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Research article

Existence theorems for Ψ-fractional hybrid systems with periodic boundary conditions

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Abstract: This research paper deals with two novel varieties of boundary value problems for nonlinear hybrid fractional differential equations involving generalized fractional derivatives known as the Ψ-Caputo fractional operators. Such operators are generated by iterating a local integral of a function with respect to another increasing positive function Ψ. The existence results to the proposed systems are obtained by using Dhage's fixed point theorem. Two pertinent examples are provided to confirm the feasibility of the obtained results. Our presented results generate many special cases with respect to different values of a Ψ function.

Keywords: hybrid fractional differential equations; existence; fixed point theorem
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1. Introduction

Fractional calculus [1, 2] is a field of mathematics that deals with integrals and derivatives of fractional orders. The most used fractional operators are the Riemann-Liouville and Caputo types. There are other types of fractional derivatives as well, we allude to [3–10] and references therein. More recently, Almeida [11] and Sousa et al. [12] have provided fractional operarors to generalize

Caputo and Hilfer types respectively with respect to another function, which have become known as ψ-Caputo and ψ-Hilfer, also Jarad and Abdeljawad in [13] have introduced interesting properties of this generalized operator in the frame of a ψ function including the generalized Laplace transform.

Due to the rapid and intense growth in fractional calculus and its applications, fractional differential equations (FDEs) have been of extraordinary interest, so, several authors have applied some generalized fractional operators to investigate the qualitative analysis of FDEs, see [14–23].

Hybrid differential equations include the fractional derivatives of an unknown function hybrid with the nonlinearity relying upon it. This class of equations emerges from a wide range of spaces of applied and physical sciences, e.g., in the redirection of a bent pillar having a consistent or changing cross-area, a three-layer shaft, electromagnetic waves, or gravity-driven streams, etc. Hybrid FDEs have been investigated using various types of fractional derivatives in literature (see, e.g. [24–39]. For instance, Dhage and Lakshmikanathm [25] investigated the existence and uniqueness results for a hybrid differential equation:

$$\begin{cases} \frac{d}{d\vartheta}\left(\frac{\upsilon(\vartheta)}{\mathsf{g}_1(\vartheta,\upsilon(\vartheta))}\right) = \mathsf{g}_2(\vartheta,\upsilon(\vartheta)), \varrho \in [0,1], \\ \upsilon(\vartheta_0) = \upsilon_0, \end{cases}$$

where $\mathsf{g}_1 : [0,1]\times\mathbb{R} \to \mathbb{R} - \{0\}$ and $\mathsf{g}_2 : [0,1]\times\mathbb{R} \to \mathbb{R}$ are continuous.

Zhao et al. [24] studied the existence and uniqueness results for a hybrid FDE in the frame of Riemann-Liouville operators:

$$\begin{cases} {}^{RL}\mathcal{D}^{\varrho}_{0^+}\left(\frac{\upsilon(\vartheta)}{\mathsf{g}_1(\vartheta,\upsilon(\vartheta))}\right) = \mathsf{g}_2(\vartheta,\upsilon(\vartheta)), \varrho \in [0,T], \\ \upsilon(0) = 0, \end{cases}$$

where $\mathsf{g}_1 : [0,T]\times\mathbb{R} \to \mathbb{R} - \{0\}$ and $\mathsf{g}_2 : [0,T]\times\mathbb{R} \to \mathbb{R}$ are continuous.

Motivated by the above investigations, we discuss two nonlinear fractional differential hybrid systems subjected to periodic boundary conditions. The first fractional nonlinear system is given by

$$\begin{cases} {}^{C}\mathcal{D}^{\varrho,\Psi}_{a^+}(\upsilon(\vartheta)\mathsf{g}_1(\vartheta,\upsilon(\vartheta))) = \mathsf{g}_2(\vartheta,\upsilon(\vartheta)),\ \varrho \in (0,1) \\ \upsilon(a) = \upsilon(b), \end{cases} \tag{1.1}$$

and the second system has the following form

$$\begin{cases} {}^{C}\mathcal{D}^{\varrho,\Psi}_{a^+}(\upsilon(\vartheta)\mathsf{g}_1(\vartheta,\upsilon(\vartheta))) = \mathsf{g}_2(\vartheta,\upsilon(\vartheta)),\ \varrho \in (1,2), \\ \upsilon(a) = \upsilon(b),\ \upsilon'(a) = \upsilon'(b), \end{cases} \tag{1.2}$$

where $\vartheta \in \mho := [a,b]$, ${}^{C}\mathcal{D}^{\varrho,\Psi}_{a^+}$ is the Ψ-Caputo fractional derivative, $\mathsf{g}_1 : \mho\times\mathbb{R} \to \mathbb{R} - \{0\}$ and $\mathsf{g}_2 : \mho\times\mathbb{R} \to \mathbb{R}$ are continuous with g_1 and g_2 are identically zero at the origin and $\mathsf{g}_2(\vartheta,0) = 0$.

In this respect, we study the existence of solutions to two types of hybrid FDEs involving generalized Caputo fractional derivatives rather than the classical Caputo one. The hybrid problems have been discussed in the literature under classical FDEs, while we investigate the generalized FDEs under similar boundary conditions which is the novel contribution of this research paper. Furthermore, the special cases generated from various values of the positive increasing function Ψ are covered in our examination.

The paper is coordinated as follows. In Section 2, we present a few documentations, definitions and lemmas. In Section 3, we demonstrate existence results for problems (1.1) and (1.2) by utilizing Dhage's fixed point theorem. In Section 4, we illustrate the acquired outcomes by examples. At last, we close our paper with a conclusion.

2. Preliminaries

Let $\Psi \in C^1(\mho, \mathbb{R})$ be an increasing differentiable function such that $\Psi'(\vartheta) \neq 0$ for all $\vartheta \in \mho$. Now, we start by defining Ψ-fractional integral and derivative:

Definition 2.1. *[1] The Ψ–Riemann-Liouville fractional integral of order $\varrho > 0$ for an integrable function $\upsilon : \mho \longrightarrow \mathbb{R}$ is given by*

$$\mathcal{I}_{a^+}^{\varrho;\Psi}\upsilon(\vartheta) = \frac{1}{\Gamma(\varrho)}\int_a^{\vartheta} \Psi'(\varsigma)(\Psi(\vartheta) - \Psi(\varsigma))^{\varrho-1}\upsilon(\varsigma)d\varsigma.$$

One can deduce that

$$D_\vartheta \left(\mathcal{I}_{a^+}^{\varrho;\Psi}\upsilon(\vartheta)\right) = \Psi'(\vartheta)\mathcal{I}_{a^+}^{\varrho-1;\Psi}\upsilon(\vartheta), \ \varrho > 1,$$

where $D_\vartheta = \frac{d}{dt}$.

Definition 2.2. *[11] For $n-1 < \varrho < n$ $(n \in \mathbb{N})$ and $\upsilon, \Psi \in C^n(\mho, \mathbb{R})$, the Ψ-Caputo fractional derivative of a function υ of order ϱ is given by*

$${}^C\mathcal{D}_{a^+}^{\varrho;\Psi}\upsilon(\vartheta) = \mathcal{I}_{a^+}^{n-\varrho;\Psi}\left(\frac{D_\vartheta}{\Psi'(\vartheta)}\right)^n \upsilon(\vartheta),$$

where $n = [\varrho] + 1$ for $\varrho \notin \mathbb{N}$, $n = \varrho$ for $\varrho \in \mathbb{N}$.

From Definition 2.2, we can express Ψ-Caputo fractional derivative by formula

$${}^C\mathcal{D}_{a^+}^{\varrho;\Psi}\upsilon(\vartheta) = \begin{cases} \int_a^{\vartheta} \frac{\Psi'(\varsigma)(\Psi(\vartheta)-\Psi(\varsigma))^{n-\varrho-1}}{\Gamma(n-\varrho)}\left(\frac{D_\vartheta}{\Psi'(\varsigma)}\right)^n \upsilon(\varsigma)d\varsigma, & \text{if } \varrho \notin \mathbb{N}, \\ \left(\frac{D_\vartheta}{\Psi'(\vartheta)}\right)^n \upsilon(\vartheta) & \text{if } \varrho \in \mathbb{N}. \end{cases}$$

Lemma 2.3. *[1] For $\varrho_1, \varrho_2 > 0$, and $\upsilon \in C(\mho, \mathbb{R})$, we have*

$$\mathcal{I}_{a^+}^{\varrho_1;\Psi}\mathcal{I}_{a^+}^{\varrho_2;\Psi}\upsilon(\vartheta) = \mathcal{I}_{a^+}^{\varrho_1+\varrho_2;\Psi}\upsilon(\vartheta), a.e. \vartheta \in \mho.$$

Lemma 2.4. *[11] Let $\varrho > 0$. If $\upsilon \in C(\mho, \mathbb{R})$, then*

$${}^C\mathcal{D}_{a^+}^{\varrho;\Psi}\mathcal{I}_{a^+}^{\varrho;\Psi}\upsilon(\vartheta) = \upsilon(\vartheta), \vartheta \in \mho,$$

and if $\upsilon \in C^{n-1}(\mho, \mathbb{R})$, then

$$\mathcal{I}_{a^+}^{\varrho;\Psi}\ {}^C\mathcal{D}_{a^+}^{\varrho;\Psi}\upsilon(\vartheta) = \upsilon(\vartheta) - \sum_{k=0}^{n-1}\frac{\left(\frac{D_\vartheta}{\Psi'(\vartheta)}\right)^k \upsilon(a)}{k!}\left[\Psi(\vartheta) - \Psi(a)\right]^k, \quad \vartheta \in \mho.$$

Lemma 2.5. *[1, 11] For $\vartheta > a$, $\varrho \geq 0, \beta > 0$. If $\varkappa_\beta(\vartheta) = (\Psi(\vartheta) - \Psi(a))^{\beta-1}$, then*

- $\mathcal{I}_{a^+}^{\varrho;\Psi}\varkappa_\beta(\vartheta) = \frac{\Gamma(\beta)}{\Gamma(\beta+\varrho)}\varkappa_{\beta+\varrho}(\vartheta)$;
- ${}^C\mathcal{D}_{a^+}^{\varrho;\Psi}\varkappa_\beta(\vartheta) = \frac{\Gamma(\beta)}{\Gamma(\beta-\varrho)}\varkappa_{\beta-\varrho}(\vartheta)$;
- ${}^C\mathcal{D}_{a^+}^{\varrho;\Psi}\varkappa_{k+1}(\vartheta) = 0$, *for all $k \in \{0, \ldots, n-1\}, n \in \mathbb{N}$.*

Now, we mention the key outcomes to the forthcoming analysis.

Theorem 2.6. *[40, 41] Let $\mathbb{X}$ be a closed convex and bounded subset of the Banach algebra $\aleph$ and let $\mathcal{A} : \aleph \to \aleph$ and $\mathcal{B} : \mathbb{X} \to \aleph$ be two operators such that*
(a) $\mathcal{A}$ is Lipschitzian with Lipschitz constant $L_{\mathcal{A}}$,
(b) $\mathcal{B}$ is compact and continuous,
(c) $\upsilon = \mathcal{A}\upsilon\mathcal{B}\upsilon^{\star} \Rightarrow \upsilon \in \mathbb{X}$ for all $\upsilon^{\star} \in \mathbb{X}$, and
(d) $L_{\mathcal{A}}M_{\mathcal{B}} < 1$, where $M_{\mathcal{B}} = \|\mathcal{B}(\mathbb{X})\| = \sup\{\|\mathcal{B}\upsilon\| : \upsilon \in \mathbb{X}\}$
Then the operator equation $\upsilon = \mathcal{A}\upsilon\mathcal{B}\upsilon$ has a solution in $\mathbb{X}$.

2.1. Solutions representation

Lemma 2.7. *A function υ is a solution of the fractional integral equation*

$$\upsilon(\vartheta) = \frac{1}{\mathsf{h}_1(\vartheta)}\left(\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{h}_2(\vartheta) + \frac{\mathsf{h}_1(a)}{\mathsf{h}_1(b) - \mathsf{h}_1(a)}\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{h}_2(b)\right) \tag{2.1}$$

if and only if υ is a solution of the periodic hybrid system

$$\begin{cases} {}^{C}\mathcal{D}_{a^+}^{\varrho,\Psi}(\upsilon(\vartheta)\mathsf{h}_1(\vartheta)) = \mathsf{h}_2(\vartheta), \varrho \in (0,1), \\ \upsilon(a) = \upsilon(b), \end{cases} \tag{2.2}$$

Proof. Applying the operator $\mathcal{I}_{a^+}^{\varrho,\Psi}$ on both sides for the first equation of (2.2) and using Lemma 2.4, we have

$$\upsilon(\vartheta)\mathsf{h}_1(\vartheta) = \mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{h}_2(\vartheta) + c_0. \tag{2.3}$$

Then, at $\vartheta = a$ and $\vartheta = b$, we get

$$\begin{aligned} \upsilon(a)\mathsf{h}_1(a) &= c_0, \\ \upsilon(b)\mathsf{h}_1(b) &= \mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{h}_2(b) + c_0. \end{aligned}$$

The periodic condition ($\upsilon(a) = \upsilon(b)$) implies that

$$\frac{c_0}{\mathsf{h}_1(a)} = \frac{\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{h}_2(b)}{\mathsf{h}_1(b)} + \frac{c_0}{\mathsf{h}_1(b)}.$$

Hence

$$c_0 = \mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{h}_2(b)\left(\frac{\mathsf{h}_1(a)}{\mathsf{h}_1(b) - \mathsf{h}_1(a)}\right).$$

Substituting the value of c_0 into (2.3), we get the solution (2.1).

Conversely, it is clear that if υ satisfies Eq (2.1), then system (2.2) is satisfied by υ, due to Lemma 2.4 and Lemma 2.5. The proof is completed. □

Lemma 2.8. *A function υ is a solution of the fractional integral equation*

$$\upsilon(\vartheta) = \frac{1}{\mathsf{h}_1(\vartheta)}\left(\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{h}_2(\vartheta) - \mu(\vartheta)\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{h}_2(b) + \nu(\vartheta)\mathcal{I}_{a^+}^{\varrho-1,\Psi}\mathsf{h}_2(b)\right) \tag{2.4}$$

if and only if υ is a solution of the periodic hybrid system

$$\begin{cases} {}^{C}\mathcal{D}_{a^+}^{\varrho,\Psi}(\upsilon(\vartheta)\mathsf{h}_1(\vartheta)) = \mathsf{h}_2(\vartheta), \varrho \in (1,2), \\ \upsilon(a) = \upsilon(b), \upsilon'(a) = \upsilon'(b), \end{cases} \tag{2.5}$$

where

$$\mu(\vartheta) := \mathsf{h}_1^{-1}(\vartheta)\frac{\left(\mathsf{h}_1(a)\left(\mathsf{h}_1(a)\Psi'(b) - \mathsf{h}_1(b)\Psi'(a)\right) - \left(\mathsf{h}_1(a)\mathsf{h}_1'(b) - \mathsf{h}_1(b)\mathsf{h}_1'(a)\right)(\Psi(\vartheta) - \Psi(a))\right)}{(\mathsf{h}_1(b) - \mathsf{h}_1(a))(\mathsf{h}_1(b)\Psi'(a) - \mathsf{h}_1(a)\Psi'(b)) + \left(\mathsf{h}_1(a)\mathsf{h}_1'(b) - \mathsf{h}_1'(a)\mathsf{h}_1(b)\right)(\Psi(b) - \Psi(a))},$$

and

$$\nu(\vartheta) := \mathsf{h}_1^{-1}(\vartheta)\frac{\mathsf{h}_1(a)\Psi'(b)\left((\Psi(b) - \Psi(\vartheta))\mathsf{h}_1(a) + (\Psi(\vartheta) - \Psi(a))\mathsf{h}_1(b)\right)}{(\mathsf{h}_1(b) - \mathsf{h}_1(a))(\mathsf{h}_1(b)\Psi'(a) - \mathsf{h}_1(a)\Psi'(b)) + \left(\mathsf{h}_1(a)\mathsf{h}_1'(b) - \mathsf{h}_1'(a)\mathsf{h}_1(b)\right)(\Psi(b) - \Psi(a))}.$$

Proof. Applying the operator $\mathcal{I}_{a^+}^{\varrho,\Psi}$ on both sides for the first equation of (2.5) and using Lemma 2.4, we have

$$\upsilon(\vartheta)\mathsf{h}_1(\vartheta) = \mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{h}_2(\vartheta) + c_0 + c_1\left[\Psi(\vartheta) - \Psi(a)\right]. \tag{2.6}$$

Differentiating Eq (2.6) with respect to ϑ and using Leibniz rule yields that

$$\upsilon'(\vartheta)\mathsf{h}_1(\vartheta) + \upsilon(\vartheta)\mathsf{h}_1'(\vartheta) = \Psi'(\vartheta)\mathcal{I}_{a^+}^{\varrho-1,\Psi}\mathsf{h}_2(\vartheta) + c_1\Psi'(\vartheta).$$

Then, at $\vartheta = a$ and $\vartheta = b$, *we get*

$$\begin{aligned} \upsilon(a)\mathsf{h}_1(a) &= c_0, \\ \upsilon(b)\mathsf{h}_1(b) &= \mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{h}_2(b) + c_0 + c_1\left[\Psi(b) - \Psi(a)\right], \end{aligned}$$

and

$$\begin{aligned} \upsilon'(a)\mathsf{h}_1(a) + \upsilon(a)\mathsf{h}_1'(a) &= c_1\Psi'(a) \\ \upsilon'(b)\mathsf{h}_1(b) + \upsilon(b)\mathsf{h}_1'(b) &= \Psi'(b)\mathcal{I}_{a^+}^{\varrho-1,\Psi}\mathsf{h}_2(b) + c_1\Psi'(b). \end{aligned}$$

The boundary conditions imply that

$$c_0 = \frac{\mathsf{h}_1(a)}{\mathsf{h}_1(b)\mathsf{h}_1(a) - \mathsf{h}_1(b)}\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{h}_2(b) + c_1\frac{\mathsf{h}_1(a)}{\mathsf{h}_1(b)\mathsf{h}_1(a) - \mathsf{h}_1(b)}\left[\Psi(b) - \Psi(a)\right], \tag{2.7}$$

and

$$c_1 = \frac{\mathsf{h}_1(a)\Psi'(b)}{\mathsf{h}_1(a)\Psi'(b) - \mathsf{h}_1(b)\Psi'(a)}\mathcal{I}_{a^+}^{\varrho-1,\Psi}\mathsf{h}_2(b) + \frac{c_0}{\mathsf{h}_1(a)}\left(\frac{\mathsf{h}_1(a)\mathsf{h}_1'(b) - \mathsf{h}_1(b)\mathsf{h}_1'(a)}{\mathsf{h}_1(a)\Psi'(b) - \mathsf{h}_1(b)\Psi'(a)}\right). \tag{2.8}$$

Solving (2.7) and (2.8) in terms of c_0 and c_1, we obtain

$$c_0 = \frac{\mathsf{h}_1(a)\left(\mathsf{h}_1(a)\Psi'(b) - \mathsf{h}_1(b)\Psi'(a)\right)\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{h}_2(b)}{(\mathsf{h}_1(b) - \mathsf{h}_1(a))(\mathsf{h}_1(b)\Psi'(a) - \mathsf{h}_1(a)\Psi'(b)) + \left(\mathsf{h}_1(a)\mathsf{h}_1'(b) - \mathsf{h}_1'(a)\mathsf{h}_1(b)\right)(\Psi(b) - \Psi(a))}$$

$$-\frac{(\mathsf{h}_1(a))^2\,\Psi'(b)\,[\Psi(b)-\Psi(a)]\,\mathcal{I}_{a^+}^{\varrho-1,\Psi}\mathsf{h}_2(b)}{(\mathsf{h}_1(b)-\mathsf{h}_1(a))\,(\mathsf{h}_1(b)\Psi'(a)-\mathsf{h}_1(a)\Psi'(b))+\left(\mathsf{h}_1(a)\mathsf{h}_1'(b)-\mathsf{h}_1'(a)\mathsf{h}_1(b)\right)(\Psi(b)-\Psi(a))},$$

and

$$\begin{aligned} c_1 &= \frac{\mathsf{h}_1(a)\Psi'(b)\,(\mathsf{h}_1(a)-\mathsf{h}_1(b))\,\mathcal{I}_{a^+}^{\varrho-1,\Psi}\mathsf{h}_2(b)}{(\mathsf{h}_1(b)-\mathsf{h}_1(a))\,(\mathsf{h}_1(b)\Psi'(a)-\mathsf{h}_1(a)\Psi'(b))+\left(\mathsf{h}_1(a)\mathsf{h}_1'(b)-\mathsf{h}_1'(a)\mathsf{h}_1(b)\right)(\Psi(b)-\Psi(a))} \\ &+\frac{\left(\mathsf{h}_1(a)\mathsf{h}_1'(b)-\mathsf{h}_1(b)\mathsf{h}_1'(a)\right)\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{h}_2(b)}{(\mathsf{h}_1(b)-\mathsf{h}_1(a))\,(\mathsf{h}_1(b)\Psi'(a)-\mathsf{h}_1(a)\Psi'(b))+\left(\mathsf{h}_1(a)\mathsf{h}_1'(b)-\mathsf{h}_1'(a)\mathsf{h}_1(b)\right)(\Psi(b)-\Psi(a))}. \end{aligned}$$

Substituting the values of c_0 and c_1 into (2.6), we get the solution (2.4).

The converse of the lemma follows by direct computation along with Lemmas 2.4 and 2.5.

This finishes the proof. □

2.2. Existence result of (1.1)

In order to achieve our main results, we list the following hypotheses:

(H1) $\mathsf{g}_2 : \mho \times \mathbb{R} \to \mathbb{R}$ and $\mathsf{g}_1 : \mho \times \mathbb{R} \to \mathbb{R} - \{0\}$ are continuous.

(H2) $\mathsf{g}_1^{-1} : \mho \times \mathbb{R} \to \mathbb{R}$ is continuous and

i) There exists a positive function ω with bounds $\|\omega\|$, such that

$$\left|\mathsf{g}_1^{-1}(\vartheta,\upsilon_1)-\mathsf{g}_1^{-1}(\vartheta,\upsilon_2)\right| \le \omega(\vartheta)\,|\upsilon_1-\upsilon_2|\,, \tag{2.9}$$

for each $(\vartheta,\upsilon_1),(\vartheta,\upsilon_2) \in \mho \times \mathbb{R}$;

ii) The mapping $\upsilon \to \mathsf{g}_1^{-1}(\vartheta,\upsilon)$ is increasing in $\mathbb{R}$ a.e. for each $\vartheta \in \mho$.

(H3) There exists constant $\mathrm{M}_{\mathsf{g}_2}$ such that

$$|\mathsf{g}_2(\vartheta,\upsilon)| \le \mathrm{M}_{\mathsf{g}_2} \text{for each } (\vartheta,\upsilon) \in \mho \times \mathbb{R}.$$

To simplify, we will use the following notations

$$\mathrm{M}_{\mathsf{g}_1} := \left|\frac{\mathsf{g}_1(a,\upsilon(a))}{\mathsf{g}_1(b,\upsilon(b))-\mathsf{g}_1(a,\upsilon(a))}\right|,$$

$$M := \left(1+\mathrm{M}_{\mathsf{g}_1}\right)\frac{(\Psi(b)-\Psi(a))^{\varrho}}{\Gamma(\varrho+1)}\mathrm{M}_{\mathsf{g}_2}, \tag{2.10}$$

$$\mathsf{s}_\Psi^{\varrho}(\vartheta,\varsigma) = \frac{\Psi'(\varsigma)(\Psi(t)-\Psi(\varsigma))^{\varrho-1}}{\Gamma(\varrho)}.$$

Theorem 2.9. *Suppose (H1)–(H3) hold. If*

$$M\,\|\omega\| < 1, \tag{2.11}$$

then hybrid problem (1.1) has a solution on $\mho$.

Proof. Define the set $\mathbb{X} = \{\upsilon \in C : \|\upsilon\| \leq R\}$. Clearly, $\mathbb{X}$ is a convex, closed, bounded subset of C. Choose

$$R \geq \frac{M\mathsf{g}_{10}}{1 - M\,\|\omega\|}, \tag{2.12}$$

where $\mathsf{g}_{10} = \sup_{\vartheta\in\mho} |\mathsf{g}_1^{-1}(\vartheta, 0)|$. From Lemma 2.7, the nonlinear hybrid problem (1.1) is equivalent to the nonlinear fractional integral equation

$$\upsilon(\vartheta) = \mathsf{g}_1^{-1}(\vartheta, \upsilon(\vartheta))\left(\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{g}_2(\vartheta, \upsilon(\vartheta)) + \frac{\mathsf{g}_1(a, \upsilon(a))}{\mathsf{g}_1(b, \upsilon(b)) - \mathsf{g}_1(a, \upsilon(a))}\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{g}_2(b, \upsilon(b))\right). \tag{2.13}$$

Define two operators $\mathcal{A} : C \to C$ and $\mathcal{B} : \mathbb{X} \to C$ by

$$\mathcal{A}\upsilon(\vartheta) = \mathsf{g}_1^{-1}(\vartheta, \upsilon(\vartheta)),\ \vartheta \in \mho,\ \text{and}$$

$$\mathcal{B}\upsilon(\vartheta) = \mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{g}_2(\vartheta, \upsilon(\vartheta)) + \frac{\mathsf{g}_1(a, \upsilon(a))}{\mathsf{g}_1(b, \upsilon(b)) - \mathsf{g}_1(a, \upsilon(a))}\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{g}_2(b, \upsilon(b)),\ \vartheta \in \mho.$$

Then, (2.13) can be express in the operator form as

$$\upsilon(\vartheta) = \mathcal{A}\upsilon(\vartheta)\mathcal{B}\upsilon(\vartheta),\ \vartheta \in \mho.$$

To achieve Theorem 2.6, we will summarize the proof in the following steps:
Step1: $\mathcal{A}$ is Lipschitzian on C.
Let $\upsilon, \upsilon^\star \in C$. Then by (H2), for $\vartheta \in \mho$

$$\begin{aligned} |\mathcal{A}\upsilon(\vartheta) - \mathcal{A}\upsilon^\star(\vartheta)| &= |\mathsf{g}_1^{-1}(\vartheta, \upsilon(\vartheta)) - \mathsf{g}_1^{-1}(\vartheta, \upsilon^\star(\vartheta))| \\ &\leq \omega(\vartheta)|\upsilon(\vartheta) - \upsilon^\star(\vartheta)|, \end{aligned}$$

which leads to

$$\|\mathcal{A}\upsilon - \mathcal{A}\upsilon^\star\| \leq \|\omega\|\|\upsilon - \upsilon^\star\|.$$

So. $\mathcal{A}$ is Lipschitzian on C with Lipschitz constant $\|\omega\|$.

Step 2: $\mathcal{B}$ is completely continuous on $\mathbb{X}$.
Firstly, $\mathcal{B}$ is continuous on C, due to the continuity of $\mathsf{g}_2, \mathsf{g}_1$ implies that $\mathcal{B}$ is continuous too. Next, we shall prove that $\mathcal{B}(\mathbb{X})$ is uniformly bounded in $\mathbb{X}$. For any $\upsilon \in \mathbb{X}$, we have

$$\begin{aligned} |\mathcal{B}\upsilon(\vartheta)| &= \left|\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{g}_2(\vartheta, \upsilon(\vartheta)) + \frac{\mathsf{g}_1(a, \upsilon(a))}{\mathsf{g}_1(b, \upsilon(b)) - \mathsf{g}_1(a, \upsilon(a))}\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{g}_2(b, \upsilon(b))\right| \\ &\leq \int_a^\vartheta \mathsf{s}_\Psi^\varrho(\vartheta, \varsigma)\,|\mathsf{g}_2\,(\varsigma, \upsilon(\varsigma))|\,d\varsigma \\ &\quad + \left|\frac{\mathsf{g}_1(a, \upsilon(a))}{\mathsf{g}_1(b, \upsilon(b)) - \mathsf{g}_1(a, \upsilon(a))}\right| \int_a^b \mathsf{s}_\Psi^\varrho(b, \varsigma)\,|\mathsf{g}_2\,(\varsigma, \upsilon(\varsigma))|\,d\varsigma \end{aligned}$$

$$\leq \mathrm{M}_{\mathrm{g}_2}\int_a^{\vartheta}\mathsf{s}_{\Psi}^{\varrho}(\vartheta,\varsigma)d\varsigma+\mathrm{M}_{\mathrm{g}_1}\mathrm{M}_{\mathrm{g}_2}\int_a^{b}\mathsf{s}_{\Psi}^{\varrho}(b,\varsigma)d\varsigma$$
$$\leq\left(1+\mathrm{M}_{\mathrm{g}_1}\right)\frac{(\Psi(b)-\Psi(a))^{\varrho}}{\Gamma(\varrho+1)}\mathrm{M}_{\mathrm{g}_2},$$

which implies

$$\|\mathcal{B}\upsilon\|\leq\left(1+\mathrm{M}_{\mathrm{g}_1}\right)\frac{(\Psi(b)-\Psi(a))^{\varrho}}{\Gamma(\varrho+1)}\mathrm{M}_{\mathrm{g}_2}=M.$$

This shows that $\{\mathcal{B}\upsilon:\upsilon\in\mathbb{X}\}$ is uniformly bounded set.
To prove that $\mathcal{B}(\mathbb{X})$ is an equicontinuous set in $\mathbb{X}$, let $\vartheta_1,\vartheta_2\in\mho$ $(\vartheta_1<\vartheta_2)$. Then for any $\upsilon\in\mathbb{X}$ and by (H3), we get

$$\begin{aligned}
&|\mathcal{B}(\upsilon)(\vartheta_2)-\mathcal{B}(\upsilon)(\vartheta_1)|\\
\leq\ &\left|\mathcal{I}_{0^+}^{\varrho,\Psi}\mathsf{g}_2(\varsigma,\upsilon(\varsigma))(\vartheta_2)-\mathcal{I}_{0^+}^{\varrho,\Psi}\mathsf{g}_2(\varsigma,\upsilon(\varsigma))(\vartheta_1)\right|\\
\leq\ &\left|\int_a^{\vartheta_2}\mathsf{s}_{\Psi}^{\varrho}(\vartheta_2,\varsigma)\mathsf{g}_2(\varsigma,\upsilon(\varsigma))\,d\varsigma-\int_a^{\vartheta_1}\mathsf{s}_{\Psi}^{\varrho}(\vartheta_1,\varsigma)\mathsf{g}_2(\varsigma,\upsilon(\varsigma))\,d\varsigma\right|\\
\leq\ &\frac{1}{\Gamma(\varrho)}\int_a^{\vartheta_1}\Psi'(\varsigma)\left|(\Psi(\vartheta_1)-\Psi(\varsigma))^{\varrho-1}-(\Psi(\vartheta_2)-\Psi(\varsigma))^{\varrho-1}\right||\mathsf{g}_2(\varsigma,\upsilon(\varsigma))|\,d\varsigma\\
&+\frac{1}{\Gamma(\varrho)}\int_{\vartheta_1}^{\vartheta_2}\Psi'(\varsigma)(\Psi(\vartheta_2)-\Psi(\varsigma))^{\varrho-1}|\mathsf{g}_2(\varsigma,\upsilon(\varsigma))|\,d\varsigma\\
\leq\ &\frac{\mathrm{M}_{\mathrm{g}_2}}{\Gamma(\varrho+1)}\left[(\Psi(\vartheta_2)-\Psi(a))^{\varrho}-(\Psi(\vartheta_1)-\Psi(a))^{\varrho}\right].
\end{aligned}$$

Distinctly, the right-hand side of the a bove inequality tends to zero independently of $\upsilon\in\mathbb{X}$ as $\vartheta_2\to\vartheta_1$. As a result of the Ascoli-Arzela theorem, $\mathcal{B}$ is a completely continuous operator on $\mathbb{X}$.

Step 3: Assumption (c) of Theorem 2.6 is satisfied.

Let $\upsilon\in C$ and $\upsilon^{\star}\in\mathbb{X}$ such that $\upsilon=\mathcal{A}\upsilon\mathcal{B}\upsilon^{\star}$. Then

$$\begin{aligned}
|\upsilon(\vartheta)|&\leq|\mathcal{A}\upsilon(\vartheta)|\left|\mathcal{B}\upsilon^{\star}(\vartheta)\right|\\
&\leq\left|\mathsf{g}_1^{-1}(\vartheta,\upsilon(\vartheta))\right|\left(\int_a^{\vartheta}\mathsf{s}_{\Psi}^{\varrho}(\vartheta,\varsigma)\left|\mathsf{g}_2(\varsigma,\upsilon^{\star}(\varsigma))\right|d\varsigma\right.\\
&\left.+\left|\frac{\mathsf{g}_1(a,\upsilon(a))}{\mathsf{g}_1(b,\upsilon(b))-\mathsf{g}_1(a,\upsilon(a))}\right|\int_a^{b}\mathsf{s}_{\Psi}^{\varrho}(b,\varsigma)\left|\mathsf{g}_2(\varsigma,\upsilon^{\star}(\varsigma))\right|d\varsigma\right)\\
&\leq\left(\left|\mathsf{g}_1^{-1}(\vartheta,\upsilon(\vartheta))-\mathsf{g}_1^{-1}(\vartheta,0)\right|+\left|\mathsf{g}_1^{-1}(\vartheta,0)\right|\right)\\
&\left(\int_a^{\vartheta}\mathsf{s}_{\Psi}^{\varrho}(\vartheta,\varsigma)\left|\mathsf{g}_2(\varsigma,\upsilon^{\star}(\varsigma))\right|d\varsigma+\mathrm{M}_{\mathrm{g}_1}\int_a^{b}\mathsf{s}_{\Psi}^{\varrho}(b,\varsigma)\left|\mathsf{g}_2(\varsigma,\upsilon^{\star}(\varsigma))\right|d\varsigma\right)\\
&\leq(\|\omega\||\upsilon(\vartheta)|+\mathsf{g}_{10})\left(1+\mathrm{M}_{\mathrm{g}_1}\right)\mathrm{M}_{\mathrm{g}_2}\frac{(\Psi(b)-\Psi(a))^{\varrho}}{\Gamma(\varrho+1)}\\
&=(\|\omega\||\upsilon(\vartheta)|+\mathsf{g}_{10})\,M,
\end{aligned}$$

which gives

$$\|\upsilon\|\leq\frac{M\mathsf{g}_{10}}{1-M\|\omega\|}\leq R.$$

Step 4: Assumption (d) of Theorem 2.6 holds.

To this end, we show that $\|\omega\| N < 1$, where $N = \|\mathcal{B}(\mathbb{X})\|$. Since

$$N = \|\mathcal{B}(\mathbb{X})\| = \sup_{\upsilon\in\mathbb{X}} \left\{ \sup_{\vartheta\in\mho} |\mathcal{B}\upsilon(\vartheta)| \right\} \le M,$$

we have

$$\|\omega\| N \le \|\omega\| M < 1.$$

Thus all the assumptions of Theorem 2.6 hold. Hence, $\upsilon = \mathcal{A}\upsilon\mathcal{B}\upsilon$ has a solution in $\mathbb{X}$. So, the hybrid problem 1.1 has a solution on $\mho$. □

2.3. Existence result of (1.2)

In view of Lemma 2.8, we have

$$\upsilon(\vartheta) = \mathsf{g}_1^{-1}(\vartheta, \upsilon(\vartheta)) \left(\mathcal{I}_{a^+}^{\varrho,\Psi} \mathsf{g}_2(\vartheta, \upsilon(\vartheta)) - \mu(\vartheta, \upsilon(\vartheta)) \mathcal{I}_{a^+}^{\varrho,\Psi} \mathsf{g}_2(b, \upsilon(b)) + \nu(\vartheta, \upsilon(\vartheta)) \mathcal{I}_{a^+}^{\varrho-1,\Psi} \mathsf{g}_2(b, \upsilon(b)) \right),$$

where

$$\mu(\vartheta, \upsilon(\vartheta)) := \mathsf{g}_1^{-1}(\vartheta, \upsilon(\vartheta)) \frac{\rho_1 - \rho_2 \left(\Psi(\vartheta) - \Psi(a)\right)}{\rho_3},$$

$$\nu(\vartheta, \upsilon(\vartheta)) := \mathsf{g}_1^{-1}(\vartheta, \upsilon(\vartheta)) \frac{\eta_1 \left(\Psi(b) - \Psi(\vartheta)\right) + \eta_2 \left(\Psi(\vartheta) - \Psi(a)\right)}{\rho_3}.$$

$$\rho_1 := \mathsf{g}_1(a, \upsilon(a)) \left(\mathsf{g}_1(a, \upsilon(a)) \Psi'(b) - \mathsf{g}_1(b, \upsilon(b)) \Psi'(a) \right),$$

$$\rho_2 := \left(\mathsf{g}_1(a, \upsilon(a)) \mathsf{g}_1'(b, \upsilon(b)) - \mathsf{g}_1(b, \upsilon(b)) \mathsf{g}_1'(a, \upsilon(a)) \right),$$

$$\begin{aligned} \rho_3 := {} & \left(\mathsf{g}_1(b, \upsilon(b)) - \mathsf{g}_1(a, \upsilon(a))\right) \left(\mathsf{g}_1(b, \upsilon(b)) \Psi'(a) - \mathsf{g}_1(a, \upsilon(a)) \Psi'(b) \right) \\ & + \left(\mathsf{g}_1(a, \upsilon(a)) \mathsf{g}_1'(b, \upsilon(b)) - \mathsf{g}_1'(a, \upsilon(a)) \mathsf{g}_1(b, \upsilon(b)) \right) \left(\Psi(b) - \Psi(a)\right), \end{aligned}$$

$$\eta_1 = \left[\mathsf{g}_1(a, \upsilon(a))\right]^2 \Psi'(b), \ \eta_2 = \mathsf{g}_1(a, \upsilon(a)) \mathsf{g}_1(b, \upsilon(b)) \Psi'(b),$$

To simplify, we will use the following notations:

$$\Omega := \left((1 + \mu^*) \frac{(\Psi(b) - \Psi(a))^{\varrho}}{\Gamma(\varrho + 1)} + \nu^* \frac{(\Psi(b) - \Psi(a))^{\varrho-1}}{\Gamma(\varrho)} \right) \mathrm{M}_{\mathsf{g}_2},$$

and

$$\mu^* = \max_{(\vartheta,\upsilon)\in\mho\times\mathbb{R}} |\mu(\vartheta, \upsilon(\vartheta))|, \text{ and } \nu^* = \max_{(\vartheta,\upsilon)\in\mho\times\mathbb{R}} |\nu(\vartheta, \upsilon(\vartheta))|.$$

Theorem 2.10. *Assume that (H1)–(H3) hold. Furthermore, if*

$$\|\omega\|\Omega < 1, \tag{2.14}$$

then the hybrid problem (1.2) has a least one solution defined on $\mho$.

Proof. Define

$$R_1 \geq \frac{g_0\Omega}{1-\|\omega\|\Omega}. \tag{2.15}$$

In the light of (2.14), $R_1 > 0$. Define a subset $\mathbb{X}$ of the Banach algebra C by

$$\mathbb{X} = \{\upsilon \in C : \|\upsilon\| \leq R_1\}.$$

Clearly, $\mathbb{X}$ is a closed, convex and bounded subset of C. Consider the operators $\mathcal{A}_1 : C \to C$ and $\mathcal{B}_1 : \mathbb{X} \to C$ defined by

$$(\mathcal{A}_1\upsilon)(\vartheta) = \mathsf{g}_1^{-1}(\vartheta, \upsilon(\vartheta)),\ \vartheta \in \mho,$$

and

$$(\mathcal{B}_1\upsilon)(\vartheta) = \mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{g}_2(\vartheta,\upsilon(\vartheta)) - \mu(\vartheta,\upsilon(\vartheta))\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{g}_2(b,\upsilon(b)) + \nu(\vartheta,\upsilon(\vartheta))\mathcal{I}_{a^+}^{\varrho-1,\Psi}\mathsf{g}_2(b,\upsilon(b)),\ \vartheta \in \mho,$$

where $\upsilon = \mathcal{A}_1\upsilon\mathcal{B}_1\upsilon,\ \upsilon \in C$.

Now, we prove that $\mathcal{A}_1$ and $\mathcal{B}_1$ fulfills assumptions of Theorem 2.6. The proof will be given in forthcoming steps.

Step I: $\mathcal{A}_1$ is lipschitzian on C with Lipschitz constants $\|\omega\|$.

Let $\upsilon, \upsilon^\star \in C$ and $\vartheta \in \mho$. Then, by using (H2), we have

$$\begin{aligned}\left|\mathcal{A}_1\upsilon(\vartheta) - \mathcal{A}_1\upsilon^\star(\vartheta)\right| &= |\mathsf{g}_1^{-1}(\vartheta,\upsilon(\vartheta)) - \mathsf{g}_1^{-1}(\vartheta,\upsilon^\star(\vartheta))| \\ &\leq \omega(\vartheta)(|\upsilon(\vartheta) - \upsilon^\star(\vartheta)|).\end{aligned}$$

Thus

$$\left\|\mathcal{A}\upsilon - \mathcal{A}\upsilon^\star\right\| \leq \|\omega\|\,\|\upsilon - \upsilon^\star\|.$$

That is, $\mathcal{A}_1$ is a Lipschitzian with Lipschitz constant $\|\omega\|$.

Step II: $\mathcal{B}_1$ is completely continuous on $\mathbb{X}$. Firstly, $\mathcal{B}_1$ is continuous on C, due to the continuity of $\mathsf{g}_2, \mathsf{g}_1, \mathsf{g}_1^{-1}$ implies that μ and ν are continuous and hence $\mathcal{B}_1$ is continuous too. Next, we shall prove that $\mathcal{B}_1(\mathbb{X})$ is uniformly bounded in $\mathbb{X}$. For any $\upsilon \in \mathbb{X}$, we have

$$\begin{aligned}|\mathcal{B}_1\upsilon(\vartheta)| &= \left|\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{g}_2(\vartheta,\upsilon(\vartheta)) - \mu(\vartheta,\upsilon(\vartheta))\mathcal{I}_{a^+}^{\varrho,\Psi}\mathsf{g}_2(b,\upsilon(b)) + \nu(\vartheta,\upsilon(\vartheta))\mathcal{I}_{a^+}^{\varrho-1,\Psi}\mathsf{g}_2(b,\upsilon(b))\right| \\ &\leq \int_a^\vartheta \mathsf{s}_\Psi^\varrho(\vartheta,\varsigma)\,|\mathsf{g}_2(\varsigma,\upsilon(\varsigma))|\,d\varsigma \\ &+ |\mu(\vartheta,\upsilon(\vartheta))| \int_a^b \mathsf{s}_\Psi^\varrho(b,\varsigma)\,|\mathsf{g}_2(\varsigma,\upsilon(\varsigma))|\,d\varsigma \\ &+ |\nu(\vartheta,\upsilon(\vartheta))| \int_a^b \mathsf{s}_\Psi^{\varrho-1}(b,\varsigma)\,|\mathsf{g}_2(\varsigma,\upsilon(\varsigma))|\,d\varsigma \\ &\leq \mathrm{M}_{\mathsf{g}_2}\int_a^\vartheta \mathsf{s}_\Psi^\varrho(\vartheta,\varsigma)d\varsigma + \mu^*\mathrm{M}_{\mathsf{g}_2}\int_a^b \mathsf{s}_\Psi^\varrho(b,\varsigma)d\varsigma + \nu^*\mathrm{M}_{\mathsf{g}_2}\int_a^b \mathsf{s}_\Psi^{\varrho-1}(b,\varsigma)d\varsigma \\ &\leq \left((1+\mu^*)\frac{(\Psi(b)-\Psi(a))^\varrho}{\Gamma(\varrho+1)} + \nu^*\frac{(\Psi(b)-\Psi(a))^{\varrho-1}}{\Gamma(\varrho)}\right)\mathrm{M}_{\mathsf{g}_2},\end{aligned}$$

which implies

$$\|\mathcal{B}_1\upsilon\| \leq \left((1+\mu^*)\,\frac{(\Psi(b)-\Psi(a))^{\varrho}}{\Gamma(\varrho+1)}+\nu^*\frac{(\Psi(b)-\Psi(a))^{\varrho-1}}{\Gamma(\varrho)}\right)\mathrm{M}_{\mathsf{g}_2}=\Omega.$$

This shows that $\{\mathcal{B}_1\upsilon : \upsilon\in\mathbb{X}\}$ is uniformly bounded set. Now, we show that $\mathcal{B}_1(\mathbb{X})$ is an equicontinuous set in $\mathbb{X}$, let $\vartheta_1,\vartheta_2\in\mho$ ($\vartheta_1<\vartheta_2$). Then for any $\upsilon\in\mathbb{X}$ and by (H3), we get

$$\begin{aligned}
&|\mathcal{B}_1(\upsilon)(\vartheta_2)-\mathcal{B}_1(\upsilon)(\vartheta_1)|\\
\leq\;& \left|\mathcal{I}_{0^+}^{\varrho,\Psi}\mathsf{g}_2\left(\vartheta_2,\upsilon(\vartheta_2)\right)-\mathcal{I}_{0^+}^{\varrho,\Psi}\mathsf{g}_2\left(\vartheta_1,\upsilon(\vartheta_1)\right)\right|\\
&+|\mu(\vartheta_2,\upsilon(\vartheta_2))-\mu(\vartheta_1,\upsilon(\vartheta_1))|\left|\mathcal{I}_{0^+}^{\varrho,\Psi}\mathsf{g}_2\left(b,\upsilon(b)\right)\right|\\
&+|\nu(\vartheta_2,\upsilon(\vartheta_2))-\nu(\vartheta_1,\upsilon(\vartheta_1))|\left|\mathcal{I}_{0^+}^{\varrho-1,\Psi}\mathsf{g}_2\left(b,\upsilon(b)\right)\right|\\
\leq\;& \left|\int_a^{\vartheta_2}\mathsf{s}_{\Psi}^{\varrho}(\vartheta_2,\varsigma)\mathsf{g}_2\left(\varsigma,\upsilon(\varsigma)\right)d\varsigma-\int_a^{\vartheta_1}\mathsf{s}_{\Psi}^{\varrho}(\vartheta_1,\varsigma)\mathsf{g}_2\left(\varsigma,\upsilon(\varsigma)\right)d\varsigma\right|\\
&+|\mu(\vartheta_2,\upsilon(\vartheta_2))-\mu(\vartheta_1,\upsilon(\vartheta_1))|\left|\int_a^{b}\mathsf{s}_{\Psi}^{\varrho}(b,\varsigma)\mathsf{g}_2\left(\varsigma,\upsilon(\varsigma)\right)d\varsigma\right|\\
&+|\nu(\vartheta_2,\upsilon(\vartheta_2))-\nu(\vartheta_1,\upsilon(\vartheta_1))|\left|\int_a^{b}\mathsf{s}_{\Psi}^{\varrho-1}(b,\varsigma)\mathsf{g}_2\left(\varsigma,\upsilon(\varsigma)\right)d\varsigma\right|\\
\leq\;& \int_a^{\vartheta_1}\left(\mathsf{s}_{\Psi}^{\varrho}(\vartheta_2,\varsigma)-\mathsf{s}_{\Psi}^{\varrho}(\vartheta_1,\varsigma)\right)|\mathsf{g}_2\left(\varsigma,\upsilon(\varsigma)\right)|\,d\varsigma+\int_{\vartheta_1}^{\vartheta_2}\mathsf{s}_{\Psi}^{\varrho}(\vartheta_2,\varsigma)\,|\mathsf{g}_2\left(\varsigma,\upsilon(\varsigma)\right)|\,d\varsigma\\
&+|\mu(\vartheta_2,\upsilon(\vartheta_2))-\mu(\vartheta_1,\upsilon(\vartheta_1))|\int_a^{b}\mathsf{s}_{\Psi}^{\varrho}(b,\varsigma)\,|\mathsf{g}_2\left(\varsigma,\upsilon(\varsigma)\right)|\,d\varsigma\\
&+|\nu(\vartheta_2,\upsilon(\vartheta_2))-\nu(\vartheta_1,\upsilon(\vartheta_1))|\int_a^{b}\mathsf{s}_{\Psi}^{\varrho-1}(b,\varsigma)\,|\mathsf{g}_2\left(\varsigma,\upsilon(\varsigma)\right)|\,d\varsigma\\
\leq\;& \frac{\mathrm{M}_{\mathsf{g}_2}}{\Gamma(\varrho+1)}\left[(\Psi(\vartheta_2)-\Psi(a))^{\varrho}-(\Psi(\vartheta_2)-\Psi(a))^{\varrho}\right]\\
&+|\mu(\vartheta_2,\upsilon(\vartheta_2))-\mu(\vartheta_1,\upsilon(\vartheta_1))|\,\frac{\mathrm{M}_{\mathsf{g}_2}}{\Gamma(\varrho+1)}\,(\Psi(b)-\Psi(a))^{\varrho}\\
&+|\nu(\vartheta_2,\upsilon(\vartheta_2))-\nu(\vartheta_1,\upsilon(\vartheta_1))|\,\frac{\mathrm{M}_{\mathsf{g}_2}}{\Gamma(\varrho)}\,(\Psi(b)-\Psi(a))^{\varrho-1}\\
\to\;& 0 \text{ as } \vartheta_2\to\vartheta_1.
\end{aligned}$$

As a result of the Ascoli-Arzela theorem, $\mathcal{B}_1$ is a completely continuous operator on $\mathbb{X}$.

Step III: We prove the third condition (c) of Theorem 2.6 holds. Let $\upsilon\in C$ and $\upsilon^\star\in\mathbb{X}$ such that $\upsilon=\mathcal{A}_1\upsilon\mathcal{B}_1\upsilon^\star$. Then

$$\begin{aligned}
|\upsilon(\vartheta)| &\leq |\mathcal{A}_1\upsilon(\vartheta)|\left|\mathcal{B}_1\upsilon^\star(\vartheta)\right|\\
&\leq \left|\mathsf{g}_1^{-1}(\vartheta,\upsilon(\vartheta))\right|\left(\int_a^{\vartheta}\mathsf{s}_{\Psi}^{\varrho}(\vartheta,\varsigma)\left|\mathsf{g}_2\left(\varsigma,\upsilon^\star(\varsigma)\right)\right|d\varsigma\right.\\
&\quad+\left|\mu(\vartheta,\upsilon^\star(\vartheta))\right|\int_a^{b}\mathsf{s}_{\Psi}^{\varrho}(b,\varsigma)\left|\mathsf{g}_2\left(\varsigma,\upsilon^\star(\varsigma)\right)\right|d\varsigma
\end{aligned}$$

$$+\left|\nu(\vartheta,\upsilon^{\star}(\vartheta))\right|\int_a^b \mathsf{s}_{\Psi}^{\varrho-1}(b,\varsigma)\left|\mathsf{g}_2\left(\varsigma,\upsilon^{\star}(\varsigma)\right)\right|d\varsigma\Big)$$
$$\leq \left(\left|\mathsf{g}_1^{-1}(\vartheta,\upsilon(\vartheta))-\mathsf{g}_1^{-1}(\vartheta,0)\right|+\left|\mathsf{g}_1^{-1}(\vartheta,0)\right|\right)$$
$$\Big(\int_a^{\vartheta}\mathsf{s}_{\Psi}^{\varrho}(\vartheta,\varsigma)\left|\mathsf{g}_2\left(\varsigma,\upsilon^{\star}(\varsigma)\right)\right|d\varsigma+\mu^*\int_a^b \mathsf{s}_{\Psi}^{\varrho}(b,\varsigma)\left|\mathsf{g}_2\left(\varsigma,\upsilon^{\star}(\varsigma)\right)\right|d\varsigma$$
$$+\nu^*\int_a^b \mathsf{s}_{\Psi}^{\varrho-1}(b,\varsigma)\left|\mathsf{g}_2\left(\varsigma,\upsilon^{\star}(\varsigma)\right)\right|d\varsigma\Big)$$
$$\leq (\|\omega\||\upsilon(\vartheta)|+\mathsf{g}_{10})$$
$$\left((1+\mu^*)\frac{(\Psi(b)-\Psi(a))^{\varrho}}{\Gamma(\varrho+1)}+\nu^*\frac{(\Psi(b)-\Psi(a))^{\varrho-1}}{\Gamma(\varrho)}\right)\mathrm{M}_{\mathsf{g}_2}$$
$$=(\|\omega\||\upsilon(\vartheta)|+\mathsf{g}_{10})\,\Omega,$$

which gives

$$\|\upsilon\|\leq\frac{\Omega\mathsf{g}_{10}}{1-\Omega\|\omega\|}\leq R_1.$$

Thus, $\|\upsilon\|\leq R_1$ and so the hypothesis (c) of Theorem 2.6 is satisfied.

Step IV: Assumption (d) of Theorem 2.6 holds.

To this end, we show that $\|\omega\|\,N<1$, where $N=\|\mathcal{B}_1(\mathbb{X})\|$. Since

$$N=\|\mathcal{B}_1(\mathbb{X})\|=\sup_{\upsilon\in\mathbb{X}}\left\{\sup_{\vartheta\in\mho}|\mathcal{B}_1\upsilon(\vartheta)|\right\}\leq\Omega,$$

we have

$$\|\omega\|\,N\leq\|\omega\|\,\Omega<1.$$

Thus all the assumptions of Theorem 2.6 hold. Hence, $\upsilon=\mathcal{A}_1\upsilon\mathcal{B}_1\upsilon$ has a solution in $\mathbb{X}$. So, the hybrid problem 1.2 has a solution on $\mho$. □

3. Examples

In this section, in order to illustrate our results, we consider two examples.

Example 3.1. *Consider the following nonlocal hybrid boundary value problem:*

$$\begin{cases} {}^{C}\mathcal{D}_{a^+}^{\varrho,\Psi}\left(\upsilon(\vartheta)\left(\frac{\vartheta^2}{10}\left(\frac{1}{2}\left(\upsilon(\vartheta)\right)+\vartheta\right)\right)^{-1}\right)=\frac{e^{(-2\vartheta)}}{\sqrt{9+\vartheta}}\left(\sin\upsilon(\vartheta)\right),\varrho\in(0,1)\\ \upsilon(0)=\upsilon(1),\end{cases}\tag{3.1}$$

From the system (3.1), and we choose $\Psi(\vartheta)=\vartheta,\ \ a=0,\ \ b=1,\ \ \varrho=4/5,\ \mathsf{g}_1^{-1}(\vartheta,\upsilon(\vartheta))=\left(\frac{\vartheta^2}{10}\left(\frac{1}{2}\left(\upsilon(\vartheta)\right)+\vartheta\right)\right)^{-1}$, $\mathsf{g}_2(\vartheta,\upsilon(\vartheta))=\frac{e^{(-2\vartheta)}}{\sqrt{9+\vartheta}}\left(\sin\upsilon(\vartheta)\right)$. *Clearly,* $\mathsf{g}_2,\mathsf{g}_1^{-1}$ *are continuous. Moreover*

$$\left|\mathsf{g}_1^{-1}(\vartheta,\upsilon_1)-\mathsf{g}_1^{-1}(\vartheta,\upsilon_2)\right|\leq\frac{1}{10}\left|\upsilon_1-\upsilon_2\right|,$$

and

$$\left|\mathsf{g}_2(\vartheta,\upsilon(\vartheta))\right|\leq\frac{1}{3},$$

with $\omega = \frac{1}{10}$, $\mathrm{M}_{\mathsf{g}_2} = \frac{1}{\sqrt{10}}$ *and* $g_0 = \sup_{\vartheta\in\mho}\left|g^{-1}(\vartheta, 0)\right| = \frac{1}{10}$. *Using these values, we get* $M\,\|\omega\| \simeq 0.35 < 1$. *As all the conditions of Theorem 2.9 are satisfied, problem 3.1 has at least one solution on* $\mho$.

Example 3.2. *Consider the following nonlocal hybrid boundary value problem:*

$$\begin{cases} {}^{C}\mathcal{D}_{a^+}^{\varrho,\Psi}\left(\upsilon(\vartheta)\left(\frac{\vartheta}{5}\left(\frac{1}{3}\left(\upsilon(\vartheta)\right)+\vartheta\right)\right)^{-1}\right) = \frac{\cos^2(2\pi\vartheta)}{5-\vartheta}\upsilon(\vartheta), \varrho\in(1,2), \\ \upsilon(1) = \upsilon(2), \upsilon^{'}(1) = \upsilon^{'}(2), \end{cases} \tag{3.2}$$

From the system (3.2), and we choose $\Psi(\vartheta) = \vartheta$, $a = 0$, $b = 1$, $\varrho = 8/5$, $g(\vartheta, \upsilon(\vartheta)) = \left(\frac{\vartheta}{5}\left(\frac{1}{3}\left(\upsilon(\vartheta)\right)+\vartheta\right)\right)^{-1}$, $\mathsf{g}_2(\vartheta, \upsilon(\vartheta)) = \frac{\cos^2(2\pi\vartheta)}{5-\vartheta}\upsilon(\vartheta)$. *Clearly,* $\mathsf{g}_2, \mathsf{g}_1^{-1}$ *are continuous. Moreover*

$$\left|\mathsf{g}_1^{-1}(\vartheta, \upsilon_1) - \mathsf{g}_1^{-1}(\vartheta, \upsilon_2)\right| \leq \frac{1}{5}\left|\upsilon_1 - \upsilon_2\right|,$$

and

$$\left|\mathsf{g}_2(\vartheta, \upsilon(\vartheta))\right| \leq \frac{1}{4},$$

with $\omega = \frac{1}{5}$, $\mathrm{M}_{\mathsf{g}_2} = \frac{1}{4}$ *and* $g_0 = \sup_{\vartheta\in\mho}\left|g^{-1}(\vartheta, 0)\right| = \frac{1}{5}$. *Using these values, we get* $\Omega\,\|\omega\| < 1$. *As all the conditions of Theorem 2.10 are satisfied, problem 3.2 has at least one solution on* $\mho$.

4. Conclusions

It is important that we examine the fractional systems of the hybrid with generalized derivatives since these derivatives cover many systems in the literature and they contain a kernel with different values that generate many special cases.

In this research work, we have investigated the sufficient conditions to the existence of solutions to two new types of boundary value problems of nonlinear hybrid fractional differential equations involving generalized fractional derivatives known as Ψ-Caputo operators. In order to achieve the objectives, we applied Dhage's fixed point theorem for the sum of three operators. Two examples are provided to confirm the feasibility of the obtained results.

Moreover, we have formulated illustrative examples for this type of hybrid fractional systems to support our main results from a numerical point of view.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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