Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(1): 156–170.
DOI: 10.3934/math.2022009
Received: 26 July 2021
Accepted: 25 September 2021
Published: 09 October 2021

Research article

Some integral inequalities for coordinated log-h-convex interval-valued functions

Fangfang Shi[1], **Guoju Ye**[1,*], **Dafang Zhao**[2] **and Wei Liu**[1]

[1] College of Science, Hohai University, 210098 Nanjing, China
[2] School of Mathematics and Statistics, Hubei Normal University, 435002 Huangshi, China

* **Correspondence:** Email: ygjhhu@163.com; Tel: 15298387799.

Abstract: We introduce and investigate the coordinated log-h-convexity for interval-valued functions. Also, we prove some new Jensen type inequalities and Hermite-Hadamard type inequalities, which generalize some known results in the literature. Moreover, some examples are given to illustrate our results.

Keywords: interval-valued functions; coordinated log-h-convex; Jensen type inequalities; Hermite-Hadamard type inequalities
Mathematics Subject Classification: 26D15, 26E25, 26B25, 28B20

1. Introduction

The convexity of function is a classical concept, since it plays a fundamental role in mathematical programming theory, game theory, mathematical economics, variational science, optimal control theory and other fields, a new branch of mathematics, convex analysis, appeared in the 1960s. However, it has been noticed that the functions encountered in a large number of theoretical and practical problems in economics are not classical convex functions, therefore, in the past decades, the generalization of function convexity has attracted the attention of many scholars and aroused great interest, such as h-convex functions [1–5], log-convex functions [6–10], log-h-convex functions [11], and especially for coordinated convex [12]. Since 2001, various extensions and generalizations of integral inequalities for coordinated convex functions have been established in [12–17].

On the other hand, calculation error has always been a troublesome problem in numerical analysis. In many problems, it is often to speculate the accuracy of calculation results or use high-precision operation as far as possible to ensure the accuracy of the results, because the accumulation of calculation errors may make the calculation results meaningless, interval analysis as a new important tool to solve uncertainty problems has attracted much attention and also has yielded fruitful results, we refer the reader to the papers [18, 19]. It is worth notion that in recent decades, many authors have combined integral inequalities with interval-valued functions(IVFs) and obtained many excellent conclusions. In [20], Costa gave Opial-type inequalities for IVFs. In [21,22], Chalco-Cano investigated Ostrowski type inequalities for IVFs by using generalized Hukuhara derivative. In [23], Román-Flores derived the Minkowski type inequalities and Beckenbach's type inequalities for IVFs. Very recently, Zhao [5, 24] established the Hermite-Hadamard type inequalities for interval-valued coordinated functions.

Motivated by these results, in the present paper, we introduce the concept of coordinated log-h-convex for IVFs, and then present some new Jensen type inequalities and Hermite-Hadamard type inequalities for interval-valued coordinated functions. Also, we give some examples to illustrate our main results.

2. Preliminaries

Let $\mathbb{R}_{\mathcal{I}}$ the collection of all closed and bounded intervals of $\mathbb{R}$. We use $\mathbb{R}_{\mathcal{I}}^{+}$ and $\mathbb{R}^{+}$ to represent the set of all positive intervals and the family of all positive real numbers respectively. The collection of all Riemann integrable real-valued functions on $[a,b]$, IVFs on $[a,b]$ and IVFs on $\triangle = [a,b]\times[c,d]$ are denoted by $\mathcal{R}_{([a,b])}$, $\mathcal{IR}_{([a,b])}$ and $\mathcal{ID}_{(\triangle)}$. For more conceptions on IVFs, see [4,25]. Moreover, we have

Theorem 1. *[4] Let $f : [a,b] \to \mathbb{R}_{\mathcal{I}}$ such that $f = [\underline{f}, \overline{f}]$. Then $f \in \mathcal{IR}_{([a,b])}$ iff $\underline{f}, \overline{f} \in \mathcal{R}_{([a,b])}$ and*

$$(\mathcal{IR})\int_a^b f(x)dx = \left[(\mathcal{R})\int_a^b \underline{f}(x)dx, (\mathcal{R})\int_a^b \overline{f}(x)dx\right].$$

Theorem 2. *[25] Let $\mathcal{F} : \triangle \to \mathbb{R}_{\mathcal{I}}$. If $\mathcal{F} \in \mathcal{ID}_{(\triangle)}$, then*

$$(\mathcal{ID})\iint_{\triangle} \mathcal{F}(x,y)dxdy = (\mathcal{IR})\int_a^b dx(\mathcal{IR})\int_c^d \mathcal{F}(x,y)dy.$$

Definition 1. [26] Let $h : [0,1] \to \mathbb{R}^{+}$. We say that $f : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$ is interval log-h-convex function or that $f \in SX(\log\text{-}h, [a,b], \mathbb{R}_{\mathcal{I}}^{+})$, if for all $x, y \in [a,b]$ and $\vartheta \in [0,1]$, we have

$$f(\vartheta x + (1-\vartheta)y) \supseteq [f(x)]^{h(\vartheta)}[f(y)]^{h(1-\vartheta)}.$$

h is called supermultiplicative if

$$h(\vartheta\tau) \geq h(\vartheta)h(\tau) \tag{2.1}$$

for all $\vartheta, \tau \in [0,1]$. If " $\geq$" in (2.1) is replaced with " $\leq$", then h is called submultiplicative.

Theorem 3. *[26] Let $\mathcal{F} : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}, h\left(\frac{1}{2}\right) \neq 0$. If $\mathcal{F} \in SX(\log\text{-}h, [a,b], \mathbb{R}_{\mathcal{I}}^{+})$ and $\mathcal{F} \in \mathcal{IR}_{([a,b])}$, then*

$$\mathcal{F}\left(\frac{a+b}{2}\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}} \supseteq \exp\left[\frac{1}{b-a}\int_a^b \ln \mathcal{F}(x)dx\right] \supseteq [\mathcal{F}(a)\mathcal{F}(b)]^{\int_0^1 h(\vartheta)d\vartheta}. \tag{2.2}$$

Theorem 4. *[27] Let $\mathcal{F} : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}, h\left(\frac{1}{2}\right) \neq 0$. If $\mathcal{F} \in SX(\log\text{-}h, [a,b], \mathbb{R}_{\mathcal{I}}^{+})$ and $\mathcal{F} \in \mathcal{IR}_{([a,b])}$, then*

$$\begin{aligned}\left[\mathcal{F}\left(\frac{a+b}{2}\right)\right]^{\frac{1}{4h^2\left(\frac{1}{2}\right)}} &\supseteq \left[\mathcal{F}\left(\frac{3a+b}{4}\right)\mathcal{F}\left(\frac{a+3b}{4}\right)\right]^{\frac{1}{4h\left(\frac{1}{2}\right)}} \\ &\supseteq \left(\int_a^b \mathcal{F}(x)dx\right)^{\frac{1}{b-a}} \\ &\supseteq \left[\mathcal{F}(a)\mathcal{F}(b)\mathcal{F}^2\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2}\int_0^1 h(\vartheta)d\vartheta} \\ &\supseteq [\mathcal{F}(a)\mathcal{F}(b)]^{\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_0^1 h(\vartheta)d\vartheta}.\end{aligned} \tag{2.3}$$

3. Main results

In this section, we define the coordinated log-h-convex for IVFs and prove some new Jensen type inequalities and Hermite-Hadamard type inequalities by using this new definition.

Definition 2. Let $h : [0,1] \to \mathbb{R}^{+}$. Then $\mathcal{F} : \triangle \to \mathbb{R}_{\mathcal{I}}^{+}$ is called a coordinated log-h-convex IVFs on $\triangle$ if the partial mappings

$$\begin{aligned}\mathcal{F}_y &: [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}, \mathcal{F}_y(x) = \mathcal{F}(x,y), \\ \mathcal{F}_x &: [c,d] \to \mathbb{R}_{\mathcal{I}}^{+}, \mathcal{F}_x(y) = \mathcal{F}(x,y)\end{aligned}$$

are log-h-convex for all $y \in [c,d]$ and $x \in [a,b]$. Then the set of all coordinated log-h-convex IVFs on $\triangle$ is denoted by $SX(\log\text{-}ch, \triangle, \mathbb{R}_{\mathcal{I}}^{+})$.

Definition 3. Let $h : [0,1] \to \mathbb{R}^{+}$. Then $\mathcal{F} : \triangle \to \mathbb{R}^{+}$ is called a coordinated log-h-convex function in $\triangle$ if for any $(x_1,y_1),(x_2,y_2) \in \triangle$ and $\vartheta \in [0,1]$ we have

$$\mathcal{F}(\vartheta x_1 + (1-\vartheta)x_2, \vartheta y_1 + (1-\vartheta)y_2) \leq [\mathcal{F}(x_1,y_1)]^{h(\vartheta)}[\mathcal{F}(x_2,y_2)]^{h(1-\vartheta)}. \tag{3.1}$$

The set of all log-h-convex functions in $\triangle$ is denoted by $SX(\log\text{-}h, \triangle, \mathbb{R}^{+})$. If inequality (3.1) is reversed, then $\mathcal{F}$ is said to be a coordinated log-h-concave function, the set of all log-h-concave functions in $\triangle$ is denoted by $SV(\log\text{-}h, \triangle, \mathbb{R}^{+})$.

Definition 4. Let $h : [0,1] \to \mathbb{R}^{+}$. Then $\mathcal{F} : \triangle \to \mathbb{R}_{\mathcal{I}}^{+}$ is called a coordinated log-h-convex IVF in $\triangle$ if for any $(x_1,y_1),(x_2,y_2) \in \triangle$ and $\vartheta \in [0,1]$ we have

$$\mathcal{F}(\vartheta x_1 + (1-\vartheta)x_2, \vartheta y_1 + (1-\vartheta)y_2) \supseteq [\mathcal{F}(x_1,y_1)]^{h(\vartheta)}[\mathcal{F}(x_2,y_2)]^{h(1-\vartheta)}.$$

The set of all log-h-convex IVFs in $\triangle$ is denoted by $SX(\log\text{-}h, \triangle, \mathbb{R}_{\mathcal{I}}^{+})$.

Theorem 5. *Let $\mathcal{F} : \triangle \to \mathbb{R}_{\mathcal{I}}^{+}$ such that $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}]$. If $\mathcal{F} \in SX(\log\text{-}h, \triangle, \mathbb{R}_{\mathcal{I}}^{+})$ iff $\underline{\mathcal{F}} \in SX(\log\text{-}h, \triangle, \mathbb{R}^{+})$ and $\overline{\mathcal{F}} \in SV(\log\text{-}h, \triangle, \mathbb{R}^{+})$.*

Proof. The proof is completed by combining the Definitions 3 and 4 above and the Theorem 3.7 of [4]. □

Theorem 6. *If* $\mathcal{F} \in SX(\log\text{-}h, \triangle, \mathbb{R}_{\mathcal{I}}^{+})$*, then* $\mathcal{F} \in SX(\log\text{-}ch, \triangle, \mathbb{R}_{\mathcal{I}}^{+})$.

Proof. Assume that $\mathcal{F} \in SX(\log\text{-}h, \triangle, \mathbb{R}_{\mathcal{I}}^{+})$. Let $\mathcal{F}_x : [c, d] \to \mathbb{R}_{\mathcal{I}}^{+}, \mathcal{F}_x(y) = \mathcal{F}(x, y)$. Then for all $\vartheta \in [0, 1]$ and $y_1, y_2 \in [c, d]$, we have

$$\begin{aligned}\mathcal{F}_x(\vartheta y_1 + (1-\vartheta)y_2) &= \mathcal{F}(x, \vartheta y_1 + (1-\vartheta)y_2)\\ &\supseteq \mathcal{F}(\vartheta x + (1-\vartheta)x, \vartheta y_1 + (1-\vartheta)y_2)\\ &\supseteq [\mathcal{F}(x, y_1)]^{h(\vartheta)} [\mathcal{F}(x, y_2)]^{h(1-\vartheta)}\\ &= [\mathcal{F}_x(y_1)]^{h(\vartheta)} [\mathcal{F}_x(y_2)]^{h(1-\vartheta)}.\end{aligned}$$

Hence $\mathcal{F}_x(y) = \mathcal{F}(x, y)$ is log-h-convex on $[c, d]$. The fact that $\mathcal{F}_y(x) = \mathcal{F}(x, y)$ is log-h-convex on $[a, b]$ goes likewise. □

Remark 1. The converse of Theorem 6 is not generally true. Let $h(\vartheta) = \vartheta$ and $\vartheta \in [0, 1]$, $\triangle_1 = \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \times \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, and $\mathcal{F} : \triangle_1 \to \mathbb{R}_{\mathcal{I}}^{+}$ be defined:

$$\mathcal{F}(x, y) = \left[e^{-sinx-siny}, 64xy\right].$$

Obviously, we have that $\mathcal{F} \in SX(\log\text{-}ch, \triangle_1, \mathbb{R}_{\mathcal{I}}^{+})$ and $\mathcal{F} \notin SX(\log\text{-}h, \triangle_1, \mathbb{R}_{\mathcal{I}}^{+})$. Indeed, if $\left(\frac{\pi}{4}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{\pi}{4}\right) \in \triangle_1$, we have

$$\mathcal{F}\left(\vartheta\frac{\pi}{4} + (1-\vartheta)\frac{\pi}{2}, \vartheta\frac{\pi}{2} + (1-\vartheta)\frac{\pi}{4}\right) = \left[e^{-sin\frac{\vartheta\pi}{4} - sin\frac{(1-\vartheta)\pi}{2}}, 8\pi^2\vartheta(1-\vartheta)\right],$$
$$\left(\mathcal{F}\left(\frac{\pi}{4}, \frac{\pi}{2}\right)\right)^{h(\vartheta)} \left(\mathcal{F}\left(\frac{\pi}{2}, \frac{\pi}{4}\right)\right)^{h(1-\vartheta)} = \left[e^{\left(1-\frac{\sqrt{2}}{2}\right)\vartheta - 1}, 2^{\vartheta+1}\pi\right].$$

If $\vartheta = 0$, then

$$\left[0, \frac{1}{e}\right] \not\supseteq \left[\frac{1}{e}, 2\pi\right].$$

Thus, $\mathcal{F} \notin SX(\log\text{-}h, \triangle_1, \mathbb{R}_{\mathcal{I}}^{+})$.

In the following, Jensen type inequalities for coordinated log-h-convex functions in $\triangle$ is considered.

Theorem 7. *Let* $p_i \in \mathbb{R}^{+}, x_i \in [a, b], y_i \in [c, d], (i = 1, 2, ..., n), \mathcal{F} : \triangle \to \mathbb{R}^{+}$. *If h is a nonnegative supermultiplicative function and* $\mathcal{F} \in SX(\log\text{-}h, \triangle, \mathbb{R}^{+})$*, then*

$$\mathcal{F}\left(\frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i x_i, \frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i y_i\right) \leq \prod_{i=1}^{n} [\mathcal{F}(x_i, y_i)]^{h\left(\frac{p_i}{\mathcal{P}_n}\right)}, \tag{3.2}$$

where $\mathcal{P}_n = \sum_{i=1}^{n} p_i$. *If h is a nonnegative submultiplicative function and* $\mathcal{F} \in SV(\log\text{-}h, \triangle, \mathbb{R}^{+})$*, then* (3.2) *is reversed.*

Proof. If $n = 2$, then from Definition 3, we have

$$\mathcal{F}\left(\frac{p_1}{\mathcal{P}_2}x_1 + \frac{p_2}{\mathcal{P}_2}x_2, \frac{p_1}{\mathcal{P}_2}y_1 + \frac{p_2}{\mathcal{P}_2}y_2\right) \leq [\mathcal{F}(x_1, y_1)]^{h\left(\frac{p_1}{\mathcal{P}_2}\right)} [\mathcal{F}(x_2, y_2)]^{h\left(\frac{p_2}{\mathcal{P}_2}\right)}.$$

Suppose (3.2) holds for $n = k$, then

$$\mathcal{F}\left(\frac{1}{\mathcal{P}_k}\sum_{i=1}^{k} p_i x_i, \frac{1}{\mathcal{P}_k}\sum_{i=1}^{k} p_i y_i\right) \leq \prod_{i=1}^{k}[\mathcal{F}(x_i, y_i)]^{h\left(\frac{p_i}{\mathcal{P}_k}\right)}.$$

Now, let us prove that (3.2) is valid when $n = k + 1$,

$$\begin{aligned}
&\mathcal{F}\left(\frac{1}{\mathcal{P}_{k+1}}\sum_{i=1}^{k+1} p_i x_i, \frac{1}{\mathcal{P}_{k+1}}\sum_{i=1}^{k+1} p_i y_i\right)\\
&= \mathcal{F}\left(\frac{1}{\mathcal{P}_{k+1}}\sum_{i=1}^{k-1} p_i x_i + \frac{p_k + p_{k+1}}{\mathcal{P}_{k+1}}\left(\frac{p_k x_k}{p_k + p_{k+1}} + \frac{p_{k+1} x_{k+1}}{p_k + p_{k+1}}\right),\right.\\
&\quad\left.\frac{1}{\mathcal{P}_{k+1}}\sum_{i=1}^{k-1} p_i y_i + \frac{p_k + p_{k+1}}{\mathcal{P}_{k+1}}\left(\frac{p_k y_k}{p_k + p_{k+1}} + \frac{p_{k+1} y_{k+1}}{p_k + p_{k+1}}\right)\right)\\
&\leq \left[\mathcal{F}\left(\frac{p_k x_k}{p_k + p_{k+1}} + \frac{p_{k+1} x_{k+1}}{p_k + p_{k+1}}, \frac{p_k y_k}{p_k + p_{k+1}} + \frac{p_{k+1} y_{k+1}}{p_k + p_{k+1}}\right)\right]^{h\left(\frac{p_k + p_{k+1}}{\mathcal{P}_{k+1}}\right)} \prod_{i=1}^{k-1}[\mathcal{F}(x_i, y_i)]^{h\left(\frac{p_i}{\mathcal{P}_{k+1}}\right)}\\
&\leq \left([\mathcal{F}(x_k, y_k)]^{h\left(\frac{p_k}{p_k + p_{k+1}}\right)}[\mathcal{F}(x_{k+1}, y_{k+1})]^{h\left(\frac{p_{k+1}}{p_k + p_{k+1}}\right)}\right)^{h\left(\frac{p_k + p_{k+1}}{\mathcal{P}_{k+1}}\right)} \prod_{i=1}^{k-1}[\mathcal{F}(x_i, y_i)]^{h\left(\frac{p_i}{\mathcal{P}_{k+1}}\right)}\\
&\leq [\mathcal{F}(x_k, y_k)]^{h\left(\frac{p_k}{\mathcal{P}_{k+1}}\right)}[\mathcal{F}(x_{k+1}, y_{k+1})]^{h\left(\frac{p_{k+1}}{\mathcal{P}_{k+1}}\right)} \prod_{i=1}^{k-1}[\mathcal{F}(x_i, y_i)]^{h\left(\frac{p_i}{\mathcal{P}_{k+1}}\right)}\\
&= \prod_{i=1}^{k+1}[\mathcal{F}(x_i, y_i)]^{h\left(\frac{p_i}{\mathcal{P}_{k+1}}\right)}.
\end{aligned}$$

This completes the proof. □

Remark 2. If $h(\vartheta) = \vartheta$, then the inequality (3.2) is the Jensen inequality for log-convex functions.

Now, we prove the Jensen inequality for log-h-convex IVFs in $\triangle$.

Theorem 8. *Let $p_i \in \mathbb{R}^+, x_i \in [a, b], y_i \in [c, d], i = 1, 2, ..., n, \mathcal{F} : \triangle \to \mathbb{R}^+_{\mathcal{I}}$ such that $\mathcal{F} = [\underline{F}, \overline{F}]$. If h is a nonnegative supermultiplicative function and $\mathcal{F} \in SX(\log\text{-}h, \triangle, \mathbb{R}^+_{\mathcal{I}})$, then*

$$\mathcal{F}\left(\frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i x_i, \frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i y_i\right) \supseteq \prod_{i=1}^{n}[\mathcal{F}(x_i, y_i)]^{h\left(\frac{p_i}{\mathcal{P}_n}\right)}, \tag{3.3}$$

where $\mathcal{P}_n = \sum_{i=1}^{n} p_i$. If $\mathcal{F} \in SV(\log\text{-}h, \triangle, \mathbb{R}^+_{\mathcal{I}})$, then (3.3) *is reversed.*

Proof. By Theorem 5 and Theorem 7, we have

$$\underline{\mathcal{F}}\left(\frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i x_i, \frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i y_i\right) \leq \prod_{i=1}^{n}[\underline{\mathcal{F}}(x_i, y_i)]^{h\left(\frac{p_i}{\mathcal{P}_n}\right)}$$

and

$$\overline{\mathcal{F}}\left(\frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i x_i, \frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i y_i\right) \geq \prod_{i=1}^{n}[\overline{\mathcal{F}}(x_i, y_i)]^{h\left(\frac{p_i}{\mathcal{P}_n}\right)}.$$

Thus,

$$\begin{aligned}
&\mathcal{F}\left(\frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i x_i, \frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i y_i\right) \\
&= \left[\underline{\mathcal{F}}\left(\frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i x_i, \frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i y_i\right), \overline{\mathcal{F}}\left(\frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i x_i, \frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i y_i\right)\right] \\
&\supseteq \left[\prod_{i=1}^{n}[\underline{\mathcal{F}}(x_i, y_i)]^{h\left(\frac{p_i}{\mathcal{P}_n}\right)}, \prod_{i=1}^{n}[\overline{\mathcal{F}}(x_i, y_i)]^{h\left(\frac{p_i}{\mathcal{P}_n}\right)}\right] \\
&= \prod_{i=1}^{n}[\mathcal{F}(x_i, y_i)]^{h\left(\frac{p_i}{\mathcal{P}_n}\right)}.
\end{aligned}$$

This completes the proof. □

Next, we prove the Hermite-Hadamard type inequalities for coordinated log-h-convex IVFs.

Theorem 9. *Let* $\mathcal{F} : \triangle \to \mathbb{R}_{\mathcal{I}}^{+}$ *and* $h : [0, 1] \to \mathbb{R}^{+}$ *be continuous. If* $\mathcal{F} \in SX(\log\text{-}ch, \triangle, \mathbb{R}_{\mathcal{I}}^{+})$*, then*

$$\begin{aligned}
&\left[\mathcal{F}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]^{\frac{1}{4h^2\left(\frac{1}{2}\right)}} \\
&\supseteq \exp\left[\frac{1}{4h\left(\frac{1}{2}\right)}\left(\frac{1}{2h\left(\frac{1}{2}\right)(b-a)}\int_a^b \ln\mathcal{F}\left(x, \frac{c+d}{2}\right)dx\right.\right. \\
&\quad \left.\left. + \frac{1}{2h\left(\frac{1}{2}\right)(d-c)}\int_c^d \ln\mathcal{F}\left(\frac{a+b}{2}, y\right)dy\right)\right] \\
&\supseteq \exp\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \ln\mathcal{F}(x, y)dxdy\right] \\
&\supseteq \exp\left[\frac{1}{2}\int_0^1 h(\vartheta)d\vartheta\left(\frac{1}{b-a}\int_a^b \ln\mathcal{F}(x, c)dx + \frac{1}{-a}\int_a^b \ln\mathcal{F}(x, d)dx\right.\right. \\
&\quad \left.\left. + \frac{1}{d-c}\int_c^d \ln\mathcal{F}(a, y)dy + \frac{1}{d-c}\int_c^d \ln\mathcal{F}(b, y)dy\right)\right] \\
&\supseteq [\mathcal{F}(a, c)\mathcal{F}(a, d)\mathcal{F}(b, c)\mathcal{F}(b, d)]^{\left(\int_0^1 h(\vartheta)d\vartheta\right)^2}.
\end{aligned} \tag{3.4}$$

Proof. Since $\mathcal{F} \in SX(\log\text{-}ch, \triangle, \mathbb{R}_{\mathcal{I}}^{+})$, we have

$$\begin{aligned}
&\mathcal{F}_x\left(\frac{c+d}{2}\right) \\
&= \mathcal{F}_x\left(\frac{\vartheta c + (1-\vartheta)d + (1-\vartheta)c + \vartheta d}{2}\right) \\
&\supseteq [\mathcal{F}_x(\vartheta c + (1-\vartheta)d)]^{h\left(\frac{1}{2}\right)}[\mathcal{F}_x((1-\vartheta)c + \vartheta d)]^{h\left(\frac{1}{2}\right)}.
\end{aligned}$$

That is,

$$\ln \mathcal{F}_x\left(\frac{c+d}{2}\right)$$
$$\supseteq h\left(\frac{1}{2}\right)\ln\left[\mathcal{F}_x(\vartheta c+(1-\vartheta)d)\mathcal{F}_x((1-\vartheta)c+\vartheta d)\right].$$

Moreover, we have

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)}\ln \mathcal{F}_x\left(\frac{c+d}{2}\right)\\
&\supseteq \left[\int_0^1 \ln \mathcal{F}_x(\vartheta c+(1-\vartheta)d)d\vartheta+\int_0^1 \ln \mathcal{F}_x((1-\vartheta)c+\vartheta d)d\vartheta\right]\\
&=\left[\int_0^1 \ln \underline{\mathcal{F}}_x(\vartheta c+(1-\vartheta)d)d\vartheta,\int_0^1 \ln \overline{\mathcal{F}}_x(\vartheta c+(1-\vartheta)d)d\vartheta\right]\\
&\quad+\left[\int_0^1 \ln \underline{\mathcal{F}}_x((1-\vartheta)c+\vartheta d)d\vartheta,\int_0^1 \ln \overline{\mathcal{F}}_x((1-\vartheta)c+\vartheta d)d\vartheta\right]\\
&=2\left[\frac{1}{d-c}\int_c^d \ln \underline{\mathcal{F}}_x(y)dy,\frac{1}{d-c}\int_c^d \ln \overline{\mathcal{F}}_x(y)dy\right]\\
&=\frac{2}{d-c}\int_c^d \ln \mathcal{F}_x(y)dy.
\end{aligned}$$

Similarly, we get

$$\frac{1}{d-c}\int_c^d \ln \mathcal{F}_x(y)dy\supseteq \ln\left[\mathcal{F}_x(c)\mathcal{F}_x(d)\right]\int_0^1 h(\vartheta)d\vartheta.$$

Then

$$\frac{1}{2h\left(\frac{1}{2}\right)}\ln \mathcal{F}_x\left(\frac{c+d}{2}\right)\supseteq \frac{1}{d-c}\int_c^d \ln \mathcal{F}_x(y)dy\supseteq \ln\left[\mathcal{F}_x(c)\mathcal{F}_x(d)\right]\int_0^1 h(\vartheta)d\vartheta.$$

That is,

$$\frac{1}{2h\left(\frac{1}{2}\right)}\ln \mathcal{F}\left(x,\frac{c+d}{2}\right)\supseteq \frac{1}{d-c}\int_c^d \ln \mathcal{F}(x,y)dy\supseteq \ln\left[\mathcal{F}(x,c)\mathcal{F}(x,d)\right]\int_0^1 h(\vartheta)d\vartheta.$$

Integrating over $[a,b]$, we have

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)(b-a)}\int_a^b \ln \mathcal{F}\left(x,\frac{c+d}{2}\right)dx\\
&\supseteq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \ln \mathcal{F}(x,y)dxdy\\
&\supseteq \left[\frac{1}{b-a}\int_a^b \ln \mathcal{F}(x,c)dx+\frac{1}{b-a}\int_a^b \ln \mathcal{F}(x,d)dx\right]\int_0^1 h(\vartheta)d\vartheta.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)(d-c)}\int_c^d \ln\mathcal{F}\left(\frac{a+b}{2},y\right)dy\\
&\supseteq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \ln\mathcal{F}(x,y)dxdy\\
&\supseteq \left[\frac{1}{d-c}\int_c^d \ln\mathcal{F}(a,y)dy+\frac{1}{d-c}\int_c^d \ln\mathcal{F}(b,y)dy\right]\int_0^1 h(\vartheta)d\vartheta.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\frac{1}{4h^2\left(\frac{1}{2}\right)}\ln\mathcal{F}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&=\frac{1}{4h\left(\frac{1}{2}\right)}\left[\frac{1}{2h\left(\frac{1}{2}\right)(b-a)}\int_a^b \ln\mathcal{F}\left(x,\frac{c+d}{2}\right)dx+\frac{1}{2h\left(\frac{1}{2}\right)(d-c)}\int_c^d \ln\mathcal{F}\left(\frac{a+b}{2},y\right)dy\right]\\
&\supseteq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \ln\mathcal{F}(x,y)dxdy\\
&\supseteq \frac{1}{2}\int_0^1 h(\vartheta)d\vartheta\left[\frac{1}{b-a}\int_a^b \ln\mathcal{F}(x,c)dx+\frac{1}{b-a}\int_a^b \ln\mathcal{F}(x,d)dx\right.\\
&\quad\left.+\frac{1}{d-c}\int_c^d \ln\mathcal{F}(a,y)dy+\frac{1}{d-c}\int_c^d \ln\mathcal{F}(b,y)dy\right]\\
&\supseteq \frac{1}{2}\left(\int_0^1 h(\vartheta)d\vartheta\right)^2[\ln\mathcal{F}(a,c)+\ln\mathcal{F}(a,d)+\ln\mathcal{F}(b,c)+\ln\mathcal{F}(b,d)\\
&\quad+\ln\mathcal{F}(a,c)+\ln\mathcal{F}(a,d)+\ln\mathcal{F}(b,c)+\ln\mathcal{F}(b,d)]\\
&\supseteq \left(\int_0^1 h(\vartheta)d\vartheta\right)^2\left[\ln\mathcal{F}(a,c)\mathcal{F}(a,d)\mathcal{F}(b,c)\mathcal{F}(b,d)\right].
\end{aligned}$$

This concludes the proof. □

Remark 3. If $\underline{\mathcal{F}}=\overline{\mathcal{F}}$ and $h(\vartheta)=\vartheta$, then Theorem 9 reduces to Corollary 3.1 of [13].

Example 1. Let $[a,b]=[c,d]=[2,3], h(\vartheta)=\vartheta$. We define $\mathcal{F}:[2,3]\times[2,3]\to\mathbb{R}_{\mathcal{I}}^{+}$ by

$$\mathcal{F}(x,y)=\left[\frac{1}{xy},e^{\sqrt{x}+\sqrt{y}}\right].$$

From Definition 2, $\mathcal{F}(x,y)\in SX(\log\text{-}ch,\triangle,\mathbb{R}_{\mathcal{I}}^{+})$.

Since

$$\left[\mathcal{F}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]^{\frac{1}{4h^2\left(\frac{1}{2}\right)}}=\left[\frac{4}{25},e^{\sqrt{10}}\right],$$

$$\exp\left[\frac{1}{4h\left(\frac{1}{2}\right)}\left(\frac{1}{2h\left(\frac{1}{2}\right)(b-a)}\int_a^b\ln\mathcal{F}\left(x,\frac{c+d}{2}\right)dx\right.\right.$$
$$\left.\left.+\frac{1}{2h\left(\frac{1}{2}\right)(d-c)}\int_c^d\ln\mathcal{F}\left(\frac{a+b}{2},y\right)dy\right)\right]=\left[\frac{8e}{135},e^{\frac{\sqrt{10}}{2}+2\sqrt{3}-\frac{4\sqrt{2}}{3}}\right],$$

$$\exp\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\ln\mathcal{F}(x,y)dxdy\right]=\left[\frac{16e^2}{729},e^{\frac{4}{3}(3\sqrt{3}-2\sqrt{2})}\right],$$

$$\exp\left[\frac{1}{2}\int_0^1h(\vartheta)d\vartheta\left(\frac{1}{b-a}\int_a^b\ln\mathcal{F}(x,c)dx+\frac{1}{b-a}\int_a^b\ln\mathcal{F}(x,d)dx\right.\right.$$
$$\left.\left.+\frac{1}{d-c}\int_c^d\ln\mathcal{F}(a,y)dy+\frac{1}{d-c}\int_c^d\ln\mathcal{F}(b,y)dy\right)\right]=\left[\frac{2\sqrt{6}e}{81},e^{\frac{15\sqrt{3}-5\sqrt{2}}{6}}\right],$$

and

$$[\mathcal{F}(a,c)\mathcal{F}(a,d)\mathcal{F}(b,c)\mathcal{F}(b,d)]^{\left(\int_0^1h(\vartheta)d\vartheta\right)^2}=\left[\frac{1}{6},e^{\sqrt{2}+\sqrt{3}}\right].$$

It follows that

$$\left[\frac{4}{25},e^{\sqrt{10}}\right]\supseteq\left[\frac{8e}{135},e^{\frac{\sqrt{10}}{2}+2\sqrt{3}-\frac{4\sqrt{2}}{3}}\right]\supseteq\left[\frac{16e^2}{729},e^{\frac{4}{3}(3\sqrt{3}-2\sqrt{2})}\right]\supseteq\left[\frac{2\sqrt{6}e}{81},e^{\frac{15\sqrt{3}-5\sqrt{2}}{6}}\right]\supseteq\left[\frac{1}{6},e^{\sqrt{2}+\sqrt{3}}\right]$$

and Theorem 9 is verified.

Theorem 10. *Let $\mathcal{F}:\triangle\to\mathbb{R}_{\mathcal{I}}^+$ and $h:[0,1]\to\mathbb{R}^+$ be continuous. If $\mathcal{F}\in SX(\log\text{-}ch,\triangle,\mathbb{R}_{\mathcal{I}}^+)$, then*

$$\begin{aligned}&\left[\mathcal{F}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]^{\frac{1}{4h^3\left(\frac{1}{2}\right)}}\\&\supseteq\exp\left[\frac{1}{4h^2\left(\frac{1}{2}\right)(b-a)}\int_a^b\ln\left(\mathcal{F}\left(x,\frac{c+d}{2}\right)\right)dx\right.\\&\qquad\left.+\frac{1}{4h^2\left(\frac{1}{2}\right)(d-c)}\int_c^d\ln\left(\mathcal{F}\left(\frac{a+b}{2},y\right)\right)dy\right]\\&\supseteq\exp\left[\frac{1}{4h\left(\frac{1}{2}\right)(b-a)}\int_a^b\ln\left(\mathcal{F}\left(x,\frac{3c+d}{4}\right)\mathcal{F}\left(x,\frac{c+3d}{4}\right)\right)dx\right.\\&\qquad\left.+\frac{1}{4h\left(\frac{1}{2}\right)(d-c)}\int_c^d\ln\left(\mathcal{F}\left(\frac{3a+b}{4},y\right)\mathcal{F}\left(\frac{a+3b}{4},y\right)\right)dy\right]\\&\supseteq\exp\left[\frac{2}{(b-a)(d-c)}\int_a^b\int_c^d\ln\mathcal{F}(x,y)dxdy\right]\end{aligned}\tag{3.5}$$

$$\supseteq \exp\left[\frac{1}{2(b-a)}\int_a^b \ln\Big(\mathcal{F}(x,c)\mathcal{F}(x,d)\mathcal{F}^2\,(x,fracc+d2)\Big)\,dx\int_0^1 h(\vartheta)d\vartheta \right.$$
$$\left. + \frac{1}{2(d-c)}\int_c^d \ln\left(\mathcal{F}(a,y)\mathcal{F}(b,y)\mathcal{F}^2\left(\frac{a+b}{2},y\right)\right)dy\int_0^1 h(\vartheta)d\vartheta\right]$$
$$\supseteq \exp\left[\left(\frac{1}{2}+h\left(\frac{1}{2}\right)\right)\frac{1}{b-a}\int_a^b \ln\left[\mathcal{F}(x,c)\mathcal{F}(x,d)\right]dx\int_0^1 h(\vartheta)d\vartheta\right.$$
$$\left.+\left(\frac{1}{2}+h\left(\frac{1}{2}\right)\right)\frac{1}{d-c}\int_c^d \ln\left[\mathcal{F}(a,y)\mathcal{F}(b,y)\right]dy\int_0^1 h(\vartheta)d\vartheta\right]$$
$$\supseteq \left[\mathcal{F}(a,c)\mathcal{F}(a,d)\mathcal{F}(b,c)\mathcal{F}(b,d)\mathcal{F}\left(\frac{a+b}{2},c\right)\mathcal{F}\left(\frac{a+b}{2},d\right)\right.$$
$$\left.\times\mathcal{F}\left(a,\frac{c+d}{2}\right)\mathcal{F}\left(b,\frac{c+d}{2}\right)\right]^{\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\left(\int_0^1 h(\vartheta)d\vartheta\right)^2}$$
$$\supseteq \left[\mathcal{F}(a,c)\mathcal{F}(a,d)\mathcal{F}(b,c)\mathcal{F}(b,d)\right]^{2\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]^2\left(\int_0^1 h(\vartheta)d\vartheta\right)^2}.$$

Proof. Since $\mathcal{F}\in SX(\log\text{-}ch,\triangle,\mathbb{R}_{\mathcal{I}}^{+})$, by using Theorem 6 and (2.3), we have

$$\frac{1}{4h^2\left(\frac{1}{2}\right)}\ln\left[\mathcal{F}_y\left(\frac{a+b}{2}\right)\right]$$
$$\supseteq \frac{1}{4h\left(\frac{1}{2}\right)}\ln\left[\mathcal{F}_y\left(\frac{3a+b}{4}\right)\mathcal{F}_y\left(\frac{a+3b}{4}\right)\right]$$
$$\supseteq \frac{1}{b-a}\int_a^b \ln\mathcal{F}_y(x)dx$$
$$\supseteq \frac{1}{2}\ln\left[\mathcal{F}_y(a)\mathcal{F}_y(b)\mathcal{F}_y^2\left(\frac{a+b}{2}\right)\right]\int_0^1 h(\vartheta)d\vartheta$$
$$\supseteq \left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\ln\left[\mathcal{F}_y(a)\mathcal{F}_y(b)\right]\int_0^1 h(\vartheta)d\vartheta.$$

That is,

$$\frac{1}{4h^2\left(\frac{1}{2}\right)}\ln\left[\mathcal{F}\left(\frac{a+b}{2},y\right)\right]$$
$$\supseteq \frac{1}{4h\left(\frac{1}{2}\right)}\ln\left[\mathcal{F}\left(\frac{3a+b}{4},y\right)\mathcal{F}\left(\frac{a+3b}{4},y\right)\right]$$
$$\supseteq \frac{1}{b-a}\int_a^b \ln\mathcal{F}(x,y)dx$$
$$\supseteq \frac{1}{2}\ln\left[\mathcal{F}(a,y)\mathcal{F}(b,y)\mathcal{F}^2\left(\frac{a+b}{2},y\right)\right]\int_0^1 h(\vartheta)d\vartheta$$
$$\supseteq \left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\ln\left[\mathcal{F}(a,y)\mathcal{F}(b,y)\right]\int_0^1 h(\vartheta)d\vartheta.$$

Moreover, we have

$$
\begin{aligned}
&\frac{1}{4h^2\left(\frac{1}{2}\right)(d-c)}\int_c^d \ln\left[\mathcal{F}\left(\frac{a+b}{2},y\right)\right]dy \\
&\supseteq \frac{1}{4h\left(\frac{1}{2}\right)(d-c)}\int_c^d \ln\left[\mathcal{F}\left(\frac{3a+b}{4},y\right)\mathcal{F}\left(\frac{a+3b}{4},y\right)\right]dy \\
&\supseteq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \ln\mathcal{F}(x,y)dxdy \\
&\supseteq \frac{1}{2(d-c)}\int_c^d \ln\left[\mathcal{F}(a,y)\mathcal{F}(b,y)\mathcal{F}^2\left(\frac{a+b}{2},y\right)\right]dy\int_0^1 h(\vartheta)d\vartheta \\
&\supseteq \left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\frac{1}{d-c}\int_c^d \ln\left[\mathcal{F}(a,y)\mathcal{F}(b,y)\right]dy\int_0^1 h(\vartheta)d\vartheta.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\frac{1}{4h^2\left(\frac{1}{2}\right)(b-a)}\int_a^b \ln\left[\mathcal{F}\left(x,\frac{c+d}{2}\right)\right]dx \\
&\supseteq \frac{1}{4h\left(\frac{1}{2}\right)(b-a)}\int_a^b \ln\left[\mathcal{F}\left(x,\frac{3+d}{4}\right)\mathcal{F}\left(x,\frac{c+3d}{4}\right)\right]dx \\
&\supseteq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \ln\mathcal{F}(x,y)dxdy \\
&\supseteq \frac{1}{2(b-a)}\int_a^b \ln\left[\mathcal{F}(x,c)\mathcal{F}(x,d)\mathcal{F}^2\left(x,\frac{c+d}{2}\right)\right]dx\int_0^1 h(\vartheta)d\vartheta \\
&\supseteq \left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\frac{1}{b-a}\int_a^b \ln\left[\mathcal{F}(x,c)\mathcal{F}(x,d)\right]dx\int_0^1 h(\vartheta)d\vartheta.
\end{aligned}
$$

We also from (2.2),

$$
\begin{aligned}
\frac{1}{2h\left(\frac{1}{2}\right)}\ln\mathcal{F}\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\supseteq \frac{1}{b-a}\int_a^b \ln\mathcal{F}\left(x,\frac{c+d}{2}\right)dx, \\
\frac{1}{2h\left(\frac{1}{2}\right)}\ln\mathcal{F}\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\supseteq \frac{1}{d-c}\int_c^d \ln\mathcal{F}\left(\frac{a+b}{2},y\right)dy.
\end{aligned}
$$

Again from (2.3),

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b \ln\mathcal{F}(x,c)\,dx &\supseteq \tfrac{1}{2}\ln\left[\mathcal{F}(a,c)\mathcal{F}(b,c)\mathcal{F}^2\left(\tfrac{a+b}{2},c\right)\right]\textstyle\int_0^1 h(\vartheta)d\vartheta \\
&\supseteq \left[\tfrac{1}{2}+h\left(\tfrac{1}{2}\right)\right]\ln\left[\mathcal{F}(a,c)\mathcal{F}(b,c)\right]\textstyle\int_0^1 h(\vartheta)d\vartheta, \\
\frac{1}{b-a}\int_a^b \ln\mathcal{F}(x,d)\,ds &\supseteq \tfrac{1}{2}\ln\left[\mathcal{F}(a,d)\mathcal{F}(b,d)\mathcal{F}^2\left(\tfrac{a+b}{2},d\right)\right]\textstyle\int_0^1 h(\vartheta)d\vartheta \\
&\supseteq \left[\tfrac{1}{2}+h\left(\tfrac{1}{2}\right)\right]\ln\left[\mathcal{F}(a,d)\mathcal{F}(b,d)\right]\textstyle\int_0^1 h(\vartheta)d\vartheta,
\end{aligned}
$$

$$\begin{aligned}\frac{1}{d-c}\int_c^d \ln \mathcal{F}(a,y)\,dy &\supseteq \tfrac{1}{2}\ln\left[\mathcal{F}(a,c)\mathcal{F}(a,d)\mathcal{F}^2\left(a,\tfrac{c+d}{2}\right)\right]\int_0^1 h(\vartheta)d\vartheta\\ &\supseteq \left[\tfrac{1}{2}+h\left(\tfrac{1}{2}\right)\right]\ln\left[\mathcal{F}(a,c)\mathcal{F}(a,d)\right]\int_0^1 h(\vartheta)d\vartheta,\\ \frac{1}{d-c}\int_c^d \ln \mathcal{F}(b,y)\,dy &\supseteq \tfrac{1}{2}\ln\left[\mathcal{F}(b,c)\mathcal{F}(b,d)\mathcal{F}^2\left(b,\tfrac{c+d}{2}\right)\right]\int_0^1 h(\vartheta)d\vartheta\\ &\supseteq \left[\tfrac{1}{2}+h\left(\tfrac{1}{2}\right)\right]\ln\left[\mathcal{F}(b,c)\mathcal{F}(b,d)\right]\int_0^1 h(\vartheta)d\vartheta\end{aligned}$$

and proof is completed. □

Example 2. Furthermore, by Example 1, we have

$$\left[\mathcal{F}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]^{\frac{1}{4h^3\left(\frac{1}{2}\right)}} = \left[\frac{16}{625}, e^{2\sqrt{10}}\right],$$

$$\exp\left[\frac{1}{4h^2\left(\frac{1}{2}\right)(b-a)}\int_a^b \ln\left(\mathcal{F}\left(x,\frac{c+d}{2}\right)\right)dx + \frac{1}{4h^2\left(\frac{1}{2}\right)(d-c)}\int_c^d \ln\left(\mathcal{F}\left(\frac{a+b}{2},y\right)\right)dy\right] = \left[\frac{64e^2}{18225}, e^{\frac{4(3\sqrt{3}-2\sqrt{2})}{3}+\sqrt{10}}\right],$$

$$\exp\left[\frac{1}{4h\left(\frac{1}{2}\right)(b-a)}\int_a^b \ln\left(\mathcal{F}\left(x,\frac{3+d}{4}\right)\mathcal{F}\left(x,\frac{c+3d}{4}\right)\right)dx + \frac{1}{4h\left(\frac{1}{2}\right)(d-c)}\int_c^d \ln\left(\mathcal{F}\left(\frac{3a+b}{4},y\right)\mathcal{F}\left(\frac{a+3b}{4},y\right)\right)dy\right] = \left[\frac{256e^2}{72171}, e^{\frac{4(3\sqrt{3}-2\sqrt{2})}{3}+\frac{3+\sqrt{11}}{2}}\right],$$

$$\exp\left[\frac{2}{(b-a)(d-c)}\int_a^b\int_c^d \ln \mathcal{F}(x,y)dxdy\right] = \left[\frac{256e^4}{531441}, e^{\frac{8(3\sqrt{3}-2\sqrt{2})}{3}}\right],$$

$$\exp\left[\frac{1}{2(b-a)}\int_a^b \ln\left(\mathcal{F}(x,c)\mathcal{F}(x,d)\mathcal{F}^2\left(x,\frac{c+d}{2}\right)\right)dx\int_0^1 h(\vartheta)d\vartheta + \frac{1}{2(d-c)}\int_c^d \ln\left(\mathcal{F}(a,y)\mathcal{F}(b,y)\mathcal{F}^2\left(\frac{a+b}{2},y\right)\right)dy\int_0^1 h(\vartheta)d\vartheta\right] = \left[\frac{16\sqrt{6}e^2}{10935}, e^{\frac{12\sqrt{3}-8\sqrt{2}+3\sqrt{10}}{6}}\right],$$

$$\exp\left[\left(\frac{1}{2}+h\left(\frac{1}{2}\right)\right)\frac{1}{b-a}\int_a^b \ln\left[\mathcal{F}(x,c)\mathcal{F}(x,d)\right]dx\int_0^1 h(\vartheta)d\vartheta + \left(\frac{1}{2}+h\left(\frac{1}{2}\right)\right)\frac{1}{d-c}\int_c^d \ln\left[\mathcal{F}(a,y)\mathcal{F}(b,y)\right]dy\int_0^1 h(\vartheta)d\vartheta\right] = \left[\frac{8e^2}{2187}, e^{\frac{15\sqrt{3}-5\sqrt{2}}{3}}\right],$$

$$\left[\mathcal{F}(a,c)\mathcal{F}(a,d)\mathcal{F}(b,c)\mathcal{F}(b,d)\mathcal{F}\left(\frac{a+b}{2},c\right)\mathcal{F}\left(\frac{a+b}{2},d\right)\times\mathcal{F}\left(a,\frac{c+d}{2}\right)\mathcal{F}\left(b,\frac{c+d}{2}\right)\right]^{\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\left(\int_0^1 h(\vartheta)d\vartheta\right)^2} = \left[\frac{\sqrt{6}}{90}, e^{\frac{3\sqrt{3}+3\sqrt{2}+\sqrt{10}}{2}}\right],$$

and

$$[\mathcal{F}(a,c)\mathcal{F}(a,d)\mathcal{F}(b,c)\mathcal{F}(b,d)]^{2\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]^2\left(\int_0^1 h(\theta)d\theta\right)^2} = \left[\frac{1}{36}, e^{2\sqrt{3}+2\sqrt{2}}\right].$$

It follows that

$$\begin{aligned}
&\left[\frac{16}{625}, e^{2\sqrt{10}}\right] \supseteq \left[\frac{64e^2}{18225}, e^{\frac{4(3\sqrt{3}-2\sqrt{2})+3\sqrt{10}}{3}}\right] \\
&\supseteq \left[\frac{256e^2}{72171}, e^{\frac{4(3\sqrt{3}-2\sqrt{2})}{3}+\frac{3+\sqrt{11}}{2}}\right] \supseteq \left[\frac{256e^4}{531441}, e^{\frac{8(3\sqrt{3}-2\sqrt{2})}{3}}\right] \\
&\supseteq \left[\frac{16\sqrt{6}e^2}{10935}, e^{\frac{12\sqrt{3}-8\sqrt{2}+3\sqrt{10}}{6}}\right] \supseteq \left[\frac{8e^2}{2187}, e^{\frac{15\sqrt{3}-5\sqrt{2}}{3}}\right] \\
&\supseteq \left[\frac{\sqrt{6}}{90}, e^{\frac{3\sqrt{3}+3\sqrt{2}+\sqrt{10}}{2}}\right] \supseteq \left[\frac{1}{36}, e^{2\sqrt{3}+2\sqrt{2}}\right]
\end{aligned}$$

and Theorem 10 is verified.

4. Conclusions

We introduced the coordinated log-h-convexity for interval-valued functions, some Jensen type inequalities and Hermite-Hadamard type inequalities are proved. Our results generalize some known inequalities and will be useful in developing the theory of interval integral inequalities and interval convex analysis. The next step in the research direction investigated inequalities for fuzzy-interval-valued functions, and some applications in interval nonlinear programming.

Acknowledgments

The first author was supported in part by the Key Projects of Educational Commission of Hubei Province of China (D20192501), the Natural Science Foundation of Jiangsu Province (BK20180500) and the National Key Research and Development Program of China (2018YFC1508100).

Conflict of interest

The authors declare no conflict of interest.

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