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Research article

Frenet curves in 3-dimensional δ-Lorentzian trans Sasakian manifolds

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Abstract: In this paper, we give some characterizations of Frenet curves in 3-dimensional δ-Lorentzian trans-Sasakian manifolds. We compute the Frenet equations and Frenet elements of these curves. We also obtain the curvatures of non-geodesic Frenet curves on 3-dimensional δ-Lorentzian trans-Sasakian manifolds. Finally, we give some results for these curves.

Keywords: Frenet curves; Frenet elements; Lorentzian metric; almost contact metric manifold; δ-Lorentzian manifold
Mathematics Subject Classification: 53A35, 53B30

1. Introduction

The differential geometry of curves especially in contact and para-contact manifolds studied by several authors. Olszak [16], derived certain necessary and sufficient conditions for an almost contact metric (a.c.m) structure on M to be normal and point out some of their consequences. Olszak completely characterized the local nature of normal a.c.m. structures on M by giving suitable examples. Moreover Olszak gave some restrictions on the scalar curvature in contact metric manifolds which are conformally flat or of constant ϕ-sectional curvature in [15].

Welyczko [21], generalized some of results for Legendre curves to the case of 3-dimensional normal almost contact metric manifolds, especially, quasi-Sasakian manifolds. Welyczko [20], studied the curvature and torsion of slant Frenet curves in 3-dimensional normal almost paracontact metric manifolds.

Curvature and torsion of Legendre curves in 3-dimensional (ε, δ) trans-Sasakian manifolds was obtained in [1]. Lee defined Lorentzian cross product in a three-dimensional almost contact Lorentzian manifold. Using a Lorentzian cross product, Lee proved that the ratio of κ and τ-1 is constant along a Frenet slant curve in a Sasakian Lorentzian three-manifold. Moreover, Lee proved that γ is a slant curve if and only if M is Sasakian for a contact magnetic curve γ in contact Lorentzian three-manifold M in [11]. Lee, also gave the properties of the generalized Tanaka-Webster connection in a contact

Lorentzian manifold in [12].

Yldrm [22] obtained curvatures of non-geodesic Frenet curves on 3-dimensional normal almost contact manifolds without neglecting α and β, and provided the results of their characterization.

Trans-Sasakian structure on a manifold with Lorentzian metric and conformally flat Lorentzian trans-Sasakian manifolds was studied in [19].

Siddiki [17] studied δ-Lorentzian trans-Sasakian manifolds with a semi-symmetric-metric connection and computed curvature tensors, Ricci curvature tensors and scalar curvature of the δ-Lorentzian trans-Sasakian manifold with a semi-symmetric-metric connection.

In this framework, the paper is organized in the following way. In section 2, we give basic definitions and propositions of a δ-Lorentzian trans-Sasakian manifold. We give the Frenet-Serret equations of a curve in Lorentzian 3-manifold. In section 3, we obtain an orthonormal basis $\{e_1, e_2, e_3\}$ by using the basis $(\zeta', \varphi\zeta', \xi)$ for the curve ζ in a 3-dimensional δ-Lorentzian trans-Sasakian manifold. Also we calculate the Frenet elements of a non-geodesic Frenet curve, slant curve and Legendre curve in this manifold. Then, we give the curvatures of the curve ζ on some kinds of δ-Lorentzian manifolds. In the last section, we give some examples for the spacelike curves on a 3-dimensional δ-Lorentzian trans-Sasakian manifold.

2. Materials and methods

2.1. δ-Lorentzian trans-Sasakian Manifolds

Let $\bar{N}$ be a δ-almost contact metric manifold equipped with δ-almost contact metric structure $(\varphi, \xi, \eta, \bar{g}, \delta)$ consisting of (1,1) tensor field φ, a vector field ξ, a 1-form η and an indefinite metric $\bar{g}$ such that

$$\varphi^2 = U + \eta(U)\xi, \quad \eta(\xi) = -1, \tag{2.1}$$

$$\varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \tag{2.2}$$

$$\bar{g}(\xi, \xi) = -\delta, \tag{2.3}$$

$$\eta(U) = \delta\bar{g}(U, \xi), \tag{2.4}$$

$$\bar{g}(\varphi U, \varphi V) = \bar{g}(U, V) + \delta\eta(U)\eta(V), \tag{2.5}$$

for all $U, V \in \bar{N}$, where $\delta^2 = 1$ so that $\delta = \mp 1$. The above structure $(\varphi, \xi, \eta, \bar{g}, \delta)$ is called the δ-Lorentzian structure on $\bar{N}$. If $\delta = 1$, then the manifold becomes the usual Lorentzian structure [2] on $\bar{N}$, the vector field ξ is timelike [18].

In the classification of almost Hermitian manifolds, there appears a class W_4 of Hermitian manifolds which are closely related to the conformal Kaehler manifolds [17]. The class $C_6 \oplus C_5$ coincides with the class of trans-Sasakian structue of type (α, β) [13]. In fact, the local nature of the two sub classes, namely C_6 and C_5 of trans-Sasakian structures are charactrized completely. An almost contact metric structure on $\bar{N}$ is called trans-Sasakian if $(\bar{N} \times \mathfrak{R}, J, G)$ belongs to the class W_4, where J is the almost complex structure on $\bar{N} \times \mathfrak{R}$ defined by

$$J(U, f\frac{d}{dt}) = \left(\varphi U - f\xi, \eta(U)\frac{d}{dt}\right), \tag{2.6}$$

for all vector fields U on $\bar{N}$ and smooth functions f on $\bar{N} \times \mathfrak{R}$ and G is the product metric on $\bar{N} \times \mathfrak{R}$. This may be expressed by the condition

$$(\nabla_U \varphi)V = \alpha(\bar{g}(U, V)\xi - \eta(V)U) + \beta(\bar{g}(\varphi U, V)\xi - \eta(V)\varphi U), \tag{2.7}$$

for any vector fields U and V on $\bar{N}$, ∇ denotes the Levi-Civita connection with respect to $\bar{g}$, α and β are smooth functions on $\bar{N}$ [17]. The existence of condition (2.3) is ensure by the above discussion.

With the above literature, the δ-Lorentzian trans-Sasakian manifolds are defined as follows.

Definition 2.1. *[2] A δ-Lorentzian manifold with structure $(\varphi, \xi, \eta, \bar{g}, \delta)$ is said to be δ-Lorentzian trans-Sasakian manifold of type (α, β) if it satisfies the condition*

$$(\nabla_U \varphi)V = \alpha(\bar{g}(U, V)\xi - \delta\eta(V)U) + \beta(\bar{g}(\varphi U, V)\xi - \delta\eta(V)\varphi U), \tag{2.8}$$

for any vector fields U and V on $\bar{N}$.

If $\delta = 1$, then the δ-Lorentzian trans-Sasakian manifold becomes the usual Lorentzian trans-Sasakian manifold of type (α, β) [17]. δ-Lorentzian trans-Sasakian manifold of type $(0, 0)$, $(0, \beta)$, $(\alpha, 0)$ are the Lorentzian cosymplectic, Lorentzian β-Kenmotsu and Lorentzian α-Sasakian manifolds respectively. In particular if $\alpha = 1, \beta = 0$ and $\alpha = 0, \beta = 1$, a δ-Lorentzian trans-Sasakian manifold reduces to a δ-Lorentzian Sasakian manifold and a δ-Lorentzian Kenmotsu manifold respectively.

From (2.4), we have

$$\nabla_U \xi = \delta(-\alpha\varphi(U) - \beta(U + \eta(U)\xi)), \tag{2.9}$$

and

$$(\nabla_U \eta)V = \alpha\bar{g}(\varphi U, V) + \beta[\bar{g}(U, V) + \delta\eta(U)\eta(V)]. \tag{2.10}$$

Further for a δ-Lorentzian trans-Sasakian manifold, we have

$$\delta\varphi(grad\alpha) = \delta(n - 2)(grad\beta), \tag{2.11}$$

and

$$2\alpha\beta - \delta(\xi\alpha) = 0. \tag{2.12}$$

2.2. Frenet curves

Let $\zeta : I \to \bar{N}$ be a unit speed curve in Lorentzian 3-manifold $\bar{N}$ such that $\bar{g}(\zeta', \zeta') = \varepsilon_1 = \mp 1$. The constant ε_1 is called the casual character of ζ. The constants ε_2 and ε_3 defined by $\bar{g}(n, n) = \varepsilon_2$ and $\bar{g}(b, b) = \varepsilon_3$ and called the second casual character and third casual character of ζ, respectively. Thus we have $\varepsilon_1\varepsilon_2 = -\varepsilon_3$.

A unit speed curve ζ is said to be spacelike or timelike if its casual character is 1 or -1, respectively. A unit speed curve ζ is said to be a Frenet curve if $\bar{g}(\zeta', \zeta') \neq 0$. A Frenet curve ζ admits an orthonormal frame field $\{t = \zeta', n, b\}$ along ζ. Then the Frenet-Serret equations are given as follows

$$\begin{aligned} \nabla_{\zeta'} t &= \varepsilon_2 \kappa n, \\ \nabla_{\zeta'} n &= -\varepsilon_1 \kappa t - \varepsilon_3 \tau b, \\ \nabla_{\zeta'} b &= \varepsilon_2 \tau n, \end{aligned} \tag{2.13}$$

where $\kappa = |\nabla_{\zeta'}\zeta'|$ is first curvature and τ is second curvature of ζ [11]. The vector fields t, n and b are called the tangent vector field, the principal normal vector field and the binormal vector field of ζ, respectively.

A Frenet curve ζ is a geodesic if and only if $\kappa = 0$. A Frenet curve ζ with constant first curvature and zero second curvature is called a pseudo-circle. A pseudo-helix is a Frenet curve ζ whose curvatures are constant.

A curve in a Lorentzian three-manifold is said to be slant if its tangent vector field has constant angle with the Reeb vector field, i.e., $\eta(\zeta') = -\bar{g}(\zeta', \xi) = constant$. If $\eta(\zeta') = -\bar{g}(\zeta', \xi) = 0$, then the curve ζ is called a Legendre curve [11].

3. Main results

In this section, we consider a 3-dimensional δ-Lorentzian trans-Sasakian manifold $\bar{N}$. Let $\zeta : I \to \bar{N}$ be a non-geodesic ($\kappa \neq 0$) Frenet curve given with the arc-parameter s and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{N}$. From the basis $(\zeta', \varphi\zeta', \xi)$ we obtain an orthonormal basis $\{e_1, e_2, e_3\}$ given by

$$\begin{aligned} e_1 &= \zeta', \\ e_2 &= \frac{\varphi\zeta'}{\sqrt{\varepsilon_1 + \delta\rho^2}}, \\ e_3 &= \frac{-\varepsilon_1\xi + \delta\rho\zeta'}{\sqrt{\varepsilon_1 + \delta\rho^2}}, \end{aligned} \tag{3.1}$$

where

$$\eta(\zeta') = \delta\bar{g}(\zeta', \xi) = \delta\rho. \tag{3.2}$$

Then if we write the covariant differentiation of ζ' as

$$\bar{\nabla}_{\zeta'} e_1 = \nu e_2 + \mu e_3, \tag{3.3}$$

where ν is a certain function.

$$\nu = \bar{g}(\bar{\nabla}_{\zeta'} e_1, e_2). \tag{3.4}$$

Moreover we obtain

$$\mu = \bar{g}(\bar{\nabla}_{\zeta'} e_1, e_3) = \frac{\delta\rho'}{\sqrt{\varepsilon_1 + \delta\rho^2}} - \varepsilon_1\delta\beta\sqrt{\varepsilon_1 + \delta\rho^2}, \tag{3.5}$$

where $\rho'(s) = \frac{d\rho(\zeta(s))}{ds}$. Then we find

$$\bar{\nabla}_{\zeta'} e_2 = -\nu e_1 + \delta\left(-\varepsilon_1\alpha + \frac{\rho\nu}{\sqrt{\varepsilon_1 + \delta\rho^2}}\right)e_3, \tag{3.6}$$

and

$$\bar{\nabla}_{\zeta'} e_3 = -\mu e_1 - \delta\left(-\varepsilon_1\alpha + \frac{\rho\nu}{\sqrt{\varepsilon_1 + \delta\rho^2}}\right)e_2. \tag{3.7}$$

The fundamental forms of the tangent vector ζ' on the basis of the Eq (3.1) is

$$[\omega_{ij}(\zeta')] = \begin{bmatrix} 0 & \nu & \mu \\ -\nu & 0 & \delta\left(-\varepsilon_1\alpha + \frac{\rho\nu}{\sqrt{\varepsilon_1+\delta\rho^2}}\right) \\ -\mu & -\delta\left(-\varepsilon_1\alpha + \frac{\rho\nu}{\sqrt{\varepsilon_1+\delta\rho^2}}\right) & 0 \end{bmatrix}, \tag{3.8}$$

and the Darboux vector connected to the vector ζ' is

$$\omega(\zeta') = \delta\left(-\varepsilon_1\alpha + \frac{\rho\nu}{\sqrt{\varepsilon_1 + \delta\rho^2}}\right)e_1 - \mu e_2 + \nu e_3. \tag{3.9}$$

So we can write

$$\bar{\nabla}_{\zeta'} e_i = \omega(\zeta') \wedge \varepsilon_i e_i \quad (1 \le i \le 3). \tag{3.10}$$

Furthermore, for any vector field $Z = \sum_{i=1}^{3} \theta^i e_i \in \chi(\bar{N})$ strictly dependent on the curve ζ on $\bar{N}$, there exists the following equation

$$\bar{\nabla}_{\zeta'} Z = \omega(\zeta') \wedge Z + \delta \sum_{i=1}^{3} \varepsilon_i e_i[\theta^i] e_i. \tag{3.11}$$

3.1. Frenet elements of ζ

Let $\zeta : I \to \bar{N}$ be a non-geodesic ($\kappa \neq 0$) Frenet curve given with the arc parameter s and the elements $\{t, n, b, \kappa, \tau\}$. The Frenet elements of this curve are calculated as follows:

If we consider the Eq (3.3), then we get

$$\varepsilon_2 \kappa n = \bar{\nabla}_{\zeta'} e_1 = \nu e_2 + \mu e_3. \tag{3.12}$$

From the Eqs (3.5) and (3.12) we find

$$\kappa = \sqrt{\nu^2 + \left(\frac{\delta\rho'}{\sqrt{\varepsilon_1 + \delta\rho^2}} - \varepsilon_1\delta\beta\sqrt{\varepsilon_1 + \delta\rho^2}\right)^2}. \tag{3.13}$$

On the other hand

$$\bar{\nabla}_{\zeta'} n = \left(\frac{\nu}{\varepsilon_2\kappa}\right)' e_2 + \frac{\nu}{\varepsilon_2\kappa}\nabla_{\zeta'} e_2 + \left(\frac{\mu}{\varepsilon_2\kappa}\right)' e_3 + \frac{\mu}{\varepsilon_2\kappa}\nabla_{\zeta'} e_3 \tag{3.14}$$

$$(3.15)$$

$$= -\varepsilon_1\kappa t - \varepsilon_3\tau B.$$

By means of the Eqs (3.6) and (3.7) we obtain

$$-\varepsilon_3\tau B = \left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)' - \frac{\delta\mu}{\varepsilon_2\kappa}\left(-\varepsilon_1\alpha + \frac{\rho\nu}{\sqrt{\varepsilon_1 + \delta\rho^2}}\right)\right] e_2 \tag{3.16}$$

$$+ \quad \left[\left(\frac{\mu}{\varepsilon_2 \kappa} \right)' + \frac{\delta \nu}{\varepsilon_2 \kappa} \left(-\varepsilon_1 \alpha + \frac{\rho \nu}{\sqrt{\varepsilon_1 + \delta \rho^2}} \right) \right] e_3.$$

By a direct computation we find

$$\left[\left(\frac{\nu}{\varepsilon_2 \kappa} \right)' \right]^2 + \left[\left(\frac{\mu}{\varepsilon_2 \kappa} \right)' \right]^2 = \left[\left(\frac{\nu}{\varepsilon_2 \kappa} \right)' \frac{\mu}{\varepsilon_2 \kappa} - \frac{\nu}{\varepsilon_2 \kappa} \left(\frac{\mu}{\varepsilon_2 \kappa} \right)' \right]^2. \tag{3.17}$$

Taking the norm of the last equation and if we consider the Eqs (3.5) and (3.16) on (3.15) we obtain

$$\tau = \left| \frac{\delta \left(-\varepsilon_1 \alpha + \frac{\rho \nu}{\sqrt{\varepsilon_1 + \delta \rho^2}} \right) -}{\sqrt{\left[\left(\frac{\nu}{\varepsilon_2 \kappa} \right)' \right]^2 + \left[\left(\frac{\frac{\delta \rho'}{\sqrt{\varepsilon_1 + \delta \rho^2}} - \varepsilon_1 \delta \beta \sqrt{\varepsilon_1 + \delta \rho^2}}{\varepsilon_2 \kappa} \right)' \right]^2}} \right|. \tag{3.18}$$

Moreover we can write the Frenet vector fields of ζ as in the following theorem.

Theorem 3.1. *Let $\bar{N}$ be a 3-dimensional δ-Lorentzian trans-Sasakian manifold and ζ be a Frenet curve on $\bar{N}$. The Frenet vector fields t, n and b are in the form of*

$$\begin{aligned} t &= \zeta' = e_1, \\ n &= \frac{\nu}{\varepsilon_2 \kappa} e_2 + \frac{\mu}{\varepsilon_2 \kappa} e_3, \\ b &= -\frac{1}{\varepsilon_3 \tau} \left[\left(\frac{\nu}{\varepsilon_2 \kappa} \right)' - \frac{\delta \mu}{\varepsilon_2 \kappa} \left(-\varepsilon_1 \alpha + \frac{\rho \nu}{\sqrt{\varepsilon_1 + \delta \rho^2}} \right) \right] e_2 \\ &- \frac{1}{\varepsilon_3 \tau} \left[\left(\frac{\mu}{\varepsilon_2 \kappa} \right)' + \frac{\delta \nu}{\varepsilon_2 \kappa} \left(-\varepsilon_1 \alpha + \frac{\rho \nu}{\sqrt{\varepsilon_1 + \delta \rho^2}} \right) \right] e_3. \end{aligned} \tag{3.19}$$

Moreover we have

$$\begin{aligned} \xi &= \delta \varepsilon_1 \rho \mathbf{t} + \frac{\delta \mu \sqrt{\varepsilon_1 + \delta \rho^2}}{\varepsilon_2 \kappa} \mathbf{n} \\ &- \frac{\delta \sqrt{\varepsilon_1 + \delta \rho^2}}{\varepsilon_3 \tau} \left[\left(\frac{\mu}{\varepsilon_2 \kappa} \right)' - \frac{\delta \nu}{\varepsilon_2 \kappa} \left(-\varepsilon_1 \alpha + \frac{\rho \nu}{\sqrt{\varepsilon_1 + \delta \rho^2}} \right) \right] \mathbf{b}. \end{aligned} \tag{3.20}$$

Let ζ be non-geodesic Frenet curve given with the arc-parameter s in 3-dimensional δ-Lorentzian trans-Sasakian manifold $\bar{N}$. So we can give the following theorem.

Theorem 3.2. *Let $\bar{N}$ be a 3-dimensional δ-Lorentzian trans-Sasakian manifold and ζ be a Frenet curve on $\bar{N}$. ζ is a slant curve ($\rho = \eta(\zeta') = cos\theta = constant$) on $\bar{N}$ if and only if the Frenet elements $\{t, n, b, \kappa, \tau\}$ of ζ are as follows.*

$$\begin{aligned} t &= e_1 = \zeta', \\ n &= e_2 = \frac{\varphi \zeta'}{\sqrt{\varepsilon_1 + \delta cos^2\theta}}, \end{aligned}$$

$$\begin{aligned} b &= e_3 = \frac{-\varepsilon_1\xi + \delta cos\theta\zeta'}{\sqrt{\varepsilon_1 + \delta cos^2\theta}}, \\ \kappa &= \sqrt{\beta^2(\varepsilon_1 + \delta cos^2\theta) + \nu^2}, \end{aligned} \tag{3.21}$$

$$\tau = \left| \frac{\delta\left(-\varepsilon_1\alpha + \frac{cos\theta\nu}{\sqrt{\varepsilon_1+\delta\cos^2\theta}}\right)}{-\sqrt{\left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)'\right]^2 + \left[\left(\frac{-\varepsilon_1\delta\beta\sqrt{\varepsilon_1+\delta\cos^2\theta}}{\varepsilon_2\kappa}\right)'\right]^2}} \right|.$$

Proof. Let the curve ζ be a slant curve in 3-dimensional δ-Lorentzian trans-Sasakian manifold $\bar{N}$. If we take into account the condition $\rho = \eta(\zeta') = cos\theta = constant$ in the Eqs (3.1), (3.13) and (3.17) we find (3.21). If the equations in (3.21) hold, from the definition of slant curves it is obvious that the curve ζ is a slant curve. □

If we consider the Theorem (3.1), we can give the following corollaries.

Corollary 3.1. *Let $\bar{N}$ be a 3-dimensional δ-Lorentzian trans-Sasakian manifold and ζ be a slant curve on $\bar{N}$. If the first curvature κ is non-zero constant, then ζ is a pseudo-helix with* $\tau = \left|\delta\left(-\varepsilon_1\alpha + \frac{cos\theta\nu}{\sqrt{\varepsilon_1+\delta\cos^2\theta}}\right)\right|$.

Corollary 3.2. *Let $\bar{N}$ be a 3-dimensional δ-Lorentzian trans-Sasakian manifold and ζ be a slant curve on this manifold $\bar{N}$. If κ is not constant and $\tau = 0$, then, ζ is a plane curve and the following equation satisfies*

$$\bar{g}(\nabla_{\zeta'}e_2, e_3) = \frac{\nu^2(\frac{\beta}{\nu})'\sqrt{\varepsilon_1 + \delta\cos^2\theta}}{\nu^2 + \beta^2(\varepsilon_1 + \delta\cos^2\theta)}. \tag{3.22}$$

Theorem 3.3. *Let $\bar{N}$ be a 3-dimensional δ-Lorentzian trans-Sasakian manifold and ζ be a spacelike Frenet curve on $\bar{N}$. ζ is a Legendre curve($\rho = \eta(\zeta') = 0$) in this manifold if and only if the Frenet elements $\{t, n, b, \kappa, \tau\}$ of ζ satisfy the following equations:*

$$\begin{aligned} t &= e_1 = \zeta', \\ n &= e_2 = \varphi\zeta', \\ b &= e_3 = -\xi, \\ \kappa &= \sqrt{\nu^2 + \beta^2}, \\ \tau &= \left|\delta\alpha + \sqrt{\left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)'\right]^2 + \left[\left(\frac{\beta}{\varepsilon_2\kappa}\right)'\right]^2}\right|. \end{aligned} \tag{3.23}$$

Proof. Let the curve ζ be a Legendre curve 3-dimensional δ-Lorentzian trans-Sasakian manifold $\bar{N}$. If we take into account the condition $\rho = \eta(\zeta') = 0$ in the Eqs (3.1), (3.13) and (3.17) we find (3.23). If the equations in (3.23) hold, from the definition of Legendre curves it is obvious that the curve ζ is a Legendre curve on $\bar{N}$. □

Corollary 3.3. *Let the curve ζ be a Legendre curve in 3-dimensional δ-Lorentzian trans-Sasakian manifold $\bar{N}$. If κ is non-zero constant and τ is equal to zero, then ζ is a plane curve and $\alpha = 0$.*

If we consider the Eqs (3.13) and (3.17) and theorem (3.1) we can give the following corollaries.

Corollary 3.4. *Let $\bar{N}$ be a 3-dimensional δ-Lorentzian trans-Sasakian manifold and ζ be a Frenet curve on this manifold $\bar{N}$. The first curvature of the curve ζ is not dependent on α and β.*

Corollary 3.5. *From the Eqs (3.13) and (3.17) the first curvature and the second curvature of ζ on 3-dimensional δ-Lorentzian cosymplectic manifold $\bar{N}$ are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\delta\rho'}{\sqrt{\varepsilon_1 + \delta\rho^2}}\right)^2}, \tag{3.24}$$

and

$$\tau = \left|\delta\frac{\rho\nu}{\sqrt{\varepsilon_1 + \delta\rho^2}} - \sqrt{\left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)'\right]^2 + \left[\left(\frac{\delta\rho'}{\varepsilon_2\kappa\sqrt{\varepsilon_1 + \delta\rho^2}}\right)'\right]^2}\right|. \tag{3.25}$$

i) *If the curve ζ in 3-dimensional δ-Lorentzian cosymplectic manifold $\bar{N}$ is a slant curve, then we have*

$$\kappa = \nu \quad and \quad \tau = \left|\delta\frac{\nu cos\theta}{\sqrt{\varepsilon_1 + \delta\cos^2\theta}}\right|. \tag{3.26}$$

ii) *If the curve ζ in 3-dimensional δ-Lorentzian cosymplectic manifold $\bar{N}$ is a Legendre curve, then we have*

$$\kappa = \nu \quad and \quad \tau = 0. \tag{3.27}$$

Corollary 3.6. *Let ζ be a curve on 3-dimensional δ-Lorentzian β-Kenmotsu manifold $\bar{N}$. Then, the first and second curvatures of ζ are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\delta\rho'}{\sqrt{\varepsilon_1 + \delta\rho^2}} - \varepsilon_1\delta\beta\sqrt{\varepsilon_1 + \delta\rho^2}\right)^2}, \tag{3.28}$$

and

$$\tau = \left|\begin{array}{c} \delta\frac{\rho\nu}{\sqrt{\varepsilon_1+\delta\rho^2}} \\ -\sqrt{\left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)'\right]^2 + \left[\left(\frac{\frac{\delta\rho'}{\sqrt{\varepsilon_1+\delta\rho^2}} - \varepsilon_1\delta\beta\sqrt{\varepsilon_1+\delta\rho^2}}{\varepsilon_2\kappa}\right)'\right]^2} \end{array}\right|.$$

If the curve ζ is a slant curve on $\bar{N}$, then we have

$$\kappa = \sqrt{\nu^2 + \beta^2(\varepsilon_1 + \delta\cos^2\theta)}, \tag{3.29}$$

$$\tau = \left|\begin{array}{c} \delta\frac{\nu cos\theta}{\sqrt{\varepsilon_1+\delta\cos^2\theta}} \\ -\sqrt{\left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)'\right]^2 + \left[\left(\frac{-\varepsilon_1\delta\beta\sqrt{\varepsilon_1+\delta\cos^2\theta}}{\varepsilon_2\kappa}\right)'\right]^2} \end{array}\right|.$$

If the curve ζ is a Legendre curve on $\bar{N}$, then we have

$$\kappa = \sqrt{\nu^2 + \varepsilon_1\beta^2} \quad and \quad \tau = \sqrt{\left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)'\right]^2 + \varepsilon_1\left[\left(\frac{\beta}{\kappa}\right)'\right]^2}. \tag{3.30}$$

Corollary 3.7. *Let ζ be a curve on 3-dimensional δ-Lorentzian α-Sasakian manifold $\bar{N}$. Then, the first curvature and the second curvature of ζ are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\delta\rho'}{\sqrt{\varepsilon_1 + \delta\rho^2}}\right)^2}, \tag{3.31}$$

and

$$\tau = \left|\frac{\delta\left(-\varepsilon_1\alpha + \frac{\rho\nu}{\sqrt{\varepsilon_1+\delta\rho^2}}\right)}{-\sqrt{\left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)'\right]^2 + \left[\left(\frac{\delta\rho'}{\varepsilon_2\kappa\sqrt{\varepsilon_1+\delta\rho^2}}\right)'\right]^2}}\right|.$$

The curvatures of ζ are

$$\kappa = \nu \quad and \quad \tau = \left|\delta\left(-\varepsilon_1\alpha + \frac{\nu cos\theta}{\sqrt{\varepsilon_1 + \delta\cos^2\theta}}\right)\right|, \tag{3.32}$$

where ζ is a slant curve in 3-dimensional δ-Lorentzian α-Sasakian manifold $\bar{N}$ and

$$\kappa = \nu \quad and \quad \tau = |\varepsilon_1\delta\alpha|, \tag{3.33}$$

where ζ is a Legendre curve in 3-dimensional δ-Lorentzian α-Sasakian manifold $\bar{N}$.

Corollary 3.8. *From the Eqs (3.13) and (3.17) the first curvature and the second curvature of ζ on 3-dimensional δ-Lorentzian Kenmotsu manifold $\bar{N}$ are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\delta\rho'}{\sqrt{\varepsilon_1 + \delta\rho^2}} - \varepsilon_1\delta\sqrt{\varepsilon_1 + \delta\rho^2}\right)^2}, \tag{3.34}$$

and

$$\tau = \left|\frac{\delta\frac{\rho\nu}{\sqrt{\varepsilon_1+\delta\rho^2}}}{-\sqrt{\left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)'\right]^2 + \left[\left(\frac{\frac{\delta\rho'}{\sqrt{\varepsilon_1+\delta\rho^2}} - \varepsilon_1\delta\sqrt{\varepsilon_1+\delta\rho^2}}{\varepsilon_2\kappa}\right)'\right]^2}}\right|.$$

i) *If the curve ζ in 3-dimensional δ-Lorentzian Kenmotsu manifold $\bar{N}$ is a slant curve, then we obtain*

$$\kappa = \sqrt{\nu^2 + \varepsilon_1 + \delta\cos^2\theta}, \tag{3.35}$$

$$\tau = \left| \begin{array}{c} \delta \frac{v cos\theta}{\sqrt{\varepsilon_1 + \delta \cos^2\theta}} \\ - \sqrt{\left[\left(\frac{\nu}{\varepsilon_2 \kappa}\right)'\right]^2 + \left[\left(\frac{-\varepsilon_1 \delta \sqrt{\varepsilon_1 + \delta \cos^2\theta}}{\varepsilon_2 \kappa}\right)'\right]^2} \end{array} \right|.$$

ii) *If the curve ζ in 3-dimensional δ-Lorentzian Kenmotsu manifold $\bar{N}$ is a Legendre curve, then we have*

$$\kappa = \sqrt{\nu^2 + \varepsilon_1} \quad and \quad \tau = \sqrt{\left[\left(\frac{\nu}{\varepsilon_2 \kappa}\right)'\right]^2 + \varepsilon_1 \left[\left(\frac{\kappa'}{\kappa}\right)'\right]^2}. \tag{3.36}$$

Corollary 3.9. *Let ζ be a curve on 3-dimensional δ-Lorentzian Sasakian manifold $\bar{N}$. Then, the first and second curvatures of ζ are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\delta \rho'}{\sqrt{\varepsilon_1 + \delta \rho^2}}\right)^2}, \tag{3.37}$$

and

$$\tau = \left| \begin{array}{c} \delta \left(-\varepsilon_1 + \frac{\rho \nu}{\sqrt{\varepsilon_1 + \delta \rho^2}}\right) \\ - \sqrt{\left[\left(\frac{\nu}{\varepsilon_2 \kappa}\right)'\right]^2 + \left[\left(\frac{\delta \rho'}{\varepsilon_2 \kappa \sqrt{\varepsilon_1 + \delta \rho^2}}\right)'\right]^2} \end{array} \right|.$$

If the curve ζ is a slant curve on $\bar{N}$, then we have

$$\kappa = \nu \quad and \quad \tau = \left| \delta \left(-\varepsilon_1 + \frac{v cos\theta}{\sqrt{\varepsilon_1 + \delta \cos^2\theta}}\right) \right|. \tag{3.38}$$

If the curve ζ is a Legendre curve on $\bar{N}$, then we obtain

$$\kappa = \nu \quad and \quad \tau = |\varepsilon_1 \delta|. \tag{3.39}$$

4. Examples

Let $\bar{N}$ be a 3-dimensional manifold given

$$\bar{N} = \left\{(x, y, z) \in \Re^3, z \neq 0\right\}, \tag{4.1}$$

where (x,y,z) denote the standart co-ordinates in $\Re^3$. Then

$$E_1 = z\frac{\partial}{\partial x}, \quad E_2 = z\frac{\partial}{\partial y}, \quad E_3 = -z\frac{\partial}{\partial z}, \tag{4.2}$$

are linearly independent of each point of $\bar{N}$ [17]. Let $\bar{g}$ be the Lorentzian metric tensor defined by

$$\begin{aligned} \bar{g}(E_1, E_1) = \bar{g}(E_2, E_2) = \bar{g}(E_3, E_3) = \delta, \\ \bar{g}(E_i, E_j) = 0, \quad i \neq j, \end{aligned} \tag{4.3}$$

for $i, j = 1, 2, 3$ and $\delta = \mp 1$. Let η be a 1-form defined by $\eta(Z) = \delta \bar{g}(Z, E_3)$ for any vector field $Z \in \Gamma(T\bar{N})$. Let φ be the (1,1)-tensor field defined by

$$\varphi E_1 = -E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = 0. \tag{4.4}$$

Then using the condition of the linearity of φ and $\bar{g}$, we obtain $\eta(E_3) = 1$ and

$$\begin{aligned} \varphi^2 Z &= Z + \eta(Z)E_3, \\ \bar{g}(\varphi Z, \varphi W) &= \bar{g}(Z, W) - \delta\eta(Z)\eta(W), \end{aligned} \tag{4.5}$$

for all $Z, W \in \Gamma(T\bar{N})$.

Now, let ∇ be the Levi-Civita connection with respect to the Lorentzian metric $\bar{g}$. Then we obtain

$$[E_1, E_2] = 0, \quad [E_1, E_3] = \delta E_1, \quad [E_2, E_3] = \delta E_2. \tag{4.6}$$

The Riemannian connection ∇ with respect to the metric $\bar{g}$ is given by

$$\begin{aligned} 2\bar{g}(\nabla_X Y, Z) &= X\bar{g}(Y, Z) + Y\bar{g}(Z, X) - Z\bar{g}(X, Y) \\ &+ \bar{g}([X, Y], Z) - \bar{g}([Y, Z], X) + \bar{g}([Z, X], Y). \end{aligned} \tag{4.7}$$

If we use this equation which is known as Koszul's formula for the Lorentzian metric tensor $\bar{g}$, we can easily calculate the covariant derivations as follows:

$$\begin{aligned} \nabla_{E_1} E_3 = \delta E_1, \quad \nabla_{E_2} E_3 = \delta E_2, \quad \nabla_{E_3} E_3 = 0, \\ \nabla_{E_1} E_2 = 0, \quad \nabla_{E_2} E_2 = -\delta E_3, \quad \nabla_{E_3} E_2 = 0, \\ \nabla_{E_1} E_1 = -\delta E_3, \quad \nabla_{E_2} E_1 = 0 \quad \nabla_{E_3} E_1 = 0. \end{aligned} \tag{4.8}$$

From the above relations, for any vector field X on $\bar{N}$, we have

$$\nabla_X \xi = \delta(X + \eta(X)\xi), \tag{4.9}$$

for $\xi = E_3$, $\alpha = 0$ and $\beta = 1$. Hence the manifold $\bar{N}$ under consideration is a δ-Lorentzian trans-Sasakian of type $(0, 1)$ manifold of dimension three.

Example 4.1. *Let γ be a spacelike curve defined as*

$$\begin{aligned} \gamma : \ & I \to \bar{N} \\ & s \to \gamma(s) = (2lns, 2, lns), \end{aligned}$$

where the curve γ parametrized by the arc length parameter t. If we differentiate $\gamma(t)$ and consider (3.1) we find

$$\frac{d\gamma}{dt} = \gamma'(t) = \frac{2}{\sqrt{3}}E_1 - \frac{1}{\sqrt{3}}E_3,$$

and

$$e_1 = \gamma'(t),$$

$$e_2 = E_1,$$
$$e_3 = -\frac{1}{\sqrt{3}}E_1 + \frac{1}{\sqrt{3}}E_3,$$

where $\rho = \eta(\gamma'(t))$. If we consider the Eqs (3.2), (3.3), (3.5), (3.13) and (3.17) we can write

$$\rho = \delta\frac{1}{\sqrt{3}}, \quad \mu = -\frac{2}{\sqrt{3}}, \quad \nu = \delta\frac{2}{3}, \qquad (4.10)$$
$$\kappa = \frac{4}{3}, \quad \tau = \frac{1}{3}.$$

Thus, the curve γ is a spacelike helix in $\bar{N}$.

Example 4.2. *Let ω be a spacelike Legendre curve defined as*

$$\begin{aligned} \omega : \ I &\to \bar{N} \\ s &\to \omega(s) = \left(\frac{s^2}{2}, \frac{s^2}{2}, 1\right). \end{aligned}$$

where the curve ω parametrized by the arc length parameter t. If we differentiate $\omega(t)$ and using (3.1) we find

$$\frac{d\omega}{dt} = \omega'(t) = \frac{\sqrt{2}}{2}E_1 + \frac{\sqrt{2}}{2}E_2,$$

and

$$e_1 = \omega'(t),$$
$$e_2 = \frac{\sqrt{2}}{2}E_1 - \frac{\sqrt{2}}{2}E_2,$$
$$e_3 = -E_3.$$

If we consider the Eqs (3.2), (3.3), (3.5), (3.13) and (3.17) we obtain

$$\rho = 0, \quad \mu = -\delta\beta, \quad \nu = 0, \qquad (4.11)$$
$$\kappa = 2|\beta|, \quad \tau = |\alpha|.$$

Conflict of interest

The author declares that there is no competing interest.

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