



Research article

Stability and bifurcation in a two-patch model with additive Allee effect

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Abstract: A two-patch model with additive Allee effect is proposed and studied in this paper. Our objective is to investigate how dispersal and additive Allee effect have an impact on the above model's dynamical behaviours. We discuss the local and global asymptotic stability of equilibria and the existence of the saddle-node bifurcation. Complete qualitative analysis on the model demonstrates that dispersal and Allee effect may lead to persistence or extinction in both patches. Also, combining mathematical analysis with numerical simulation, we verify that the total population abundance will increase when the Allee effect constant a increases or m decreases. And the total population density increases when the dispersal rate D_1 increases or the dispersal rate D_2 decreases.

Keywords: additive Allee effect; dispersal; stability; bifurcation; extinction

Mathematics Subject Classification: 34C25, 92D25, 34D20, 34D40

1. Introduction

With the continuous development of the country and on-going progress in society, there are more and more empirical results show that the destruction of nature by human has affected the habitat of the population [1]. The habitat of the population is first fragmented, then splits into patches, and finally becomes extinct, which will seriously threaten the survival and diversity of biological populations. However, if ecological corridor can be established to allow the species to migrate between different patches, it may become one of the meaningful measures to protect the species from extinction. Therefore, there have been a large body of literatures in modeling population migration and analyzing the protection and development of ecosystem, which provide biological insight for ecological management.

The dynamics of plaque population can be traced back to the year of 1951 (see [2]). Subsequently, in 1977, Freedman and Waltman [3] consider a two-patch model with a single species in logistic

population growth as follows.

$$\begin{aligned}\frac{dN_1}{dt} &= r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) + D(N_2 - N_1), \\ \frac{dN_2}{dt} &= r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) + D(N_1 - N_2),\end{aligned}\tag{1.1}$$

where $N_i (i = 1, 2)$ represents the population density in patch i , parameter r_i is the intrinsic growth rate, K_i is carrying capacity and D is the dispersal rate. Freedman and Waltman show that under certain conditions, the total population abundance of a spatially distributed population with dispersal can be larger than the total carrying capacities $K_1 + K_2$. After that, a more in-depth study of this model is derived by DeAngelis et al. [4], Holt [5], DeAngelis and Zhang [6] and Roger Arditi et al. [7].

However, in nature, the dispersal between patches is often asymmetric. It is necessary to consider asymmetric dispersal with general dispersal rates. Arditi et al. [8] extend the model (1.1) by considering asymmetric dispersals. Moreover, recently, Wu et al. [9] characterize the following two-patch model:

$$\begin{aligned}\frac{dN_1}{dt} &= r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) + D(N_2 - sN_1), \\ \frac{dN_2}{dt} &= -\bar{r}_2 N_2 + D(sN_1 - N_2).\end{aligned}\tag{1.2}$$

Parameter D represents the dispersal intensity and s reflects the dispersal asymmetry. The authors show that the population persists in both patches but reaches a density which is less (higher) than that without dispersal at intermediate (small) dispersal rates, while extremely large dispersal could result in the population extinction. And asymmetric dispersal is more favorable than symmetric dispersal under certain conditions. For more relevant works, one can see [10, 11]. In recent years, there has been growing interest in understanding the roles that Allee effect [12] plays in the interaction between population. Due to its strong possible impact on population dynamics, Allee effect has been observed extensively in ecology, biology and epidemiology and so on [13–15]. For example, Kang et al. [16] have considered two-patch model with Allee effect and dispersal

$$\begin{aligned}\frac{dx}{dt} &= rx(x - \theta)(1 - x) + \mu(y - x), \\ \frac{dy}{dt} &= ry(y - \theta)(1 - y) + \mu(x - y)\end{aligned}\tag{1.3}$$

where $x(t)$ and $y(t)$ denote the population density at time t in different patches. The parameters $\mu \in [0, 1]$ and θ represent the dispersal intensity and Allee threshold, respectively. It was shown that the dispersal parameter μ and the Allee threshold θ will affect the global dynamics. Subsequently, about two-patch models with dispersal, studies of Allee effect on interacting population have been done recently in [17, 18].

Particularly, the equation

$$\frac{dx}{dt} = r \left[\left(1 - \frac{x}{K}\right) - \frac{m}{x+a} \right] x\tag{1.4}$$

is suffered from the additive Allee effect which was firstly introduced in [19]. The positive parameters m and a are the Allee effect constants. The additive Allee effect consists of two cases, i.e., weak and strong Allee effects. That is, if $0 < m < a$, it is the weak Allee effect; if $m > a$, it is the strong

Allee effect. Consequently, there are a lot of excellent works in mathematical, biology and ecology models with additive Allee effect [20–23]. For example, in [20], the authors propose a single species logistic model with feedback control and additive Allee effect in the growth of species. The result shows that Allee effect can make the above system's dynamic behaviors more complex. In [21], the authors consider an SI epidemic model incorporating additive Allee effect and time delay. They obtain that both additive Allee effect and time delay have vital effect on the prevalence of the disease. It brings to our attention that, the literature which has investigated the two-patch model with additive Allee effect is rather few. To the best of the authors' knowledge, in [24], analytical conditions for the global stability of the patchy model in the case of weak Allee effect. However, in the case of strong Allee effect, only numerical simulations are provided. In other words, the author [24] has not provided a complete qualitative analysis since it involves an algebraic equation of nine-degree. Moreover, the occurrence of bifurcation phenomenon due to the Allee effect still remain unknown. Motivated by the above, in this paper, we will consider the following one species model with additive Allee effect and dispersal:

$$\begin{aligned}\frac{dx_1}{dt} &= -x_1 + D_2x_2 - D_1x_1, \\ \frac{dx_2}{dt} &= x_2\left(1 - x_2 - \frac{m}{x_2 + a}\right) + D_1x_1 - D_2x_2\end{aligned}\tag{1.5}$$

where x_1 and x_2 represent the densities of the species in patch 1 and 2, respectively. Similar to that in [9], in this paper, we assume that patch 1 is the sink (i.e., the coefficient of x_1 is negative). And the species in patch 2 is affected by the additive Allee effect (patch 2 is the source, i.e., the coefficient of x_2 is positive). From the view point of biology, we only need to focus on the nonnegative solution of system (1.5). Thus it is assumed that the initial conditions of (1.5) are $x_1(0) \geq 0$, $x_2(0) \geq 0$.

Different from the strong Allee effect in [16–18], this paper will pay attention to investigate the patchy model with additive Allee effect, i.e., the system (1.5). We will try our best to present the possible qualitative behavior and bifurcation phenomena. Firstly, we discuss the existence and stability of all nonnegative equilibria. Throughout this paper, we will denote local asymptotically stable by *LAS*. Then, by the Bendixson-Dulac discriminant, if certain condition holds, the unique positive equilibrium in system (1.5) is globally asymptotically stable which indicates that the species will persist in both patches. Secondly, we obtain that system (1.5) undergoes the saddle-node bifurcation. Finally, we investigate the effect of Allee effect and dispersal on total population abundance. We notice that comparing with [9, 24], system (1.5) is experiencing more complex dynamics such as saddle-node bifurcation when incorporating the additive Allee effect. The main result in this paper is a good extension and supplement that of [9, 24].

The paper is organized as follows. Section 2 represents the existence and stability of the equilibria. In Section 3, we will discuss the saddle-node bifurcation of the system. In Section 4, we show the effect of Allee effect and dispersal on total population abundance. Brief discussions are presented finally in Section 5.

2. Existence and stability of equilibria

In this section, we shall discuss the existence and stability of all nonnegative equilibria for system (1.5). It is easy to check that the system (1.5) always has a trivial equilibrium $E_0(0, 0)$. For other non-trivial equilibria, we shall investigate them as follows.

The equilibrium of system (1.5) is determined by the following equation:

$$\begin{cases} -x_1 + D_2x_2 - D_1x_1 = 0, \\ x_2\left(1 - x_2 - \frac{m}{x_2 + a}\right) + D_1x_1 - D_2x_2 = 0. \end{cases} \quad (2.1)$$

For the possible equilibria, we consider the positive solution of the following equation:

$$x_2^2 - \left(1 - \frac{D_2}{1+D_1} - a\right)x_2 + [m - a(1 - \frac{D_2}{1+D_1})] = 0. \quad (2.2)$$

Let $\Delta(m)$ be the discriminant of Eq (2.2), i.e.,

$$\Delta(m) = \left(1 - \frac{D_2}{1+D_1} + a\right)^2 - 4m. \quad (2.3)$$

From the above, we have the unique root of $\Delta(m) = 0$ and denote it by m^* as follows.

$$m^* = \frac{\left(1 - \frac{D_2}{1+D_1} + a\right)^2}{4}. \quad (2.4)$$

It follows that $\Delta(m) > 0$, $\Delta(m) = 0$ and $\Delta(m) < 0$ when $m < m^*$, $m = m^*$ and $m > m^*$, respectively. The existence of the positive equilibria is obtained as follows.

Theorem 2.1. Suppose that $a < 1 - \frac{D_2}{1+D_1}$.

- (i) If $m \leq a(1 - \frac{D_2}{1+D_1})$, system (1.5) has a unique positive equilibrium $E_1(x_{11}, x_{21})$.
- (ii) If $a(1 - \frac{D_2}{1+D_1}) < m < m^*$, system (1.5) has two distinct positive equilibria $E_1(x_{11}, x_{21})$ and $E_2(x_{12}, x_{22})$.
- (iii) If $m = m^*$, system (1.5) has a unique positive equilibrium $E_3(x_{13}, x_{23})$.
- (iv) If $m > m^*$, system (1.5) has no positive equilibria.

Theorem 2.2. Suppose that $a \geq 1 - \frac{D_2}{1+D_1}$.

- (i) If $m < a(1 - \frac{D_2}{1+D_1})$, then system (1.5) has a unique positive equilibrium $E_1(x_{11}, x_{21})$.
- (ii) If $m \geq a(1 - \frac{D_2}{1+D_1})$, then system (1.5) has no positive equilibrium.

In the above, $x_{21} = \frac{1}{2}(1 - \frac{D_2}{1+D_1} - a + \sqrt{\Delta(m)})$, $x_{22} = \frac{1}{2}(1 - \frac{D_2}{1+D_1} - a - \sqrt{\Delta(m)})$, $x_{23} = \frac{1}{2}(1 - \frac{D_2}{1+D_1} - a)$ and $x_{1i} = \frac{D_2}{1+D_1}x_{2i}$, $i = 1, 2, 3$.

Moreover, the Jacobian matrix at $E(x_1, x_2)$ of system (1.5) is

$$J_E = \begin{bmatrix} -1 - D_1 & D_2 \\ D_1 & 1 - 2x_2 - \frac{ma}{(x_2+a)^2} - D_2 \end{bmatrix}. \quad (2.5)$$

Thus, the determinant and the trace of the Jacobian matrix at $E_0(0, 0)$ is

$$\text{Det}[J_{E_0}] = \left(\frac{m}{a} - 1\right)(1 + D_1) + D_2,$$

$$\text{Tr}[J_{E_0}] = -\frac{m}{a} - D_1 - D_2 < 0.$$

So if $m < a(1 - \frac{D_2}{1+D_1})$, we have $\text{Det}[J_{E_0}] < 0$, then $E_0(0, 0)$ is a saddle. If $m > a(1 - \frac{D_2}{1+D_1})$, we have $\text{Det}[J_{E_0}] > 0$, then $E_0(0, 0)$ is stable. If $m = a(1 - \frac{D_2}{1+D_1})$, we have $\text{Det}[J_{E_0}] = 0$, i.e., J_{E_0} has a unique zero eigenvalue. We will analyze that E_0 is an attracting saddle-node as follows. For system (1.5), we apply the Taylor expansion of $\frac{m}{x_2+a}$, it can be rewritten as

$$\begin{cases} \dot{x}_1 = (-1 - D_1)x_1 + D_2x_2, \\ \dot{x}_2 = -\frac{D_1D_2}{1+D_1}x_2 + D_1x_1 + \left(-1 + \frac{m}{a^2}\right)x_2^2 - \frac{m}{a^3}x_2^3 + Q(x_2) \end{cases} \quad (2.6)$$

where $Q(x_2)$ denotes the power sery with term x_2^j satisfying $j \geq 4$. Moreover, we make the following transformation:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1+D_1}{D_1} \\ \frac{1+D_1}{D_2} & 1 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}, \quad (2.7)$$

then system (1.5) becomes

$$\begin{cases} \dot{\bar{x}}_1 = p_{20}\bar{x}_1^2 + p_{11}\bar{x}_1\bar{x}_2 + p_{02}\bar{x}_2^2 + p_{30}\bar{x}_1^3 + p_{12}\bar{x}_1\bar{x}_2^2 + p_{21}\bar{x}_1^2\bar{x}_2 + p_{03}\bar{x}_2^3 + P_1(\bar{x}_1, \bar{x}_2), \\ \dot{\bar{x}}_2 = q_{01}\bar{x}_2 + q_{02}\bar{x}_2^2 + q_{11}\bar{x}_1\bar{x}_2 + q_{20}\bar{x}_1^2 + q_{21}\bar{x}_1^2\bar{x}_2 + q_{12}\bar{x}_1\bar{x}_2^2 + q_{03}\bar{x}_2^3 + q_{30}\bar{x}_1^3 + P_2(\bar{x}_1, \bar{x}_2) \end{cases} \quad (2.8)$$

where $P_1(\bar{x}_1, \bar{x}_2)$, $P_2(\bar{x}_1, \bar{x}_2)$ denote the power series with term $\bar{x}_1^i \bar{x}_2^j$ satisfying $i + j \geq 4$ and

$$\begin{aligned} p_{20} &= \frac{(1+D_1)^3(m-a^2)}{a^2D_2[D_1D_2+(1+D_1)^2]}, p_{11} = \frac{2(1+D_1)^2(m-a^2)}{a^2[D_1D_2+(1+D_1)^2]}, p_{02} = \frac{D_2(1+D_1)(m-a^2)}{a^2[D_1D_2+(1+D_1)^2]}, \\ p_{30} &= -\frac{m(1+D_1)^4}{a^3D_2^2[D_1D_2+(1+D_1)^2]}, p_{12} = -\frac{3m(1+D_1)^2}{a^3[D_1D_2+(1+D_1)^2]}, p_{21} = -\frac{3m(1+D_1)^3}{a^3D_2[D_1D_2+(1+D_1)^2]}, \\ p_{03} &= -\frac{m(1+D_1)D_2}{a^3[D_1D_2+(1+D_1)^2]}, q_{01} = -\frac{D_1D_2+(1+D_1)^2}{1+D_1}, q_{02} = \frac{D_1D_2(m-a^2)}{a^2[D_1D_2+(1+D_1)^2]}, \\ q_{11} &= \frac{2(D_1+D_1^2)(m-a^2)}{a^2[D_1D_2+(1+D_1)^2]}, q_{20} = \frac{(1+D_1)^2D_1(m-a^2)}{a^2D_2[D_1D_2+(1+D_1)^2]}, q_{21} = -\frac{3m(1+D_1)^2D_1}{a^3D_2[D_1D_2+(1+D_1)^2]}, \\ q_{12} &= -\frac{3m(1+D_1)D_1}{a^3[D_1D_2+(1+D_1)^2]}, q_{03} = -\frac{mD_1D_2}{a^3[D_1D_2+(1+D_1)^2]}, q_{30} = -\frac{m(1+D_1)^3D_1}{a^3D_2^2[D_1D_2+(1+D_1)^2]}. \end{aligned}$$

Introducing a new time variable $\tau = -\frac{D_1D_2+(1+D_1)^2}{1+D_1}t$, for the sake of simplicity, we still retain t to τ and obtain

$$\begin{cases} \dot{\bar{x}}_1 = v_{20}\bar{x}_1^2 + v_{11}\bar{x}_1\bar{x}_2 + v_{02}\bar{x}_2^2 + v_{30}\bar{x}_1^3 + v_{12}\bar{x}_1\bar{x}_2^2 + v_{21}\bar{x}_1^2\bar{x}_2 + v_{03}\bar{x}_2^3 + Q_1(\bar{x}_1, \bar{x}_2), \\ \dot{\bar{x}}_2 = \bar{x}_2 + w_{02}\bar{x}_2^2 + w_{11}\bar{x}_1\bar{x}_2 + w_{20}\bar{x}_1^2 + w_{21}\bar{x}_1^2\bar{x}_2 + w_{12}\bar{x}_1\bar{x}_2^2 + w_{03}\bar{x}_2^3 + w_{30}\bar{x}_1^3 + Q_2(\bar{x}_1, \bar{x}_2) \end{cases} \quad (2.9)$$

where $v_{ij} = -\frac{p_{ij}(1+D_1)}{D_1D_2+(1+D_1)^2}$, $w_{ij} = -\frac{q_{ij}(1+D_1)}{D_1D_2+(1+D_1)^2}$, p_{ij} , q_{ij} ($i + j = 2, 3$) are the same as those in (2.8). When $m = a(1 - \frac{D_2}{1+D_1})$, the coefficient of $\bar{x}_1^2$ can be simplified as $v_{20} = -\frac{(1+D_1)^4(1-\frac{D_2}{1+D_1}-a)}{aD_2[D_1D_2+(1+D_1)^2]^2} < 0$. Considering the new time variable τ and using Theorem 7.1 in [25], we can conclude that the equilibrium E_0 is an attracting saddle-node. Here that the domain of E_3 consists of a parabolic sector and a hyperbolic sector. Also, in the parabolic sector, the solution trajectory tends to the equilibrium E_3 when t is large sufficiently.

Following, we concentrate on the stability of coexistence equilibria $E_i(x_{1i}, x_{2i})$, $i = 1, 2$. The corresponding Jacobian matrix at $E_i(x_{1i}, x_{2i})$ is given by

$$J_{E_i} = \begin{bmatrix} -1 - D_1 & D_2 \\ D_1 & 1 - 2x_{2i} - \frac{ma}{(x_{2i}+a)^2} - D_2 \end{bmatrix}. \quad (2.10)$$

Combining with (2.1), the determinant and trace of the Jacobian matrix can be simplified as

$$\text{Det}[J_{E_i}] = \frac{x_{2i}}{(x_{2i} + a)^2}((x_{2i} + a)^2 - m)(1 + D_1),$$

$$\text{Tr}[J_{E_i}] = -D_1 - 2x_{2i} - \frac{ma}{(x_{2i} + a)^2} - D_2 < 0.$$

Furthermore, we can get

$$\text{Det}[J_{E_1}] = \frac{x_{21} \sqrt{\Delta(m)} (\sqrt{\Delta(m)} + \sqrt{\Delta(m) + 4m})(1 + D_1)}{2(x_{21} + a)^2} > 0,$$

$$\text{Det}[J_{E_2}] = \frac{x_{22} \sqrt{\Delta(m)} (\sqrt{\Delta(m)} - \sqrt{\Delta(m) + 4m})(1 + D_1)}{2(x_{22} + a)^2} < 0.$$

Thus E_1 is *LAS* and E_2 is a saddle.

We present the above discussion in Theorem 2.3 and Theorem 2.4.

Theorem 2.3. Suppose that $a < 1 - \frac{D_2}{1+D_1}$.

- (i) If $m < a(1 - \frac{D_2}{1+D_1})$, then $E_0(0, 0)$ is a saddle and $E_1(x_{11}, x_{21})$ is *LAS*;
- (ii) If $m = a(1 - \frac{D_2}{1+D_1})$, then $E_0(0, 0)$ is an attracting saddle-node and $E_1(x_{11}, x_{21})$ is *LAS*;
- (iii) If $a(1 - \frac{D_2}{1+D_1}) < m < m^* = \frac{(1 - \frac{D_2}{1+D_1} + a)^2}{4}$, then $E_0(0, 0)$ and $E_1(x_{11}, x_{21})$ are *LAS* and $E_2(x_{12}, x_{22})$ is a saddle;
- (iv) If $m = m^*$, then $E_0(0, 0)$ is *LAS* and $E_3(x_{13}, x_{23})$ is an attracting saddle-node;
- (v) If $m > m^*$, then $E_0(0, 0)$ is *LAS* and there exists no positive equilibrium.

Proof. We will only need to prove that E_3 is an attracting saddle-node when $m = m^*$. Let $X = x_1 - x_{13}$, $\bar{X} = x_2 - x_{23}$, system (1.5) can be transformed to the following system:

$$\begin{cases} \dot{X} = -X + D_2\bar{X} - D_1X, \\ \dot{\bar{X}} = (\bar{X} + x_{23}) \left[1 - \bar{X} - x_{23} - \frac{m}{\bar{X} + x_{23} + a} \right] + D_1(X + x_{13}) - D_2(\bar{X} + x_{23}). \end{cases} \quad (2.11)$$

Applying the Taylor expansion of $\frac{m}{\bar{X} + x_{23} + a}$, system (2.11) can be rewritten as

$$\begin{cases} \dot{X} = -X + D_2\bar{X} - D_1X, \\ \dot{\bar{X}} = -\frac{D_1D_2}{1+D_1}\bar{X} + D_1X + \left(-1 + \frac{ma}{(x_{23} + a)^3} \right) \bar{X}^2 - \frac{m}{(x_{23} + a)^3} \bar{X}^3 + Q(\bar{X}) \end{cases} \quad (2.12)$$

where $Q(\bar{X})$ denotes the power sery with term $\bar{X}^j$ satisfying $j \geq 4$. The Jacobian matrix of system (2.12) evaluated at the origin can be calculated as

$$J_{E_3} = \begin{bmatrix} -1 - D_1 & D_2 \\ D_1 & -\frac{D_1D_2}{1+D_1} \end{bmatrix}, \quad (2.13)$$

then the eigenvalues of J_{E_3} are $\lambda_1 = 0$ and $\lambda_2 = -1 - D_1 - \frac{D_1 D_2}{1+D_1}$. We make the following transformation:

$$\begin{pmatrix} X \\ \bar{X} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1+D_1}{D_1} \\ \frac{1+D_1}{D_2} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad (2.14)$$

then system (2.12) becomes

$$\begin{cases} \dot{x}_1 = c_{20}x_1^2 + c_{11}x_1x_2 + c_{02}x_2^2 + c_{30}x_1^3 + c_{12}x_1x_2^2 + c_{21}x_1^2x_2 + c_{03}x_2^3 + P_1(x_1, x_2), \\ \dot{x}_2 = d_{01}x_2 + d_{02}x_2^2 + d_{11}x_1x_2 + d_{20}x_1^2 + d_{21}x_1^2x_2 + d_{12}x_1x_2^2 + d_{03}x_2^3 + d_{30}x_1^3 + P_2(x_1, x_2) \end{cases} \quad (2.15)$$

where $P_1(x_1, x_2)$, $P_2(x_1, x_2)$ denote the power series with term $x_1^i x_2^j$ satisfying $i + j \geq 4$ and

$$\begin{aligned} c_{20} &= \frac{(1+D_1)^3[-(x_{23}+a)^3+ma]}{(x_{23}+a)^3 D_2 [D_1 D_2 + (1+D_1)^2]}, c_{11} = \frac{2(1+D_1)^2[-(x_{23}+a)^3+ma]}{(x_{23}+a)^3 [D_1 D_2 + (1+D_1)^2]}, c_{02} = \frac{D_2(1+D_1)[-(x_{23}+a)^3+ma]}{(x_{23}+a)^3 [D_1 D_2 + (1+D_1)^2]}, \\ c_{30} &= -\frac{m(1+D_1)^4}{(x_{23}+a)^3 D_2^2 [D_1 D_2 + (1+D_1)^2]}, c_{12} = -\frac{3m(1+D_1)^2}{(x_{23}+a)^3 [D_1 D_2 + (1+D_1)^2]}, c_{21} = -\frac{3m(1+D_1)^3}{(x_{23}+a)^3 D_2 [D_1 D_2 + (1+D_1)^2]}, \\ c_{03} &= -\frac{m(1+D_1)D_2}{(x_{23}+a)^3 [D_1 D_2 + (1+D_1)^2]}, d_{01} = -\frac{1+2D_1+D_1 D_2+D_1^2}{1+D_1}, d_{02} = \frac{D_1 D_2[-(x_{23}+a)^3+ma]}{(x_{23}+a)^3 [D_1 D_2 + (1+D_1)^2]}, \\ d_{11} &= \frac{2(D_1+D_1^2)[-(x_{23}+a)^3+ma]}{(x_{23}+a)^3 [D_1 D_2 + (1+D_1)^2]}, d_{20} = \frac{(1+D_1)^2 D_1[-(x_{23}+a)^3+ma]}{(x_{23}+a)^3 D_2 [D_1 D_2 + (1+D_1)^2]}, d_{21} = -\frac{3m(1+D_1)^2 D_1}{(x_{23}+a)^3 D_2 [D_1 D_2 + (1+D_1)^2]}, \\ d_{12} &= -\frac{3m(1+D_1)D_1}{(x_{23}+a)^3 [D_1 D_2 + (1+D_1)^2]}, d_{03} = -\frac{m D_1 D_2}{(x_{23}+a)^3 [D_1 D_2 + (1+D_1)^2]}, d_{30} = -\frac{m(1+D_1)^3 D_1}{(x_{23}+a)^3 D_2^2 [D_1 D_2 + (1+D_1)^2]}. \end{aligned}$$

Introducing a new time variable $\tau = -\frac{D_1 D_2 + (1+D_1)^2}{1+D_1} t$, for the sake of simplicity, we still retain t to denote τ and obtain

$$\begin{cases} \dot{x}_1 = e_{20}x_1^2 + e_{11}x_1x_2 + e_{02}x_2^2 + e_{30}x_1^3 + e_{12}x_1x_2^2 + e_{21}x_1^2x_2 + e_{03}x_2^3 + Q_1(x_1, x_2), \\ \dot{x}_2 = x_2 + f_{02}x_2^2 + f_{11}x_1x_2 + f_{20}x_1^2 + f_{21}x_1^2x_2 + f_{12}x_1x_2^2 + f_{03}x_2^3 + f_{30}x_1^3 + Q_2(x_1, x_2) \end{cases} \quad (2.16)$$

where $e_{ij} = -\frac{c_{ij}(1+D_1)}{D_1 D_2 + (1+D_1)^2}$, $f_{ij} = -\frac{d_{ij}(1+D_1)}{D_1 D_2 + (1+D_1)^2}$, c_{ij} , d_{ij} ($i + j = 2, 3$) are the same as those in (2.15).

When $m = m^*$, the coefficient of x^2 can be simplified as $e_{20} = -\frac{(1+D_1)^4[-(x_{23}+a)^3+m^*a]}{(x_{23}+a)^3 D_2 [D_1 D_2 + (1+D_1)^2]} > 0$. Here we use $-(x_{23}+a)^3 + m^*a = \frac{1}{8}(a-1 + \frac{D_2}{1+D_1})(a+1 - \frac{D_2}{1+D_1})^2 < 0$. Considering the new time variable τ and using Theorem 7.1 in [25], we can conclude that the equilibrium E_3 is an attracting saddle-node. This completes the proof. $\square$

Similarly, we can obtain the result as follows.

Theorem 2.4. Suppose that $a \geq 1 - \frac{D_2}{1+D_1}$.

- (i) If $m < a(1 - \frac{D_2}{1+D_1})$, then $E_0(0, 0)$ is a saddle and $E_1(x_{11}, x_{21})$ is *LAS*;
- (ii) If $m = a(1 - \frac{D_2}{1+D_1})$, then $E_0(0, 0)$ is an attracting saddle-node and $E_1(x_{11}, x_{21})$ is *LAS*;
- (iii) if $m > a(1 - \frac{D_2}{1+D_1})$, then $E_0(0, 0)$ is *LAS* and there exists no positive equilibrium.

Corollary 2.1. When $m \leq a(1 - \frac{D_2}{1+D_1})$, for system (1.5), the unique positive equilibrium $E_1(x_{11}, x_{21})$ is globally asymptotically stable.

Proof. As was shown in Theorems 2.3 and 2.4, E_1 is *LAS* when $m \leq a(1 - \frac{D_2}{1+D_1})$. To prove

that E_1 is globally asymptotically stable, we consider the Dulac function $u(x_1, x_2) = 1$. We can get

$$\frac{\partial(uP)}{\partial x_1} + \frac{\partial(uQ)}{\partial x_2} = -2 - D_1 - 2x_2 - \frac{ma}{(x_2 + a)^2} - D_2 < 0$$

where

$$P(x_1, x_2) = -x_1 + D_2x_2 - D_1x_1,$$

$$Q(x_1, x_2) = x_2(1 - x_2 - \frac{m}{x_2 + a}) + D_1x_1 - D_2x_2.$$

According to the Bendixson-Dulac discriminant, system (1.5) has no limit cycle in the first quadrant. And when $m \leq a(1 - \frac{D_2}{1+D_1})$, E_0 is unstable in the first quadrant. Thus one can deduce that $E_1(x_{11}, x_{21})$ is global asymptotically stable when $m \leq a(1 - \frac{D_2}{1+D_1})$. The proof of Corollary 2.1 is finished.

Corollary 2.2. When $a < 1 - \frac{D_2}{1+D_1}$ and $m > m^*$, for system (1.5), the trivial equilibrium $E_0(0, 0)$ is globally asymptotically stable.

Proof. As was mentioned in Theorem 2.3, E_0 is LAS when $a < 1 - \frac{D_2}{1+D_1}$ and $m > m^*$. Also, for system (1.5), we can declare that under the initial conditions $x_1(0) \geq 0$ and $x_2(0) \geq 0$, the solution $(x_1(t), x_2(t))$ satisfies $x_i(t) \geq 0 (i = 1, 2)$ when $t \geq 0$. Actually, we can prove $x_1(t) \geq 0 (t \geq 0)$ as follows. If not, one can suppose that there exists $t_0 > 0$ such that $x_1(t_0) = 0$, $x_1(t) \geq 0 (t \in (0, t_0))$ and $\dot{x}_1(t_0) < 0$. Thus, from the second equation of system (1.5), we have $\dot{x}_2(t) \geq x_2(t)(1 - D_2 - x_2(t) - \frac{m}{x_2(t)+a})$ when $t \in (0, t_0)$. As a result, $x_2(t) \geq x_2(0) \exp \int_0^t (1 - D_2 - x_2(s) - \frac{m}{x_2(s)+a}) ds \geq 0$, $t \in (0, t_0)$. From the continuity of $x_2(t)$, it follows that $x_2(t_0) \geq 0$. Moreover, from the first equation of system (1.5), we can obtain that $\dot{x}_1(t_0) = D_2x_2(t_0) \geq 0$. The above leads to a contradiction with $\dot{x}_1(t_0) < 0$. So it follows that $x_1(t) \geq 0 (t \geq 0)$. Similarly, we have $x_2(t) \geq 0 (t \geq 0)$. Moreover, we have $\frac{d(x_1+x_2)}{dt} = -x_1 + x_2(1 - x_2 - \frac{m}{x_2+a}) \leq -x_1 + x_2(1 - x_2) = -(x_1 + x_2) + x_2(2 - x_2) \leq 1 - (x_1 + x_2)$ which shows that $\limsup_{t \rightarrow +\infty} x_i(t) \leq 1$, $i = 1, 2$. In other words, the solution of system (1.5) is ultimately bounded and then the solution can't extend to the infinity when t is large sufficiently. Also, there exists no limit cycle since system (1.5) has no other equilibrium. Thus $E_0(0, 0)$ is global asymptotically stable. The proof of Corollary 2.2 is finished.

Similarly, we can obtain the result as follows.

Corollary 2.3. When $a \geq 1 - \frac{D_2}{1+D_1}$ and $m > a(1 - \frac{D_2}{1+D_1})$, for system (1.5), the trivial equilibrium $E_0(0, 0)$ is globally asymptotically stable.

From Corollaries 2.1–2.3, we can conclude that the additive Allee effect m and a is closely related with the population survival. In detail, when $m \leq a(1 - \frac{D_2}{1+D_1})$, the species in both patches can survive. However, when $m > a(1 - \frac{D_2}{1+D_1})$, the species may still be persistent or become extinguished. Actually, when $a \geq 1 - \frac{D_2}{1+D_1}$, $m > a(1 - \frac{D_2}{1+D_1})$ or $a < 1 - \frac{D_2}{1+D_1}$, $m > m^*$, both species will be extinct. In other words, the larger the Allee effect constant m , the more likely the species will become extinct. The phase ports of system (1.5) is given in Figure 1.

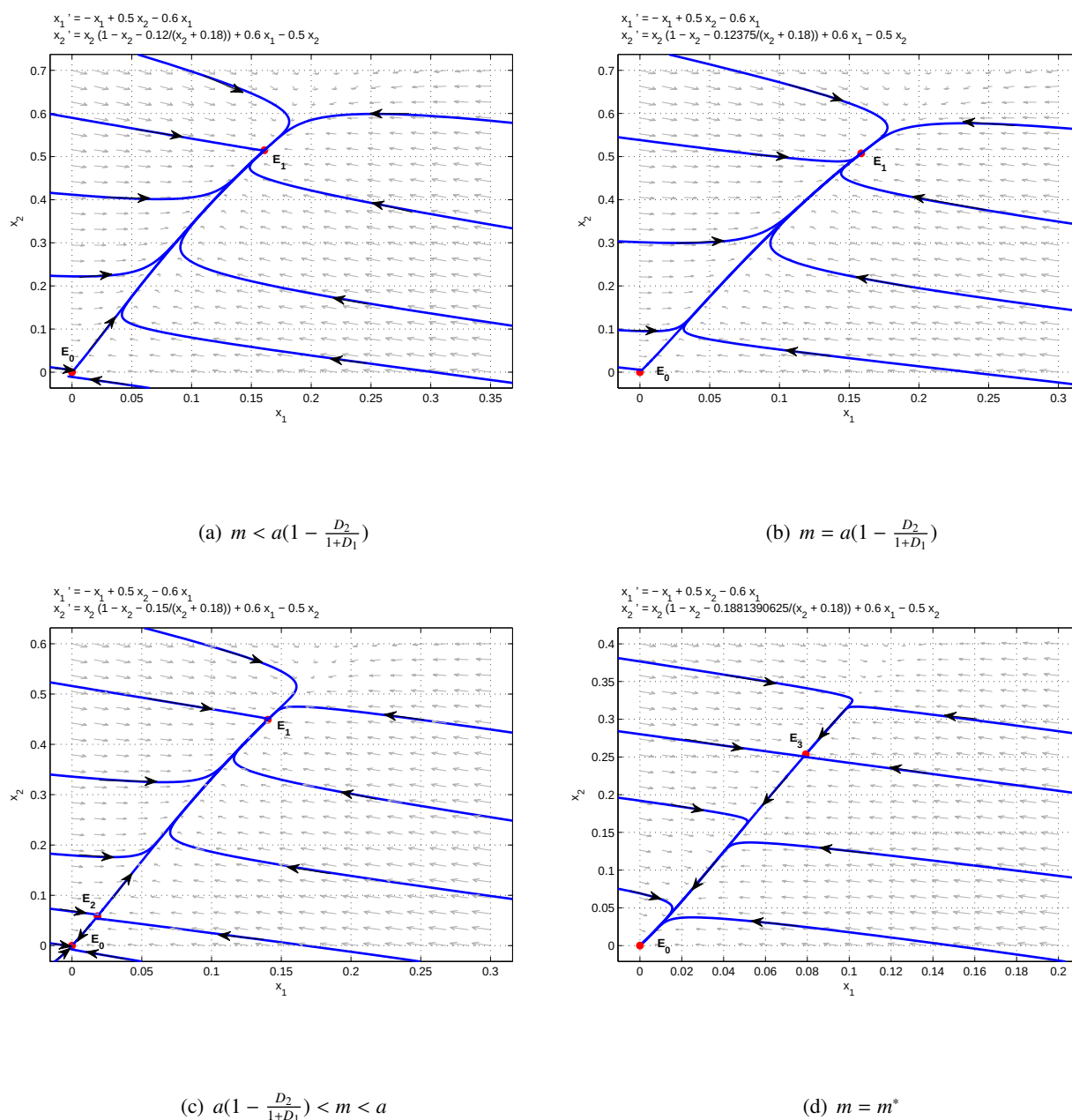


Figure 1. The phase portraits of system (1.5). (a) E_0 is a saddle and E_1 is LAS; (b) E_0 is an attracting saddle-node and E_1 is LAS; (c) E_0 , E_1 are LAS and E_2 is a saddle; (d) E_0 is LAS and E_3 is an attracting saddle-node.

3. Saddle-node bifurcation

From Section 2, we know that if $a < m < m^*$, system (1.5) has two positive equilibria, i.e., $E_1(x_{11}, x_{21})$ and $E_2(x_{12}, x_{22})$; if $m = m^*$, the system has a unique positive equilibrium $E_3(x_{13}, x_{23})$; if $m > m^*$, the system has no positive equilibria. The above indicates that there will be the occurrence

of saddle-node bifurcation. Next we will prove that system (1.5) undergoes the saddle-node bifurcation when $m = m_{SN} \triangleq \frac{(1 - \frac{D_2}{1+D_1} + a)^2}{4}$. And the numerical simulation of the saddle-node bifurcation is shown in Figure 2.

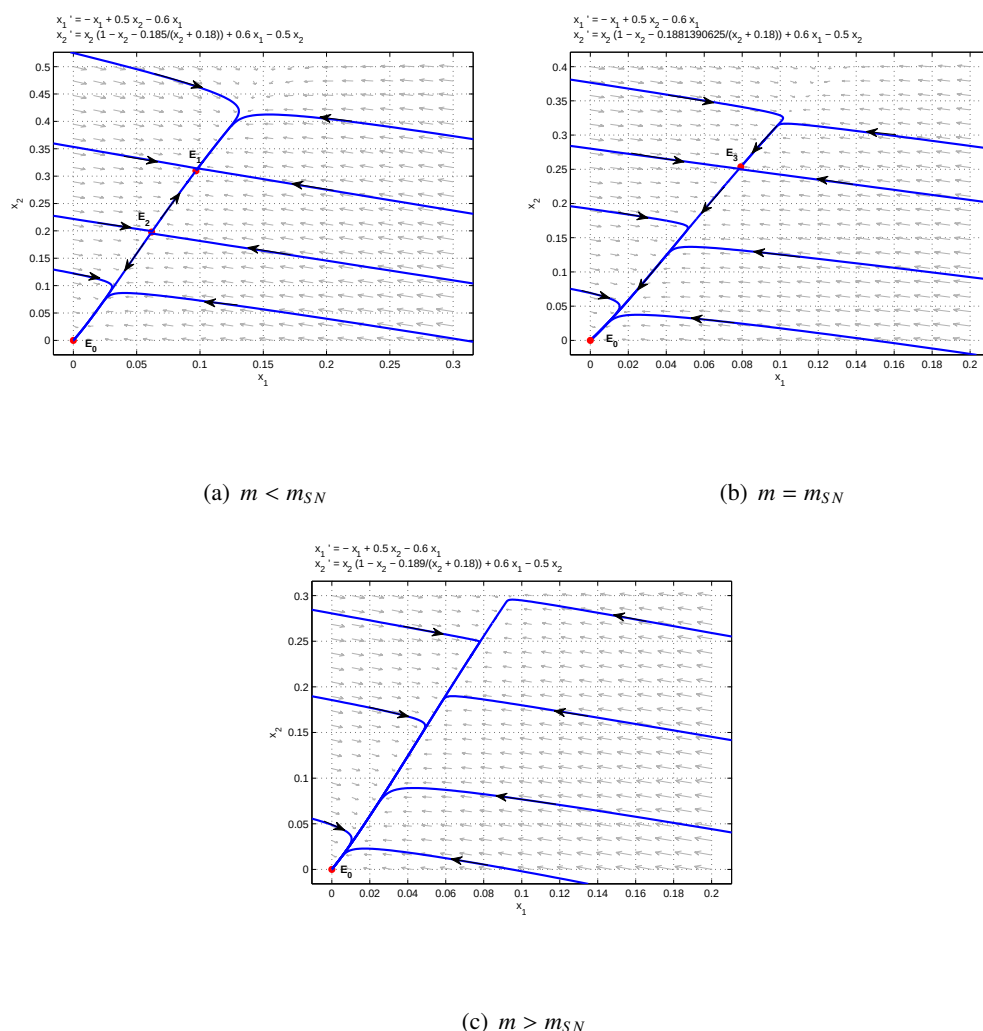


Figure 2. The saddle-node bifurcation of system (1.5).

Theorem 3.1. System (1.5) undergoes the saddle-node bifurcation when $m = m_{SN}$.

Proof. We use Sotomayor's theorem in [26] to verify the transversality condition for the occurrence of saddle-node bifurcation at $m = m_{SN}$. Notice that the determinant and the trace of the corresponding Jacobian matrix can be given by $\text{Det}[J_{E_3}] = 0$ and $\text{Tr}[J_{E_3}] = -1 - D_1 - \frac{D_1 D_2}{1+D_1} < 0$. Then J_{E_3} has a unique zero eigenvalue, named by λ_1 .

For the matrices J_{E_3} and $J_{E_3}^T$, let V and W represent two eigenvectors corresponding to the eigenvalue λ_1 , respectively. Then we can get

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1+D_1}{D_2} \end{pmatrix}, \quad W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1+D_1}{D_1} \end{pmatrix}. \quad (3.1)$$

Furthermore, we obtain

$$F_m(E_3; m_{SN}) = \begin{pmatrix} 0 \\ \frac{(1 - \frac{D_2}{1+D_1} - a)[2(-1 + \frac{D_2}{1+D_1} - a) + (1 - \frac{D_2}{1+D_1} + a)^2]}{2(1 - \frac{D_2}{1+D_1} + a)^2} \end{pmatrix}, \quad (3.2)$$

$$\begin{aligned} D^2 F(E_3; m_{SN})(V, V) &= \begin{pmatrix} \frac{\partial^2 F_1}{\partial x_1^2} V_1^2 + 2 \frac{\partial^2 F_1}{\partial x_1 \partial x_2} V_1 V_2 + \frac{\partial^2 F_1}{\partial x_2^2} V_2^2 \\ \frac{\partial^2 F_2}{\partial x_1^2} V_1^2 + 2 \frac{\partial^2 F_2}{\partial x_1 \partial x_2} V_1 V_2 + \frac{\partial^2 F_2}{\partial x_2^2} V_2^2 \end{pmatrix}_{(E_3; m_{SN})} \\ &= \begin{pmatrix} 0 \\ \frac{2(-1 + \frac{D_2}{1+D_1} + a)(1+D_1)^2}{(1 - \frac{D_2}{1+D_1} + a)D_2^2} \end{pmatrix}. \end{aligned} \quad (3.3)$$

Here $F_1 = -x_1 + D_2 x_2 - D_1 x_1$ and $F_2 = x_2(1 - x_2 - \frac{m}{x_2 + a}) + D_1 x_1 - D_2 x_2$. According to (3.2) and (3.3), it follows that

$$\begin{aligned} W^T F_m(E_3; m_{SN}) &= \frac{(1+D_1)(1 - \frac{D_2}{1+D_1} - a)[2(-1 + \frac{D_2}{1+D_1} - a) + (1 - \frac{D_2}{1+D_1} + a)^2]}{2D_1(1 - \frac{D_2}{1+D_1} + a)^2} \neq 0, \\ W^T [D^2 F(E_3; m_{SN})(V, V)] &= \frac{2(-1 + \frac{D_2}{1+D_1} + a)(1+D_1)^3}{(1 - \frac{D_2}{1+D_1} + a)D_1 D_2^2} \neq 0 \end{aligned}$$

which means that the saddle-node bifurcation occurs at E_3 when $m = m_{SN}$. Thus we complete the proof. $\square$

4. Effect of Allee effect and dispersal

Through the previous analysis, we can see that when the parameters satisfy certain conditions, the unique positive equilibrium $E_1(x_{11}, x_{21})$ is globally asymptotically stable. At this time, the total population abundance is

$$T_1 = x_{11} + x_{21} = \frac{(1 + D_1 + D_2)(1 - \frac{D_2}{1+D_1} - a + \sqrt{\Delta(m)})}{2(1 + D_1)}.$$

Following, we will explore the effect of Allee effect and dispersal on the total population abundance, respectively.

Firstly, by simple calculation, we have

$$\begin{aligned} \frac{dT_1}{da} &= \frac{1 + D_1 + D_2}{2(1 + D_1)} \left(-1 + \frac{1 - \frac{D_2}{1+D_1} + a}{\sqrt{\Delta(m)}} \right) > 0, \\ \frac{dT_1}{dm} &= -\frac{1 + D_1 + D_2}{(1 + D_1)\sqrt{\Delta(m)}} < 0. \end{aligned}$$

Thus we can obtain that the total population abundance will increase when the Allee effect constant a increases or m decreases. Let $D_1 = 1$, $D_2 = 0.5$. In Figure 3(a),(b), we have used the software

of matlab to plot the bifurcation diagram when $a = 0.2$ or $m = 0.2$, respectively. The bifurcation diagram shows the maximum and minimum density of species 1, i.e., x_{11} and x_{21} as Allee effect m or a is increasing. Here x_{11} and x_{12} are regarded as functions of m , a which are plotted in Figure 3(a),(b), respectively. It follows that Allee effect has a vital influence on the species. In detail, the density of the species decreases when the Allee effect coefficients m increases or a decreases.

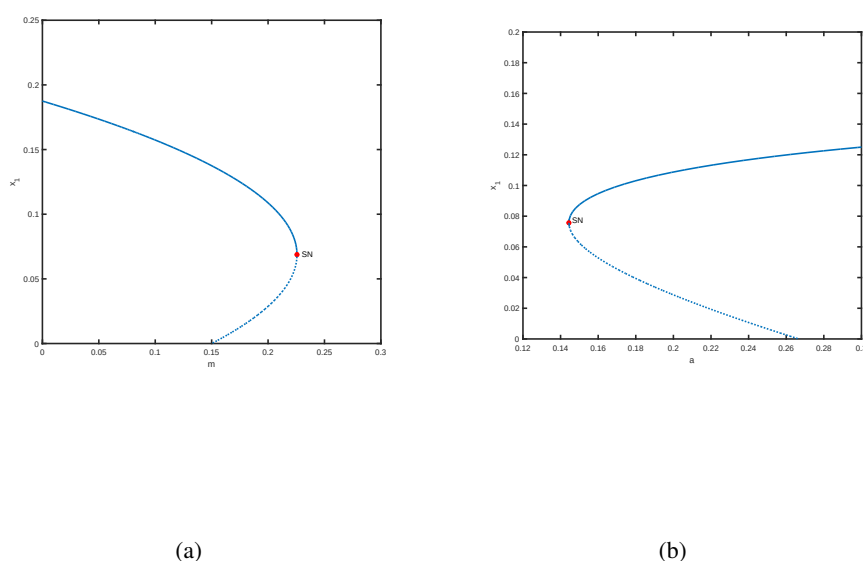


Figure 3. The bifurcation diagram showing the maximum and minimum density of species 1, i.e., x_{11} and x_{21} as Allee effect m or a is increasing. Solid(dotted) curves represent the stable(unstable) singularities, i.e., $x_{11}(x_{21})$, respectively. SN denotes the saddle-node bifurcation.

Secondly, we will explore the possible impact of dispersal on the total population abundance through numerical simulation. We regard the total population abundance T_1 as a function of D_1 or D_2 , respectively. In Figure 4(a), let $m = 0.2$, $a = 0.5$, $D_2 = 1$, it follows that T_1 increases when D_1 increases. In Figure 4(b), let $m = 0.02$, $a = 0.5$, $D_1 = 10$, it follows T_1 decreases when D_2 increases. Especially, when $m = m^*$, the system (1.5) has a unique positive equilibrium E_3 . Then the corresponding total population abundance $T_1 = x_{13} + x_{23} = \frac{(1+D_1+D_2)(1-\frac{D_2}{1+D_1}-a)}{2(1+D_1)}$. A direct computation shows $\frac{dT_1}{dD_1} = \frac{D_2}{2(1+D_1)^2}[a + \frac{2D_2}{1+D_1}] > 0$, $\frac{dT_1}{dD_2} = -\frac{1}{2(1+D_1)}(a + \frac{2D_2}{1+D_1}) < 0$. This observation is consistent with the numerical simulation in Figure 4. Actually, it is not difficult to understand the above result biologically. Notice that $D_1(D_2)$ is the dispersal rate of the species from patch 1(2) to patch 2(1) and the intrinsic growth rate of the species is negative (positive) in patch 1(2) when there is no dispersal. As a result, once D_1 increases, the species easily migrates from patch 1 to patch 2, that is, migrates to a patch with a larger growth rate. The migration will result in the increase of the total population

abundance. On the contrary, when D_2 increases, the species easily migrates from patch 2 to patch 1, which makes the species more likely to be extinct and then reduce the total population abundance.

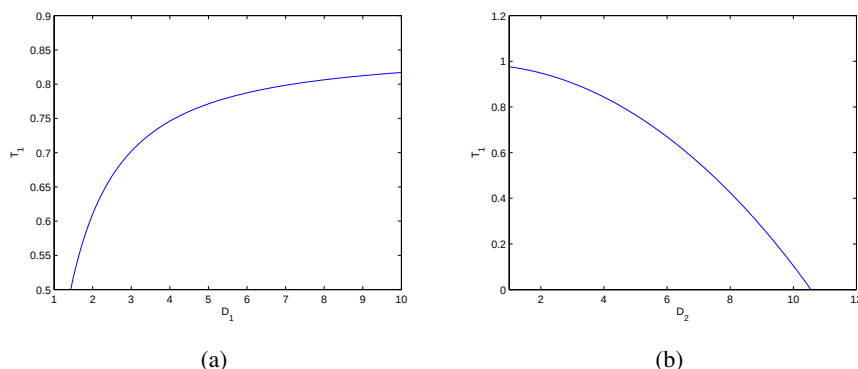


Figure 4. The total population abundance T_1 as D_1 or D_2 is increasing.

5. Conclusions

In this paper, we have proposed and studied a two-patch model with additive Allee effect and dispersal. Our goal of this investigation is to discuss how the above model's dynamical behaviour such as the persistence and extinction is influenced by the additive Allee effect and dispersal. The model always has a trivial equilibrium $E_0(0, 0)$ which is a saddle or a stable node when $m < a(1 - \frac{D_2}{1-D_1})$ or $m > a(1 - \frac{D_2}{1-D_1})$, respectively. For the existence and stability of the positive equilibrium, we have considered two cases, i.e., $a < 1 - \frac{D_2}{1-D_1}$ and $a \geq 1 - \frac{D_2}{1-D_1}$ in Theorems 2.3 and 2.4. In detail, when $a < 1 - \frac{D_2}{1-D_1}$, there may exist a stable positive equilibrium E_1 , two different positive equilibria E_1, E_2 (E_1 is stable and E_2 is a saddle), an attracting saddle-node E_3 or no positive equilibrium under the case $m \leq a(1 - \frac{D_2}{1-D_1})$, $a(1 - \frac{D_2}{1-D_1}) < m < m^*$, $m = m^*$, $m > m^*$, respectively. On the other hand, when $a \geq 1 - \frac{D_2}{1-D_1}$, there may exist a stable positive equilibrium E_1 or no positive equilibrium under the case $m \leq a(1 - \frac{D_2}{1-D_1})$, $m > a(1 - \frac{D_2}{1-D_1})$, respectively. Furthermore, in Theorem 3.1, we have revealed that the system can undergo the saddle-node bifurcation around the unique positive equilibrium E_3 .

In all, the above results indicate that the additive Allee effect and dispersal have great impact on the species permanence, extinction and stability. Especially, according to Corollary 2.1, E_1 is globally asymptotically stable which shows that the species in both patches coexist when $m \leq a(1 - \frac{D_2}{1-D_1})$. From Corollaries 2.2 and 2.3, one can find that both species go extinct when $a < 1 - \frac{D_2}{1-D_1}$, $m > m^*$ or $a \geq 1 - \frac{D_2}{1-D_1}$, $m > a(1 - \frac{D_2}{1-D_1})$. Furthermore, in order to study the effect of the Allee effect and dispersal on the population more concisely, we also investigate the total population abundance and have found that it will increase when the Allee effect constant a increases or m decreases. Also, we find that the total population density increases when the dispersal rate D_1 increases or D_2 decreases. Additive Allee effect can reduce complex dynamics and bifurcation. Actually, many literatures have investigated system (1.5) when $m = 0$. For example, in [9], system (1.5) with $m = 0$ has at most a positive equilibrium. However, when $m > 0$, there may exist one or two positive equilibria and even the emergence of the bifurcation. Also, for the strong Allee effect case in (1.5), we provide a rigorous

qualitative analysis other than numerical simulation which has been done in [24]. The main results in this paper can be seen as a good supplement to the work in [9] and [24].

Here we have to point out that this paper investigates a patchy model in which the species in patch 1 is assumed to die exponentially and the system only admits saddle-node bifurcation. As we know, it is also meaningful to consider the logistic growth rate for the species in one of the patches and the species in the other patch is suffered from the additive Allee effect. In this case, the above system may exhibit at most six positive equilibria and even undergo complex bifurcation such as Hopf bifurcation and Bogdanov-Takens bifurcation. It is rather difficult for us to present a rigorous qualitative analysis at present. We hope to consider this aspect for future work by use of “bifurcation control” and “parametric normal forms” in [27–33].

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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