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Research article

Positive ground state solutions for asymptotically periodic generalized quasilinear Schrödinger equations

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Abstract: In this paper, we study the existence of a positive ground state solution for a class of generalized quasilinear Schrödinger equations with asymptotically periodic potential. By the variational method, a positive ground state solution is obtained. Compared with the existing results, our results improve and generalize some existing related results.

Keywords: generalized quasilinear Schrödinger equations; asymptotically periodic; positive ground state solution; Nehari manifold method
Mathematics Subject Classification: 35J60, 35J50

1. Introduction

In this paper, we are concerned with the following generalized quasilinear Schrödinger equations

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $g \in C^1(\mathbb{R}, \mathbb{R}^+)$ is an even function with $g'(t) \geq 0$ for all $t \geq 0$ and $g(0) = 1$, $h \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $V(x)$ is 1-periodic or asymptotically periodic potential.

The equations are related to the solitary wave solutions for quasilinear Schrödinger equations

$$iz_t = -\triangle z + W(x)z - h(x,z) - \triangle(l(|z|^2))l'(|z|^2)z, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, h and l can be used to model physical phenomenon. The form of (1.2) has many applications in physics. For instance, the case $l(s) = s$ models the time evolution of the condensate wave function in superfluid film [13, 21]. In the case of $l(s) = \sqrt{1+s}$, problem (1.2) models the self-channeling of a high-power ultrashort laser in matter [1, 30]. For more physical background, we can refer to [2, 5, 20, 29] and references therein.

Putting $z(t,x) = \exp(-iEt)u(x)$ in (1.2), where $E \in \mathbb{R}$ and u is a real function, we obtain the following elliptic equation

$$-\triangle u + V(x)u - \triangle(l(u^2))l'(u^2)u = h(x,u), \quad x \in \mathbb{R}^N. \tag{1.3}$$

When $l(s) = s$, Eq (1.3) is the superfluid film equation in plasma physics

$$-\triangle u + V(x)u - \triangle(u^2)u = h(x,u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

Note that Eq (1.4) is a special case of (1.1) if we choose $g^2(u) = 1 + 2u^2$. Equation (1.4) has been extensively studied, to see [3, 23, 27, 33, 34].

When $l(s) = \sqrt{1+s}$, (1.3) derives the following equation

$$-\triangle u + V(x)u - [\triangle(1+u^2)^{\frac{1}{2}}]\frac{u}{2(1+u^2)^{\frac{1}{2}}} = h(x,u), \quad x \in \mathbb{R}^N, \tag{1.5}$$

which is used as a model of the self-channeling of a high-power ultrashort laser in matter. We obverse that Eq (1.5) is also a particular case of (1.1) if we take $g^2(u) = 1 + \frac{u^2}{2(1+u^2)}$.

For (1.5), there were many papers studying the existence of solutions, to see [9, 12, 32, 39] and references therein. In [19], when the nonlinear term is autonomous, the authors obtained a positive ground state solution by a perturbation approach. In [4], the authors proved that (1.5) has at least a positive solution by using a change of variable, monotonicity trick developed by Jeanjean and a priori estimate. In [16], the authors got the existence of infinitely many nontrivial solutions by using a revised Clark theorem and a priori estimate of the solution.

Furthermore, if we take $g^2(u) = 1 + \frac{[(l(u^2))']^2}{2}$, then (1.3) turns into (1.1).

In recent years, many scholars have studied (1.1), for example [6, 7, 10, 11, 22, 24, 28, 31, 35]. Particularly, in [31], the authors obtained a positive solution with the autonomous nonlinear term. In [6], the authors acquired the existence of ground state solutions with periodic potential. In [10, 11], the authors established the existence of positive solutions with the critical exponents, where critical exponents are 2^* and $\alpha 2^*$, respectively. With regard to generalized quasilinear Schrödinger-Maxwell systems and generalized quasilinear Schrödinger equation of Kirchhoff type, we can refer to [8, 26, 40] and references therein. To the best of our knowledge, there is no work concerning with the unified asymptotic process on V and h at infinity for general quasilinear Schrödinger equations.

Motivated by above papers, we establish the existence of a positive ground state solution for Eq (1.1) and the corresponding periodic equation. We point out that Eq (1.1) is more general than (1.4) and (1.5). There are several difficulties in dealing with Eq (1.1). The first one is that the energy functional associated with Eq (1.1) is not well defined in the whole space $H^1(\mathbb{R}^N)$ due to the presence of second order nonhomogeneous term. Another difficulty is the lack of compactness owing to the unboundedness of the domain. The final difficulty is that the functional loses the translation invariance because of the asymptotically periodic potential. To overcome the above difficulties, we firstly introduce a change of variable and we reformulate quasilinear elliptic Eq (1.1) into semilinear elliptic equation, whose associated functional is well defined in our working space. Secondly, we employ the energy comparison method to overcome the loss of translation invariance. Finally, to find a positive ground state solution, we use the Concentration-Compactness Principle, Nehari manifold method and the strong maximun principle. Our results are a complement and generalization of some results obtained by [18, 34, 37].

Now, let us recall some basic facts. Set

$$H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\},$$

with the norm

$$\|u\|_H = \Big(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)dx\Big)^{\frac{1}{2}}.$$

It is well known that in the study of the elliptic equations, the potential function V plays an important role in dealing with compactness problem. Let us introduce the following working space

$$X = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 dx < \infty\},$$

endowed with the norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx\Big)^{\frac{1}{2}}, u \in X.$$

Then, the subsequent condition (V) implies that the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_H$ (see [18]).

We define the energy functional associated with (1.1) by

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} H(x,u)dx, \tag{1.6}$$

where $H(x,u) = \int_0^u h(x,\tau)d\tau$. However, Ψ is not well defined in the usual Sobolev space $H^1(\mathbb{R}^N)$ because of the term $\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 dx$. To overcome this difficulty, we make use of the change of variable introduced by [35],

$$v = G(u) = \int_0^u g(t)dt.$$

After the change of variable, we can obtain the following functional

$$\Phi(v) = \frac{1}{2}\int_{\mathbb{R}^N} \Big[|\nabla v|^2 + V(x)(G^{-1}(v))^2\Big]dx - \int_{\mathbb{R}^N} H(x, G^{-1}(v))dx, \tag{1.7}$$

where $G^{-1}(v)$ is the inverse function of $G(u)$. Since g is a nondecreasing positive function, we can get $|G^{-1}(v)| \leq \frac{1}{g(0)}|v|$. From this and our hypotheses, it is easy to verify that Φ is well defined on X and $\Phi \in C^1$.

In order to obtain a critical point of (1.1), it suffices to study the following semilinear equations

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} - \frac{h(x, G^{-1}(v))}{g(G^{-1}(v))} = 0, \ x \in \mathbb{R}^N. \tag{1.8}$$

If v is said to be a weak solution for Eq (1.8), then it should satisfy

$$\langle \Phi'(v), \varphi\rangle = \int_{\mathbb{R}^N} \Big[\nabla v \nabla\varphi + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi - \frac{h(x, G^{-1}(v))}{g(G^{-1}(v))}\varphi\Big]dx = 0, \tag{1.9}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. We note that if v is a critical point of the functional Φ, then $u = G^{-1}(v)$ is a critical point of the functional Ψ, i.e. $u = G^{-1}(v)$ is a solution of Eq (1.1).

Next, we give the following condition on potential V:

$$(V)\ 0 < V_0 \le V(x) \le V_p(x) \in L^\infty(\mathbb{R}^N) \text{ and } V(x) - V_p(x) \in \mathcal{F}_0, \text{ where}$$
$$\mathcal{F}_0 := \Big\{k(x) : \forall \varepsilon > 0,\ \lim_{|y|\to\infty} \text{meas}\,\{x \in B_1(y) : |k(x)| \ge \varepsilon\} = 0\Big\},$$
$$V_0 \text{ is a positive constant and } V_p \text{ is } 1-\text{periodic in } x_i, 1 \le i \le N.$$

On nonlinearity term h, since we look for a positive ground state solution, we assume that $h(x,t) = 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^-$. Moreover, the function h satisfies the following assumptions:

(H_1) $\displaystyle\lim_{t\to 0^+} \frac{h(x,t)}{g(t)G(t)} = 0$ uniformly for $x \in \mathbb{R}^N$.

(H_2) $\displaystyle\lim_{t\to +\infty} \frac{h(x,t)}{g(t)G(t)^{2^*-1}} = 0$ uniformly for $x \in \mathbb{R}^N$, where $2^* = \dfrac{2N}{N-2}$ for $N \ge 3$.

(H_3) $t \mapsto \dfrac{h(x,t)}{g(t)G(t)^3}$ is nondecreasing on $(0,+\infty)$.

(H_4) There exists a periodic function $h_p \in C(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R})$ and $h_p(x,t) = 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^-$, which is $1-$periodic in x_i, $1 \le i \le N$, such that

(1) $h(x,t) \ge h_p(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$ and $h(x,t) - h_p(x,t) \in \mathcal{F}$, where

$\mathcal{F} := \Big\{k(x,t) : \forall\varepsilon > 0, \displaystyle\lim_{|y|\to\infty} \text{meas}\{x \in B_1(y) : |k(x,t)| \ge \varepsilon\} = 0 \text{ uniformly for } |t| \text{ bound}\Big\}.$

(2) $t \mapsto \dfrac{h_p(x,t)}{g(t)G(t)^3}$ is nondecreasing on $(0,+\infty)$.

(3) $\displaystyle\lim_{t\to+\infty} \frac{H_p(x,t)}{G(t)^2} = +\infty$ uniformly for $x \in \mathbb{R}^N$, where $H_p(x,t) = \displaystyle\int_0^t h_p(x,\tau)d\tau$.

We employ (H_1), (H_2) and (H_4) conditions to ensure that the energy functional Φ has the mountain pass geometry structure. However, under these hypotheses, this functional does not satisfy the Cerami compactness condition, since the domain is all $\mathbb{R}^N$. We observe that the condition (H_3) and (3) of (H_4) are used in the proof of the boundedness of the Cerami sequence of the functional associated with (1.1). We also observe that the asymptotic process of (V) and (1) of (H_4) is uniform at infinity due to [18], and then was used in [17, 25, 38].

Now, we present the first result of this paper.

Theorem 1.1. (Asymptotically periodic case). Assume that (V) and $(H_1) - (H_4)$ hold. Then, Eq (1.1) has a positive ground state solution.

Remark 1.2. Compared with the known results in [6, 18, 24, 34, 37], our results are new and different due to the following some facts:

(1) While [18, 34, 37], our model is more general and they are our special case. In our results, there is no need to assume $h(x,t) \in C^1(\mathbb{R}^N, \mathbb{R})$. The constrained manifold also need not be of class C^1.

(2) Contrasting with [34], we give a new asymptotic process of potential and nonlinearity term at infinity. To some extent, we extend and complement their results.

(3) In [6, 24], they verified the mountain pass geometry structure under the following condition

$$(H_2')\ |h(x,t)| \le C(1 + g(t)|G(t)|^\alpha) \text{ for some } C > 0 \text{ and } \alpha \in (2, 2^*).$$

Our condition (H_2) is somewhat weaker than (H_2').

(4) We choose condition (H_3) and (3) of (H_4) to be weaker than Ambrosetti-Rabinowitz type condition.

In the special case: $V = V_p, h = h_p$, Theorem 1.1 clearly gives us a solution for the periodic equation. Indeed, considering the following equation

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V_p(x)u = h_p(x,u), \quad x \in \mathbb{R}^N. \tag{1.10}$$

under the hypothesis:

(V_1) The function V_p is 1-periodic in $x_i, 1 \leq i \leq N$ and there exists a constant $V_0 > 0$ such that

$$0 < V_0 \leq V_p(x) \in L^\infty(\mathbb{R}^N), \ \text{ for all } x \in \mathbb{R}^N.$$

Corollary 1.3. (Periodic case). Assume that (V_1) holds, h_p satisfies $(H_1), (H_2)$, (2) of (H_4) and (3) of (H_4). Then, Eq (1.10) has a positive ground state solution.

Remark 1.4. To the best of our knowledge, even for the periodic case, our method is new. In [26], under condition (V_1), h_p satisfies $(H_1), (H_2)$ and other conditions. They could get at least one nontrivial ground state solution by the Mountain Pass Theorem. While, we obtain the existence of a positive ground state solution with Eq (1.10) by Nehari manifold method.

The rest of this article is organized as follows: In Section 2, we present some preliminary lemmas. In Section 3, we give the proof for our result.

Notations

- $L^s(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the norm

$$\|u\|_s = \Big(\int_{\mathbb{R}^N} |u|^s dx\Big)^{\frac{1}{s}}, \forall s \in [1, +\infty).$$

- $\|u\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^N} |u(x)|$ denotes the usual norm in $L^\infty(\mathbb{R}^N)$.
- $(X^*, \|\cdot\|_*)$ is the dual space of $(X, \|\cdot\|)$.
- $B_R(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < R\}$.
- C represent different positive constants.

2. Some preliminary lemmas

Before we prove the existence of a positive ground state solution for Eq (1.1), we present some useful lemmas.

Lemma 2.1. [24, 40] The functions g, G, G^{-1} satisfy the following properties:

(1) the functions $G(\cdot)$ and $G^{-1}(\cdot)$ are strictly increasing and odd;
(2) $G(s) \leq g(s)s$ for all $s \geq 0; G(s) \geq g(s)s$ for all $s \leq 0$;
(3) $g(G^{-1}(s)) \geq g(0) = 1$ for all $s \in \mathbb{R}$;
(4) $|((G^{-1}(s))'| = |\frac{1}{g(G^{-1}(s))}| < 1$;
(5) $\frac{G^{-1}(s)}{s}$ is decreasing on $(0, +\infty)$ and increasing on $(-\infty, 0)$;
(6) $|G^{-1}(s)| \leq \frac{1}{g(0)}|s| = |s|$ for all $s \in \mathbb{R}$;
(7) $\frac{|G^{-1}(s)|}{g(G^{-1}(s))} \leq \frac{1}{g^2(0)}|s| = |s|$ for all $s \in \mathbb{R}$;
(8) $\frac{G^{-1}(s)s}{g(G^{-1}(s))} \leq |G^{-1}(s)|^2$ for all $s \in \mathbb{R}$;
(9) $\frac{G^{-1}(s)}{s^3 g(G^{-1}(s))}$ is nonincreasing on $(0, \infty)$;
(10) $\lim_{|s| \to 0} \frac{G^{-1}(s)}{s} = \frac{1}{g(0)} = 1$ and

$$\lim_{|s| \to \infty} \frac{G^{-1}(s)}{s} = \begin{cases} \frac{1}{g(\infty)}, & \text{if } g \text{ is bounded,} \\ 0, & \text{if } g \text{ is unbounded.} \end{cases}$$

Lemma 2.2. Assume that $(H_1) - (H_4)$ hold, then for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$, we have

$$\frac{h(x, G^{-1}(s))}{4g(G^{-1}(s))} - H(x, G^{-1}(s)) \geq 0, \quad \frac{h_p(x, G^{-1}(s))}{4g(G^{-1}(s))} - H_p(x, G^{-1}(s)) \geq 0. \tag{2.1}$$

For any $\delta > 0$, there exist $r_\delta > 0, C_\delta > 0$ and $\alpha \in (2, 2^*)$ such that

$$0 \leq \frac{h_p(x, G^{-1}(s))}{g(G^{-1}(s))} \leq \frac{h(x, G^{-1}(s))}{g(G^{-1}(s))} \leq \delta|s|, \ \forall (x, s) \in \mathbb{R}^N \times [-r_\delta, r_\delta], \tag{2.2}$$

$$0 \leq \frac{h_p(x, G^{-1}(s))}{g(G^{-1}(s))} \leq \frac{h(x, G^{-1}(s))}{g(G^{-1}(s))} \leq \delta|s| + C_\delta|s|^{2^*-1}, \ \forall (x, s) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.3}$$

$$0 \leq \frac{h_p(x, G^{-1}(s))}{g(G^{-1}(s))} \leq \frac{h(x, G^{-1}(s))}{g(G^{-1}(s))} \leq C_\delta|s| + \delta|s|^{2^*-1}, \ \forall (x, s) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.4}$$

$$0 \leq \frac{h_p(x, G^{-1}(s))}{g(G^{-1}(s))} \leq \frac{h(x, G^{-1}(s))}{g(G^{-1}(s))} \leq \delta(|s| + |s|^{2^*-1}) + C_\delta|s|^{\alpha-1}, \ \forall (x, s) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.5}$$

$$0 \leq H_p(x, G^{-1}(s)) \leq H(x, G^{-1}(s)) \leq \frac{\delta}{2}|s|^2 + \frac{C_\delta}{2^*}|s|^{2^*}, \ \forall (x, s) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.6}$$

$$0 \leq H_p(x, G^{-1}(s)) \leq H(x, G^{-1}(s)) \leq \frac{\delta}{2}|s|^2 + \frac{\delta}{2^*}|s|^{2^*} + \frac{C_\delta}{\alpha}|s|^\alpha, \ \forall (x, s) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.7}$$

Proof. We give the proof the first inequality of (2.1). From (H_3), we have

$$\begin{aligned} H(x,t) &= \int_0^t h(x,\tau)d\tau = \int_0^t \frac{h(x,\tau)}{g(\tau)G(\tau)^3} g(\tau)G(\tau)^3 d\tau \\ &\leq \frac{h(x,t)}{g(t)G(t)^3}\int_0^t g(\tau)G(\tau)^3 d\tau = \frac{h(x,t)}{g(t)G(t)^3}\int_0^t G(\tau)^3 dG(\tau) = \frac{h(x,t)G(t)}{4g(t)}, \end{aligned}$$

that is

$$H(x,t) \leq \frac{h(x,t)G(t)}{4g(t)},$$

then, taking $t = G^{-1}(s)$, (2.1) is proved. According to $(H_1) - (H_4)$ conditions, it is easy to deduce the inequalities of (2.2)–(2.7).

Lemma 2.3. [18] Assume that condition (V) holds. Then, there are two positive constants d_1 and d_2 such that $d_1\|u\|_H \leq \|u\| \leq d_2\|u\|_H$ for all $u \in X$.

Define the Nehari manifold

$$\mathcal{N} := \{v \in X \setminus \{0\} : \langle \Phi'(v), v\rangle = 0\},$$

and set $c := \inf_{v\in\mathcal{N}} \Phi(v)$.

Lemma 2.4. Assume that (V) and $(H_1) - (H_4)$ are satisfied, then for any $v \in X, v \neq 0$, there exists a unique $t_v > 0$ such that $t_v v \in \mathcal{N}$. Moreover, the maximun of $\Phi(tv)$ for $t \geq 0$ is arrived at t_v.

Proof. Let $v \in X \setminus \{0\}$ and define a function $f(t) := \Phi(tv)$ on $[0, \infty)$. It follows from Lemma 2.1-(10), (H_1) and the Lebesgue dominated convergence theorem that

$$\begin{aligned} \frac{f(t)}{t^2} &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)\frac{(G^{-1}(tv))^2}{t^2}dx - \int_{\mathbb{R}^N} \frac{H(x,G^{-1}(tv))}{t^2}dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)\frac{(G^{-1}(tv))^2}{t^2v^2}v^2dx - \int_{\mathbb{R}^N} \frac{H(x,G^{-1}(tv))}{t^2v^2}v^2dx \\ &\to \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)v^2dx, \end{aligned}$$

when $t \to 0$, which shows that $f(t) > 0$ for $t > 0$ small enough. Set $\Omega = \{x \in \mathbb{R}^N : v(x) > 0\}$. Using the fact that (1), (3) of (H_4) and the Fatou lemma, We have

$$\liminf_{t\to+\infty} \int_\Omega \frac{H(x,G^{-1}(tv))}{t^2v^2}v^2dx \geq \liminf_{t\to+\infty} \int_\Omega \frac{H_p(x,G^{-1}(tv))}{t^2v^2}v^2dx = +\infty. \tag{2.8}$$

Hence, from Lemma 2.1-(6) and (2.8), we can get

$$\limsup_{t\to+\infty}\frac{f(t)}{t^2} \leq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)v^2dx - \liminf_{t\to+\infty}\int_\Omega \frac{H_p(x,G^{-1}(tv))}{t^2v^2}v^2dx,$$

which deduces $f(t) \to -\infty$ as $t \to +\infty$. So there exists $t_v > 0$ such that $f(t_v) = \max_{t>0} f(t)$ and $f'(t_v) = 0$, i.e., $\Phi(t_v v) = \max_{t>0}\Phi(tv)$ and $t_v v \in \mathcal{N}$.

The condition $f'(t) = 0$ is equivalent to

$$\frac{1}{t^2}\|\nabla v\|_2^2 = \int_{v\neq 0}\Big[\frac{h(x, G^{-1}(tv))}{g(G^{-1}(tv))t^3v^3} - V(x)\frac{G^{-1}(tv)}{g(G^{-1}(tv))t^3v^3}\Big]v^4dx. \tag{2.9}$$

By (H_3) and Lemma 2.1-(9), the right side of (2.9) is increasing for $t > 0$. So, there is a unique $t_v > 0$ such that $f'(t_v) = 0$.

From Lemma 2.4, we can obtain the following lemma easily.

Lemma 2.5. Assume that (V) and $(H_1) - (H_4)$ are satisfied. Then, the functional Φ satisfies the following mountain pass geometry structure:

(i) there exist positive constants ρ and b such that $\Phi(v) \geq b$ for $\|v\| = \rho$;

(ii) there exists a function $v_0 \in X$ such that $\|v_0\| > \rho$ and $\Phi(v_0) < 0$.

Lemma 2.6. Assume that (V) and $(H_1) - (H_4)$ hold, then there exists a bounded Cerami sequence $\{v_n\} \subset X$ for Φ.

Proof. From Lemma 2.5, we know that Φ satisfies the mountain pass geometry structure. Thus, the mountain pass theorem deduces that there exists a Cerami sequence $\{v_n\} \subset X$ such that

$$\Phi(v_n) \to c \text{ and } (1 + \|v_n\|)\|\Phi'(v_n)\|_* \to 0, \tag{2.10}$$

where $c := \inf_{v\in\mathcal{N}} \Phi(v)$.

We claim that a Cerami sequence $\{v_n\} \subset X$ satisfies

$$\int_{\mathbb{R}^N} |\nabla v_n|^2dx + \int_{\mathbb{R}^N} V(x)(G^{-1}(v_n))^2dx \leq M, \tag{2.11}$$

for some $M > 0$.

From (2.10), we have

$$\Phi(v_n) = \frac{1}{2}\int_{\mathbb{R}^N}\Big[|\nabla v_n|^2 + V(x)(G^{-1}(v_n))^2\Big]dx - \int_{\mathbb{R}^N} H(x, G^{-1}(v_n))dx \to c, \tag{2.12}$$

and

$$\langle\Phi'(v_n), v_n\rangle = \int_{\mathbb{R}^N}\Big[|\nabla v_n|^2 + V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n - \frac{h(x, G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n\Big]dx = o_n(1)\|v_n\|. \tag{2.13}$$

From (2.1), (2.12), (2.13) and Lemma 2.1-(8), we obtain

$$\begin{aligned}
c + 1 &\geq \Phi(v_n) - \frac{1}{4}\langle\Phi'(v_n), v_n\rangle \\
&= \frac{1}{4}\int_{\mathbb{R}^N} |\nabla v_n|^2dx + \frac{1}{4}\int_{\mathbb{R}^N} V(x)(G^{-1}(v_n))^2dx + \frac{1}{4}\int_{\mathbb{R}^N} V(x)\Big[(G^{-1}(v_n))^2 - \frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))}\Big] \\
&\quad + \int_{\mathbb{R}^N}\Big[\frac{h(x, G^{-1}(v_n))v_n}{4g(G^{-1}(v_n))} - H(x, G^{-1}(v_n))\Big]dx \\
&\geq \frac{1}{4}\int_{\mathbb{R}^N} |\nabla v_n|^2dx + \frac{1}{4}\int_{\mathbb{R}^N} V(x)(G^{-1}(v_n))^2dx,
\end{aligned}$$

that is

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 + \int_{\mathbb{R}^N} V(x)(G^{-1}(v_n))^2dx \leq 4(c + 1) := M. \tag{2.14}$$

Thus, (2.11) holds.

Next, we claim that $\int_{\mathbb{R}^N} v_n^2 dx$ is bounded. In fact, by Lemma 2.1-(2) and (2.14), we have

$$\begin{aligned}\int_{|G^{-1}(v_n)|\le 1} v_n^2 dx &\le g^2(1)\int_{|G^{-1}(v_n)|\le 1}(G^{-1}(v_n))^2 dx\\ &\le \frac{C^2}{V_0}\int_{|G^{-1}(v_n)|\le 1} V(x)(G^{-1}(v_n))^2 dx \le \frac{C^2}{V_0}M.\end{aligned} \tag{2.15}$$

Moreover, by the Sobolev inequality and (2.14), we deduce that

$$\begin{aligned}\int_{|G^{-1}(v_n)|>1} v_n^2 dx &\le \int_{|G^{-1}(v_n)|>1} v_n^{2^*} dx \le C\Big(\int_{|G^{-1}(v_n)|>1}|\nabla v_n|^2 dx\Big)^{\frac{2^*}{2}}\\ &\le C\Big(\int_{\mathbb{R}^N}|\nabla v_n|^2 dx\Big)^{\frac{2^*}{2}} \le CM^{\frac{2^*}{2}}.\end{aligned} \tag{2.16}$$

Obviously, there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} v_n^2 dx = \int_{|G^{-1}(v_n)|\le 1} v_n^2 dx + \int_{|G^{-1}(v_n)|>1} v_n^2 dx \le C. \tag{2.17}$$

The claim is proved, then combining (2.14) with (2.17), $\{v_n\} \subset X$ is bounded.

3. Proof of Theorem 1.1

In this section, before proving Theorem 1.1, we give three important lemmas to help us complete the proof of Theorem 1.1.

Lemma 3.1. [18] Assume that (V) and $(H_1) - (H_4)$ are satisfied. If $v \in \mathcal{N}$ and $\Phi(v) = c$, then v is a solution of problem (1.1).

Since $V(x)$ is asymptotically periodic. In this case, the functional Φ loses the translation invariance. The following two lemmas give careful estimates among V and V_p, h and h_p, H and H_p, which is inspired by [18, 37].

Lemma 3.2. Assume that $(V), (H_1), (H_2)$ and (1) of (H_4) hold. Suppose that $\{v_n\}$ is bounded in X and $v_n \to 0$ in $L^{\gamma}_{loc}(\mathbb{R}^N)$, for any $\gamma \in [2, 2^*)$, then up to a subsequence, one has

$$\text{(i)} \int_{\mathbb{R}^N}\Big(V(x) - V_p(x)\Big)(G^{-1}(v_n))^2 dx = o_n(1). \tag{3.1}$$

$$\text{(ii)} \int_{\mathbb{R}^N}\Big[H(x, G^{-1}(v_n)) - H_p(x, G^{-1}(v_n))\Big]dx = o_n(1). \tag{3.2}$$

Proof. (i) The proof of (3.1). Firstly, we assert that for any given $k(x) \in \mathcal{F}_0$ and $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\int_{\{x:|k(x)|\ge\varepsilon\}}|v|^2 dx \le C\int_{B_{R_\varepsilon+1}(0)}|v|^2 dx + C\varepsilon^{\frac{2}{N}}\|v\|_H^2, \quad \forall v \in X, \tag{3.3}$$

where C is a constant and independent on ε. (3.3) has already been proved in [18]. For the convenience of readers, we give a brief proof as following. According to the definition of $\mathcal{F}_0$, for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\text{meas}\{x \in B_1(y) : |k(x)| \ge \varepsilon\} < \varepsilon, \quad \forall |y| \ge R_\varepsilon.$$

Covering $\mathbb{R}^N$ by balls $B_1(y_i), i \in \mathbb{N}$, where each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls. Without loss of generality, we assume that $|y_i| < R_\varepsilon$, $i = 1, 2, \cdots, n_\varepsilon$ and $|y_i| \geq R_\varepsilon$, $i = n_\varepsilon + 1, n_\varepsilon + 2, \cdots, +\infty$. By the Hölder and Sobolev inequalities, we have

$$\begin{aligned}
\int_{\{x:|k(x)|\geq\varepsilon\}} |v|^2 dx &\leq \sum_{i=1}^{+\infty} \int_{\Omega_i} |v|^2 dx \\
&= \sum_{i=1}^{n_\varepsilon} \int_{\Omega_i} |v|^2 dx + \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_i} |v|^2 dx \\
&\leq (N+1) \int_{\{x\in B_{R_\varepsilon+1}(0):k(x)\geq\varepsilon\}} |v|^2 dx + \sum_{i=n_\varepsilon+1}^{+\infty} (\text{meas}\Omega_i)^{\frac{2}{N}} \Big(\int_{\Omega_i} |v|^{2^*} dx \Big)^{\frac{N-2}{N}} \\
&\leq (N+1) \int_{B_{R_\varepsilon+1}(0)} |v|^2 dx + C\varepsilon^{\frac{2}{N}} \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_i} (|\nabla v|^2 + v^2) dx \\
&\leq C \int_{B_{R_\varepsilon+1}(0)} |v|^2 dx + C\varepsilon^{\frac{2}{N}} \|v\|_H^2,
\end{aligned}$$

where $\Omega_i = \{x \in B_1(y_i) : |k(x)| \geq \varepsilon\}$.

Set $k(x) := V(x) - V_p(x) \in \mathcal{F}_0$, by Lemma 2.1-(6), we have

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^N} \Big(V(x) - V_p(x)\Big)(G^{-1}(v_n))^2 dx \Big| \leq \int_{\mathbb{R}^N} |k(x)||G^{-1}(v_n)|^2 dx \\
&\leq \int_{\mathbb{R}^N} |k(x)v_n^2| dx = \int_{\{x:|k(x)|\geq\varepsilon\}} |k(x)v_n^2| dx + \int_{\{x:|k(x)|<\varepsilon\}} |k(x)v_n^2| dx \\
&\leq 2\|V_p\|_\infty \Big[C \int_{B_{R_\varepsilon+1}(0)} v_n^2 dx + C\varepsilon^{\frac{2}{N}} \|v_n\|_H^2 \Big] + \varepsilon \int_{\mathbb{R}^N} |v_n|^2 dx \\
&= C\varepsilon^{\frac{2}{N}} + C\varepsilon + o_n(1).
\end{aligned}$$

Let $\varepsilon \to 0$, this completes the proof of (3.1).

(ii) The proof of (3.2). Let $\overline{h}(x, s) := h(x, s) - h_p(x, s) \in \mathcal{F}$. By the definition of $\mathcal{F}$, for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\text{meas}\,\{x \in B_1(y) : |\bar{h}(x, s)| \geq \varepsilon\} < \varepsilon, \quad \forall\, |y| \geq R_\varepsilon,\ |s| \leq \frac{1}{\varepsilon},$$

covering $\mathbb{R}^N$ by balls $B_1(y_i), i \in \mathbb{N}$, where each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls. Without loss of generality, we assume that $|y_i| < R_\varepsilon$, $i = 1, 2, \cdots, n_\varepsilon$ and $|y_i| \geq R_\varepsilon$, $i = n_\varepsilon + 1, n_\varepsilon + 2, \cdots, +\infty$. Using the mean value theorem and Lemma 2.1-(4), there exists $t_n \in (0, 1)$ such that

$$\begin{aligned}
H(x, G^{-1}(v_n)) - H_p(x, G^{-1}(v_n)) &= \Big[h(x, G^{-1}(t_n v_n)) - h_p(x, G^{-1}(t_n v_n))\Big](G^{-1}(t_n v_n))' v_n \\
&= \frac{\Big[h(x, G^{-1}(t_n v_n)) - h_p(x, G^{-1}(t_n v_n))\Big] v_n}{g(G^{-1}(t_n v_n))}.
\end{aligned}$$

Set

$$\Omega_1 := \{x \in B_1(y_i) : |\overline{h}(x, G^{-1}(t_n v_n))| < \varepsilon\},$$
$$\Omega_2 := \{x \in B_1(y_i) : |G^{-1}(t_n v_n)| \le \frac{1}{\varepsilon}, |\overline{h}(x, G^{-1}(t_n v_n))| \ge \varepsilon\},$$
$$\Omega_3 := \{x \in B_1(y_i) : |G^{-1}(t_n v_n)| > \frac{1}{\varepsilon}, |\overline{h}(x, G^{-1}(t_n v_n))| \ge \varepsilon\}.$$

Then, we obtain

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^N} \Big[H(x, G^{-1}(v_n)) - [H_p(x, G^{-1}(v_n))\Big] dx\Big| \\
&\le \sum_{i=1}^{+\infty} \int_{B_1(y_i)} \frac{|h(x, G^{-1}(t_n v_n)) - h_p(x, G^{-1}(t_n v_n))||v_n|}{g(G^{-1}(t_n v_n))} dx \\
&= \sum_{i=1}^{n_\varepsilon} \int_{B_1(y_i)} \frac{|h(x, G^{-1}(t_n v_n)) - h_p(x, G^{-1}(t_n v_n))||v_n|}{g(G^{-1}(t_n v_n))} dx \\
&\quad + \sum_{i=n_\varepsilon+1}^{+\infty} \int_{B_1(y_i)} \frac{|h(x, G^{-1}(t_n v_n)) - h_p(x, G^{-1}(t_n v_n))||v_n|}{g(G^{-1}(t_n v_n))} dx \\
&= \sum_{i=1}^{n_\varepsilon} \int_{B_1(y_i)} \frac{|h(x, G^{-1}(t_n v_n)) - h_p(x, G^{-1}(t_n v_n))||v_n|}{g(G^{-1}(t_n v_n))} dx \\
&\quad + \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_1} \frac{|h(x, G^{-1}(t_n v_n)) - h_p(x, G^{-1}(t_n v_n))||v_n|}{g(G^{-1}(t_n v_n))} dx \\
&\quad + \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_2} \frac{|h(x, G^{-1}(t_n v_n)) - h_p(x, G^{-1}(t_n v_n))||v_n|}{g(G^{-1}(t_n v_n))} dx \\
&\quad + \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_3} \frac{|h(x, G^{-1}(t_n v_n)) - h_p(x, G^{-1}(t_n v_n))||v_n|}{g(G^{-1}(t_n v_n))} dx \\
&:= \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4.
\end{aligned}$$

From (2.4), we have

$$\begin{aligned}
\Phi_1 &\le (N+1) \int_{B_{R_\varepsilon+1}(0)} \frac{|h(x, G^{-1}(t_n v_n)) - h_p(x, G^{-1}(t_n v_n))||v_n|}{g(G^{-1}(t_n v_n)} dx \\
&\le (N+1) \int_{B_{R_\varepsilon+1}(0)} 2(C_\delta |t_n v_n| + \delta |t_n v_n|^{2^*-1})|v_n| dx \\
&\le 2(N+1) C_\delta \int_{B_{R_\varepsilon+1}(0)} |v_n|^2 dx + 2(N+1)\delta \int_{B_{R_\varepsilon+1}(0)} |v_n|^{2^*} dx \\
&\le C\delta + o_n(1).
\end{aligned}$$

Set

$$\Omega_{11} := \{x \in B_1(y_i) : |\overline{h}(x, G^{-1}(t_n v_n))| < \varepsilon, |G^{-1}(t_n v_n)| \le r_\delta\},$$
$$\Omega_{12} := \{x \in B_1(y_i) : |\overline{h}(x, G^{-1}(t_n v_n))| < \varepsilon, |G^{-1}(t_n v_n)| > r_\delta\}.$$

By (2.2), we obtain

$$
\begin{aligned}
\Phi_2 &= \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_{11}} \frac{|h(x, G^{-1}(t_n v_n)) - h_p(x, G^{-1}(t_n v_n))||v_n|}{g(G^{-1}(t_n v_n))} dx \\
&\quad + \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_{12}} \frac{|h(x, G^{-1}(t_n v_n)) - h_p(x, G^{-1}(t_n v_n))||v_n|}{g(G^{-1}(t_n v_n))} dx \\
&\leq \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_{11}} 2\delta |t_n v_n||v_n| dx + \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_{12}} \frac{\varepsilon}{r_\delta} |t_n v_n||v_n| dx \\
&\leq 2\delta \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_{11}} |v_n|^2 dx + \frac{\varepsilon}{r_\delta} \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_{12}} |v_n|^2 dx \\
&\leq 2(N+1)\delta \int_{\mathbb{R}^N} |v_n|^2 dx + \frac{(N+1)\varepsilon}{r_\delta} \int_{\mathbb{R}^N} |v_n|^2 dx \\
&\leq C\delta + C\varepsilon.
\end{aligned}
$$

From (2.4), Hölder and Sobolev inequalities, we get

$$
\begin{aligned}
\Phi_3 &\leq \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_2} 2\Big[(C_\delta |t_n v_n| + \delta |t_n v_n|^{2^*-1}) v_n\Big] dx \\
&\leq \sum_{i=n_\varepsilon+1}^{+\infty} \Big[2C_\delta \int_{\Omega_2} |v_n|^2 dx + 2\delta \int_{\Omega_2} |v_n|^{2^*} dx\Big] \\
&\leq 2C_\delta \sum_{i=n_\varepsilon+1}^{+\infty} (\text{meas } \Omega_2)^{\frac{2}{N}} \Big(\int_{\Omega_2} |v_n|^{2^*} dx\Big)^{\frac{N-2}{N}} + 2(N+1)\delta \int_{\mathbb{R}^N} |v_n|^{2^*} dx \\
&\leq 2C_\delta (N+1) \varepsilon^{\frac{2}{N}} \int_{\mathbb{R}^N} (|\nabla v_n|^2 + |v_n|^2) dx + C\delta \\
&\leq C\varepsilon^{\frac{2}{N}} + C\delta.
\end{aligned}
$$

Due to (2.5), one has

$$
\begin{aligned}
\Phi_4 &\leq \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_3} 2\Big[(\delta |t_n v_n| + \delta |t_n v_n|^{2^*-1} + C_\delta |t_n v_n|^{\alpha-1}) v_n\Big] dx \\
&\leq \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_3} 2\Big(\delta |v_n|^2 + \delta |v_n|^{2^*} + C_\delta |v_n|^\alpha\Big) dx \\
&\leq 2\delta(N+1) \int_{\mathbb{R}^N} (|v_n|^2 + |v_n|^{2^*}) dx + 2C_\delta \varepsilon^{2^*-\alpha} \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_3} |v_n|^{2^*} dx \\
&\leq C\delta + C\varepsilon^{2^*-\alpha}.
\end{aligned}
$$

Now, let $\varepsilon \to 0$ and then $\delta \to 0$, we have $\Phi_i \to 0, i = 1, 2, 3, 4$, the proof of (3.2) is completed.

Lemma 3.3. Assume that $(V), (H_1), (H_2)$ and (1) of (H_4) are satisfied. If $\{v_n\}$ is bounded in X and $|y_n| \to +\infty$. Then, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, one has

$$\text{(i)} \int_{\mathbb{R}^N} \Big(V(x) - V_p(x)\Big) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} \varphi(x - y_n)dx = o_n(1). \tag{3.4}$$

$$\text{(ii)} \int_{\mathbb{R}^N} \frac{h(x, G^{-1}(v_n)) - h_p(x, G^{-1}(v_n))}{g(G^{-1}(v_n))} \varphi(x - y_n)dx = o_n(1). \tag{3.5}$$

Proof. (i) The proof of (3.4). By Lemma 2.1-(3) and (6), we have

$$\Big| \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} \varphi \Big| \le |G^{-1}(v_n)\varphi| \le |v_n \varphi|. \tag{3.6}$$

Due to $\varphi \in C_0^\infty(\mathbb{R}^N)$, one has

$$\int_{B_{R_\varepsilon+1}(0)} |\varphi(x - y_n)|^2 dx = o_n(1). \tag{3.7}$$

Set $k(x) := V(x) - V_p(x) \in \mathcal{F}$, in view of (3.3), (3.6), (3.7) and the Hölder inequality, we have

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^N} \Big(V(x) - V_p(x)\Big) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} \varphi(x - y_n)dx \Big| \le \int_{\mathbb{R}^N} \Big| \Big(V(x) - V_p(x)\Big) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} \varphi(x - y_n) \Big| dx \\
&= \int_{\{x:|k(x)| \ge \varepsilon\}} \Big| k(x) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} \varphi(x - y_n) \Big| dx + \int_{\{x:|k(x)| < \varepsilon\}} \Big| k(x) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} \varphi(x - y_n) \Big| dx \\
&\le 2\|V_p\|_\infty \int_{\{x:|k(x)| \ge \varepsilon\}} |v_n \varphi(x - y_n)| dx + \varepsilon \int_{\{x:|k(x)| < \varepsilon\}} |v_n \varphi(x - y_n)| dx \\
&\le 2\|V_p\|_\infty \|v_n\|_2 \Big(\int_{\{x:|k(x)| \ge \varepsilon\}} |\varphi(x - y_n)|^2 dx \Big)^{\frac{1}{2}} + \varepsilon \|v_n\|_2 \|\varphi\|_2 \\
&\le 2\|V_p\|_\infty \|v_n\|_2 \Big(C \int_{B_{R_\varepsilon+1}(0)} |\varphi(x - y_n)|^2 dx + C\varepsilon^{\frac{2}{N}} \|\varphi\|_H^2 \Big)^{\frac{1}{2}} + \varepsilon \|v_n\|_2 \|\varphi\|_2 \\
&\le C\varepsilon^{\frac{1}{N}} + C\varepsilon + o_n(1).
\end{aligned}$$

Let $\varepsilon \to 0$, (3.4) is proved.

(ii) The proof of (3.5). Let $\overline{h}(x, s) := h(x, s) - h_p(x, s) \in \mathcal{F}$. As the proof of Lemma 3.2, covering $\mathbb{R}^N$ by balls $B_1(y_i)$. Set

$$\begin{aligned}
\Omega_4 &:= \{x \in B_1(y_i) : |\overline{h}(x, G^{-1}(v_n))| < \varepsilon\}, \\
\Omega_5 &:= \{x \in B_1(y_i) : |G^{-1}(v_n)| \le \frac{1}{\varepsilon}, |\overline{h}(x, G^{-1}(v_n))| \ge \varepsilon\}, \\
\Omega_6 &:= \{x \in B_1(y_i) : |G^{-1}(v_n)| > \frac{1}{\varepsilon}, |\overline{h}(x, G^{-1}(v_n))| \ge \varepsilon\}.
\end{aligned}$$

We obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\Big|\frac{\overline{h}(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi(x-y_n)\Big|dx\\
&\le \sum_{i=1}^{n_\varepsilon}\int_{B_1(y_i)}\Big|\frac{\overline{h}(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi(x-y_n)\Big|dx+\sum_{i=n_\varepsilon+1}^{+\infty}\int_{B_1(y_i)}\Big|\frac{\overline{h}(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi(x-y_n)\Big|dx\\
&= \sum_{i=1}^{n_\varepsilon}\int_{B_1(y_i)}\Big|\frac{\overline{h}(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi(x-y_n)\Big|dx+\sum_{i=n_\varepsilon+1}^{+\infty}\int_{\Omega_4}\Big|\frac{\overline{h}(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi(x-y_n)\Big|dx\\
&\quad+\sum_{i=n_\varepsilon+1}^{+\infty}\int_{\Omega_5}\Big|\frac{\overline{h}(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi(x-y_n)\Big|dx+\sum_{i=n_\varepsilon+1}^{+\infty}\int_{\Omega_6}\Big|\frac{\overline{h}(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi(x-y_n)\Big|dx\\
&:=\Phi_5+\Phi_6+\Phi_7+\Phi_8.
\end{aligned}
$$

From (2.4) and (3.7), we have

$$
\begin{aligned}
\Phi_5&\le (N+1)\int_{B_{R_\varepsilon+1}(0)}\Big|\frac{\overline{h}(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi(x-y_n)\Big|dx\\
&\le (N+1)\int_{B_{R_\varepsilon+1}(0)}2(C_\delta|v_n|+\delta|v_n|^{2^*-1})|\varphi(x-y_n)|dx\\
&\le 2(N+1)C_\delta\int_{B_{R_\varepsilon+1}(0)}|v_n||\varphi(x-y_n)|dx+2(N+1)\delta\int_{B_{R_\varepsilon+1}(0)}|v_n|^{2^*-1}|\varphi(x-y_n)|dx\\
&\le 2C_\delta\|v_n\|_2\Big(\int_{B_{R_\varepsilon+1}(0)}|\varphi(x-y_n)|^2\Big)^{\frac12}+2(N+1)\delta\|v_n\|_{2^*}^{2^*-1}\|\varphi\|_{2^*}\\
&\le C\delta+o_n(1).
\end{aligned}
$$

On account of Lemma 2.1-(3), we obtain

$$
\begin{aligned}
\Phi_6&=\sum_{i=n_\varepsilon+1}^{+\infty}\int_{\Omega_4}\Big|\frac{\overline{h}(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi(x-y_n)\Big|dx\\
&\le \varepsilon\sum_{i=n_\varepsilon+1}^{+\infty}\int_{\Omega_4}|\varphi(x-y_n)|dx\\
&\le (N+1)\varepsilon\int_{\mathbb{R}^N}|\varphi(x-y_n)|dx\\
&\le C\varepsilon.
\end{aligned}
$$

From (2.4), the Hölder, Yong and Sobolev inequalities, one has

$$
\begin{aligned}
\Phi_7 &\le \sum_{i=n_\varepsilon+1}^{+\infty}\int_{\Omega_5} 2(C_\delta|v_n|+\delta|v_n|^{2^*-1})|\varphi(x-y_n)|dx\\
&\le \sum_{i=n_\varepsilon+1}^{+\infty}\int_{\Omega_5} 2C_\delta|v_n\varphi(x-y_n)|dx+\sum_{i=n_\varepsilon+1}^{+\infty}\int_{\Omega_5} 2\delta|v_n|^{2^*-1}|\varphi(x-y_n|dx\\
&\le 2C_\delta\sum_{i=n_\varepsilon+1}^{+\infty}(\text{meas }\Omega_5)^{\frac{2}{N}}\times\Big(\int_{\Omega_5}(|v_n\varphi(x-y_n)|)^{\frac{N}{N-2}}dx\Big)^{\frac{N-2}{N}}+2(N+1)\delta\|v_n\|_{2^*}^{2^*-1}\|\varphi\|_{2^*}\\
&\le 2C_\delta\varepsilon^{\frac{2}{N}}\sum_{i=n_\varepsilon+1}^{+\infty}\Big(\int_{\Omega_5}(\frac{|v_n|^{2^*}}{2}+\frac{|\varphi(x-y_n)|^{2^*}}{2})dx\Big)^{\frac{N-2}{N}}+C\delta\\
&\le 2C_\delta\varepsilon^{\frac{2}{N}}\sum_{i=n_\varepsilon+1}^{+\infty}\Big[\Big(\frac{1}{2}\int_{\Omega_5}|v_n|^{2^*}dx\Big)^{\frac{N-2}{N}}+\Big(\frac{1}{2}\int_{\Omega_5}|\varphi(x-y_n)|^{2^*}dx\Big)^{\frac{N-2}{N}}\Big]+C\delta\\
&\le 2C_\delta\varepsilon^{\frac{2}{N}}(N+1)(\frac{1}{2})^{\frac{N-2}{N}}C\Big[\int_{\mathbb{R}^N}(|\nabla v_n|^2+|v_n|^2)dx+\int_{\mathbb{R}^N}(|\nabla\varphi(x-y_n)|^2+|\varphi(x-y_n)|^2)dx\Big]+C\delta\\
&= C\varepsilon^{\frac{2}{N}}+C\delta.
\end{aligned}
$$

By using (2.5) and Hölder inequality, we have

$$
\begin{aligned}
\Phi_8 &\le \sum_{i=n_\varepsilon+1}^{+\infty}\int_{\Omega_6} 2\Big(\delta|v_n|+\delta|v_n|^{2^*-1}+C_\delta|v_n|^{\alpha-1}\Big)|\varphi(x-y_n)|dx\\
&\le 2\delta(N+1)\int_{\mathbb{R}^N}|v_n\varphi(x-y_n)|)|dx+2\delta(N+1)\int_{\mathbb{R}^N}|v_n|^{2^*-1}|\varphi(x-y_n)|dx\\
&\quad+2C_\delta\varepsilon^{2^*-\alpha}\sum_{i=n_\varepsilon+1}^{+\infty}\int_{\Omega_6}|v_n|^{2^*-1}|\varphi(x-y_n)|dx\\
&\le 2\delta(N+1)(\|v_n\|_2\|\varphi\|_2+\|v_n\|_{2^*}^{2^*-1}\|\varphi\|_{2^*})+2(N+1)C_\delta\varepsilon^{2^*-\alpha}\|v_n\|_{2^*}^{2^*-1}\|\varphi\|_{2^*}\\
&\le C\delta+C_\delta\varepsilon^{2^*-\alpha}.
\end{aligned}
$$

As before, let $\varepsilon\to 0$ and then $\delta\to 0$, we get $\Phi_i\to 0, i=5,6,7,8$, we complete the proof of (3.5).

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We can follow the energy comparison method in [15] to give the proof of Theorem 1.1. The functional Φ is rewritten as

$$\Phi(v)=\frac{1}{2}\int_{\mathbb{R}^N}\Big[|\nabla v|^2+V(x)(G^{-1}(v))^2\Big]dx-\int_{\mathbb{R}^N}H(x,G^{-1}(v))dx.$$

We also need to consider the corresponding periodic functional $\Phi_p: X\to\mathbb{R}$,

$$\Phi_p(v)=\frac{1}{2}\int_{\mathbb{R}^N}\Big[|\nabla v|^2+V_p(x)(G^{-1}(v))^2\Big]dx-\int_{\mathbb{R}^N}H_p(x,G^{-1}(v))dx.$$

By Lemma 2.5, there exists a Cerami sequence $\{v_n\}\subset X$, such that

$$\Phi(v_n)\to c \text{ and } (1+\|v_n\|)\|\Phi'(v_n)\|_*\to 0.$$

According to Lemma 2.6, the Cerami sequence $\{v_n\}$ is bounded in X. Thus, there exists $v \in X$ such that $v_n \rightharpoonup v$ in X and then $\Phi'(v) = 0$, that is v is a weak solution of problem (1.8). The proof of this result is standard, so we omit here.

(i) The case $v = 0$. Then $v_n \rightharpoonup 0$ in X, $v_n \to 0$ in $L^{\gamma}_{loc}(\mathbb{R}^N), 2 \le \gamma < 2^*$ and $v_n(x) \to 0$ a.e. on $\mathbb{R}^N$. From Lemma 3.2 and Lemma 3.3, we can deduce that

$$\begin{aligned}\Big|\Phi(v_n) - \Phi_p(v_n)\Big| \le \frac{1}{2}\int_{\mathbb{R}^N}\Big|\Big(V(x) - V_p(x)\Big)(G^{-1}(v_n))^2\Big|dx \\ + \int_{\mathbb{R}^N}\Big|H(x, G^{-1}(v_n)) - H_p(x, G^{-1}(v_n))\Big|dx = o_n(1),\end{aligned} \tag{3.8}$$

and taking $\varphi \in X$ with $\|\varphi\| = 1$, we obtain that

$$\begin{aligned}\Big\|\langle\Phi'(v_n) - \Phi'_p(v_n)\Big\|_* &\le \sup_{\varphi\in X,\|\varphi\|=1}\Big[\int_{\mathbb{R}^N}\Big|\Big(V(x) - V_p(x)\Big)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\varphi\Big|dx \\ &\quad + \int_{\mathbb{R}^N}\Big|\frac{h(x, G^{-1}(v_n)) - h_p(x, G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi\Big|dx\Big] \\ &= o_n(1).\end{aligned} \tag{3.9}$$

From (3.8) and (3.9), we can get that $\{v_n\}$ is also a Cerami sequence for Φ_p. Namely,

$$\Phi_p(v_n) \to c \text{ and } (1 + \|v_n\|)\|\Phi'_p(v_n)\|_* \to 0. \tag{3.10}$$

Define

$$\beta := \lim_{n\to\infty}\sup\sup_{y\in\mathbb{R}^N}\int_{B_1(y)} v_n^2 dx,$$

if $\beta = 0$, the Lions lemma [14], we have $v_n \to 0$ in $L^{\alpha}(\mathbb{R}^N)$ for all $\alpha \in (2, 2^*)$.

From (2.5) and (2.7), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} H(x, G^{-1}(v_n))dx = \lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{h(x, G^{-1}(v_n))v_n}{g(G^{-1}(v_n)}dx = 0. \tag{3.11}$$

It is implied by Lemma 2.1-(8) that

$$0 \le |G^{-1}(s)|^2 - \frac{G^{-1}(s)s}{g(G^{-1}(s))} \to 0(s \to 0). \tag{3.12}$$

Then, combining with (3.11) and (3.12), we obtain

$$\begin{aligned}c &= \Phi(v_n) - \frac{1}{2}\langle\Phi'(v_n), v_n\rangle + o_n(1) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} V(x)\Big[(G^{-1}(v_n))^2 - \frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n)}\Big]dx + \int_{\mathbb{R}^N}\Big[\frac{h(x, G^{-1}(v_n))v_n}{2g(G^{-1}(v_n))} - H(x, G^{-1}(v_n))\Big]dx + o_n(1) \\ &\to 0.\end{aligned}$$

This contradiction shows $\beta > 0$. Up to a subsequence, there exist a sequence $\{y_n\} \subset \mathbb{Z}^N$ and $r > 0$ such that $|y_n| \to \infty$,

$$\int_{B_{r(0)}} w_n^2 dx = \int_{B_{r(y_n)}} v_n^2 dx \ge \frac{\beta}{2} > 0, \tag{3.13}$$

where $w_n(x) := v_n(x+y_n)$. Since $\|w_n\| = \|v_n\|$, we see that $\{w_n\}$ is bounded. Going if necessary to a subsequence, we have $w_n \rightharpoonup w$ in X, $w_n \to w$ in $L^{\gamma}_{loc}(\mathbb{R}^N), 2 \le \gamma < 2^*$ and $w_n(x) \to w(x)$ a.e. on $\mathbb{R}^N$. Thus, (3.13) implies that $w \neq 0$. For any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\langle \Phi'_p(w), \varphi\rangle = \lim_{n\to\infty}\langle \Phi'_p(w_n), \varphi\rangle = 0.$$

Hence $\Phi'_p(w) = 0$. Next, our task is to verify that $\Phi_p(w) \le c$. Since

$$\|w_n\| = \|v_n\|, \ \ \Phi_p(v_n) = \Phi_p(w_n), \ \ \Phi'_p(v_n) = \Phi'_p(w_n). \tag{3.14}$$

From (2.1), (3.10), (3.14), Lemma 2.1-(8), (1) of (H_4) and the Fatou lemma, we obtain

$$\begin{aligned}
c &= \liminf_{n\to\infty}\Big[\Phi_p(w_n) - \frac{1}{4}\langle \Phi'_p(w_n), w_n\rangle\Big]\\
&= \liminf_{n\to\infty}\frac{1}{4}\int_{\mathbb{R}^N}\Big[|\nabla w_n|^2 + V_p(x)(G^{-1}(w_n))^2\Big]dx + \liminf_{n\to\infty}\frac{1}{4}\int_{\mathbb{R}^N} V_p(x)\Big[(G^{-1}(w_n))^2 - \frac{G^{-1}(w_n)w_n}{g(G^{-1}(w_n)}\Big]dx\\
&\quad + \liminf_{n\to\infty}\int_{\mathbb{R}^N}\Big[\frac{h_p(x, G^{-1}(w_n))w_n}{4g(G^{-1}(w_n))} - H_p(x, G^{-1}(w_n))\Big]dx\\
&\ge \frac{1}{4}\int_{\mathbb{R}^N}\Big[|\nabla w|^2 + V_p(x)(G^{-1}(w))^2\Big]dx + \frac{1}{4}\int_{\mathbb{R}^N} V_p(x)\Big[(G^{-1}(w))^2 - \frac{G^{-1}(w)w}{g(G^{-1}(w)}\Big]dx\\
&\quad + \int_{\mathbb{R}^N}\Big[\frac{h_p(x, G^{-1}(w))w_n}{4g(G^{-1}(w))} - H_p(x, G^{-1}(w))\Big]dx\\
&= \Phi_p(w) - \frac{1}{4}\langle \Phi'_p(w), w\rangle = \Phi_p(w).
\end{aligned} \tag{3.15}$$

By (3.15), (1) of (H_4) and the definition of c

$$c \le \Phi(t_w w) \le \Phi_p(t_w w) \le \Phi_p(w) \le c.$$

This implies that $\Phi(t_w w) = c$. Let $w_0 = t_w w$, then $w_0 \in \mathcal{N}$ and $\Phi(w_0) = c$. In view of Lemma 3.1, $\Phi'(w_0) = 0$. This shows that w_0 is a ground state solution of problem (1.8).

(ii) The case $v \neq 0$. Since v is a weak solution of problem (1.8), $\Phi(v) \ge c$. It follows from (2.1), Lemma 2.1-(8) and Fatou's lemma that

$$\begin{aligned}
c &= \liminf_{n\to\infty}\Big[\Phi(v_n) - \frac{1}{4}\langle \Phi'(v_n), v_n\rangle\Big]\\
&= \liminf_{n\to\infty}\frac{1}{4}\int_{\mathbb{R}^N}\Big[|\nabla v_n|^2 + V(x)(G^{-1}(v_n))^2\Big]dx + \liminf_{n\to\infty}\frac{1}{4}\int_{\mathbb{R}^N} V(x)\Big[(G^{-1}(v_n))^2 - \frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n)}\Big]dx\\
&\quad + \liminf_{n\to\infty}\int_{\mathbb{R}^N}\Big[\frac{h(x, G^{-1}(v_n))v_n}{4g(G^{-1}(v_n))} - H(x, G^{-1}(v_n))\Big]dx\\
&\ge \frac{1}{4}\int_{\mathbb{R}^N}\Big[|\nabla v|^2 + V(x)(G^{-1}(v))^2\Big]dx + \frac{1}{4}\int_{\mathbb{R}^N} V(x)\Big[(G^{-1}(v))^2 - \frac{G^{-1}(v)v}{g(G^{-1}(v)}\Big]dx\\
&\quad + \int_{\mathbb{R}^N}\Big[\frac{h(x, G^{-1}(v))v}{4g(G^{-1}(v))} - H(x, G^{-1}(v))\Big]dx\\
&= \Phi(v) - \frac{1}{4}\langle \Phi'(v), v\rangle = \Phi(v).
\end{aligned}$$

This shows that $\Phi(v) = c$ and $\Phi'(v) = 0$, which implies that v is a ground state solution of problem (1.8).

From (i) and (ii), we can obtain that Eq (1.8) has a ground state solution $v \in X$. By using the strong maximum principle [36], we can get that v is a positive ground state solution of (1.8). Namely, Eq (1.1) possesses a positive ground state solution $u = G^{-1}(v)$ and the proof is completed.

4. Conclusions

In this paper, we have established the existence of a positive ground state solution for Eqs (1.1) and (1.10) by the variational method. In comparison with previous works, this paper has several new features. Firstly, we consider a more general model than the literature [18, 34, 37]. Secondly, compared to the literature [34], we give a new asymptotic process of potential and nonlinearity term. Finally, we choose the more general nonlinear term than Ambrosetti-Rabinowitz condition. Therefore, to some extent, we have improved and extended the results of the existing literature.

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Conflict of interest

The author declares that he has no competing interests concerning the publication of this article.

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