v(a)} (v(b) - v(a))^\beta \Big[\frac{|\lambda| \mathcal{P}(\beta, \frac{1-\rho}{\rho}(v(b) - v(a)))}{(1-\rho)^\beta} N \\ &\quad + \frac{\mathcal{P}(\alpha+\beta, \frac{1-\rho}{\rho}(v(b) - v(a)))}{(1-\rho)^{\alpha+\beta}} (\varsigma_0 + \varsigma_1 N) + \frac{|\lambda| |\xi| \mathcal{P}(\beta, \frac{1-\rho}{\rho}(v(\eta) - v(a)))}{(1-\rho)^\beta} N \\ &\quad + \frac{|\xi| \mathcal{P}(\alpha+\beta, \frac{1-\rho}{\rho}(v(\eta) - v(a)))}{(1-\rho)^{\alpha+\beta}} (\varsigma_0 + \varsigma_1 N) \Big] \\ &< +\infty. \end{aligned}$$

Consequently, $\|x\| < +\infty$ for any $x \in \Upsilon$. Therefore, $T(\Upsilon)$ is uniformly bounded.

Now, we shadow forth the equicontinuity of T on Υ. Let $x \in \Upsilon$.

For any $t_1, t_2 \in [a, b]$, where $t_2 > t_1$, we have

$$\begin{aligned} |Tx(t_2) - Tx(t_1)| &\leq |\lambda| \left| \left(\mathcal{J}_a^{\beta,\rho,v} x\right)(t_2) - \left(\mathcal{J}_a^{\beta,\rho,v} x\right)(t_1) \right| \\ &\quad + \left| \left(\mathcal{J}_a^{\alpha+\beta,\rho,v} f(\cdot, x(\cdot))\right)(t_2) - \left(\mathcal{J}_a^{\alpha+\beta,\rho,v} f(\cdot, x(\cdot))\right)(t_1) \right| \\ &= |\lambda| \Big| \frac{1}{\rho^\beta \Gamma(\beta)} \int_a^{t_2} (v(t_2) - v(\tau))^{\beta-1} e^{\frac{\rho-1}{\rho}(v(t_2) - v(\tau))} x(\tau) v'(\tau) d\tau \\ &\quad - \frac{1}{\rho^\beta \Gamma(\beta)} \int_a^{t_1} (v(t_1) - v(\tau))^{\beta-1} e^{\frac{\rho-1}{\rho}(v(t_1) - v(\tau))} x(\tau) v'(\tau) d\tau \Big| \end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^{t_2}(v(t_2)-v(\tau))^{\alpha+\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_2)-v(\tau))}f\left((\tau),x(\tau)\right)v'(\tau)d\tau\right.\\
&\left.-\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^{t_1}(v(t_1)-v(\tau))^{\alpha+\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_1)-v(\tau))}f\left((\tau),x(\tau)\right)v'(\tau)d\tau\right|.
\end{aligned}
$$

Taking the advantage of the relation $\int_a^{t_2}=\int_a^{t_1}+\int_{t_1}^{t_2}$, we acquire that

$$
\begin{aligned}
&|Tx(t_2)-Tx(t_1)|\\
=\;&\frac{|\lambda|}{\rho^{\beta}\Gamma(\beta)}\left|\int_a^{t_1}(v(t_2)-v(\tau))^{\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_2)-v(\tau))}x(\tau)\,v'(\tau)d\tau\right.\\
&+\int_{t_1}^{t_2}(v(t_2)-v(\tau))^{\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_2)-v(\tau))}x(\tau)\,v'(\tau)d\tau\\
&\left.-\int_a^{t_1}(v(t_1)-v(\tau))^{\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_1)-v(\tau))}x(\tau)\,v'(\tau)d\tau\right|+\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\\
&\times\left|\int_a^{t_1}(v(t_2)-v(\tau))^{\alpha+\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_2)-v(\tau))}f\left((\tau),x(\tau)\right)v'(\tau)d\tau\right.\\
&+\int_{t_1}^{t_2}(v(t_2)-v(\tau))^{\alpha+\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_2)-v(\tau))}f\left((\tau),x(\tau)\right)v'(\tau)d\tau\\
&\left.-\int_a^{t_1}(v(t_1)-v(\tau))^{\alpha+\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_1)-v(\tau))}f\left((\tau),x(\tau)\right)v'(\tau)d\tau\right|\\
=\;&\frac{|\lambda|}{\rho^{\beta}\Gamma(\beta)}\left|\int_a^{t_1}\left((v(t_2)-v(\tau))^{\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_2)-v(\tau))}-(v(t_1)-v(\tau))^{\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_1)-v(\tau))}\right)\right.\\
&\left.\times x(\tau)\,v'(\tau)d\tau+\int_{t_1}^{t_2}(v(t_2)-v(\tau))^{\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_2)-v(\tau))}x(\tau)\,v'(\tau)d\tau\right|+\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\\
&\times\left|\int_a^{t_1}\left((v(t_2)-v(\tau))^{\alpha+\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_2)-v(\tau))}-(v(t_1)-v(\tau))^{\alpha+\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_1)-v(\tau))}\right)\right.\\
&\left.\times f\left((\tau),x(\tau)\right)v'(\tau)d\tau+\int_{t_1}^{t_2}(v(t_2)-v(\tau))^{\alpha+\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_2)-v(\tau))}f\left((\tau),x(\tau)\right)v'(\tau)d\tau\right|\\
\le\;&\frac{|\lambda|N}{\rho^{\beta}\Gamma(\beta)}\left\{\int_a^{t_1}\left|\left(V_\beta(t_2,\tau)-V_\beta(t_1,\tau)\right)v'(\tau)\right|d\tau\right.\\
&\left.+\int_{t_1}^{t_2}\left|(v(t_2)-v(\tau))^{\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_2)-v(\tau))}v'(\tau)\right|d\tau\right\}+\frac{\varsigma_0+\varsigma_1 N}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\\
&\times\left\{\left|\int_a^{t_1}\left|\left(V_{\alpha+\beta}(t_2,\tau)-V_{\alpha+\beta}(t_1,\tau)\right)v'(\tau)v'(\tau)\right|d\tau\right.\right.\\
&\left.+\int_{t_1}^{t_2}\left|(v(t_2)-v(\tau))^{\alpha+\beta-1}\,e^{\frac{\rho-1}{\rho}(v(t_2)-v(\tau))}v'(\tau)\right|d\tau\right\},
\end{aligned}
$$

where the function V_δ (here $\delta=\beta,\alpha+\beta$) is given by (3.9). Thus, from Lemma 15

$$
|Tx(t_2)-Tx(t_1)|\;\le\;\frac{|\lambda|N}{\rho^{\beta}\Gamma(\beta)}\left\{\int_a^{t_1}\left|\left(V_\beta(t_2,\tau)-V_\beta(t_1,\tau)\right)v'(\tau)\right|d\tau\right.
$$

$$+\frac{\rho^{\beta}\Gamma(\beta)}{(1-\rho)^{\beta}}\mathcal{P}(\beta,\frac{1-\rho}{\rho}(v(t_2)-v(t_1)))\Bigg\}+\frac{\varsigma_0+\varsigma_1 N}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}$$
$$\times\left\{\left|\int_a^{t_1}\right|\Big(V_{\alpha+\beta}(t_2,\tau)-V_{\alpha+\beta}(t_1,\tau)\Big)v'(\tau)\Big|d\tau\right.$$
$$\left.+\frac{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}{(1-\rho)^{\alpha+\beta}}\mathcal{P}(\alpha+\beta,\frac{1-\rho}{\rho}(v(t_2)-v(t_1)))\right\}.$$

Then, by making use of Lemma 16, we achieve

$$\lim_{t_2\to t_1}|Tx(t_2)-Tx(t_1)|=0.$$

Thus, the operator T is equicontinuous. Hence, by Arzela-Ascoli theorem, we deduce that the operator T is completely continuous.

Finally, we will verify that the set $\Phi(T)=\{x\in X : x=mTx \text{ for some } 0<m<1\}$ is bounded.

For all $x\in\Phi(T)$, and for any $t\in[a,b]$, we have

$$\begin{aligned}|x(t)| &= m|Tx(t)|\\ &\le \frac{|\lambda|\mathcal{P}(\beta,\frac{1-\rho}{\rho}(v(b)-v(a)))}{(1-\rho)^{\beta}}\|x\|+\frac{\mathcal{P}(\alpha+\beta,\frac{1-\rho}{\rho}(v(b)-v(a)))}{(1-\rho)^{\alpha+\beta}}(\varsigma_0+\varsigma_1\|x\|)\\ &\quad+|Q|e^{\frac{\rho-1}{\rho}v(a)}(v(b)-v(a))^{\beta}\left[\frac{|\lambda|\mathcal{P}(\beta,\frac{1-\rho}{\rho}(v(b)-v(a)))}{(1-\rho)^{\beta}}\|x\|\right.\\ &\quad+\frac{\mathcal{P}(\alpha+\beta,\frac{1-\rho}{\rho}(v(b)-v(a)))}{(1-\rho)^{\alpha+\beta}}(\varsigma_0+\varsigma_1\|x\|)+\frac{|\lambda||\xi|\mathcal{P}(\beta,\frac{1-\rho}{\rho}(v(\eta)-v(a)))}{(1-\rho)^{\beta}}\|x\|\\ &\quad\left.+\frac{|\xi|\mathcal{P}(\alpha+\beta,\frac{1-\rho}{\rho}(v(\eta)-v(a)))}{(1-\rho)^{\alpha+\beta}}(\varsigma_0+\varsigma_1\|x\|)\right].\end{aligned}$$

Then, we obtain the following after simplifications

$$\|x\|\le\Big(\varsigma_1 S_{\alpha+\beta}+|\lambda|S_{\beta}\Big)\|x\|+\varsigma_0 S_{\alpha+\beta}.$$

This brings forth to

$$\|x\|\le\frac{\varsigma_0 S_{\alpha+\beta}}{1-\Big(\varsigma_1 S_{\alpha+\beta}+|\lambda|S_{\beta}\Big)},$$

which proves that $\Phi(T)$ is bounded. Thus, by Leray-Schauder alternative theorem, the operator T has at least one fixed point. Hence, the initial value problem (1.1) and (1.2) has at least one solution on $[a,b]$. The proof is completed. □

7. Special cases

In this section, we elaborate some special cases. From Lemma (13), in the case $\rho=1$ we can replace the formulas $\frac{\mathcal{P}(\delta,\frac{1-\rho}{\rho}(v(b)-v(a)))}{(1-\rho)^{\delta}}$ and $\frac{\mathcal{P}(\delta,\frac{1-\rho}{\rho}(v(\eta)-v(a)))}{(1-\rho)^{\delta}}$ by the formulas $\frac{(v(b)-v(a))^{\delta}}{\Gamma(\delta+1)}$ and $\frac{(v(\eta)-v(a))^{\delta}}{\Gamma(\delta+1)}$ resprectively.

Thereof, we conclude that

$$\begin{aligned}\lim_{\rho\to 1^-} S_\delta &= \left(\frac{(v(b)-v(a))^\delta}{\Gamma(\delta+1)}\right)\left(1+|\widehat{Q}|(v(b)-v(a))^\beta\right)\\ &\quad +|\widehat{Q}`|(v(b)-v(a))^\beta\,|\xi|\left(\frac{(v(\eta)-v(a))^\delta}{\Gamma(\delta+1)}\right) := \widehat{S}_\delta,\quad \delta\in\{\beta,\alpha+\beta\},\end{aligned} \tag{7.1}$$

where

$$\widehat{Q} = \left[(v(b)-v(a))^\beta - \xi(v(\eta)-v(a))^\beta\right]^{-1}.$$

Accordingly, we can state the following result.

Theorem 20. *Let $\rho = 1$ and $f : [a,b]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying assumption (H_1). Then problem (1.2)–(1.4) has a unique solution on $[a,b]$ if*

$$K\widehat{S}_{\alpha+\beta} + |\lambda|\,\widehat{S}_\beta < 1, \tag{7.2}$$

where $\widehat{S}_\delta$ ($\delta = \alpha+\beta,\beta$) is given by (7.1).

Because $P(\alpha,x)\in[0,1]$ for all $\alpha, x\in\mathbb{R}^+$, we obtain the inequalities:

$$S_{\alpha+\beta} \le \frac{S^*}{(1-\rho)^\alpha},\quad \text{and}\quad S_\beta \le S^*, \tag{7.3}$$

where

$$S^* = \frac{1+(1+|\xi|)\,|Q|\,e^{\frac{\rho-1}{\rho}v(a)}(v(b)-v(a))^\beta}{(1-\rho)^\beta}. \tag{7.4}$$

So, from Theorem 18 and Theorem 19 we obtain the following results:

Corollary 21. *Let $\rho\in(0,1)$ and $f : [a,b]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the assumption (H_1). Then the problem (1.1) and (1.2) has a unique solution on $[a,b]$ if*

$$\frac{KS^*}{(1-\rho)^\alpha} + |\lambda|\,S^* < 1, \tag{7.5}$$

where S^* is given by (7.4).

Corollary 22. *Let $\rho\in(0,1)$, and assume that (H_2) holds. If*

$$\frac{\varsigma_1 S^*}{(1-\rho)^\alpha} + |\lambda|\,S^* < 1, \tag{7.6}$$

then the boundary value problem (1.1) and (1.2) has at least one solution on $[a,b]$.

8. Applications

In this section, we bring in two examples in order to corroborate our theoretical results.

Example 23. *Consider the following problem*

$$ {}^{C}_{p}\mathfrak{D}_{0}^{\frac{3}{4},\frac{3}{4},t}\left({}^{C}_{p}\mathfrak{D}_{0}^{\frac{1}{2},\frac{3}{4},t}+\frac{1}{8}\right)x(t)=1-t+\frac{\sin x(t)}{11},\, t\in[0,1], \tag{8.1}$$

$$ x(0)=0,\quad x(1)=\frac{1}{2}x(\tfrac{1}{2}), \tag{8.2}$$

Here $v(t)=t, a=0, b=1, \eta=0.5, \alpha=0.75, \beta=0.5, \rho=0.75, \xi=1/2, \lambda=1/8,$ *and* $f(t,x(t))=1-t+\frac{\sin x(t)}{11}$.

So, we get $|f(t,x)-f(t,y)|\le K|x-y|$, *where* $K=\frac{1}{11}$.

By using Matlab program with the given value, we obtain

$$S_{\beta}=4.681316269082853,$$

$$S_{\alpha+\beta}=4.216579478045753,$$

and

$$KS_{\alpha+\beta}+|\lambda|S_{\beta}=0.968489940730425<1.$$

By virtue of Theorem 18, we conclude that problem (8.1) and (8.2) has a unique solution on $[0,1]$.

Example 24. *Consider the following problem*

$$ {}^{C}_{p}\mathfrak{D}_{1}^{\frac{1}{4},\frac{1}{2},\ln(t)}\left({}^{C}_{p}\mathfrak{D}_{1}^{\frac{1}{2},\frac{1}{2},\ln(t)}+\frac{1}{10}\right)x(t)=\frac{e-t+|x(t)|+\ln|x(t)|}{24},\, t\in[1,e], \tag{8.3}$$

$$ x(1)=0,\quad x(e)=\frac{1}{7}x(\tfrac{3}{2}), \tag{8.4}$$

Here $v(t)=\ln(t), a=1, b=e, \eta=1.5, \alpha=0.25, \beta=0.5, \rho=0.5, \xi=1/7, \lambda=1/10,$ *and* $f(t,x(t))=\frac{e-t+|x(t)|+\ln|x(t)|}{24}$.

So, we get $|f(t,x)|\le\frac{e-1}{24}+\frac{1}{12}|x|$ *,(i.e.,* $\varsigma_1=\frac{1}{12}$*).*

By using Matlab program with the given value, we obtain

$$S_{\beta}=5.295418315878468,$$

$$S_{\alpha+\beta}=5.529735638675511,$$

and

$$\varsigma_1 S_{\alpha+\beta}+|\lambda|S_{\beta}=0.990353134810806<1.$$

In so far as Theorem 19, we go through that problem (8.3) and (8.4) has at least one solution on $[1,e]$.

9. Conclusions

In this article, we discussed the existence and uniqueness of solutions to a certain type of Langevin equation subject to nonlocal boundary conditions with the assistance of the lower regularized incomplete Gamma function. The derivative involved in this type of Langevin equation is the generalized Caputo propotional fractional derivative which encloses many of the known fractional derivatives. To the best of our knowledge, this article is the first to handle the existence and uniqueness of solutions to differential equations in the frame of such generalized fractional derivatives of a function with respect to another function.

Conflict of interest

The authors declare there is no conflict of interests.

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