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Research article

Boundedness of some operators on grand generalized Morrey spaces over non-homogeneous spaces

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Abstract: The aim of this paper is to obtain the boundedness of some operator on grand generalized Morrey space $\mathcal{L}^{p),\varphi,\phi}_{\mu}(G)$ over non-homogeneous spaces, where $G \subset \mathbb{R}^n$ is a bounded domain. Under assumption that functions φ and ϕ satisfy certain conditions, the authors prove that the Hardy-Littlewood maximal operator, fractional integral operators and θ-type Calderón-Zygmund operators are bounded on the non-homogeneous grand generalized Morrey space $\mathcal{L}^{p),\varphi,\phi}_{\mu}(G)$. Moreover, the boundedness of commutator $[b, T^G_\theta]$ which is generated by θ-type Calderón-Zygmund operator T_θ and $b \in \mathrm{RBMO}(\mu)$ on spaces $\mathcal{L}^{p),\varphi,\phi}_{\mu}(G)$ is also established.

Keywords: non-doubling measure; maximal operator; fractional integral operator; θ-type Calderón -Zygmund operator; grand generalized Morrey space
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1. Introduction

Let G be a bounded domain in $\mathbb{R}^n$. Recall that a Radon measure μ on the domain G is said to satisfy the polynomial growth condition, if there exists a positive constant C_0 such that, for all $x \in G$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq C_0 r^d, \tag{1.1}$$

where d is a fixed number in $(0, n]$ and $B(x, r) := \{y \in G : |x - y| < r\}$. The bounded domain G with a such Radon measure is also called a non-homogeneous space. Moreover, Tolsa [24] showed that the analysis associated with the non-homogeneous space over Euclidean space $\mathbb{R}^n$ plays a key role in solving the long-standing open Painlevé's problem and Vitushkin's conjecture. On the development and research of the operators and function spaces over non-homogeneous spaces, we refer readers to see [5, 7, 17, 19, 21–23, 25].

On the other hand, Iwaniec and Sbordone [9] introduced the theory of grand Lebesgue space $L^{p)}$, which is one of the intensively developing directions in Modern analysis. What's more, the grand

Lebesgue spaces have important applications in geometric function theory, Sobolev spaces theory and PDEs; for example, see [1–3, 6, 10], respectively. Since then, many papers focus on the grand spaces and the boundedness of operators on these spaces. For example, Kokilashvili [11] obtained the boundedness of several well-known operators on weighted grand Lebesgue spaces. In 2019, Kokilashvili et al. established the weighted extrapolation results in grand Morrey spaces and obtained some applications in PDE (see [15]). In 2021, Kokilashvili and Meskhi [12] obtain the boundedness of maximal operators, fractional integral operators and singular integral operators on generalized weighted grand Lebesgue spaces over non-doubling measures. More researches on the boundedness of integral operators in grand spaces can be seen [13, 14, 16, 20] and the references therein. The interpolation result in grand spaces can be seen in [4, 8].

In this paper, we will consider the boundedness of maximal operators, fractional integral operators and θ-type Calderón-Zygmund operators in grand generalized Morrey spaces $\mathcal{L}_\mu^{p),\varphi,\phi}(G)$ over non-homogeneous spaces. For the study of maximal operators, fractional integral operators and θ-type Calderón-Zygmund operators in generalized Morrey spaces defined on non-homogeneous spaces, we rely on the results of references [5, 18, 21].

Now let us begin to recall some necessary notions. The following definitions of the coefficient $K_{B,S}$ and (α,β)-doubling ball are from [23], also see [5].

Definition 1.1. For any two balls $B \subset S$, define

$$K_{B,S} := 1 + \sum_{k=1}^{N_{B,S}} \frac{\mu(2^k B)}{(2^k r_B)^n}, \tag{1.2}$$

where r_B and r_S respectively denote the radii of the balls B and S, and $N_{B,S}$ the smallest integer satisfying $2^{N_{B,S}} r_B \geq r_S$.

Definition 1.2. Let $\alpha,\beta \in (1,\infty)$. A ball $B \subset G$ is said to be (α,β)-doubling if $\mu(\alpha B) \leq \beta\mu(B)$.

In [23], Tolsa showed that there exists a lot of "big" doubling balls. To be precise, given any point $x \in \text{supp}(\mu)$ and $c > 0$, there exists some (α,β)-doubling ball B centered at x with radius $r_B \geq c$ due to the growth condition (1.1).

Let $1 < p < \infty$ and φ be a function on $(0, p-1]$ which is a positive bounded and satisfies $\lim\limits_{x\to 0}\varphi(x) = 0$. The class of such functions will be simply denoted by Φ_p. Then the norm of functions f in grand Lebesgue space $L_\mu^{p),\varphi}(G)$ is defined by

$$\|f\|_{L_\mu^{p),\varphi}(G)} = \sup_{0<\varepsilon<p-1} [\varphi(\varepsilon)]^{\frac{1}{p-\varepsilon}} \|f\|_{L_\mu^{p-\varepsilon}(G)}, \tag{1.3}$$

where $L_\mu^r(G)$ is the classical Lebesgue space with respect to a measure μ, and defined by the norm:

$$\|f\|_{L_\mu^r(G)} := \left(\int_G |f(x)|^r \mathrm{d}\mu(x)\right)^{\frac{1}{r}}, \quad 1 \leq r < \infty.$$

On the base of grand Lebesgue space $L_\mu^{p),\varphi}(G)$, we recall the definition of grand generalized Morrey spaces as follows.

Definition 1.3. Let $1 < p < \infty$ and $\varphi \in \Phi_p$. Suppose that $\phi : (0,\infty) \to (0,\infty)$ is an increasing function. Then grand generalized Morrey space $\mathcal{L}_\mu^{p),\varphi,\phi}(G)$ is defined by

$$\|f\|_{\mathcal{L}_\mu^{p),\varphi,\phi}(G)} := \left\{ f \in L^1_{\text{loc}}(\mu, G) : \|f\|_{\mathcal{L}_\mu^{p),\varphi,\phi}(G)} < \infty \right\},$$

where

$$
\begin{aligned}
&\|f\|_{\mathcal{L}_\mu^{p),\varphi,\phi}(G)}\\
&:= \sup_{0<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B\subset G}[\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}} \Big(\int_B |f(x)|^{p-\varepsilon} \mathrm{d}\mu(x) \Big)^{\frac{1}{p-\varepsilon}}\\
&= \sup_{0<\varepsilon<p-1} \varphi(\varepsilon) \|f\|_{\mathcal{L}_\mu^{p-\varepsilon,\phi}(G)}.
\end{aligned} \tag{1.4}
$$

Especially, if we take $\varphi(\varepsilon) = \varepsilon^\theta$ with $\theta > 0$ in (1.4), then we can denote

$$
\|f\|_{\mathcal{L}_\mu^{p),\varphi,\phi}(G)} := \|f\|_{\mathcal{L}_\mu^{p),\theta,\phi}(G)}.
$$

Remark 1.4. (1) If we take the bounded domain $G = \mathbb{R}^n$ in Definition 1.1, then generalized Morrey space $\mathcal{L}_\mu^{r,\phi}(G)$ is just the generalized Morrey space $\mathcal{L}_\mu^{r,\phi}(\mathbb{R}^n)$ (see [21]), namely,

$$
\|f\|_{\mathcal{L}_\mu^{r,\phi}(\mathbb{R}^n)} := \sup_B [\phi(\mu(B))]^{-\frac{1}{r}} \Big(\int_B |f(x)|^r \mathrm{d}\mu(x) \Big)^{\frac{1}{r}}, \quad 1 \le r < \infty. \tag{1.5}
$$

(2) If we take function $\phi(t) = t^{\frac{p}{q}-1}$ for $t > 0$ and $1 < p \le q < \infty$, then grand generalized Morrey space $\mathcal{L}_\mu^{p),\varphi,\phi}(G)$ defined as in (1.4) is just the grand Morrey space $L_\mu^{p),q}(G)$ which is sightly modified in [15], that is,

$$
\|f\|_{\mathcal{L}_\mu^{p),q,\varphi}(G)} = \sup_{0<\varepsilon<p-1} \varphi(\varepsilon) \sup_B [\mu(B)]^{\frac{1}{q}-\frac{1}{p-\varepsilon}} \|f\|_{L_\mu^{p-\varepsilon}(B)}. \tag{1.6}
$$

Throughout the whole paper, C represents a positive constant which is independent of the main parameters. Given any $q \in (1, \infty)$, let $q' := q/(q-1)$ denote its conjugate index. Furthermore, $m_B(f)$ denotes the mean value of function f over ball B, that is, $m_B(f) = \frac{1}{\mu(B)} \int_B f(y)\mathrm{d}\mu(y)$.

2. Hardy-Littlewood maximal operator on $\mathcal{L}_\mu^{p),\varphi,\phi}(G)$

Let G be a bounded set in $\mathbb{R}^n$ with non-negative Radon measure μ without mass-point and $\mu(G) < \infty$. Throughout the paper, for a function $f : G \to \mathbb{R}$, we denote

$$
\overline{f}(x) := \begin{cases} f(x), & \text{if } x \in G \\ 0, & \text{if } x \notin G. \end{cases} \tag{2.1}
$$

For a μ-integral function $f : G \to \mathbb{R}$, the Hardy-Littlewood center maximal function in [12] is defined by

$$
M_G f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)\bigcap G} |f(y)|\mathrm{d}\mu(y), \tag{2.2}
$$

By (2.1), if the bounded set G tends to space $\mathbb{R}^n$, then Hardy-Littlewood center maximal function M_G defined as in (2.2) is denoted by

$$
M\overline{f}(x) = M_G f(x), \quad x \in G. \tag{2.3}
$$

The main result of this section is stated as follows.

Theorem 2.1. Let $1 < p < \infty$, $\varphi \in \Phi_p$ and $\phi : (0,\infty) \to (0,\infty)$ be an increasing function. Assume that the mapping $t \mapsto \frac{\phi(t)}{t}$ is almost decreasing: there exists a positive constant C such that

$$\frac{\phi(t)}{t} \le C\frac{\phi(s)}{s}. \tag{2.4}$$

for $s \ge t$. Then M_G defined as in (2.2) is bounded on $\mathcal{L}_\mu^{p),\varphi,\phi}(G)$.

Proof. Choosing a number δ such that $0 < \varepsilon \le \delta < p-1$, then, by applying Definition 1.3, write

$$\begin{aligned}
&\|M_G f\|_{\mathcal{L}_\mu^{p),\varphi,\phi}(G)} \\
&\quad = \sup_{0<\varepsilon<p-1} \varphi(\varepsilon) \sup_B [\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}} \|M_G f\|_{L_\mu^{p-\varepsilon}(B)} \\
&\quad \le \sup_{0<\varepsilon\le\delta} \varphi(\varepsilon) \sup_B [\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}} \|M_G f\|_{L_\mu^{p-\varepsilon}(B)} \\
&\qquad\quad + \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_B [\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}} \|M_G f\|_{L_\mu^{p-\varepsilon}(B)} \\
&\quad =: \mathrm{D}_1 + \mathrm{D}_2.
\end{aligned}$$

For D_1, by applying the $\mathcal{L}_\mu^{p,\phi}(\mathbb{R}^n)$-boundedness of M (see [21]) and (1.4), we can deduce that

$$\begin{aligned}
&\sup_{0<\varepsilon\le\delta} \varphi(\varepsilon) \sup_B [\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}} \|M_G f\|_{L_\mu^{p-\varepsilon}(B)} \\
&\quad = \sup_{0<\varepsilon\le\delta} \varphi(\varepsilon) \|M_G f\|_{\mathcal{L}_\mu^{p-\varepsilon,\phi}(B)} \le \sup_{0<\varepsilon\le\delta} \varphi(\varepsilon) \|M\overline{f}\|_{\mathcal{L}_\mu^{p-\varepsilon,\phi}(\mathbb{R}^n)} \\
&\quad \le C \sup_{0<\varepsilon\le\delta} \varphi(\varepsilon) \|f\|_{\mathcal{L}_\mu^{p-\varepsilon,\phi}(G)} \le C\|f\|_{\mathcal{L}_\mu^{p),\varphi,\phi}(G)}.
\end{aligned}$$

Now let us estimate D_2. Since $\delta < \varepsilon < p-1$, then we have $\frac{p-\delta}{p-\varepsilon} > 1$. Further, by applying Hölder's inequality and $\mathcal{L}_\mu^{p,\phi}(\mathbb{R}^n)$-boundedness of M, we have

$$\begin{aligned}
\mathrm{D}_2 &= \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_B [\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}} \|M\overline{f}\|_{L_\mu^{p-\varepsilon}(B)} \\
&\le \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_B [\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}} \|M\overline{f}\|_{L_\mu^{p-\delta}(B)} [\mu(B)]^{\frac{1}{p-\varepsilon}-\frac{1}{p-\delta}} \\
&= \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon)[\varphi(\delta)]^{-1}\varphi(\delta) \sup_B [\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}} [\phi(\mu(B))]^{\frac{1}{p-\delta}} \\
&\qquad \times [\phi(\mu(B))]^{-\frac{1}{p-\delta}} \|M\overline{f}\|_{L_\mu^{p-\delta}(B)} [\mu(B)]^{\frac{1}{p-\varepsilon}-\frac{1}{p-\delta}} \\
&= \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon)[\varphi(\delta)]^{-1}\varphi(\delta) \sup_B [\phi(\mu(B))]^{\frac{1}{p-\delta}-\frac{1}{p-\varepsilon}} [\mu(B)]^{\frac{1}{p-\varepsilon}-\frac{1}{p-\delta}} \\
&\qquad \times [\phi(\mu(B))]^{-\frac{1}{p-\delta}} \|M\overline{f}\|_{L_\mu^{p-\delta}(B)} \\
&\le \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon)[\varphi(\delta)]^{-1}[\mu(G)]^{\frac{1}{p-\varepsilon}-\frac{1}{p-\delta}}
\end{aligned}$$

$$\times\varphi(\delta)\sup_B[\phi(\mu(B))]^{-\frac{1}{p-\delta}}\|M\overline{f}\|_{L_\mu^{p-\delta}(B)}$$
$$\leq \|f\|_{\mathcal{L}_\mu^{p),\varphi,\phi}(G)}\sup_{\delta<\varepsilon<p-1}\varphi(\varepsilon)[\varphi(\delta)]^{-1}[\mu(G)]^{\frac{\varepsilon-\delta}{(p-\varepsilon)(p-\delta)}}$$
$$\leq C\|f\|_{\mathcal{L}_\mu^{p),\varphi,\phi}(G)}.$$

Which, together with the estimate for D_1, the proof of Theorem 2.1 is finished. □

With an argument similar to that used in the proof of Theorem 2.1, it is easy to obtain the following result on the maximal operator $\widetilde{M}_{r,G}$.

Corollary 2.2. Let $1 < p < \infty$, $\varphi \in \Phi_p$ and $\phi : (0,\infty) \to (0,\infty)$ be an increasing function. Assume that the mapping $t \mapsto \frac{\phi(t)}{t}$ is almost decreasing function satisfying (2.4). Then non-centered maximal operator $\widetilde{M}_{r,G}$ is bounded on $\mathcal{L}_\mu^{p),\varphi,\phi}(G)$, where $\widetilde{M}_{r,G}$ is defined by

$$\widetilde{M}_r(\overline{f})(x) = \widetilde{M}_{r,G}(f)(x) := \sup_{x\in B}\left(\frac{1}{\mu(B)}\int_{B\bigcap G}|f(y)|^r d\mu(y)\right)^{\frac{1}{r}}. \tag{2.5}$$

3. Fractional integral operator on $\mathcal{L}_\mu^{p),\varphi,\phi}(G)$

Let G be a bounded domain in $\mathbb{R}^n$, then fractional integral operator I_α^G being associated with G is defined by

$$I_\alpha^G f(x) := \int_G \frac{f(y)}{|x-y|^{n-\alpha}} d\mu(y), \quad 0 < \alpha < n. \tag{3.1}$$

By applying (2.1), it is easy to see that (3.1) is equivalent to the following form

$$I_\alpha^G f(x) = I_\alpha \overline{f}(x). \tag{3.2}$$

Theorem 3.1. Let G be a bounded domain in $\mathbb{R}^n$, $0 < \alpha < n$, $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose that measure μ satisfies (1.1), and ϕ satisfies (2.4) and the following inequality

$$\int_r^\infty \frac{\phi(t)}{t^{\frac{p}{q}}}\frac{dt}{t} \leq C\frac{\phi(r)}{r^{\frac{q}{p}}}. \tag{3.3}$$

We set

$$\psi(\varepsilon) = \left[\varphi\left(p - \frac{n(q-\varepsilon)}{n+\alpha(q-\varepsilon)}\right)\right]^{\frac{n}{\alpha(q-\varepsilon)+n}}, \quad 0 < \varepsilon < q-1,$$

where $\varphi \in \Phi_p$. Then I_α^G is bounded from $\mathcal{L}_\mu^{p),\varphi,\phi}(G)$ to $\mathcal{L}_\mu^{q),\psi,\phi^{\frac{p}{q}}}(G)$.

To prove the above theorem, we need the following lemma in [21].

Lemma 3.2. Let $0 < \alpha < n$, $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose that measure ϕ satisfies (2.4) and (3.3). Then I_α is bounded from $\mathcal{L}_\mu^{p,\varphi,\phi}(G)$ to $\mathcal{L}_\mu^{q,\psi,\phi^{\frac{p}{q}}}(G)$.

Proof of Theorem 3.1. Via the definition of ψ and $\varphi \in \Phi_p$, it is easy to see that $\psi \in \Phi_q$. Let us fix σ with $\sigma \in (0, q-1)$, then write

$$\|I_\alpha^G f\|_{\mathcal{L}_\mu^{q),\psi,\phi^{\frac{p}{q}}}(G)}$$

$$
\begin{aligned}
&= \sup_{0<\varepsilon<q-1} \psi(\varepsilon) \sup_B [\phi(\mu(B))]^{-\frac{p}{q-\varepsilon}} \Big(\int_B |I_\alpha^G f(x)|^{q-\varepsilon} \mathrm{d}\mu(x) \Big)^{\frac{1}{q-\varepsilon}} \\
&= \max \Bigg\{ \sup_{0<\varepsilon<\sigma} \psi(\varepsilon) \sup_B [\phi(\mu(B))]^{-\frac{p}{q-\varepsilon}} \Big(\int_B |I_\alpha^G f(x)|^{q-\varepsilon} \mathrm{d}\mu(x) \Big)^{\frac{1}{q-\varepsilon}}, \\
&\quad \sup_{\sigma\le\varepsilon<q-1} \psi(\varepsilon) \sup_B [\phi(\mu(B))]^{-\frac{p}{q-\varepsilon}} \Big(\int_B |I_\alpha^G f(x)|^{q-\varepsilon} \mathrm{d}\mu(x) \Big)^{\frac{1}{q-\varepsilon}} \Bigg\} \\
&= \max\{\mathrm{E}_1, \mathrm{E}_2\}.
\end{aligned}
$$

For E_2. By applying Hölder's inequality and Definition 1.3, we obtain that

$$
\begin{aligned}
&\sup_{\sigma\le\varepsilon<q-1} \psi(\varepsilon) \sup_B [\phi(\mu(B))]^{-\frac{p}{q-\varepsilon}} \Big(\int_B |I_\alpha^G f(x)|^{q-\varepsilon} \mathrm{d}\mu(x) \Big)^{\frac{1}{q-\varepsilon}} \\
&\le \sup_{\sigma\le\varepsilon<q-1} \psi(\varepsilon) \sup_B [\phi(\mu(B))]^{-\frac{p}{q-\varepsilon}} \Bigg\{ \Big(\int_B |I_\alpha^G f(x)|^{q-\sigma} \mathrm{d}\mu(x) \Big)^{\frac{q-\varepsilon}{q-\sigma}} [\mu(B)]^{1-\frac{q-\varepsilon}{q-\sigma}} \Bigg\}^{\frac{1}{q-\varepsilon}} \\
&= \sup_{\sigma\le\varepsilon<q-1} \psi(\varepsilon) \sup_B [\phi(\mu(B))]^{-\frac{p(\varepsilon-\sigma)}{(q-\varepsilon)(q-\sigma)}} [\mu(B)]^{\frac{\varepsilon-\sigma}{(q-\varepsilon)(q-\sigma)}} \\
&\qquad \times [\phi(\mu(B))]^{-\frac{p}{q-\sigma}} \Big(\int_B |I_\alpha^G f(x)|^{q-\sigma} \mathrm{d}\mu(x) \Big)^{\frac{1}{q-\sigma}} \\
&\le \sup_{\sigma\le\varepsilon<q-1} \psi(\varepsilon) [\mu(G)]^{\frac{\varepsilon-\sigma}{(q-\varepsilon)(q-\sigma)}} \sup_B [\phi(\mu(B))]^{-\frac{p}{q-\sigma}} \Big(\int_B |I_\alpha \overline{f}(x)|^{q-\sigma} \mathrm{d}\mu(x) \Big)^{\frac{1}{q-\sigma}} \\
&\le [\mu(G)]^{q-1-\sigma} \sup_{\sigma\le\varepsilon<q-1} \psi(\varepsilon) \|I_\alpha \overline{f}\|_{\mathcal{L}_\mu^{q-\sigma,\phi^{\frac{p}{q-\sigma}}}(G)} \\
&\le C_{\sigma,q} \sup_{0<\varepsilon\le\sigma} \psi(\varepsilon) \|I_\alpha \overline{f}\|_{\mathcal{L}_\mu^{q-\varepsilon,\phi^{\frac{p}{q-\varepsilon}}}(G)}.
\end{aligned}
$$

Thus, for small constant $\sigma > 0$,

$$
\|I_\alpha^G f\|_{\mathcal{L}_\mu^{q),\psi,\phi^{\frac{p}{q}}}(G)} \le C_{\sigma,q} \sup_{0<\varepsilon\le\sigma} \psi(\varepsilon) \|I_\alpha \overline{f}\|_{\mathcal{L}_\mu^{q-\varepsilon,\phi^{\frac{p}{q-\varepsilon}}}(G)}.
$$

By applying Lemma 3.2 and (1.4) , we find that there exist $0 < \varepsilon \le \sigma$ and $0 < \eta \le \delta$ such that

$$
\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q}.
$$

holds. So we have

$$
\begin{aligned}
\psi(\varepsilon)\|I_\alpha \overline{f}\|_{\mathcal{L}_\mu^{q-\varepsilon,\phi^{\frac{p}{q-\varepsilon}}}(G)} &\le \psi(\varepsilon)\|I_\alpha \overline{f}\|_{\mathcal{L}_\mu^{q-\varepsilon,\phi^{\frac{p}{q-\varepsilon}}}(\mathbb{R}^n)} \le C\psi(\varepsilon)\|\overline{f}\|_{\mathcal{L}_\mu^{q-\varepsilon,\phi^{\frac{p}{q-\varepsilon}}}(\mathbb{R}^n)} \\
&\le C\psi\Big(q - \frac{n(p-\eta)}{n-\alpha(p-\eta)}\Big)\|f\|_{\mathcal{L}_\mu^{p-\eta,\phi}(G)} \\
&\le C[\varphi(\eta)]^{\frac{p-\eta}{q-\varepsilon}} [\varphi(\eta)]^{-1} \varphi(\eta) \|f\|_{\mathcal{L}_\mu^{p-\eta,\phi}(G)} \\
&\le C\|f\|_{\mathcal{L}_\mu^{p),\phi}(G)}.
\end{aligned}
$$

Further, taking the supremum on ε, we can obtain that

$$\|I_\alpha^G f\|_{\mathcal{L}_\mu^{q),\psi,\phi^{\frac{p}{q}}}(G)} \le C\|f\|_{\mathcal{L}_\mu^{p),\phi}(G)}.$$

Thus, the proof of Theorem 3.1 is completed. □

4. θ-Type Calderón-Zygmund operators on $\mathcal{L}_\mu^{p),\varphi,\phi}(G)$

Definition 4.1. Let θ be a non-negative and non-decreasing function on $(0,\infty)$ with satisfying

$$\int_0^1 \frac{\theta(t)}{t}\mathrm{d}t < \infty. \tag{4.1}$$

A kernel $K_\theta \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\})$ is called a θ-type kernel if there exists a constant $C > 0$ such that

$$|K(x,y)| \le \frac{C}{|x-y|^n}. \tag{4.2}$$

for all $x, y \in \mathbb{R}^n$ with $x \ne y$, and for all $x, x', y \in \mathbb{R}^n$ with $|x-y| \ge 2|x-x'|$,

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \le C\theta\Big(\frac{|x-x'|}{|x-y|}\Big)\frac{1}{|x-y|^n}. \tag{4.3}$$

Remark 4.2. If we take the function $\theta(t) = t^\epsilon$ with $t > 0$ and $\epsilon > 0$, then θ-type kernel K_θ defined as in Definition 4.1 is just the standard kernel K on non-doubling measure space (see [21]).

Definition 4.3. Let $\rho \in (1,\infty)$ and G be a bounded domain in $\mathbb{R}^n$. A locally integrable function f is said to be in the *space* RBMO(μ) if there exist a positive constant C and, for any ball $B \subset G$, a number f_B such that

$$\frac{1}{\mu(\rho B)}\int_B |f(x) - f_B|\mathrm{d}\mu(x) \le C. \tag{4.4}$$

and, for any two balls B and S such that $B \subset S$,

$$|f_B - f_S| \le CK_{B,S}, \tag{4.5}$$

where f_B represents the mean value of function f over ball B, that is,

$$f_B := \frac{1}{\mu(B)}\int_B f(y)\mathrm{d}\mu(y).$$

The infimum of the positive constants C satisfying both (4.4) and (4.5) is defined to be the RBMO(μ) norm of f, and it will be denoted by $\|f\|_{\mathrm{RBMO}(\mu)}$ (or $\|f\|_*$).

Let $L_b^\infty(\mu)$ be the space of all $L^\infty(\mu)$ functions with bounded support. A linear operator T_θ^G is called a θ-type Calderón-Zygmund operator T_θ^G with kernel K_θ satisfying (4.2) and (4.3) if, for all $f \in L_b^\infty(\mu)$ and $x \notin \mathrm{supp}(f)$,

$$T_\theta^G(f)(x) := \int_G K_\theta(x,y)f(y)\mathrm{d}\mu(y) = \int_{\mathbb{R}^n} K_\theta(x,y)\overline{f}(y)\mathrm{d}\mu(y). \tag{4.6}$$

Thus, we also denote $T_\theta^G(f)(x) = T_\theta(\overline{f})(x)$.
Given a function $b \in \mathrm{RBMO}(\mu)$, the commutator $[b, T_\theta^G]$ which is generated by b and T_θ^G is defined by

$$[b, T_\theta](\overline{f})(x) = [b, T_\theta^G](f)(x) := \int_G (b(x) - b(y))K_\theta(x, y)f(y)\mathrm{d}\mu(y). \tag{4.7}$$

The main theorems of this section is stated as follows.
Theorem 4.4. Let $p \in (1, \infty)$, $\varphi \in \Phi_p$ and μ satisfy condition (1.1). Suppose that ϕ is a function satisfying (2.4), the doubling condition

$$\sup_{0<r\le s\le 2r} \frac{\phi(r)}{\phi(s)} < \infty. \tag{4.8}$$

and

$$\int_r^\infty \frac{\phi(t)}{t}\frac{\mathrm{d}t}{t} \le C\frac{\phi(r)}{r}. \tag{4.9}$$

Then T_θ^G defined as in (4.6) is bounded on $\mathcal{L}_\mu^{p),\varphi,\phi}(G)$, that is, there exists a constant $C > 0$ such that, for all $f \in \mathcal{L}_\mu^{p),\varphi,\phi}(G)$,

$$\|T_\theta^G(f)\|_{\mathcal{L}_\mu^{p),\varphi,\phi}(G)} \le C\|f\|_{\mathcal{L}_\mu^{p),\varphi,\phi}(G)}.$$

Theorem 4.5. Let θ be a non-negative and non-decreasing function on $(0, \infty)$ with satisfying (4.1), $p \in (1, \infty)$ and K satisfy (4.2) and (4.3). Suppose that μ satisfies condition (1.1) and G is a bounded domain in $\mathbb{R}^n$. Then $T_{\theta,\varepsilon}^G$ is bounded on $\mathcal{L}_\mu^{p),\varphi,\phi}(G)$, that is, there exists a constant $C > 0$ such that, for any $f \in \mathcal{L}_\mu^{p),\varphi,\phi}(G)$,

$$\|T_{\theta,\varepsilon}^G(f)\|_{\mathcal{L}_\mu^{p),\varphi,\phi}(G)} \le C\|f\|_{\mathcal{L}_\mu^{p),\varphi,\phi}(G)},$$

where the truncated operator $T_{\theta,\varepsilon}^G$ is defined by

$$T_{\theta,\varepsilon}^G f(x) := \int_{\{y\in G:|x-y|>\varepsilon\}} K(x, y)f(y)\mathrm{d}\mu(y), \quad x \in G. \tag{4.10}$$

Moreover, we also denote $T_{\theta,\varepsilon}^G f(x) = T_{\theta,\varepsilon}\overline{f}(x)$.
Remark 4.6. Once we prove that $\{T_{\theta,\varepsilon}^G\}_{\varepsilon>0}$ is bounded in grand generalized Morrey space $\mathcal{L}_\mu^{p,\varphi,\phi}(G)$ uniformly on $\varepsilon > 0$, then it is east to obtain that T_θ^G is bounded in grand generalized Morrey space $\mathcal{L}_\mu^{p,\varphi,\phi}(G)$. Thus, we only need to prove the Theorem 4.5.
Theorem 4.7. Let $p \in (1, \infty)$, $b \in \mathrm{RBMO}(\mu)$, $\varphi \in \Phi_p$ and μ satisfy condition (1.1). Suppose that T_θ^G is bounded on $L_\mu^2(G)$, and ϕ is a function satisfying (2.4), (4.8) and (4.9). Then, the commutator $[b, T_\theta^G]$ defined as in (4.7) is bounded on $\mathcal{L}_\mu^{p),\varphi,\phi}(G)$, that is, there exists a constant $C > 0$ such that, for all $f \in \mathcal{L}_\mu^{p),\varphi,\phi}(G)$,

$$\|[b, T_\theta^G]f\|_{\mathcal{L}_\mu^{p),\varphi,\phi}(G)} \le C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{\mathcal{L}_\mu^{p),\varphi,\phi}(G)}.$$

To prove the above theorems, we should recall the following lemma which is slightly modified in [5].
Lemma 4.8. Let θ be a non-negative and non-decreasing function on $(0, \infty)$ with satisfying (4.1),

$p \in (1, \infty)$ and K satisfy (4.2) and (4.3). Suppose that μ satisfies condition (1.1). Then $T_{\theta,\varepsilon}$ is bounded on $L^p_\mu(\mathbb{R}^n)$, that is, there exists a constant $C > 0$ such that, for all $f \in L^p_\mu(\mathbb{R}^n)$,

$$\|T_{\theta,\varepsilon}(f)\|_{L^p_\mu(\mathbb{R}^n)} \leq C\|f\|_{L^p_\mu(\mathbb{R}^n)},$$

where the truncated operator $T_{\theta,\varepsilon}$ is defined by

$$T_{\theta,\varepsilon}f(x) = \int_{|x-y|>\varepsilon} K(x,y)f(y)\mathrm{d}\mu(y), \quad x \in \mathbb{R}^n. \tag{4.11}$$

Also, we should establish the following lemmas about $T_{\theta,\varepsilon}$ and commutator $[b, T_\theta]$.

Lemma 4.9. Let θ be a non-negative and non-decreasing function on $(0, \infty)$ with satisfying (4.1), $p \in (1, \infty)$, and K satisfy (4.2) and (4.3). Suppose that ϕ is a function satisfying (2.4), (4.8) and (4.9). Then $T_{\theta,\varepsilon}$ defined as in (4.11) is bounded on $\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)$.

Proof. Let $B := B(c_B, r_B)$ be a fixed ball with center at c_B and radius r_B, and set $\varepsilon < r_B$. Decompose function f as

$$f := f_1 + f_2 := f\chi_{2B} + f\chi_{\mathbb{R}^n\setminus(2B)}.$$

Then write

$$\|T_{\theta,\varepsilon}(f)\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)} \leq \|T_{\theta,\varepsilon}(f_1)\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)} + \|T_{\theta,\varepsilon}(f_2)\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)} =: \mathrm{F}_1 + \mathrm{F}_2.$$

With an argument similar to that used in the estimate of Theorem 1.1 in [21], it is easy to see that

$$\mathrm{F}_2 = \|T_{\theta,\varepsilon}(f_2)\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)}.$$

For F_1. By applying (1.3), (2.4) and Lemma 4.8, we can deduce that

$$\begin{aligned}
\|T_{\theta,\varepsilon}(f_1)\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)} &= \sup_{B\subset\mathbb{R}^n}[\phi(\mu(B))]^{-\frac{1}{p}}\Big(\int_B |T_{\theta,\varepsilon}(f_1)(x)|^p\mathrm{d}\mu(x)\Big)^{\frac{1}{p}} \\
&\leq \sup_{B\subset\mathbb{R}^n}[\phi(\mu(B))]^{-\frac{1}{p}}\Big(\int_{\mathbb{R}^n} |T_{\theta,\varepsilon}(f_1)(x)|^p\mathrm{d}\mu(x)\Big)^{\frac{1}{p}} \\
&\leq C\sup_{B\subset\mathbb{R}^n}[\phi(\mu(B))]^{-\frac{1}{p}}\Big(\int_{2B} |f(x)|^p\mathrm{d}\mu(x)\Big)^{\frac{1}{p}} \\
&\leq C\sup_{B\subset\mathbb{R}^n}[\phi(\mu(B))]^{-\frac{1}{p}}[\phi(\mu(2B))]^{\frac{1}{p}}[\phi(\mu(2B))]^{-\frac{1}{p}}\Big(\int_{2B} |f(x)|^p\mathrm{d}\mu(x)\Big)^{\frac{1}{p}} \\
&\leq C\|f\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)}\sup_{B\subset\mathbb{R}^n}\Big[\frac{\mu(2B)}{\mu(B)}\Big]^{\frac{1}{p}} \\
&\leq C\|f\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)}.
\end{aligned}$$

Which, combing the estimate of F_1, the proof of Lemma 4.9 is finished. □

Moreover, we say that T_θ is bounded in Lebesgue space $L^p_\mu(\mathbb{R}^n)$ if the family of truncate operators $\{T_{\theta,\varepsilon}\}_{\varepsilon>0}$ is bounded in $L^p_\mu(\mathbb{R}^n)$ uniformly on $\varepsilon > 0$, and T_θ is bounded in generalized Morrey space $\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)$ if $\{T_{\theta,\varepsilon}\}_{\varepsilon>0}$ is bounded in $\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)$ uniformly on $\varepsilon > 0$, where T_θ is defined by

$$T_\theta f(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\mathrm{d}\mu(y), \quad x \in \mathbb{R}^n. \tag{4.12}$$

Respectively, given a function $b \in \mathrm{RBMO}(\mu)$, commutator $[b, T_\theta]$ which is generated by T_θ and b is defined by

$$[b, T_\theta]f(x) = b(x)T_\theta f(x) - T_\theta(bf)(x). \tag{4.13}$$

Lemma 4.10. Let $b \in \mathrm{RBMO}(\mu)$, θ be a non-negative and non-decreasing function on $(0, \infty)$ with satisfying (4.1), $p \in (1, \infty)$, and K satisfy (4.2) and (4.3). Suppose that ϕ is a function satisfying (2.4), (4.8) and (4.9). Then $[b, T_\theta]$ defined as in (4.13) is bounded on $\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)$.

To prove Lemma 4.10, we should recall the sharp maximal function $M^\sharp$

$$M^\sharp f(x) = \sup_{B \ni x} \frac{1}{\mu(\frac{3}{2}B)} \int_B |f(y) - m_{\widetilde{B}}(f)| \mathrm{d}\mu(y) + \sup_{\substack{B \subset S:\ x \in B \\ B,S \text{ doubling}}} \frac{|m_B(f) - m_S(f)|}{K_{B,S}},$$

and the non-centered doubling maximal operator

$$Nf(x) = \sup_{\substack{B \ni x \\ B \text{ doubling}}} \frac{1}{\mu(B)} \int_B |f(y)| \mathrm{d}\mu(y).$$

By applying Lebesgue differential theorem, it is easy to see that for any $f \in L^1_{\mathrm{loc}}(\mu)$,

$$|f(x)| \le Nf(x), \tag{4.14}$$

for μ-a.e. $x \in \mathbb{R}^n$; see [22].

Proof of Lemma 4.10. With a slightly modified argument similar to that used in the estimate of (9.4) in [22], we also obtain the following pointwise inequality on the commutator $[b, T_\theta]f$, that is,

$$M^\sharp([b, T_\theta]f)(x) \le C\|b\|_{\mathrm{RBMO}(\mu)}\{\widetilde{M}_{r,(9/8)}f(x) + \widetilde{M}_{r,(3/2)}(T_\theta f)(x) + T_{\theta,\varepsilon}(f)(x)\}, \tag{4.15}$$

where, for any $\rho > 1$, $\widetilde{M}_{r,(\rho)}$ is the non-centered maximal operator defined by

$$\widetilde{M}_{r,(\rho)}(f)(x) = \sup_B \left(\frac{1}{\mu(\rho B)} \int_B |f(y)|^r \mathrm{d}\mu(y) \right)^{\frac{1}{r}}.$$

From (4.14), the $\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)$-boundedness of $\widetilde{M}_{r,(\rho)}$ (see [21]) and Lemma 4.9, it follows that

$$\begin{aligned}
&\|[b, T_\theta]f\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)} \\
&= \sup_B [\phi(B)]^{-\frac{1}{p}} \|[b, T_\theta]f\|_{L^p_\mu(B)} \le \sup_B [\phi(B)]^{-\frac{1}{p}} \|N([b, T_\theta]f)\|_{L^p_\mu(B)} \\
&\le C \sup_B [\phi(B)]^{-\frac{1}{p}} \|M^\sharp([b, T_\theta]f)\|_{L^p_\mu(B)} \le C\|M^\sharp([b, T_\theta]f)\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)} \\
&\le C\|b\|_{\mathrm{RBMO}(\mu)} \left\{ \|\widetilde{M}_{r,(9/8)}f\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)} + \|\widetilde{M}_{r,(3/2)}(T_\theta f)\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)} + \|T_{\theta,\varepsilon}(f)\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)} \right\} \\
&\le C\|b\|_{\mathrm{RBMO}(\mu)} \left\{ \|f\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)} + \|T_\theta f\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)} \right\} \\
&\le C\|b\|_{\mathrm{RBMO}(\mu)} \|f\|_{\mathcal{L}^{p,\phi}_\mu(\mathbb{R}^n)}.
\end{aligned}$$

which is our desired result. □

Now we state the proofs of Theorems 4.5 and 4.7 as follows.

Proof of Theorem 4.5. Let δ be a fixed constant satisfying $0<\varepsilon<\delta<p-1$. By applying Definition 1.3, write

$$\begin{aligned}\|T^G_{\theta,\varepsilon}(f)\|_{\mathcal{L}^{p),\varphi,\phi}_\mu(G)} &= \sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\|T^G_{\theta,\varepsilon}(f)\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(G)}\\ &\le \sup_{0<\varepsilon<\delta}\varphi(\varepsilon)\|T^G_{\theta,\varepsilon}(f)\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(G)}+\sup_{\delta<\varepsilon<p-1}\varphi(\varepsilon)\|T^G_{\theta,\varepsilon}(f)\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(G)}\\ &= \mathrm{G}_1+\mathrm{G}_2.\end{aligned}$$

The estimates for G_1 goes as follows. From Definition 1.3 and Lemma 4.9, it follows that

$$\begin{aligned}&\sup_{0<\varepsilon<\delta}\varphi(\varepsilon)\|T^G_{\theta,\varepsilon}(f)\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(G)}\le\sup_{0<\varepsilon<\delta}\varphi(\varepsilon)\|T_{\theta,\varepsilon}(\overline{f})\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(\mathbb{R}^n)}\\ &\le C\sup_{0<\varepsilon<\delta}\varphi(\varepsilon)\|\overline{f}\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(\mathbb{R}^n)}\le C\sup_{0<\varepsilon<\delta}\varphi(\varepsilon)\|f\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(G)}\\ &\le C\|f\|_{\mathcal{L}^{p),\varphi,\phi}_\mu(G)}.\end{aligned}$$

Since $0<\delta<\varepsilon<p-1$, we notice that $\frac{p-\delta}{p-\varepsilon}>1$. Applying Hölder inequality and the boundedness of $T^G_{\theta,\varepsilon}$ in $L^p_\mu(\mathbb{R}^n)$ (see Lemma 4.8), we can deduce that

$$\begin{aligned}&\sup_{\delta<\varepsilon<p-1}\varphi(\varepsilon)\|T^G_{\theta,\varepsilon}(f)\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(G)}\\ &=\sup_{\delta<\varepsilon<p-1}\varphi(\varepsilon)\|T_{\theta,\varepsilon}(\overline{f})\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(G)}\\ &=\sup_{\delta<\varepsilon<p-1}\varphi(\varepsilon)\sup_{B\subset G}[\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}}\Big(\int_B|T_{\theta,\varepsilon}(\overline{f})(x)|^{p-\varepsilon}\mathrm{d}\mu(x)\Big)^{\frac{1}{p-\varepsilon}}\\ &\le\sup_{\delta<\varepsilon<p-1}\varphi(\varepsilon)\sup_{B\subset G}[\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}}\Big(\int_B|T_{\theta,\varepsilon}(\overline{f})(x)|^{p-\delta}\mathrm{d}\mu(x)\Big)^{\frac{1}{p-\delta}}[\mu(B)]^{\frac{1}{p-\varepsilon}-\frac{1}{p-\delta}}\\ &\le C\sup_{\delta<\varepsilon<p-1}\varphi(\varepsilon)\sup_{B\subset G}[\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}}\Big(\int_B|\overline{f}(x)|^{p-\delta}\mathrm{d}\mu(x)\Big)^{\frac{1}{p-\delta}}[\mu(B)]^{\frac{1}{p-\varepsilon}-\frac{1}{p-\delta}}\\ &\le C\sup_{\delta<\varepsilon<p-1}\varphi(\varepsilon)\sup_{B\subset G}[\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}}[\phi(\mu(B))]^{\frac{1}{p-\delta}}[\mu(B)]^{\frac{1}{p-\varepsilon}-\frac{1}{p-\delta}}\\ &\quad\times[\phi(\mu(B))]^{-\frac{1}{p-\delta}}\Big(\int_B|f(x)|^{p-\delta}\mathrm{d}\mu(x)\Big)^{\frac{1}{p-\delta}}\\ &\le C\sup_{\delta<\varepsilon<p-1}\varphi(\varepsilon)\sup_{B\subset G}[\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}}[\phi(\mu(B))]^{\frac{1}{p-\delta}}[\mu(B)]^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}}\\ &\quad\times[\phi(\mu(B))]^{-\frac{1}{p-\delta}}\Big(\int_B|f(x)|^{p-\delta}\mathrm{d}\mu(x)\Big)^{\frac{1}{p-\delta}}.\end{aligned}$$

For the above result, we divide into the following cases.

Case I If $r_B>1$, then, by applying (2.4), we obtain that

$$[\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}}[\phi(\mu(B))]^{\frac{1}{p-\delta}}[\mu(B)]^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}}=\Big[\frac{\mu(B)}{\phi(\mu(B))}\Big]^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}}$$

$$= \left[\frac{\phi(\mu(B(c_B,1)))}{\phi(\mu(B))} \times \frac{\mu(B)}{\phi(\mu(B(c_B,1)))}\right]^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}}$$
$$\leq C\left[\frac{\mu(B(c_B,1))}{\phi(\mu(B(c_B,1)))}\right]^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}} \leq C.$$

Case II If $r_B \leq 1$, then, by applying the monotonicity of ϕ, we can deduce that

$$[\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}}[\phi(\mu(B))]^{\frac{1}{p-\delta}}[\mu(B)]^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}} \leq [\phi(\mu(B))]^{-\frac{1}{p-\delta}}[\phi(\mu(B))]^{\frac{1}{p-\delta}}[\mu(B)]^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}}$$
$$\leq [\mu(B(c_B,1))]^{p-1-\delta} \leq C.$$

Combing the cases **I** and **II**, we further obtain that

$$\begin{aligned}
&\sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon)\|T^G_{\theta,\varepsilon}(f)\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(G)} \\
&\leq C \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B\subset G}[\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}}[\phi(\mu(B))]^{\frac{1}{p-\delta}}[\mu(B)]^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}} \\
&\quad \times[\phi(\mu(B))]^{-\frac{1}{p-\delta}}\left(\int_B |f(x)|^{p-\delta}\mathrm{d}\mu(x)\right)^{\frac{1}{p-\delta}} \\
&\leq C \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B\subset G}[\phi(\mu(B))]^{-\frac{1}{p-\delta}}\left(\int_B |f(x)|^{p-\delta}\mathrm{d}\mu(x)\right)^{\frac{1}{p-\delta}} \\
&\leq C \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon)[\varphi(\delta)]^{-1}\varphi(\delta) \sup_{B\subset G}[\phi(\mu(B))]^{-\frac{1}{p-\delta}}\left(\int_B |f(x)|^{p-\delta}\mathrm{d}\mu(x)\right)^{\frac{1}{p-\delta}} \\
&\leq C\varphi(p-1)[\varphi(\delta)]^{-1}\|f\|_{L^{p),\varphi,\phi}_\mu(\mathbb{R}^n)} \\
&\leq C\|f\|_{L^{p),\varphi,\phi}_\mu(\mathbb{R}^n)}.
\end{aligned}$$

Which, together with estimate of G_1, we obtain the desired result. □

Proof of Theorem 4.7. Let δ be a fixed constant satisfying $0 < \varepsilon < \delta < p-1$. By applying Definition 1.3, write

$$\begin{aligned}
&\|[b, T^G_\theta](f)\|_{\mathcal{L}^{p),\varphi,\phi}_\mu(G)} \\
&= \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\|[b, T^G_{\theta,}](f)\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(G)} \\
&\leq \sup_{0<\varepsilon<\delta} \varphi(\varepsilon)\|[b, T^G_\theta](f)\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(G)} + \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon)\|[b, T^G_\theta](f)\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(G)} \\
&= F_1 + F_2.
\end{aligned}$$

The estimates for F_1 is given as follows. By applying Lemma 4.10 and Definition 1.3, we obtain that

$$\begin{aligned}
&\sup_{0<\varepsilon<\delta} \varphi(\varepsilon)\|[b, T^G_\theta](f)\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(G)} \\
&= \sup_{0<\varepsilon<\delta} \varphi(\varepsilon)\|[b, T_\theta](\overline{f})\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(G)} \leq \sup_{0<\varepsilon<\delta} \varphi(\varepsilon)\|[b, T_\theta](\overline{f})\|_{\mathcal{L}^{p-\varepsilon,\phi}_\mu(\mathbb{R}^n)}
\end{aligned}$$

$$\leq C\|b\|_{\mathrm{RBMO}(\mu)} \sup_{0<\varepsilon<\delta} \varphi(\varepsilon)\|\overline{f}\|_{\mathcal{L}_{\mu}^{p-\varepsilon,\phi}(\mathbb{R}^n)} \leq C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{\mathcal{L}_{\mu}^{p),\varphi,\phi}(G)},$$

where $\overline{f}$ is defined as in (2.1).

Now let us estimate F_2. By virtue of Hölder's inequality, the $L^p_\mu(\mathbb{R}^n)$-boundedness of $[b, T^G_\theta]$ (see Lemma 4.10) and **Cases** I and II, it then follows that

$$\begin{aligned}
&\sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon)\|[b, T^G_\theta](f)\|_{\mathcal{L}_{\mu}^{p-\varepsilon,\phi}(G)} \\
&= \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon)\|[b, T_\theta](\overline{f})\|_{\mathcal{L}_{\mu}^{p-\varepsilon,\phi}(G)} \\
&= \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B\subset G}[\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}}\left(\int_B |[b, T_\theta](\overline{f})(x)|^{p-\varepsilon}\mathrm{d}\mu(x)\right)^{\frac{1}{p-\varepsilon}} \\
&\leq C\|b\|_{\mathrm{RBMO}(\mu)} \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B\subset G}[\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}}\left(\int_B |\overline{f}(x)|^{p-\delta}\mathrm{d}\mu(x)\right)^{\frac{1}{p-\delta}}[\mu(B)]^{\frac{1}{p-\varepsilon}-\frac{1}{p-\delta}} \\
&\leq C\|b\|_{\mathrm{RBMO}(\mu)} \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B\subset G}[\phi(\mu(B))]^{-\frac{1}{p-\varepsilon}}[\phi(\mu(B))]^{\frac{1}{p-\delta}}[\mu(B)]^{\frac{1}{p-\varepsilon}-\frac{1}{p-\delta}} \\
&\qquad \times[\phi(\mu(B))]^{-\frac{1}{p-\delta}}\left(\int_B |f(x)|^{p-\delta}\mathrm{d}\mu(x)\right)^{\frac{1}{p-\delta}} \\
&\leq C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{\mathcal{L}_{\mu}^{p),\varphi,\phi}(G)}.
\end{aligned}$$

Which, together with the estimate of F_1, we complete the proof of Theorem 4.7 □

5. Conclusions

In this paper, we mainly obtain the boundedness of Hardy-Littlewood maximal operator, fractional integral operators and θ-type Calderón-Zygmund operators on the non-homogeneous grand generalized Morrey space $\mathcal{L}_{\mu}^{p),\varphi,\phi}(G)$. In addition, the boundedness of commutator $[b, T^G_\theta]$ which is generated by θ-type Calderón-Zygmund operator T_θ and b on spaces $\mathcal{L}_{\mu}^{p),\varphi,\phi}(G)$ is also established.

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Conflict of interest

The authors declare that they have no conflict of interest.

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