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Research article

An optimal Z-eigenvalue inclusion interval for a sixth-order tensor and its an application

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Abstract: An optimal Z-eigenvalue inclusion interval for a sixth-order tensor is presented. As an application, a sufficient condition for the positive definiteness of a sixth-order real symmetric tensor (also a homogeneous polynomial form) is obtained, which is used to judge the asymptotically stability of time-invariant polynomial systems.

Keywords: sixth-order tensors; Z-eigenvalues; inclusion intervals; positive definiteness
Mathematics Subject Classification: 15A18, 15A42, 15A69

1. Introduction

Let m and n be two positive integers, $m \geq 2$ and $n \geq 2$, $[n]$ be the set $\{1, 2, \ldots, n\}$, $\mathbb{C}$ (resp. $\mathbb{R}$) be the set of all complex (resp. real) numbers, $\mathbb{R}^n$ be the set of all dimension n real vectors, $\mathbb{R}^{[m,n]}$ be the set of all order m dimension n real tensors. Let $x = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^n$. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, i.e.,

$$a_{i_1 i_2 \cdots i_m} \in \mathbb{R}, \quad i_j \in [n], \quad j \in [m].$$

Furthermore, $\mathcal{A}$ is called symmetric [15] if $a_{i_1 i_2 \cdots i_m} = a_{i_{\pi(1)} \cdots i_{\pi(m)}}$ for $\pi \in \Pi_m$, where Π_m is the permutation group of m indices.

Given a tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$, if there are $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n \backslash \{0\}$ such that

$$\mathcal{A}x^{m-1} = \lambda x \quad \text{and} \quad x^\top x = 1,$$

where $\mathcal{A}x^{m-1}$ is an n-dimensional vector whose ith component is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \ldots, i_m \in [n]} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m},$$

then λ is called an E-eigenvalue of $\mathcal{A}$ and x an E-eigenvector of $\mathcal{A}$ associated with λ. If both λ and x are real, then λ is called a Z-eigenvalue of $\mathcal{A}$ and x a Z-eigenvector of $\mathcal{A}$ associated with λ. Let $\sigma(\mathcal{A})$ be the set of all Z-eigenvalues of $\mathcal{A}$.

The Z-eigenvalues of an even order real symmetric tensor $\mathcal{A}$ is introduced by Qi in [15] in order to identify the positive definiteness of an m-th degree homogeneous polynomial form

$$f(x) = \mathcal{A}x^m = \sum_{i_1,\ldots,i_m\in[n]} a_{i_1i_2\cdots i_m}x_{i_1}x_{i_2}\cdots x_{i_m}, \tag{1.1}$$

and $f(x)$ is positive definite, i.e., $f(x) > 0$ for any $x \in \mathbb{R}^n\backslash\{0\}$, if and only if $\mathcal{A}$ is positive definite [15]. Furthermore, $\mathcal{A}$ is positive definite if and only if all of its Z-eigenvalues are positive. The positive definiteness of $f(x)$ is widely used in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control [1–4, 17].

Next, a special tensor, the Z-identity tensor, is recalled.

Definition 1.1. [10, 11, 15] *A tensor $\mathcal{I} = (e_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ with m even is called a Z-identity tensor if for any vector $x \in \mathbb{R}^n$,*

$$\mathcal{I}x^{m-1} = x \quad \text{and} \quad x^\top x = 1.$$

Note here that an even order n dimension Z-identity tensor is not unique in general. For instance, the following two tensors are both Z-identity tensors:

Case I. ([11, Definition 2.1]): Let $\mathcal{I}_1 = (e_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$, where

$$e_{i_1i_1i_2i_2\cdots i_ki_k} = 1, \quad i_1, i_2, \ldots, i_k \in [n], \ \text{and} \ m = 2k;$$

Case II. ([10, Property 2.4]): Let $\mathcal{I}_2 = (e_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$, where

$$e_{i_1\cdots i_m} = \frac{1}{m!}\sum_{\pi\in\Pi_m} \delta_{i_{\pi(1)}i_{\pi(2)}}\delta_{i_{\pi(3)}i_{\pi(4)}}\cdots\delta_{i_{\pi(m-1)}i_{\pi(m)}},$$

where δ is the standard Kronecker delta, i,e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

For convenient applications, the Z-identity tensor $\mathcal{I}_2 = (e_{i_1i_2\cdots i_6}) \in \mathbb{R}^{[6,n]}$ is listed as follows:

$$e_{i_1i_2\cdots i_6} = \begin{cases} 1, & \text{if } i_1 = i_2 = \cdots = i_6, \\ 1/5, & \text{if } (i_1, i_2, \ldots, i_6) \in \bigcup\limits_{\substack{i\neq j,\\ i,j\in[n]}} \{\pi(i,i,i,i,j,j)\}, \\ 1/15, & \text{if } (i_1, i_2, \ldots, i_6) \in \bigcup\limits_{\substack{i\neq j\neq k,\\ i,j,k\in[n]}} \{\pi(i,i,j,j,k,k)\}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\{\pi(i_1, i_2, \ldots, i_6)\}$ is the set of all combinations of $i_1, i_2, \ldots, i_6$; also see [22] for details.

According to Theorem 8.5 in [16], in order to judge the strong ellipticity condition of anisotropic elastic materials, it is necessary to judge the positive definiteness of three second order symmetric tensors, a fourth-order symmetric tensor and a sixth-order symmetric tensor based on Z-eigenvalues of these tensors. Hence, one can calculate the smallest Z-eigenvalue or all Z-eigenvalues of an even order tensor to judge its positive definiteness.

However, when m and n are large, it is not easy to compute all Z-eigenvalues or even the smallest Z-eigenvalue of a tensor. Fortunately, for the problem of judging the positive definiteness of an even order real symmetric tensor $\mathcal{A}$, we do not need to calculate all Z-eigenvalues of $\mathcal{A}$, but only need to know the symbols of all Z-eigenvalues. In view of this, a general approach is adopted: one can construct a set including all Z-eigenvalues of $\mathcal{A}$, and if this set is just in the right-half complex plane, then he can conclude that all Z-eigenvalues are positive, and consequently, the tensor $\mathcal{A}$ is positive definite. The related results are showed in [6–8, 11–14, 18–31].

In order to obtain many sufficient conditions for the positive definiteness of an even order real symmetric tensor, Li et al. [11] and Sang and Chen [22], respectively, presented their Z-eigenvalue inclusion intervals with n parameters for an even order real tensor as follows:

Theorem 1.2. [11, Theorem 2.2] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ with m being even. Then for any vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$,*

$$\sigma(\mathcal{A}) \subseteq \mathcal{G}(\mathcal{A}, \alpha) := \bigcup_{i \in [n]} \Big(\mathcal{G}_i(\mathcal{A}, \alpha) := \{z \in \mathbb{R} : |z - \alpha_i| \le R_i(\mathcal{A}, \alpha_i)\} \Big), \tag{1.2}$$

where

$$R_i(\mathcal{A}, \alpha_i) = \sum_{(i_2,\ldots,i_m) \in \Lambda_i} |a_{ii_2\cdots i_m} - \alpha_i e_{ii_2\cdots i_m}| + \sum_{(i_2,\ldots,i_m) \in \overline{\Lambda}_i} |a_{ii_2\cdots i_m}|, \tag{1.3}$$

and

$$\Lambda_i = \{(i_2, \ldots, i_m) : e_{ii_2\cdots i_m} \neq 0, \quad i_2, \ldots, i_m \in [n]\},$$
$$\overline{\Lambda}_i = \{(i_2, \ldots, i_m) : e_{ii_2\cdots i_m} = 0, \quad i_2, \ldots, i_m \in [n]\}.$$

Theorem 1.3. [22, Theorem 3.1] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ with m being even, and $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$. Then*

$$\sigma(\mathcal{A}) \subseteq \Upsilon(\mathcal{A}, \alpha) := \bigcup_{i \in [n]} \Big(\Upsilon_i(\mathcal{A}, \alpha) := \{z \in \mathbb{R} : |z - \alpha_i| \le r_i(\mathcal{A}, \alpha_i)\} \Big), \tag{1.4}$$

where

$$r_i(\mathcal{A}, \alpha_i) = r_i^{\Delta \cap \Lambda_i}(\mathcal{A}, \alpha_i) + r_i^{\overline{\Delta} \cap \Lambda_i}(\mathcal{A}, \alpha_i) + r_i^{\Delta \cap \overline{\Lambda}_i}(\mathcal{A}) + r_i^{\overline{\Delta} \cap \overline{\Lambda}_i}(\mathcal{A}),$$

$$r_i^{\Delta \cap \Lambda_i}(\mathcal{A}, \alpha_i) = \frac{1}{(m-2)^{\frac{m-2}{2}}} \sum_{(i_2,\ldots,i_m) \in \Delta \cap \Lambda_i} |a_{ii_2\cdots i_m} - \alpha_i e_{ii_2\cdots i_m}|,$$
$$r_i^{\overline{\Delta} \cap \Lambda_i}(\mathcal{A}, \alpha_i) = \sum_{(i_2,\ldots,i_m) \in \overline{\Delta} \cap \Lambda_i} |a_{ii_2\cdots i_m} - \alpha_i e_{ii_2\cdots i_m}|,$$
$$r_i^{\Delta \cap \overline{\Lambda}_i}(\mathcal{A}) = \frac{1}{(m-2)^{\frac{m-2}{2}}} \sum_{(i_2,\ldots,i_m) \in \Delta \cap \overline{\Lambda}_i} |a_{ii_2\cdots i_m}|,$$
$$r_i^{\overline{\Delta} \cap \overline{\Lambda}_i}(\mathcal{A}) = \sum_{(i_2,\ldots,i_m) \in \overline{\Delta} \cap \overline{\Lambda}_i} |a_{ii_2\cdots i_m}|,$$

and

$$\Delta = \{(i_2, \dots, i_m) : i_2 \neq \cdots \neq i_m, \textit{or only two of } i_2, \dots, i_m \in [n] \textit{ are the same}\},$$
$$\overline{\Delta} = \{(i_2, \dots, i_m) : (i_2, \dots, i_m) \notin \Delta,\ i_2, \dots, i_m \in [n]\}.$$

Furthermore, $\Upsilon(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha)$.

From Theorems 1.2 and 1.3, it can be seen that the forms of $R_i(\mathcal{A}, \alpha_i)$ and $r_i(\mathcal{A}, \alpha_i)$ are closely related to the Z-identify tensor $\mathcal{I}$. For convenient applications, when the Z-identify tensor $\mathcal{I}$ is taken as $\mathcal{I}_1$, the specific forms of $R_i(\mathcal{A}, \alpha_i)$ and $r_i(\mathcal{A}, \alpha_i)$ had been given in [11, Corollary 1] and [22, Corollary 3]. However, when the Z-identify tensor $\mathcal{I}$ is taken as $\mathcal{I}_2$, the specific forms of $R_i(\mathcal{A}, \alpha_i)$ and $r_i(\mathcal{A}, \alpha_i)$ have not been given.

Hence, we in this paper focus on giving the specific forms of $R_i(\mathcal{A}, \alpha_i)$ and $r_i(\mathcal{A}, \alpha_i)$ when the Z-identify tensor $\mathcal{I}$ is taken as $\mathcal{I}_2$ and partially answering the two questions proposed in [22]:

Question 1: What is the specific form of Theorem 1.3 for $m \geq 6$ and m is even if the Z-identity tensor $\mathcal{I}$ as $\mathcal{I}_2$?

Question 2: What is the specific form of the Z-identity tensor $\mathcal{I}_2$ for the order $m \geq 8$ and m is even?

The remaining chapters are arranged as follows. In Section 2, for a sixth-order tensor $\mathcal{A}$, the specific forms of the inclusion intervals $\mathcal{G}(\mathcal{A}, \alpha)$ and $\Upsilon(\mathcal{A}, \alpha)$ with a parameter vector α are given. Subsequently, by selecting appropriate parameter vector α, the optimal interval of $\Upsilon(\mathcal{A}, \alpha)$ is presented. In Section 3, an application of the optimal Z-eigenvalue inclusion interval is considered. This optimal interval is used to present a sufficient condition for the positive definiteness of sixth-order real symmetric tensors (also homogeneous polynomial forms), which is used to judge the asymptotically stability of time-invariant polynomial systems.

2. An optimal Z-eigenvalue inclusion interval for a sixth-order tensor with $\mathcal{I}_2$

The specific forms of Theorems 1.2 and 1.3 are firstly listed.Subsequently, an appropriate parameter vector α is taken to optimize the interval of $\Upsilon(\mathcal{A}, \alpha)$ in Theorem 1.3.

Let $m = 6$ and the Z-identity tensor $\mathcal{I}$ be $\mathcal{I}_2$. Consider $R_i(\mathcal{A}, \alpha_i)$ in (1.3). Then, for each $i \in [n]$,

$$\Lambda_i = \{(i, i, i, i, i), \pi(i, i, i, j, j), \pi(i, j, j, j, j), \pi(i, j, j, k, k)\},$$

and $e_{iiiiii} = 1$, $e_{\pi(i,i,i,i,j,j)} = e_{\pi(i,i,j,j,j,j)} = \frac{1}{5}$ and $e_{\pi(i,i,j,j,k,k)} = \frac{1}{15}$ for $j, k \in [n]$ and $j \neq k \neq i$. Consequently,

$$\sum_{(i_2,\dots,i_m)\in\Lambda_i} |a_{ii_2\cdots i_m} - \alpha_i e_{ii_2\cdots i_m}| = |a_{iiiiii} - \alpha_i| + \sum_{j\neq i}\left(\sum_{\upsilon\in\{\pi(i,i,i,j,j)\}} |a_{i\upsilon} - \frac{1}{5}\alpha_i| + \sum_{\upsilon\in\{\pi(i,j,j,j,j)\}} |a_{i\upsilon} - \frac{1}{5}\alpha_i|\right)$$
$$+ \sum_{j\neq k\neq i}\sum_{\upsilon\in\{\pi(i,j,j,k,k)\}} |a_{i\upsilon} - \frac{1}{15}\alpha_i|, \tag{2.1}$$

and

$$\sum_{(i_2,\dots,i_m)\in\overline{\Lambda}_i} |a_{ii_2\cdots i_m}| = \sum_{i_2,\dots,i_m\in[n]} |a_{ii_2\cdots i_m}| - \sum_{(i_2,\dots,i_m)\in\Lambda_i} |a_{ii_2\cdots i_m}|$$

$$= \sum_{i_2,\dots,i_m\in[n]} |a_{ii_2\cdots i_m}| - |a_{iiiiii}| - \sum_{j\neq i}\left(\sum_{\upsilon\in\{\pi(i,i,i,j,j)\}} |a_{i\upsilon}| + \sum_{\upsilon\in\{\pi(i,j,j,j,j)\}} |a_{i\upsilon}|\right) - \sum_{j\neq k\neq i}\sum_{\upsilon\in\{\pi(i,j,j,k,k)\}} |a_{i\upsilon}|,$$

where, in order to shorten formulas, $\sum\limits_{(i_2,\dots,i_6)\in S} a_{ii_2\cdots i_6}$ is written as $\sum\limits_{\upsilon\in S} a_{i\upsilon}$ for $i, i_2, \dots, i_6 \in [n]$ and a set S.

Hence, the specific form of Theorem 1.2 for $m = 6$ is listed as follows:

Corollary 2.1. *Let $\mathcal{A} = (a_{i_1i_2\cdots i_6}) \in \mathbb{R}^{[6,n]}$ and $\alpha = (\alpha_1, \dots, \alpha_n)^\top \in \mathbb{R}^n$. Then (1.2) holds, where*

$$\begin{aligned} R_i(\mathcal{A}, \alpha_i) =& |a_{iiiiii} - \alpha_i| + \sum_{j\neq i}\left(\sum_{\upsilon\in\{\pi(i,i,i,j,j)\}} |a_{i\upsilon} - \frac{1}{5}\alpha_i| + \sum_{\upsilon\in\{\pi(i,j,j,j,j)\}} |a_{i\upsilon} - \frac{1}{5}\alpha_i|\right) \\ &+ \sum_{j\neq k\neq i}\sum_{\upsilon\in\{\pi(i,j,j,k,k)\}} |a_{i\upsilon} - \frac{1}{15}\alpha_i| + \sum_{(i_2,\dots,i_m)\in\overline{\Lambda}_i} |a_{ii_2\cdots i_m}|. \end{aligned}$$

Next, the specific form of Theorem 1.3 for $m = 6$ is considered. Let $m = 6$. Then

$$\Delta = \{\pi(j, k, l, s, t), \pi(j, j, k, l, s), \ \text{where} \ j, k, l, s, t \in [n] \ \text{and} \ j \neq k \neq l \neq s \neq t\},$$

and

$$\sum_{(i_2,\dots,i_6)\in\Delta} |a_{ii_2\cdots i_6}| = \sum_{j\neq k\neq l\neq s\neq t}\sum_{\upsilon\in\{\pi(j,k,l,s,t)\}} |a_{i\upsilon}| + \sum_{j\neq k\neq l\neq s}\sum_{\upsilon\in\{\pi(j,j,k,l,s)\}} |a_{i\upsilon}|.$$

Corollary 2.2. *Let $\mathcal{A} = (a_{i_1i_2\cdots i_6}) \in \mathbb{R}^{[6,n]}$ and $\alpha = (\alpha_1, \dots, \alpha_n)^\top \in \mathbb{R}^n$. Then (1.4) holds, where*

$$\begin{aligned} r_i(\mathcal{A}, \alpha_i) =& |a_{iiiiii} - \alpha_i| + \sum_{j\neq i}\left(\sum_{\upsilon\in\{\pi(i,i,i,j,j)\}} |a_{i\upsilon} - \frac{1}{5}\alpha_i| + \sum_{\upsilon\in\{\pi(i,j,j,j,j)\}} |a_{i\upsilon} - \frac{1}{5}\alpha_i|\right) \\ &+ \sum_{j\neq k\neq i}\sum_{\upsilon\in\{\pi(i,j,j,k,k)\}} |a_{i\upsilon} - \frac{1}{15}\alpha_i| + \eta_i(\mathcal{A}), \end{aligned} \tag{2.2}$$

where

$$\eta_i(\mathcal{A}) = \begin{cases} \sum\limits_{(i_2,\dots,i_6)\in\overline{\Lambda}_i} |a_{ii_2\cdots i_6}|, & 2 \leq n \leq 3; \quad (2.3) \\ \sum\limits_{(i_2,\dots,i_6)\in\overline{\Lambda}_i} |a_{ii_2\cdots i_6}| - \frac{15}{16}\sum\limits_{(i_2,\dots,i_6)\in\Delta} |a_{ii_2\cdots i_6}|, & n \geq 4. \quad (2.4) \end{cases}$$

Moreover, when $2 \leq n \leq 3$, then $\Upsilon(\mathcal{A}, \alpha) = \mathcal{G}(\mathcal{A}, \alpha)$; when $n \geq 4$, then $\Upsilon(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha)$.

Proof. Let

$$N = \{(i_2, \dots, i_6) : i_2, \dots, i_6 \in [n]\}.$$

The proof is divided into two parts depending on the difference of dimension.

(i) When $2 \le n \le 3$, then $\Delta = \emptyset$ and $\overline{\Delta} = N$, consequently, $\Delta \cap \Lambda_i = \Delta \cap \overline{\Lambda}_i = \emptyset$, $\overline{\Delta} \cap \Lambda_i = \Lambda_i$ and $\overline{\Delta} \cap \overline{\Lambda}_i = \overline{\Lambda}_i$. Hence,

$$r_i^{\Delta\cap\Lambda_i}(\mathcal{A}, \alpha_i) = r_i^{\Delta\cap\overline{\Lambda}_i}(\mathcal{A}) = 0, \quad r_i^{\overline{\Delta}\cap\overline{\Lambda}_i}(\mathcal{A}) = \sum_{(i_2,\dots,i_6)\in\overline{\Lambda}_i} |a_{ii_2\cdots i_6}|,$$

$$r_i^{\overline{\Delta}\cap\Lambda_i}(\mathcal{A}, \alpha_i) = \sum_{(i_2,\dots,i_6)\in\Lambda_i} |a_{ii_2\cdots i_6} - \alpha_i e_{ii_2\cdots i_6}|, \tag{2.5}$$

and consequently,

$$\begin{aligned} r_i(\mathcal{A}, \alpha_i) =& r_i^{\Delta\cap\Lambda_i}(\mathcal{A}, \alpha_i) + r_i^{\overline{\Delta}\cap\Lambda_i}(\mathcal{A}, \alpha_i) + r_i^{\Delta\cap\overline{\Lambda}_i}(\mathcal{A}) + r_i^{\overline{\Delta}\cap\overline{\Lambda}_i}(\mathcal{A}) \\ =& \sum_{(i_2,\dots,i_6)\in\Lambda_i} |a_{ii_2\cdots i_6} - \alpha_i e_{ii_2\cdots i_6}| + \sum_{(i_2,\dots,i_6)\in\overline{\Lambda}_i} |a_{ii_2\cdots i_6}| = R_i(\mathcal{A}, \alpha_i). \end{aligned}$$

By (2.1) and (2.3), (2.2) follows, which implies $\Upsilon(\mathcal{A}, \alpha) = \mathcal{G}(\mathcal{A}, \alpha)$.

(ii) If $n \ge 4$, then $\Delta \ne \emptyset$, but $\Delta \cap \Lambda_i = \emptyset$, consequently, $\overline{\Delta} \cap \Lambda_i = \Lambda_i$ and $\Delta \cap \overline{\Lambda}_i = \Delta$, which implies that (2.5) holds, $r_i^{\Delta\cap\Lambda_i}(\mathcal{A}, \alpha_i) = 0$ and

$$r_i^{\Delta\cap\overline{\Lambda}_i}(\mathcal{A}) = \frac{1}{16} \sum_{(i_2,\dots,i_6)\in\Delta} |a_{ii_2\cdots i_6}|.$$

By $\overline{\Lambda}_i = N \cap \overline{\Lambda}_i = (\Delta \cup \overline{\Delta}) \cap \overline{\Lambda}_i = (\Delta \cap \overline{\Lambda}_i) \cup (\overline{\Delta} \cap \overline{\Lambda}_i) = \Delta \cup (\overline{\Delta} \cap \overline{\Lambda}_i)$, we have $\overline{\Delta} \cap \overline{\Lambda}_i = \overline{\Lambda}_i - \Delta$. Hence,

$$r_i^{\overline{\Delta}\cap\overline{\Lambda}_i}(\mathcal{A}) = \sum_{(i_2,\dots,i_6)\in\overline{\Lambda}_i} |a_{ii_2\cdots i_6}| - \sum_{(i_2,\dots,i_6)\in\Delta} |a_{ii_2\cdots i_6}|.$$

Consequently,

$$\begin{aligned} r_i(\mathcal{A}, \alpha_i) =& r_i^{\Delta\cap\Lambda_i}(\mathcal{A}, \alpha_i) + r_i^{\overline{\Delta}\cap\Lambda_i}(\mathcal{A}, \alpha_i) + r_i^{\Delta\cap\overline{\Lambda}_i}(\mathcal{A}) + r_i^{\overline{\Delta}\cap\overline{\Lambda}_i}(\mathcal{A}) \\ =& \sum_{(i_2,\dots,i_6)\in\Lambda_i} |a_{ii_2\cdots i_6} - \alpha_i e_{ii_2\cdots i_6}| + \frac{1}{16} \sum_{(i_2,\dots,i_6)\in\Delta} |a_{ii_2\cdots i_6}| + \sum_{(i_2,\dots,i_6)\in\overline{\Lambda}_i} |a_{ii_2\cdots i_6}| - \sum_{(i_2,\dots,i_6)\in\Delta} |a_{ii_2\cdots i_6}| \\ =& \sum_{(i_2,\dots,i_6)\in\Lambda_i} |a_{ii_2\cdots i_6} - \alpha_i e_{ii_2\cdots i_6}| + \sum_{(i_2,\dots,i_6)\in\overline{\Lambda}_i} |a_{ii_2\cdots i_6}| - \frac{15}{16} \sum_{(i_2,\dots,i_6)\in\Delta} |a_{ii_2\cdots i_6}| \\ =& R_i(\mathcal{A}, \alpha_i) - \frac{15}{16} \sum_{(i_2,\dots,i_6)\in\Delta} |a_{ii_2\cdots i_6}|. \end{aligned}$$

By (2.1) and (2.4), (2.2) follows, which implies $\Upsilon(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha)$ by $\sum_{(i_2,\dots,i_6)\in\Delta} |a_{ii_2\cdots i_6}| \ge 0$ for $i \in [n]$. □

It is showed in Theorem 1.3 that $\Upsilon(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha)$. When $m = 6$, it is easy to see the relationship $\Upsilon(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha)$ from Corollaries 2.1 and 2.2. Next, we considered this problem: How to choose a parameter vector α to optimize the inclusion interval $\Upsilon(\mathcal{A}, \alpha)$ in Corollary 2.2. Before that, two lemmas are listed.

Lemma 2.1. [22, Lemma 4.2] *Let*

$$f(x) = x - \frac{1}{a}\sum_{i\in[n]} |x - b_i| - c$$

be a real valued function about x, where a is a positive integer, $b_i \in \mathbb{R}$ *and* $b_1 \le b_2 \le \cdots \le b_n$ *with* $n \ge a$, *and* $c \in \mathbb{R}$. *Assume that a is odd.*

(i) If n is odd, then

$$\max_{x\in\mathbb{R}} f(x) = \frac{1}{a}\Bigg(\sum_{i=1}^{\frac{n+a}{2}} b_i - \sum_{i=\frac{n+a}{2}+1}^{n} b_i\Bigg) - c,$$

and this takes place for every $x \in [b_{\frac{n+a}{2}}, b_{\frac{n+a}{2}+1}]$ *if* $b_{\frac{n+a}{2}} \neq b_{\frac{n+a}{2}+1}$, *and only for* $x = b_{\frac{n+a}{2}}$ *if* $b_{\frac{n+a}{2}} = b_{\frac{n+a}{2}+1}$. *Note that let* $[b_{\frac{n+a}{2}}, b_{\frac{n+a}{2}+1}]$ *be* $[b_{\frac{n+a}{2}}, +\infty)$ *if* $b_{\frac{n+a}{2}+1}$ *does not exist.*

(ii) If n is even, then

$$\max_{x\in\mathbb{R}} f(x) = \frac{1}{a}\Bigg(\sum_{i=1}^{\frac{n+a-1}{2}} b_i - \sum_{i=\frac{n+a+3}{2}}^{n} b_i\Bigg) - c,$$

and this maximum is reached when $x = b_{\frac{n+a+1}{2}}$.

Lemma 2.2. [22, Lemma 4.1] *Let*

$$g(x) = x + \frac{1}{a}\sum_{i\in[n]} |x - b_i| + c$$

be a real valued function about x, where a is a positive integer, $b_i \in \mathbb{R}$ *and* $b_1 \le b_2 \le \cdots \le b_n$ *with* $n \ge a + 1$, *and* $c \in \mathbb{R}$. *Assume that a is odd.*

(i) If n is odd, then

$$\min_{x\in\mathbb{R}} g(x) = \frac{1}{a}\Bigg(\sum_{i=\frac{n-a}{2}+1}^{n} b_i - \sum_{i=1}^{\frac{n-a}{2}} b_i\Bigg) + c$$

and this takes place for every $x \in [b_{\frac{n-a}{2}}, b_{\frac{n-a}{2}+1}]$ *if* $b_{\frac{n-a}{2}} \neq b_{\frac{n-a}{2}+1}$, *and only for* $x = b_{\frac{n-a}{2}}$ *if* $b_{\frac{n-a}{2}} = b_{\frac{n-a}{2}+1}$.

(ii) If n is even, then

$$\min_{x\in\mathbb{R}} g(x) = \frac{1}{a}\Bigg(\sum_{i=\frac{n-a+3}{2}}^{n} b_i - \sum_{i=1}^{\frac{n-a-1}{2}} b_i\Bigg) + c$$

and this minimum is reached when $x = b_{\frac{n-a+1}{2}}$.

Now, the optimal inclusion interval of the interval $\Upsilon(\mathcal{A}, \alpha)$ for sixth-order tensors is presented.

Theorem 2.3. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_6}) \in \mathbb{R}^{[6,n]}$. Then*

$$\sigma(\mathcal{A}) \subseteq \Upsilon(\mathcal{A}) := \bigcup_{i\in[n]} [l_i, u_i],$$

where l_i and u_i are taken by the following methods:

(i) If n is odd, then

$$l_i = \frac{1}{15}\Big(\sum_{k=1}^{\frac{15n^2+15}{2}} b_{i,k} - \sum_{k=\frac{15n^2+17}{2}}^{15n^2} b_{i,k} \Big) - \eta_i(\mathcal{A}) \quad \textit{and} \quad u_i = \frac{1}{15}\Big(\sum_{k=\frac{15n^2-13}{2}}^{15n^2} b_{i,k} - \sum_{k=1}^{\frac{15n^2-15}{2}} b_{i,k} \Big) + \eta_i(\mathcal{A});$$

(ii) If n is even, then

$$l_i = \frac{1}{15}\Big(\sum_{k=1}^{\frac{15n^2+14}{2}} b_{i,k} - \sum_{k=\frac{15n^2+18}{2}}^{15n^2} b_{i,k} \Big) - \eta_i(\mathcal{A}) \quad \textit{and} \quad u_i = \frac{1}{15}\Big(\sum_{k=\frac{15n^2-12}{2}}^{15n^2} b_{i,k} - \sum_{k=1}^{\frac{15n^2-16}{2}} b_{i,k} \Big) + \eta_i(\mathcal{A}).$$

Here, for each $i \in [n]$, $b_{i,1} \le b_{i,2} \le \cdots \le b_{i,15n^2}$ is an arrangement in non-decreasing order of a_{iiiiii} with its number 15, $5a_{i\upsilon}$ with its number 3 for $\upsilon \in \{\pi(i,i,i,j,j)\}$ and $j \neq i$, $5a_{i\upsilon}$ with its number 3 for $\upsilon \in \{\pi(i,j,j,j,j)\}$ and $j \neq i$, $15a_{i\upsilon}$ with its number 1 for $\upsilon \in \{\pi(i,j,j,k,k)\}$ and $j \neq k \neq i$, for $j,k \in [n]$.

Proof. Let $\lambda \in \sigma(\mathcal{A})$. By Corollary 2.2, there exists $i \in [n]$ such that

$$|\lambda - \alpha_i| \le r_i(\mathcal{A}, \alpha_i), \quad \text{i.e.,} \quad \lambda \in [f(\alpha_i), g(\alpha_i)] \tag{2.6}$$

for any real number α_i, where

$$\begin{aligned}
f(\alpha_i) =& \alpha_i - r_i(\mathcal{A}, \alpha_i) \\
=& \alpha_i - |a_{iiiiii} - \alpha_i| - \sum_{j\neq i}\Bigg(\sum_{\upsilon\in\{\pi(i,i,i,j,j)\}} |a_{i\upsilon} - \frac{1}{5}\alpha_i| + \sum_{\upsilon\in\{\pi(i,j,j,j,j)\}} |a_{i\upsilon} - \frac{1}{5}\alpha_i| \Bigg) \\
& - \sum_{j\neq k\neq i} \sum_{\upsilon\in\{\pi(i,j,j,k,k)\}} |a_{i\upsilon} - \frac{1}{15}\alpha_i| - \eta_i(\mathcal{A}) \\
=& \alpha_i - \frac{1}{15}\Bigg\{ 15|a_{iiiiii} - \alpha_i| + 3\sum_{j\neq i}\Big(\sum_{\upsilon\in\{\pi(i,i,i,j,j)\}} |5a_{i\upsilon} - \alpha_i| + \sum_{\upsilon\in\{\pi(i,j,j,j,j)\}} |5a_{i\upsilon} - \alpha_i| \Big) \\
& + \sum_{j\neq k\neq i} \sum_{\upsilon\in\{\pi(i,j,j,k,k)\}} |15a_{i\upsilon} - \alpha_i| \Bigg\} - \eta_i(\mathcal{A}) \\
=& \alpha_i - \frac{1}{15} \sum_{k\in[15n^2]} |b_{i,k} - \alpha_i| - \eta_i(\mathcal{A})
\end{aligned}$$

and

$$g(\alpha_i) = \alpha_i + r_i(\mathcal{A}, \alpha_i)$$

$$
\begin{aligned}
=&\alpha_i + |a_{iiiiii} - \alpha_i| + \sum_{j\neq i}\left(\sum_{\upsilon\in\{\pi(i,i,i,j,j)\}} |a_{i\upsilon} - \frac{1}{5}\alpha_i| + \sum_{\upsilon\in\{\pi(i,j,j,j,j)\}} |a_{i\upsilon} - \frac{1}{5}\alpha_i|\right)\\
&+ \sum_{j\neq k\neq i}\sum_{\upsilon\in\{\pi(i,j,j,k,k)\}} |a_{i\upsilon} - \frac{1}{15}\alpha_i| + \eta_i(\mathcal{A})\\
=&\alpha_i + \frac{1}{15}\Bigg\{15|a_{iiiiii} - \alpha_i| + 3\sum_{j\neq i}\Bigg(\sum_{\upsilon\in\{\pi(i,i,i,j,j)\}} |5a_{i\upsilon} - \alpha_i| + \sum_{\upsilon\in\{\pi(i,j,j,j,j)\}} |5a_{i\upsilon} - \alpha_i|\Bigg)\\
&\quad + \sum_{j\neq k\neq i}\sum_{\upsilon\in\{\pi(i,j,j,k,k)\}} |15a_{i\upsilon} - \alpha_i|\Bigg\} + \eta_i(\mathcal{A})\\
=&\alpha_i + \frac{1}{15}\sum_{k\in[15n^2]} |b_{i,k} - \alpha_i| + \eta_i(\mathcal{A}).
\end{aligned}
$$

Note here that $b_{i,1} \le b_{i,2} \le \cdots \le b_{i,15n^2}$ is an arrangement in non-decreasing order of

$$
\underbrace{a_{iiiiii}, \ldots, a_{iiiiii}}_{\text{the number is 15}}, \underbrace{5a_{i\upsilon}, 5a_{i\upsilon}, 5a_{i\upsilon}}_{\upsilon\in\{\pi(i,i,i,j,j)\}, j\neq i}, \underbrace{5a_{i\upsilon}, 5a_{i\upsilon}, 5a_{i\upsilon}}_{\upsilon\in\{\pi(i,j,j,j,j)\}, j\neq i}, \underbrace{15a_{i\upsilon}}_{\upsilon\in\{\pi(i,j,j,k,k)\}, j\neq k\neq i} \tag{2.7}
$$

for $j, k \in [n]$. By the fact that there are $n-1$ ways to pick $j \in [n]$ with $j \neq i$ and

$$
\begin{aligned}
\{\pi(i,i,i,j,j)\} =&\{(i,i,i,j,j), (i,i,j,i,j), (i,i,j,j,i), (i,j,i,i,j), (i,j,i,j,i),\\
&(i,j,j,i,i), (j,i,i,i,j), (j,i,i,j,i), (j,i,j,i,i), (j,j,i,i,i)\},\\
\{\pi(i,j,j,j,j)\} =&\{(i,j,j,j,j), (j,i,j,j,j), (j,j,i,j,j), (j,j,j,i,j), (j,j,j,j,i)\},
\end{aligned}
$$

it can be seen that the number of elements in $\underbrace{5a_{i\upsilon}, 5a_{i\upsilon}, 5a_{i\upsilon}}_{\upsilon\in\{\pi(i,i,i,j,j)\}, j\neq i}$ is $3 \times 10 \times (n-1)$ and the number of elements in $\underbrace{5a_{i\upsilon}, 5a_{i\upsilon}, 5a_{i\upsilon}}_{\upsilon\in\{\pi(i,j,j,j,j)\}, j\neq i}$ is $3 \times 5 \times (n-1)$. By the fact that there are $\frac{(n-1)(n-2)}{2}$ ways to pick $j, k \in [n]$ with $j \neq k \neq i$ and

$$
\begin{aligned}
\{\pi(i,j,j,k,k)\} =&\{(i,j,j,k,k), (j,i,j,k,k), (j,j,i,k,k), (j,j,k,i,k), (j,j,k,k,i), (i,j,k,j,k),\\
&(j,i,k,j,k), (j,k,i,j,k), (j,k,j,i,k), (j,k,j,k,i), (i,j,k,k,j), (j,i,k,k,j),\\
&(j,k,i,k,j), (j,k,k,i,j), (j,k,k,j,i), (i,k,j,k,j), (k,i,j,k,j), (k,j,i,k,j),\\
&(k,j,k,i,j), (k,j,k,j,i), (i,k,j,j,k), (k,i,j,j,k), (k,j,i,j,k), (k,j,j,i,k),\\
&(k,j,j,k,i), (i,k,k,j,j), (k,i,k,j,j), (k,k,i,j,j), (k,k,j,i,j), (k,k,j,j,i)\},
\end{aligned}
$$

it can be seen that the number of elements in $\underbrace{15a_{i\upsilon}}_{\upsilon\in\{\pi(i,j,j,k,k)\}, j\neq k\neq i}$ is $1 \times 30 \times \frac{(n-1)(n-2)}{2}$. Hence, the number of elements in (2.7) is

$$
15 + 3 \times 10 \times (n-1) + 3 \times 5 \times (n-1) + 1 \times 30 \times \frac{(n-1)(n-2)}{2} = 15n^2.
$$

Next, the maximum of $f(\alpha_i)$ and the minimum of $g(\alpha_i)$ for $\alpha_i \in \mathbb{R}$ are considered for two cases: n is odd or even.

(i) Let n be odd. Then $15n^2$ is odd. By Lemma 2.1 (taking $a = 15$), we have

$$\max_{\alpha_i\in\mathbb{R}} f(\alpha_i) = f(b_{i,\frac{15n^2+15}{2}}) = \frac{1}{15}\Bigg(\sum_{k=1}^{\frac{15n^2+15}{2}} b_{i,k} - \sum_{k=\frac{15n^2+17}{2}}^{15n^2} b_{i,k}\Bigg) - \eta_i(\mathcal{A}) \geq f(b_{i,\frac{15n^2-13}{2}}). \tag{2.8}$$

By Lemma 2.2 (taking $a = 15$), we have

$$\min_{\alpha_i\in\mathbb{R}} g(\alpha_i) = g(b_{i,\frac{15n^2-13}{2}}) = \frac{1}{15}\Bigg(\sum_{k=\frac{15n^2-13}{2}}^{15n^2} b_{i,k} - \sum_{k=1}^{\frac{15n^2-15}{2}} b_{i,k}\Bigg) + \eta_i(\mathcal{A}) \leq g(b_{i,\frac{15n^2+15}{2}}). \tag{2.9}$$

Taking $\alpha_i = b_{i,\frac{15n^2-13}{2}}$ and $\alpha_i = b_{i,\frac{15n^2+15}{2}}$ in (2.6), respectively, we have $\lambda \in \Big[f(b_{i,\frac{15n^2-13}{2}}), g(b_{i,\frac{15n^2-13}{2}})\Big]$ and $\lambda \in \Big[f(b_{i,\frac{15n^2+15}{2}}), g(b_{i,\frac{15n^2+15}{2}})\Big]$. By (2.8), (2.9) and the existence of λ, we have

$$\lambda \in \Big[f(b_{i,\frac{15n^2+15}{2}}), g(b_{i,\frac{15n^2-13}{2}})\Big],$$

which implies that $\lambda \in [l_i, u_i] \subseteq \bigcup\limits_{i\in[n]} [l_i, u_i]$.

(ii) Let n be even. Then $15n^2$ is even. By Lemma 2.1 (taking $a = 15$), we have

$$\max_{\alpha_i\in\mathbb{R}} f(\alpha_i) = f(b_{i,\frac{15n^2+16}{2}}) = \frac{1}{15}\Bigg(\sum_{k=1}^{\frac{15n^2+14}{2}} b_{i,k} - \sum_{k=\frac{15n^2+18}{2}}^{15n^2} b_{i,k}\Bigg) - \eta_i(\mathcal{A}) \geq f(b_{i,\frac{15n^2-14}{2}}). \tag{2.10}$$

By Lemma 2.2 (taking $a = 15$), we have

$$\min_{\alpha_i\in\mathbb{R}} g(\alpha_i) = g(b_{i,\frac{15n^2-14}{2}}) = \frac{1}{15}\Bigg(\sum_{k=\frac{15n^2-12}{2}}^{15n^2} b_{i,k} - \sum_{k=1}^{\frac{15n^2-16}{2}} b_{i,k}\Bigg) + \eta_i(\mathcal{A}) \leq g(b_{i,\frac{15n^2+16}{2}}). \tag{2.11}$$

Taking $\alpha_i = b_{i,\frac{15n^2-14}{2}}$ and $\alpha_i = b_{i,\frac{15n^2+16}{2}}$ in (2.6), respectively, we have $\lambda \in \Big[f(b_{i,\frac{15n^2-14}{2}}), g(b_{i,\frac{15n^2-14}{2}})\Big]$ and $\lambda \in \Big[f(b_{i,\frac{15n^2+16}{2}}), g(b_{i,\frac{15n^2+16}{2}})\Big]$. Furthermore, by (2.10), (2.11) and the existence of λ, we have

$$\lambda \in \Big[f(b_{i,\frac{15n^2+16}{2}}), g(b_{i,\frac{15n^2-14}{2}})\Big],$$

i.e., $\lambda \in [l_i, u_i]$, and consequently, $\lambda \in \bigcup\limits_{i\in[n]} [l_i, u_i]$. □

By Corollary 2.2 and the proof of Theorem 2.3, the following comparison theorem among Corollary 2.1, Corollary 2.2 and Theorem 2.3 is given easily.

Theorem 2.4. *Let $\mathcal{A} = (a_{i_1 i_2\cdots i_6}) \in \mathbb{R}^{[6,n]}$. Then, for any vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$,*

$$\Upsilon(\mathcal{A}) \subseteq \Upsilon(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha).$$

3. An application of the optimal Z-eigenvalue inclusion interval for sixth-order tensors

In this section, we give the application of the optimal Z-eigenvalue inclusion interval $\Upsilon(\mathcal{A})$ in Theorem 2.3 for a sixth-order tensor $\mathcal{A}$ in determining the positive definiteness of a sixth-order tensor and the asymptotically stability of time-invariant polynomial systems.

3.1. The positive definiteness of homogeneous polynomial forms

As shown in [11,20–22,30], a Z-eigenvalue inclusion interval can provide a sufficient condition for the positive definiteness of tensors. Based on the inclusion interval $\Upsilon(\mathcal{A})$ in Theorem 2.3, a sufficient condition for the positive definiteness of a sixth-order real symmetric tensor is given.

Corollary 3.1. *Let $\mathcal{A} = (a_{i_1\cdots i_6}) \in \mathbb{R}^{[6,n]}$ and λ be a Z-eigenvalue of $\mathcal{A}$.*
(i) If $l_i > 0$ for each $i \in [n]$, then $\lambda > 0$, where

$$l_i = \begin{cases} \frac{1}{15}\Big(\sum\limits_{k=1}^{\frac{15n^2+15}{2}} b_{i,k} - \sum\limits_{k=\frac{15n^2+17}{2}}^{15n^2} b_{i,k}\Big) - \eta_i(\mathcal{A}), & n \text{ is odd,} \\ \frac{1}{15}\Big(\sum\limits_{k=1}^{\frac{15n^2+14}{2}} b_{i,k} - \sum\limits_{k=\frac{15n^2+18}{2}}^{15n^2} b_{i,k}\Big) - \eta_i(\mathcal{A}), & n \text{ is even,} \end{cases}$$

$b_{i,1} \le b_{i,2} \le \cdots \le b_{i,15n^2}$ is an arrangement in non-decreasing order of a_{iiiiii} with its number 15, $5a_{i\upsilon}$ with its number 3 for $\upsilon \in \{\pi(i,i,i,j,j)\}$ and $j \neq i$, $5a_{i\upsilon}$ with its number 3 for $\upsilon \in \{\pi(i,j,j,j,j)\}$ and $j \neq i$, $15a_{i\upsilon}$ with its number 1 for $\upsilon \in \{\pi(i,j,j,k,k)\}$ and $j \neq k \neq i$, for $j,k \in [n]$, and $\eta_i(\mathcal{A})$ is defined in (2.3) and (2.4).
(ii) Furthermore, if $\mathcal{A}$ is symmetric, then $\mathcal{A}$ is positive definite, consequently, $f(x)$ defined by (1.1) is positive definite.

In order to judge the positive definiteness of an order 6 dimension 2 or 3 real symmetric tensor for convenience, the conditions of Corollary 3.1 and the interval $\Upsilon(\mathcal{A})$ in Theorem 2.3 are listed.

Let $\mathcal{A} = (a_{i_1\cdots i_6}) \in \mathbb{R}^{[6,2]}$ be a symmetric tensor with elements defined as follows:

$$a_{111111} = d_1,\ a_{111112} = d_2,\ a_{111122} = d_3,\ a_{111222} = d_4,\ a_{112222} = d_5,\ a_{122222} = d_6,\ a_{222222} = d_7.$$

By Theorem 2.3, we have

$$\begin{aligned} l_1 &= \frac{1}{15}(b_{1,1} + \cdots + b_{1,37} - b_{1,39} - \cdots - b_{1,60}) - (5|d_2| + 10|d_4| + |d_6|), \\ u_1 &= \frac{1}{15}(-b_{1,1} - \cdots - b_{1,22} + b_{1,24} + \cdots + b_{1,60}) + (5|d_2| + 10|d_4| + |d_6|), \\ l_2 &= \frac{1}{15}(b_{2,1} + \cdots + b_{2,37} - b_{2,39} - \cdots - b_{2,60}) - (|d_2| + 10|d_4| + 5|d_6|), \\ u_2 &= \frac{1}{15}(-b_{2,1} - \cdots - b_{2,22} + b_{2,24} + \cdots + b_{2,60}) + (|d_2| + 10|d_4| + 5|d_6|), \end{aligned}$$

where $b_{1,1} \leq b_{1,2} \leq \cdots \leq b_{1,60}$ is an arrangement in non-decreasing order of d_1 with its number 15, $5d_3$ with its number 30, $5d_5$ with its number 15; $b_{2,1} \leq b_{2,2} \leq \cdots \leq b_{2,60}$ is an arrangement in non-decreasing order of d_7 with its number 15, $5d_5$ with its number 30, $5d_3$ with its number 15.

Let $\mathcal{A} = (a_{i_1\cdots i_6}) \in \mathbb{R}^{[6,3]}$ be a symmetric tensor with elements defined as follows:

$$a_{111111} = d_1,\ a_{111112} = d_2,\ a_{111122} = d_3,\ a_{111222} = d_4,\ a_{112222} = d_5,\ a_{122222} = d_6,\ a_{222222} = d_7,$$
$$a_{111113} = d_8,\ a_{111123} = d_9,\ a_{111133} = d_{10},\ a_{111223} = d_{11},\ a_{111233} = d_{12},\ a_{111333} = d_{13},\ a_{112223} = d_{14},$$
$$a_{112233} = d_{15},\ a_{112333} = d_{16},\ a_{113333} = d_{17},\ a_{122223} = d_{18},\ a_{122233} = d_{19},\ a_{122333} = d_{20},\ a_{123333} = d_{21},$$
$$a_{133333} = d_{22},\ a_{222223} = d_{23},\ a_{222233} = d_{24},\ a_{222333} = d_{25},\ a_{223333} = d_{26},\ a_{233333} = d_{27},\ a_{333333} = d_{28}.$$

By Theorem 2.3, we have

$$\begin{aligned}
l_1 &= \frac{1}{15}(b_{1,1} + \cdots + b_{1,75} - b_{1,76} - \cdots - b_{1,135}) - \eta_1(\mathcal{A}),\\
u_1 &= \frac{1}{15}(-b_{1,1} - \cdots - b_{1,60} + b_{1,61} + \cdots + b_{1,135}) + \eta_1(\mathcal{A}),\\
l_2 &= \frac{1}{15}(b_{2,1} + \cdots + b_{2,75} - b_{2,76} - \cdots - b_{2,135}) - \eta_2(\mathcal{A}),\\
u_2 &= \frac{1}{15}(-b_{2,1} - \cdots - b_{2,60} + b_{2,61} + \cdots + b_{2,135}) + \eta_2(\mathcal{A}),\\
l_3 &= \frac{1}{15}(b_{3,1} + \cdots + b_{3,75} - b_{3,76} - \cdots - b_{3,135}) - \eta_3(\mathcal{A}),\\
u_3 &= \frac{1}{15}(-b_{3,1} - \cdots - b_{3,60} + b_{3,61} + \cdots + b_{3,135}) + \eta_3(\mathcal{A}),
\end{aligned}$$

where

$$\begin{aligned}
\eta_1(\mathcal{A}) =& 5|d_2| + 10|d_4| + |d_6| + 5|d_8| + 20|d_9| + 30|d_{11}| + 30|d_{12}| + 10|d_{13}|\\
&+ 20|d_{14}| + 20|d_{16}| + 5|d_{18}| + 10|d_{19}| + 10|d_{20}| + 5|d_{21}| + |d_{22}|,\\
\eta_2(\mathcal{A}) =& |d_2| + 10|d_4| + 5|d_6| + 5|d_9| + 20|d_{11}| + 10|d_{12}| + 30|d_{14}| + 10|d_{16}|\\
&+ 20|d_{18}| + 30|d_{19}| + 20|d_{20}| + 5|d_{21}| + 5|d_{23}| + 10|d_{25}| + |d_{27}|,\\
\eta_3(\mathcal{A}) =& |d_8| + 5|d_9| + 10|d_{11}| + 20|d_{12}| + 10|d_{13}| + 10|d_{14}| + 30|d_{16}| + 5|d_{18}|\\
&+ 20|d_{19}| + 30|d_{20}| + 20|d_{21}| + 5|d_{22}| + |d_{23}| + 10|d_{25}| + 5|d_{27}|,
\end{aligned}$$

and $b_{1,1} \leq b_{1,2} \leq \cdots \leq b_{1,135}$ is an arrangement in non-decreasing order of d_1 with its number 15, $5d_3$ with its number 30, $5d_{10}$ with its number 30, $5d_5$ with its number 15, $5d_{17}$ with its number 15, $15d_{15}$ with its number 30; $b_{2,1} \leq b_{2,2} \leq \cdots \leq b_{2,135}$ is an arrangement in non-decreasing order of d_7 with its number 15, $5d_5$ with its number 30, $5d_{24}$ with its number 30, $5d_3$ with its number 15, $5d_{26}$ with its number 15, $15d_{15}$ with its number 30; $b_{3,1} \leq b_{3,2} \leq \cdots \leq b_{3,135}$ is an arrangement in non-decreasing order of d_{28} with its number 15, $5d_{17}$ with its number 30, $5d_{26}$ with its number 30, $5d_{10}$ with its number 15, $5d_{24}$ with its number 15, $15d_{15}$ with its number 30.

Example 3.1. Let $\mathcal{A} = (a_{i_1 i_2\cdots i_6}) \in \mathbb{R}^{[6,3]}$ be a symmetric tensor with elements defined as follows:

$$a_{111111} = a_{222222} = a_{333333} = 20, \quad a_{112233} = 1.3,$$

$$\begin{aligned}a_{111112} &= a_{111222} = a_{122222} = a_{111113} = a_{111123} = a_{111223} = a_{111233} = a_{111333} = a_{112223} = a_{112333}\\ &= a_{122223} = a_{122233} = a_{122333} = a_{123333} = a_{133333} = a_{222223} = a_{222333} = a_{233333} = -0.1\\ a_{111122} &= a_{111133} = a_{112222} = a_{113333} = a_{222233} = a_{223333} = 3.9.\end{aligned}$$

Our goal is to judge the positive definiteness of $\mathcal{A}$. Firstly, the method in Corollary 2 of [21] is considered. By computations, we have

$$a_{111111} = 20 < 22.2 = \sum_{(i_2,\ldots,i_6)\in\overline{\Lambda}_1} |a_{1i_2\cdots i_6}| + r_1^{\Lambda_1,a_{111111}}(\mathcal{A}),$$

where

$$\begin{aligned}r_1^{\Lambda_1,a_{111111}}(\mathcal{A}) =&|5a_{112222} - a_{111111}| + |10a_{111122} - 2a_{111111}| + |5a_{113333} - a_{111111}|\\ &+ |10a_{111133} - 2a_{111111}| + |30a_{112233} - 2a_{111111}|,\end{aligned}$$

which shows that the conditions of Corollary 2 of [21] are not satisfied. Hence, we do not use Corollary 2 of [21] to judge the the positive definiteness of $\mathcal{A}$. Moreover, by

$$a_{111111} = 20 \neq 19.5 = 5a_{111122} = 5a_{112222} = 15a_{112233},$$

it can be seen that Proposition 1 of [21] is also not used to judge the the positive definiteness of $\mathcal{A}$.

Next, we use Corollaries 2.1 and 2.2 to judge the positive definiteness of $\mathcal{A}$. Corollary 2.2 shows that $\Upsilon(\mathcal{A},\alpha) = \mathcal{G}(\mathcal{A},\alpha)$ when $n = 3$ and hence only $\mathcal{G}(\mathcal{A},\alpha)$ is showed. By computations, we have

$$\begin{aligned}&d_1 = d_7 = d_{28} = 20, \quad d_3 = d_5 = d_{10} = d_{17} = d_{24} = d_{26} = 3.9, \quad d_{15} = 1.3,\\ &d_2 = d_4 = d_6 = d_8 = d_9 = d_{11} = d_{12} = d_{13} = d_{14} = d_{16} = d_{18} = d_{19} = d_{20}\\ &\quad = d_{21} = d_{22} = d_{23} = d_{25} = d_{27} = -0.1,\end{aligned}$$

and hence

$$\eta_i(\mathcal{A}) = \sum_{(i_2,\ldots,i_6)\in\overline{\Lambda}_i} |a_{ii_2\cdots i_6}| = 18.2, \quad i = 1,2,3.$$

Let $\alpha = (\alpha_1,\alpha_2,\alpha_3)^\top \in \mathbb{R}^3$. By Corollary 2.1, we have

$$\begin{aligned}R_1(\mathcal{A},\alpha_1) =&|d_1 - \alpha_1| + |10d_3 - 2\alpha_1| + |5d_5 - \alpha_1| + |10d_{10} - 2\alpha_1| + |5d_{17} - \alpha_1|\\ &+ |30d_{15} - 2\alpha_1| + \eta_1(\mathcal{A}),\\ R_2(\mathcal{A},\alpha_2) =&|d_7 - \alpha_2| + |10d_5 - 2\alpha_2| + |5d_3 - \alpha_2| + |10d_{24} - 2\alpha_2| + |5d_{26} - \alpha_2|\\ &+ |30d_{15} - 2\alpha_2| + \eta_2(\mathcal{A}),\\ R_3(\mathcal{A},\alpha_3) =&|d_{28} - \alpha_3| + |10d_{17} - 2\alpha_3| + |5d_{10} - \alpha_3| + |10d_{26} - 2\alpha_3| + |5d_{24} - \alpha_3|\\ &+ |30d_{15} - 2\alpha_3| + \eta_3(\mathcal{A}),\end{aligned}$$

and

$$\mathcal{G}(\mathcal{A},\alpha) := \bigcup_{i\in[3]}[\alpha_i - R_i(\mathcal{A},\alpha_i), \alpha_i + R_i(\mathcal{A},\alpha_i)].$$

In order to judge the positive definiteness of $\mathcal{A}$ by Corollary 2.1, we need to take a specific α to obtain the Z-eigenvalue inclusion interval $\mathcal{G}(\mathcal{A},\alpha)$ and observe the position of $\mathcal{G}(\mathcal{A},\alpha)$ in the complex plane. If $\mathcal{G}(\mathcal{A},\alpha)$ is just in the right-half complex plane, then we can conclude that $\mathcal{A}$ is positive definite. However, taking $\alpha = (10,10,10)^\top \in \mathbb{R}^3$, we have

$$\mathcal{G}(\mathcal{A},\alpha) = [-94.2, 114.2] \cup [-94.2, 114.2] \cup [-94.2, 114.2] = [-94.2, 114.2];$$

and taking $\alpha = (20,20,20)^\top \in \mathbb{R}^3$, we have

$$\mathcal{G}(\mathcal{A},\alpha) = [-2.2, 42.2] \cup [-2.2, 42.2] \cup [-2.2, 42.2] = [-2.2, 42.2].$$

From $-94.2 < 0$ and $-2.2 < 0$, it can be seen that $\mathcal{G}(\mathcal{A},\alpha)$ is not used to judge the positive definiteness of $\mathcal{A}$ when α is taken as such two vectors and that it is not easy to choose the optimal parameter vector α^* to minimize the interval $\mathcal{G}(\mathcal{A},\alpha)$.

By (2.8) and (2.9) in Theorem 2.3, we can calculate that $b_{i,\frac{15n^2+15}{2}} = b_{i,75} = 19.5 = b_{i,61} = b_{i,\frac{15n^2-13}{2}}$ for $i \in [n]$ and $n = 3$, and hence the optimal parameter vector is

$$\alpha^* = (19.5, 19.5, 19.5)^\top \in \mathbb{R}^3,$$

and the minimize interval of $\mathcal{G}(\mathcal{A},\alpha)$ for any $\alpha \in \mathbb{R}^3$ is

$$\Upsilon(\mathcal{A}) = \mathcal{G}(\mathcal{A},\alpha^*) = [0.8, 38.2] \cup [0.8, 38.2] \cup [0.8, 38.2] = [0.8, 38.2].$$

Because the interval $\Upsilon(\mathcal{A})$ is in the right-half complex plane, which implies that all Z-eigenvalues of $\mathcal{A}$ lie in the interval [0.8,38.2], we can conclude that $\mathcal{A}$ is positive definite.

Finally, we use Corollary 3.1 to judge the positive definiteness of $\mathcal{A}$. By computations, we have

$$l_i = \frac{1}{15}(b_{i,1} + \cdots + b_{i,75} - b_{i,76} - \cdots - b_{i,135}) - \eta_i(\mathcal{A}) = 0.8 > 0,$$

where $b_{i,1} \leq b_{i,2} \leq \cdots \leq b_{i,135}$ is an arrangement in non-decreasing order of 19.5 with its number 120 and 20 with its number 15 for $i \in [3]$. Hence, by Corollary 3.1, $\mathcal{A}$ is positive definite. In fact, all different Z-eigenvalues of $\mathcal{A}$ are 17.5333, 20.0250, 20.0618, 20.2302. □

3.2. The asymptotically stability of the time-invariant polynomial system

As shown in Section 3.2 of [4] that any time-invariant polynomial system can be written as

$$\begin{aligned} \Sigma : \dot{x}_1 &= \sum_{i_2\in[n]} a_{1i_2}x_{i_2} + \sum_{i_2,i_3\in[n]} a_{1i_2i_3}x_{i_2}x_{i_3} + \cdots + \sum_{i_2,\ldots,i_m\in[n]} a_{1i_2\cdots i_m}x_{i_2}\cdots x_{i_m}, \\ &\vdots \\ \dot{x}_n &= \sum_{i_2\in[n]} a_{ni_2}x_{i_2} + \sum_{i_2,i_3\in[n]} a_{ni_2i_3}x_{i_2}x_{i_3} + \cdots + \sum_{i_2,\ldots,i_m\in[n]} a_{ni_2\cdots i_m}x_{i_2}\cdots x_{i_m}, \end{aligned} \tag{3.1}$$

where $a_{i_1i_2\cdots i_m} \in \mathbb{R}$ are invariant under any permutation of indices $i_2,\ldots,i_m$. Particularly, when $m = 3$, the system Σ can be regarded as the epidemic model; for details, see [5]. The stability is a basic property of a system. Deng, Li and Bu in [4] represented the time-invariant polynomial system (3.1) by tensors as follows

$$\Sigma : \dot{x} = \mathcal{A}_2 x + \mathcal{A}_3 x^2 + \cdots + \mathcal{A}_m x^{m-1},$$

where $\mathcal{A}_t = (a_{i_1 i_2 \cdots i_t}) \in \mathbb{R}^{[t,n]}$ $(t = 2, \ldots, m)$ and $x = (x_1, \ldots, x_n)^\top$, and gave the analysis of stability of the following nonlinear system

$$\Sigma : \dot{x} = \mathcal{A}_2 x + \mathcal{A}_4 x^3 + \cdots + \mathcal{A}_{2k} x^{2k-1} \tag{3.2}$$

by Lyapunov stability theorem [9] and the positive definiteness of tensors as follows.

Theorem 3.1. [4, Theorem 3.3] *For the nonlinear system Σ in (3.2), if $-\mathcal{A}_t$, $t = 2, 4, \ldots, 2k$, is positive definite, then the equilibrium point of Σ is asymptotically stable.*

Next, we give a nonlinear polynomial system and write it in the form of (3.2). By the positive definiteness of tensors, we analyse the stability of the system.

Example 3.2. Let

$$\begin{aligned}
\Sigma : \dot{x}_1 =& -3x_1 + x_2 + x_3 - 4.5x_1^3 - 0.3x_1^2x_2 - 0.3x_1x_3^2 - 1.5x_1x_2^2 - 0.3x_2x_3^2 \\
& - 20x_1^5 + 0.5x_1^4x_2 + 0.5x_1^4x_3 - 39x_1^3x_2^2 - 39x_1^3x_3^2 + 2x_1^3x_2x_3 + x_1^2x_2^3 + x_1^2x_3^3 \\
& + 3x_1^2x_2^2x_3 + 3x_1^2x_2x_3^2 - 19.5x_1x_2^4 - 19.5x_1x_3^4 + 2x_1x_2^3x_3 - 39x_1x_2^2x_3^2 + 2x_1x_2x_3^3 \\
& + 0.1x_2^5 + 0.5x_2^4x_3 + x_2^3x_3^2 + x_2^2x_3^3 + 0.5x_2x_3^4 + 0.1x_3^5, \\
\dot{x}_2 =& x_1 - 3x_2 + x_3 - 0.1x_1^3 - 1.5x_1^2x_2 - 0.3x_1x_3^2 - 3.2x_2^3 - 0.3x_2^2x_3 - 1.5x_2x_3^2 \\
& + 0.1x_1^5 - 19.5x_1^4x_2 + 0.5x_1^4x_3 + x_1^3x_2^2 + x_1^3x_3^2 + 2x_1^3x_2x_3 - 39x_1^2x_2^3 + x_1^2x_3^3 \\
& + 3x_1^2x_2^2x_3 - 39x_1^2x_2x_3^2 + 0.5x_1x_2^4 + 2x_1x_2^3x_3 + 3x_1x_2^2x_3^2 + 2x_1x_2x_3^3 + 0.5x_1x_3^4 \\
& - 20x_2^5 + 0.5x_2^4x_3 - 39x_2^3x_3^2 + x_2^2x_3^3 - 19.5x_2x_3^4 + 0.1x_3^5, \\
\dot{x}_3 =& x_1 + x_2 - 3x_3 - 0.3x_1^2x_3 - 0.6x_1x_2x_3 - 0.1x_2^3 - 1.5x_2^2x_3 - 4.4x_3^3 \\
& + 0.1x_1^5 + 0.5x_1^4x_2 - 19.5x_1^4x_3 + x_1^3x_2^2 + x_1^3x_3^2 + 2x_1^3x_2x_3 + x_1^2x_2^3 - 39x_1^2x_3^3 \\
& - 39x_1^2x_2^2x_3 + 3x_1^2x_2x_3^2 + 0.5x_1x_2^4 + 2x_1x_2^3x_3 + 3x_1x_2^2x_3^2 + 2x_1x_2x_3^3 + 0.5x_1x_3^4 \\
& + 0.1x_2^5 - 19.5x_2^4x_3 + x_2^3x_3^2 - 39x_2^2x_3^3 + 0.5x_2x_3^4 - 20x_3^5.
\end{aligned}$$

Then Σ can be written as $\dot{x} = \mathcal{A}_2 x + \mathcal{A}_4 x^3 + \mathcal{A}_6 x^5$, where $x = (x_1, x_2, x_3)^\top$,

$$\mathcal{A}_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{pmatrix},$$

$$\mathcal{A}_4 = \left(\begin{array}{ccc|ccc|ccc}
a_{1111} & a_{1112} & a_{1113} & a_{1211} & a_{1212} & a_{1213} & a_{1311} & a_{1312} & a_{1313} \\
a_{1121} & a_{1122} & a_{1123} & a_{1221} & a_{1222} & a_{1223} & a_{1321} & a_{1322} & a_{1323} \\
a_{1131} & a_{1132} & a_{1133} & a_{1231} & a_{1232} & a_{1233} & a_{1331} & a_{1332} & a_{1333} \\
\hline
a_{2111} & a_{2112} & a_{2113} & a_{2211} & a_{2212} & a_{2213} & a_{2311} & a_{2312} & a_{2313} \\
a_{2121} & a_{2122} & a_{2123} & a_{2221} & a_{2222} & a_{2223} & a_{2321} & a_{2322} & a_{2323} \\
a_{2131} & a_{2132} & a_{2133} & a_{2231} & a_{2232} & a_{2233} & a_{2331} & a_{2332} & a_{2333} \\
\hline
a_{3111} & a_{3112} & a_{3113} & a_{3211} & a_{3212} & a_{3213} & a_{3311} & a_{3312} & a_{3313} \\
a_{3121} & a_{3122} & a_{3123} & a_{3221} & a_{3222} & a_{3223} & a_{3321} & a_{3322} & a_{3323} \\
a_{3131} & a_{3132} & a_{3133} & a_{3231} & a_{3232} & a_{3233} & a_{3331} & a_{3332} & a_{3333}
\end{array}\right)$$

$$= -\left(\begin{array}{ccc|ccc|ccc} 4.5 & 0.1 & 0 & 0.1 & 0.5 & 0 & 0 & 0 & 0.1 \\ 0.1 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0.1 & 0 & 0 & 0.1 & 0.1 & 0.1 & 0 \\ \hline 0.1 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.1 \\ 0.5 & 0 & 0 & 0 & 3.2 & 0.1 & 0 & 0.1 & 0.5 \\ 0 & 0 & 0.1 & 0 & 0.1 & 0.5 & 0.1 & 0.5 & 0 \\ \hline 0 & 0 & 0.1 & 0 & 0 & 0.1 & 0.1 & 0.1 & 0 \\ 0 & 0 & 0.1 & 0 & 0.1 & 0.5 & 0.1 & 0.5 & 0 \\ 0.1 & 0.1 & 0 & 0.1 & 0.5 & 0 & 0 & 0 & 4.4 \end{array}\right),$$

and $\mathcal{A}_6 = (a_{i_1 i_2 \cdots i_6}) \in \mathbb{R}^{[6,3]}$ is a symmetric tensor with elements defined as follows:

$$\begin{aligned} & a_{111111} = a_{222222} = a_{333333} = -20, \quad a_{112233} = -1.3, \\ & a_{111112} = a_{111222} = a_{122222} = a_{111113} = a_{111123} = a_{111223} = a_{111233} = a_{111333} = a_{112223} = a_{112333} \\ & \qquad = a_{122223} = a_{122233} = a_{122333} = a_{123333} = a_{133333} = a_{222223} = a_{222333} = a_{233333} = 0.1 \\ & a_{111122} = a_{111133} = a_{112222} = a_{113333} = a_{222233} = a_{223333} = -3.9. \end{aligned}$$

It is proved in Example 3 of [4] that both $-\mathcal{A}_2$ and $-\mathcal{A}_4$ are positive definite. Example 3.1 shows that $-\mathcal{A}_6$ is positive definite. Furthermore, by Theorem 3.1, it can be seen that the polynomial system Σ is asymptotically stable. □

4. Conclusions

In this paper, we in Corollaries 2.1 and 2.2 gave the specific forms of two Geršgorin-type Z-eigenvalue inclusion intervals $\mathcal{G}(\mathcal{A}, \alpha)$ in Theorem 1.2 (i.e., Theorem 2.2 in [11]) and $\Upsilon(\mathcal{A}, \alpha)$ in Theorem 1.3 (i.e., Theorem 3.1 in [22]) with a parameter vector α for a sixth-order tensor $\mathcal{A}$. Subsequently, we chose an appropriate parameter vector α to minimize the interval $\Upsilon(\mathcal{A}, \alpha)$ and hence derived an optimal interval $\Upsilon(\mathcal{A})$. As an application, we used the interval $\Upsilon(\mathcal{A})$ to obtain a sufficient condition for the positive definiteness of a sixth-order real symmetric tensor (also a homogeneous polynomial form), which is used to judge the asymptotically stability of time-invariant polynomial systems.

Now, we answer Question 2: What is the specific form of the Z-identity tensor $\mathcal{I}_2$ for the order $m \geq 8$ and m is even? This question is answered only for $m = 8$. By calculation, the specific form of the Z-identity tensor $\mathcal{I}_2 = (e_{i_1 i_2 \cdots i_8}) \in \mathbb{R}^{[8,n]}$ is as follows:

$$e_{i_1 i_2 \cdots i_8} = \begin{cases} 1, & \text{if } i_1 = i_2 = \cdots = i_8, \\ 1/7, & \text{if } (i_1, i_2, \ldots, i_8) \in \bigcup\limits_{\substack{i \neq j,\\ i,j \in [n]}} \{\pi(i, i, j, j, j, j, j, j)\}, \\ 3/35, & \text{if } (i_1, i_2, \ldots, i_8) \in \bigcup\limits_{\substack{i \neq j,\\ i,j \in [n]}} \{\pi(i, i, i, i, j, j, j, j)\}, \\ 1/35, & \text{if } (i_1, i_2, \ldots, i_8) \in \bigcup\limits_{\substack{i \neq j \neq k,\\ i,j,k \in [n]}} \{\pi(i, i, j, j, k, k, k, k)\}, \\ 1/105, & \text{if } (i_1, i_2, \ldots, i_8) \in \bigcup\limits_{\substack{i \neq j \neq k \neq l,\\ i,j,k \in [n]}} \{\pi(i, i, j, j, k, k, l, l)\}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\{\pi(i_1, i_2, \ldots, i_8)\}$ is the set of all combinations of $i_1, i_2, \ldots, i_8$.

Let $m = 8$. Using the Z-identity tensor $\mathcal{I}_2$ and the same method as Corollaries 2.1 and 2.2, the specific forms of the Z-eigenvalue inclusion intervals $\mathcal{G}(\mathcal{A}, \alpha)$ in Theorem 1.2 and $\Upsilon(\mathcal{A}, \alpha)$ in Theorem 1.3 can be derived. And by using the similar methods as in Theorem 2.3, we can also choose an appropriate parameter vector α to optimize the interval $\Upsilon(\mathcal{A}, \alpha)$ and present a sufficient condition for the positive definiteness of eighth-order real symmetric tensors. This can be taken as a further question.

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Conflict of interest

The author declares no conflict of interest.

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