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Research article

Explicit formulas of alternating multiple zeta star values $\zeta^\star(\bar{1}, \{1\}_{m-1}, \bar{1})$ and $\zeta^\star(2, \{1\}_{m-1}, \bar{1})$

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Abstract: In a recent paper [4], Xu studied some alternating multiple zeta values. In particular, he gave two recurrence formulas of alternating multiple zeta values $\zeta^\star(\bar{1}, \{1\}_{m-1}, \bar{1})$ and $\zeta^\star(2, \{1\}_{m-1}, \bar{1})$. In this paper, we will give the closed forms representations of $\zeta^\star(\bar{1}, \{1\}_{m-1}, \bar{1})$ and $\zeta^\star(2, \{1\}_{m-1}, \bar{1})$ in terms of single zeta values and polylogarithms.

Keywords: multiple harmonic (star) sums; multiple zeta (star) values; multiple polylogarithm function; alternating multiple zeta values; closed forms
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1. Introduction

We begin with some basic notation. Let $\mathbf{s} := (s_1, \ldots, s_k) \in \mathbb{N}^k$, the (alternating) multiple harmonic (star) sums are defined by

$$\zeta_n(\mathbf{s}) \equiv \zeta_n(s_1, \ldots, s_k) := \sum_{n \ge n_1 > \cdots > n_k > 0} \prod_{j=1}^{k} n_j^{-|s_j|} \operatorname{sgn}(s_j)^{n_j}, \tag{1.1}$$

$$\zeta_n^\star(\mathbf{s}) \equiv \zeta_n^\star(s_1, \ldots, s_k) := \sum_{n \ge n_1 \ge \cdots \ge n_k \ge 1} \prod_{j=1}^{k} n_j^{-|s_j|} \operatorname{sgn}(s_j)^{n_j}, \tag{1.2}$$

where s_j stands for non-zero integer, and

$$\operatorname{sgn}(s_j) := \begin{cases} 1, & s_j > 0, \\ -1, & s_j < 0. \end{cases}$$

We may compactly indicate the presence of an alternating sign. When $\operatorname{sgn}(s_j) = -1$, by placing a bar over the corresponding integer exponent s_j. For example,

$$\zeta_n\left(\bar{2},3,\bar{1},4\right)=\zeta_n\left(-2,3,-1,4\right)=\sum_{n\geq n_1>n_2>n_3>n_4\geq 1}\frac{(-1)^{n_1+n_3}}{n_1^2n_2^3n_3n_4^4}.$$

Moreover, for convenience, by $\left\{s_1,\ldots,s_j\right\}_d$ we denote the sequence of depth dj with d repetitions of $\left\{s_1,\ldots,s_j\right\}$. For example,

$$\zeta_n(2,3,\{1\}_2)=\zeta_n(2,3,1,1),\ \zeta_n^\star(5,2,\{1\}_3)=\zeta_n^\star(5,2,1,1,1).$$

The sums of types (1.1) and (1.2) (one of more the s_j barred) are called the alternating multiple harmonic sums and alternating multiple harmonic star sums, respectively. Conventionally, we call $\mathrm{dep}(\mathbf{s})=k$ the depth and $w\equiv\mathrm{wt}(\mathbf{s}):=s_1+\cdots+s_k$ the weight.

Obviously, the limit cases of alternating multiple harmonic (star) sums give rise to alternating multiple zeta (star) values, for example,

$$\zeta\left(\bar{2},3,\bar{1},4\right)=\lim_{n\to\infty}\zeta_n\left(\bar{2},3,\bar{1},4\right).$$

The systematic study of multiple zeta values began in the early 1990s with the works of Hoffman [2] and Zagier [6]. After that it has been attracted a lot of research on them in the last three decades (see, for example, the book of Zhao [7]).

The main purpose of the present paper is to obtain the explicit formulas of alternating multiple zeta star values $\zeta^\star(\bar{1},\{1\}_{m-1},\bar{1})$ and $\zeta^\star(2,\{1\}_{m-1},\bar{1})$.

2. Main results

In this section we first give a lemma, which will be useful in the development of the main results of this paper.

Lemma 2.1. *(see [3, 4]) For $n,m\in\mathbb{N}$ and $x\in[-1,1)$, the following relation holds:*

$$\begin{aligned}\int_0^x t^{n-1}\ln^m(1-t)dt&=\frac{1}{n}\ln^m(1-x)(x^n-1)+m!\frac{(-1)^m}{n}\zeta_n^\star(\{1\}_m;x)\\&\quad-\frac{1}{n}\sum_{i=1}^{m-1}(-1)^{i-1}i!\binom{m}{i}\ln^{m-i}(1-x)\{\zeta_n^\star(\{1\}_i;x)-\zeta_n^\star(\{1\}_i)\},\end{aligned}\tag{2.1}$$

where the parametric multiple harmonic star sum $\zeta_n^\star(s_1,\cdots,s_{k-1},s_k;x)$ is defined by

$$\zeta_n^\star(s_1,\cdots,s_{k-1},s_k;x):=\sum_{n\geq n_1\geq\cdots\geq n_k\geq 1}\frac{x^{n_k}}{n_1^{s_1}\cdots n_{k-1}^{s_{k-1}}n_k^{s_k}},$$

and $\zeta_n^\star(\emptyset;x):=x^n$.

We note the fact that $\int_0^x = \int_0^1 + \int_1^x$ and $\int_0^1 t^{n-1}\ln^m(1-t)dt = m!\frac{(-1)^m}{n}\zeta_n^\star(\{1\}_m)$, then (2.1) can be rewritten as

$$\int_1^x t^{n-1}\ln^m(1-t)dt = \frac{1}{n}\sum_{i=0}^m (-1)^i i!\binom{m}{i}\ln^{m-i}(1-x)\left\{\zeta_n^\star(\{1\}_i;x) - \zeta_n^\star(\{1\}_i)\right\}. \tag{2.2}$$

From (2.2), by a direct calculation, we can find that

$$\begin{aligned}\zeta_n^\star(\{1\}_m;x) &= \frac{n}{m!}\sum_{i=0}^m (-1)^i\binom{m}{i}\ln^{m-i}(1-x)\int_1^x t^{n-1}\ln^i(1-t)dt + \zeta_n^\star(\{1\}_m)\\ &= \frac{n(-1)^{m-1}}{m!}\int_x^1 t^{n-1}\ln^m\left(\frac{1-t}{1-x}\right)dt + \zeta_n^\star(\{1\}_m).\end{aligned} \tag{2.3}$$

Theorem 2.2. *For positive integers k, m and real $x \in [-1, 1)$,*

$$\begin{aligned}\sum_{n=1}^\infty \frac{\zeta_{n-1}(\{1\}_{k-1})\zeta_n^\star(\{1\}_m;x)}{n^2} &= \sum_{j-0}^m \binom{k+j}{j}\frac{\ln^{m-j}(1-x)}{(m-j)!}\mathrm{Li}_{2,\{1\}_{k+j-1}}(x)\\ &\quad - \sum_{j=0}^{m-1}\binom{k+j}{j}\frac{\ln^{m-j}(1-x)}{(m-j)!}\zeta(k+j+1),\end{aligned} \tag{2.4}$$

where $\mathrm{Li}_{s_1,s_2,\cdots,s_k}(x)$ is multiple polylogarithm function defined by

$$\mathrm{Li}_{s_1,s_2,\cdots,s_k}(x) := \sum_{n_1>n_2>\cdots>n_k\geq 1}\frac{x^{n_1}}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}}, \quad x \in [-1,1). \tag{2.5}$$

Proof. Multiplying (2.3) by $\frac{\zeta_{n-1}(\{1\}_{k-1})}{n^2}$, then summing both sides of it over n yields

$$\begin{aligned}&\sum_{n=1}^\infty \frac{\zeta_{n-1}(\{1\}_{k-1})\zeta_n^\star(\{1\}_m;x)}{n^2} - \sum_{n=1}^\infty \frac{\zeta_{n-1}(\{1\}_{k-1})\zeta_n^\star(\{1\}_m)}{n^2}\\ &= \frac{(-1)^{k+m-1}}{k!m!}\int_x^1 \frac{\ln^m\left(\frac{1-t}{1-x}\right)\ln^k(1-t)}{t}dt\\ &= \frac{(-1)^{k-1}}{k!m!}\sum_{j=0}^m\binom{m}{j}(-1)^j\ln^{m-j}(1-x)\int_x^1\frac{\ln^{j+k}(1-x)}{x}dx.\end{aligned} \tag{2.6}$$

From [3, Eq (2.24)], an elementary calculation shows that

$$\int_0^x \frac{\ln^k(1-t)}{t}dt = (-1)^k k!\mathrm{Li}_{2,\{1\}_{k-1}}(x).$$

From [1, 3], we have the duality relation $\zeta(m+1,\{1\}_{k-1}) = \zeta(k+1,\{1\}_{m-1})$. Hence, letting $m = 1$ yields $\zeta(2,\{1\}_{k-1}) = \zeta(k+1)$. Thus, the formula (2.4) holds. □

Similarly, we can also give the following results of alternating zeta type values:

Theorem 2.3. *For positive integer m and $x \in [-1, 1]$,*

$$\sum_{n=1}^{\infty} \frac{\zeta_n^{\star}(\{1\}_m; x)}{n}(-1)^n = \mathrm{Li}_{m+1}\left(\frac{1-x}{2}\right) - \mathrm{Li}_{m+1}\left(\frac{1}{2}\right). \tag{2.7}$$

Proof. We note that the formula (2.3) can be rewritten as

$$\begin{aligned}\zeta_n^{\star}(\{1\}_m; x) =& \frac{n}{m!}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\ln^{m-i}(1-x)\int_0^x t^{n-1}\ln^i(1-t)dt \\ &- \sum_{i=0}^{m-1}\frac{\zeta_n^{\star}(\{1\}_i)}{(m-i)!}\ln^{m-i}(1-x).\end{aligned} \tag{2.8}$$

Multiplying (2.8) by $\frac{(-1)^n}{n}$ and summing with respect to n, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\zeta_n^{\star}(\{1\}_m; x)}{n}(-1)^n =& -\frac{1}{m!}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\ln^{m-i}(1-x)\int_0^x \frac{\ln^i(1-t)}{1+t}dt \\ &- \sum_{i=0}^{m-1}\frac{\zeta^{\star}(\bar{1}, \{1\}_i)}{(m-i)!}\ln^{m-i}(1-x).\end{aligned} \tag{2.9}$$

In [5], Xu proved the result

$$\begin{aligned}\int_0^x \frac{\ln^m(1-t)}{1+t}dt =& (-1)^m m!\mathrm{Li}_{m+1}\left(\frac{1}{2}\right) + \ln^m(1-x)\ln\left(\frac{1+x}{2}\right) \\ &+ \sum_{l=1}^{m}(-1)^{l+1}\ln^{m-l}(1-x)\, l!\binom{m}{l}\mathrm{Li}_{l+1}\left(\frac{1-x}{2}\right).\end{aligned} \tag{2.10}$$

Combining (2.9) with (2.10), by an elementary calculation, we have the result (2.7). □

Corollary 2.4. *For a positive integer m,*

$$\zeta^{\star}(\bar{1}, \{1\}_{m-1}, \bar{1}) = \zeta(m+1) - \mathrm{Li}_{m+1}\left(\frac{1}{2}\right), \tag{2.11}$$

$$\zeta^{\star}(2, \{1\}_{m-1}, \bar{1}) = \sum_{i=0}^{m}\frac{i+1}{(m-i)!}\ln^{m-i}(2)\zeta(\bar{2}, \{1\}_i) - \sum_{i=0}^{m-1}\frac{i+1}{(m-i)!}\zeta(i+2)\ln^{m-i}(2), \tag{2.12}$$

where ([1])

$$\begin{aligned}\zeta(\bar{2}, \{1\}_{m-1}) =& \frac{(-1)^m}{(m+1)!}\ln^{m+1}(2) + (-1)^m\left(\zeta(m+1) - \mathrm{Li}_{m+1}\left(\frac{1}{2}\right)\right) \\ &- (-1)^m\sum_{j=1}^{m}\frac{\ln^{m+1-j}(2)}{(m+1-j)!}\mathrm{Li}_j\left(\frac{1}{2}\right).\end{aligned}$$

Proof. Letting $k = 1, x = -1$ in (2.4) and $x = -1$ in (2.7) and noting the facts that (according to the definitions of alternating multiple zeta star values)

$$\zeta^\star(\bar{1},\{1\}_{m-1},\bar{1}) = \sum_{n=1}^{\infty} \frac{\zeta_n^\star(\{1\}_{m-1},\bar{1})}{n}(-1)^n \quad \text{and} \quad \zeta^\star(2,\{1\}_{m-1},\bar{1}) = \sum_{n=1}^{\infty} \frac{\zeta_n^\star(\{1\}_{m-1},\bar{1})}{n^2}$$

yield the desired results. □

Hence, from Corollary 2.4, we obtain the explicit evaluations of alternating multiple zeta values $\zeta^\star(\bar{1},\{1\}_{m-1},\bar{1})$ and $\zeta^\star(2,\{1\}_{m-1},\bar{1})$.

3. Conclusions

In this paper, we use the integrals of logarithmic function to establish explicit formulas of the two (alternating) multiple zeta type values involving multiple harmonic star sum and parametric multiple harmonic star sum

$$\sum_{n=1}^{\infty} \frac{\zeta_{n-1}(\{1\}_{k-1})\zeta_n^\star(\{1\}_m;x)}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\zeta_n^\star(\{1\}_m;x)}{n}(-1)^n$$

in terms of zeta values and multiple polylogarithms. Further, by applying these formulas obtained, we obtain the explicit evaluations of alternating multiple zeta values $\zeta^\star(\bar{1},\{1\}_{m-1},\bar{1})$ and $\zeta^\star(2,\{1\}_{m-1},\bar{1})$.

It is possible that some similar sums involving (parametric) multiple harmonic star sum can be computed by the methods and techniques given in this paper. For example, we can get the following theorem:

Theorem 3.1. *For positive integers m, k and $x \in [-1, 1)$,*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\left(\zeta_n^\star(\{1\}_m;x) - \zeta_n^\star(\{1\}_m)\right)\zeta_n(\bar{1},\{1\}_{k-1})}{n}(-1)^n \\
&= \frac{(-1)^{k-1}}{m!k!} \sum_{j=0}^{m} \binom{m}{j} (-1)^j \ln^{m-j}(1-x) \int_x^1 \frac{\ln^{j+k}(1-t)}{t} dt \\
&\quad - \frac{(-1)^{k-1}}{m!k!} \sum_{j=0}^{m} \binom{m}{j} (-1)^j \ln^{m-j}(1-x) \int_x^1 \frac{\ln^{j+k}(1-t)}{1+t} dt.
\end{aligned} \tag{3.1}$$

Proof. The proof of Theorem 3.1 is similar as the proof of Theorem 2.3. Multiplying (2.3) by $\frac{\zeta_n(\bar{1},\{1\}_{k-1})}{n}(-1)^n$ and summing with respect to n, and applying the identity

$$\frac{\ln^k(1-t)}{1+t} = (-1)^k k! \sum_{n=1}^{\infty} \zeta_n(\bar{1},\{1\}_{k-1})(-t)^n \quad (k \in \mathbb{N},\ t \in (-1,1)),$$

we have

$$\sum_{n=1}^{\infty} \frac{\left(\zeta_n^\star(\{1\}_m;x) - \zeta_n^\star(\{1\}_m)\right)\zeta_n(\bar{1},\{1\}_{k-1})}{n}(-1)^n = \frac{(-1)^{m+k-1}}{m!k!} \int_x^1 \frac{\ln^m\left(\frac{1-t}{1-x}\right)\ln^k(1-t)}{t(1+t)} dt.$$

Hence, we may easily deduce the desired result by a direct calculation. □

In particular, setting $x = 0$ in (3.1) yields

$$\sum_{n=1}^{\infty} \frac{\zeta_n^\star(\{1\}_m)\zeta_n(\bar{1},\{1\}_{k-1})}{n}(-1)^n = \binom{m+k}{k}\Big(\zeta(m+k+1) + \zeta(\bar{1},\bar{1},\{1\}_{m+k-1})\Big). \tag{3.2}$$

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Conflict of interest

The author declares no conflict of interest in this paper.

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